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MALLIAVIN CALCULUS AND MARTINGALE EXPANSION *

BY

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ABSTRACT. – We proved the validity of the asymptotic expansion for the distribution of a martingale with jumps. A sufficient condition is presented in terms of the decay of certain integrations of Fourier type. In order to estimate such Fourier type integrals, we use the Malliavin calculus and show that it becomes a key to our program. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Asymptotic expansion is now admitted to be fundaments of the higher-order asymptotic statistical theory beyond the first-order theory; cf. Akahira and Takeuchi [1], Bhattacharya and Ghosh [2], Ghosh [5], Pfanzagl [14,15], Taniguchi [20]. This is also the case when we consider statistical problems for semimartingales. In spite of the importance, the asymptotic expansion of martingales is rather a new topic discussed in the latest mathematical statistics. Mykland [11] is a prominent work which presented, for continuous martingales $(M_{n,t}\colon\ 0 \leqslant t \leqslant T_n)$, an asymptotic expansion of the expectation $E[g(M_{n,T_n})]$ for a class of C^2-functions g. Consecutively, in [12,13], he presented expansions for martingales with discrete time parameter and for martingales with jumps, in a class of C^2-functions g.

In certain problems, for instance, problems of confidence intervals, statisticians need asymptotic expansion without the C^2-regularity condition.

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Generally speaking, it is possible to remove such a regularity condition if we may assume a condition which ensures the regularity of the distribution of M_{n,T_n}. It is well known that the so-called Cramér condition serves as such a condition for independent observations.

In this article, we will consider asymptotic expansion for martingales with jumps. In order to validate the expansion for martingales it will be necessary to examine the order of the decay of certain Fourier integrals. For this purpose, we will take advantage of the Malliavin calculus. In fact, by means of the associated integration-by-parts setting, it is possible to prove the validity of the expansion of $E[g(M_{n,T_n})]$ for any measurable function g. It seems to be the only handy method which universally applies to functionals appearing in statistics for stochastic processes with continuous-time parameter. In the previous paper [26], continuous martingales were treated, however the same methodology can be applied to general martingales with jumps. The aim of this article is to clarify the idea and to carry out this program for martingales admitting jumps.

For functionals of ε-Markov processes with mixing property, the *local approach* provides an efficient way to the asymptotic expansion (cf. Götze and Hipp [6,7], Kusuoka and Yoshida [10], Yoshida [28] and Sakamoto and Yoshida [18,19]). However, the present martingale approach (*global approach*) still has advantages of being widely applied: a class of diffusion processes and estimation of a diffusion coefficient ([26]), the M-estimator over Wiener space (Sakamoto and Yoshida [17]), and the second order expansions in the small σ-theory ([22–25], Dermoune and Kutoyants [4]). Besides, in order to illustrate another merit, we will in Section 3 give a simple example of a long-range-dependent-energy martingale to which the geometric-mixing approach cannot apply but our martingale method is still applicable.

The organization of the present article is as follows. First, we will summarize the Malliavin calculus for jump processes in Section 2. The main result will be stated in Section 3 and the proof of it will be presented in Section 4. Since it is not yet clear whether the embedding technique for martingales preserves the smoothness of functionals in the sense of the Malliavin calculus, we do not use the embedding technique as Mykland did (of course, in his case, such smoothness is not necessary because of the smoothness of functions g). The method here is based on the Fourier analysis rather classical.

2. Malliavin calculus for jump processes

In this section, we give a review of the Malliavin calculus for jump processes formulated by Bichteler, Gravereaux and Jacod [3] among many other possible formulations. Any formulation is possible to use if a sufficiently higher order integration-by-parts formula in differential form or in difference form is available in it.

Let $(\Theta, \mathcal{B}, \Pi)$ be a probability space. The following formulation of the Malliavin calculus was adopted by Bichteler et al. [3].

DEFINITION. – *A linear operator $\mathcal{L}$ on $\mathcal{D}(\mathcal{L}) \in \bigcap_{p>1} L^p(\Pi)$ into $\bigcap_{p>1} L^p(\Pi)$ is called a* Malliavin operator *if*:

(1) *$\mathcal{B}$ is generated by $\mathcal{D}(\mathcal{L})$.*
(2) *For $f \in C^2_\uparrow(\mathbf{R}^n)$, $n \in \mathbf{N}$, and $F \in \mathcal{D}(\mathcal{L})^n$, $f \circ F \in \mathcal{D}(\mathcal{L})$.*
(3) *For any $F, G \in \mathcal{D}(\mathcal{L})$, $E^\Pi[F\mathcal{L}G] = E^\Pi[G\mathcal{L}F]$.*
(4) *For $F \in \mathcal{D}(\mathcal{L})$, $\mathcal{L}(F^2) \geqslant 2F\mathcal{L}F$. Namely, the bilinear operator Γ on $\mathcal{D}(\mathcal{L}) \times \mathcal{D}(\mathcal{L})$ associated with $\mathcal{L}$ by $\Gamma(F,G) = \mathcal{L}(FG) - F\mathcal{L}G - G\mathcal{L}F$ is nonnegative definite.*
(5) *For $F = (F^1, \ldots, F^n) \in \mathcal{D}(\mathcal{L})^n$, $n \in \mathbf{N}$, and $f \in C^2_\uparrow(\mathbf{R}^n)$,*

$$\mathcal{L}(f \circ F) = \sum_{i=1}^n \partial_i f \circ F\mathcal{L}F^i + \frac{1}{2}\sum_{i,j=1}^n \partial_i\partial_j f \circ F\Gamma\left(F^i, F^j\right).$$

Let $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ be a Malliavin operator. We define $\|F\|_{D^2_p}$ by

$$\|F\|_{D^2_p} = \|F\|_p + \|\mathcal{L}F\|_p + \left\|\Gamma^{1/2}(F,F)\right\|_p$$

for $p \geqslant 2$. Denote by D^2_p the completion of $\mathcal{D}(\mathcal{L})$ with respect to $\|\cdot\|_{D^2_p}$. Then spaces $(D^2_p, \|\cdot\|_{D^2_p})$ become Banach spaces with natural inclusions between them. Let

$$D^2_{\infty-} = \bigcap_{p\geqslant 2} D^2_p.$$

There exists an integration-by-parts setting (IBPS). See Theorem 8-18 of [3], p. 107, for details. We shall rewrite the integration-by-parts formula for a partially nondegenerate situation. Let $\Delta = \det \sigma_F$, $\sigma_F = (\sigma_F^{i,j})_{i,j=1}^d$, where $\sigma_F^{i,j} = \Gamma(F^i, F^j)$ and $\sigma_{[i,i']}$ is the (i,i')-cofactor of σ_F. Assume that $F \in D^2_{\infty-}(\mathbf{R}^d)$. Also we suppose that a trunction functional ψ satisfies the condition that $\Delta = 0 \Rightarrow \Delta^{-1}\psi = 0$ a.s., hence $\Delta \cdot \Delta^{-1}\psi = \psi$ a.s. If $\sigma_F^{i,j} \in D^2_{\infty-}$ and $\Delta^{-1}\psi \in D^2_{\infty-}$, then the integration-by-parts formula holds, i.e., for $f \in C^2_{\uparrow}(\mathbf{R}^d)$,

$$E^{\Pi}\left[\partial_i f(F)\psi\right] = E^{\Pi}\left[f(F)\mathcal{J}_i^F \psi\right], \tag{1}$$

where the operator

$$\mathcal{J}_i^F : \{\psi\colon \Theta \to \overline{\mathbf{R}} \text{ such that } \Delta^{-1}\psi \in D^2_{\infty-}\} \to \bigcap_{p>1} L_p(\Pi)$$

is defined by

$$\mathcal{J}_i^F \psi = -\sum_{i'=1}^{d} \left\{2\Delta^{-1}\psi\sigma_{[i,i']}\mathcal{L}F^{i'} + \Gamma\left(\Delta^{-1}\psi\sigma_{[i,i']}, F^{i'}\right)\right\}.$$

For $k \in \mathbf{N}$, define $S'_k[F]$ and $S''_k[\psi]$ as follows:

$S'_1[F] := \{\sigma_F^{i,j}\colon i, j = 1, \ldots, d\}$ if $F \in D^2_{\infty-}(\mathbf{R}^d)$;

$S'_k[F] := \{\sigma_F^{i,j}, \mathcal{L}F^i, \Gamma(S'_{k-1}[F], F^i)\colon i, j = 1, \ldots, d\}$ if $F \in D^2_{\infty-}(\mathbf{R}^d)$ and $S'_{k-1}[F] \subset D^2_{\infty-}$;

$S''_1[\psi; F] := \{\Delta^{-1}\psi\}$ if $\Delta = 0$ implies $\Delta^{-1}\psi = 0$;

$S''_k[\psi; F] := \{\Delta^{-1}S''_{k-1}[\psi; F], \Delta^{-1}\Gamma(S''_{k-1}[\psi; F], F^i)\colon\ i = 1, \ldots, d\}$ if $S''_{k-1}[\psi; F] \subset D^2_{\infty-}$ and if $\Delta = 0$ implies $\Delta^{-1}S''_{k-1}[\psi; F] \cup \Delta^{-1}\Gamma(S''_{k-1}[\psi; F], F) = \{0\}$.

Moreover, we prepare the following notation:

$\Delta := \Delta_F$.

$S_1[\psi; F] := S'_1[F] \cup S''_1[\psi; F]$ if $F \in D^2_{\infty-}(\mathbf{R}^d)$;

$S_k[\psi; F] := S_{k-1}[\psi; F] \cup S'_k[F] \cup S''_k[\psi; F]$ if $F \in D^2_{\infty-}(\mathbf{R}^d)$, $S'_{k-1}[F] \subset D^2_{\infty-}$ and $S''_{k-1}[\psi; F] \subset D^2_{\infty-}$, and if $\Delta = 0$ implies $\Delta^{-1}S''_{k-1}[\psi; F] \cup \Delta^{-1}\Gamma(S''_{k-1}[\psi; F], F) = \{0\}$. Here we denoted $\Gamma(A, B) = \{\Gamma(a, b)\colon a \in A, b \in B\}$ for function sets A and B.

PROPOSITION 1. – *Suppose that $F \in D^2_{\infty-}$. If $S_k[\psi; F] \subset D^2_{\infty-}$, then for $f \in C^{k+1}_{\uparrow}(\mathbf{R}^d)$,*

$$E^{\Pi}\left[\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k} f(F)\psi\right] = E^{\Pi}\left[f(F)\mathcal{J}^F_{i_k}\cdots\mathcal{J}^F_{i_2}\mathcal{J}^F_{i_1}\psi\right].$$

Proof. – From assumption that $S_k[\psi; F] \subset D^2_{\infty-}$, it follows that $\Delta^{-1}\mathcal{J}^F_{i_j}\cdots\mathcal{J}^F_{i_1}(\psi)$ $(1 \leqslant j \leqslant k-1)$ are well defined, and that $\Delta^{-1}\mathcal{J}^F_{i_j}\cdots\mathcal{J}^F_{i_1}(\psi) \in L\{S''_{j+1}[\psi; F]\} \otimes Alg_{\mathbf{R}}(S'_{j+1}[F])$. The assertion of the proposition can be proved by this fact and by induction with the aid of the integration-by-parts formula (1). □

3. Main result

We consider a sequence of d-dimensional random vectors $\{X_n\}$ with stochastic expansion:

$$X_n = M_{n,T_n} + r_n N_n,$$

where M_{n,T_n} and N_n are d-dimensional random vectors, both of which are defined on a probability space $(\Omega^n, \mathcal{F}^n, P^n)$, and $M_n = (M_n)_{t\in[0,T_n]}$ is a d-dimensional local martingale with respect to a filtration $(\mathcal{F}^n_t)_{t\in[0,T_n]}$ over $(\Omega^n, \mathcal{F}^n, P^n)$. We assume that $M_{n,0} = 0$ and denote by M_n^c the continuous part of M_n and by M_n^d the purely discontinuous part of M_n. $\{r_n\}$ is a sequence of positive numbers satisfying $\lim_{n\to\infty} r_n = 0$. Here we use the parameter $n \in \mathbf{N}$, however it is clearly possible to consider any directed set in place of $\mathbf{N}$ for our results.

The optional quadratic covariation process of the local martingale M_n is denoted by $[M_n] = [M_n, M_n] = ([M_n^i, M_n^j])_{i,j=1}^d$, and the predictable covariation process is denoted by $\langle M_n\rangle = \langle M_n, M_n\rangle = (\langle M_n^i, M_n^j\rangle)_{i,j=1}^d$. Moreover we put $\Xi_n = \alpha^*[M_n] + \beta^*\langle M_n\rangle$ and $\widetilde{\Xi}_n = \Xi_n - I$, where I is the identity matrix. Though we will take $\alpha^* = 1/3$ and $\beta^* = 2/3$ later as Mykland found it, we will leave them for a while to see why those numbers should be chosen in this way. Put $\xi_n = r_n^{-1}\widetilde{\Xi}_{n,T_n}$.

For semimartingale X, μ^X denotes the integer-valued random measure on $\mathbf{R}_+ \times \mathbf{R}^d$, and it is defined by

$$\mu^X(\omega; \mathrm{d}t, \mathrm{d}x) = \sum_{s>0} 1_{\{\Delta X_s \neq 0\}} \delta_{(s,\Delta X_s)}(\mathrm{d}t, \mathrm{d}x).$$

It is well known that there exists a unique random measure $\nu = \nu^X$ which compensates μ^X. The random measure ν_n denotes the compensator corresponding to $\mu_n = \mu^{M_n}$.

In this section, we assume that each $(\Omega^n, \mathcal{F}^n, P^n)$ has a Malliavin operator $\mathcal{L}^n$ with corresponding Γ^n and $D^{n,2}_{\infty-}$. Let $\kappa_n = r_n^{-2}|x|^4 * \mu_{n,T_n}$ and let $\lambda_n = r_n^{-2}|x|^4 * \nu_{n,T_n}$. Put $\mathcal{S}^n_{-1} = \{M_{n,T_n}, N_n\}$. Given a nonnegative random variable s_n on Ω^n, $\mathcal{S}^n_0$ is defined only when $\mathcal{S}^n_{-1} \subset D^{n,2}_{\infty-}$ by

$$\mathcal{S}^n_0 = \{s_n, \sigma_{M_{n,T_n}}, \Gamma^n(M_{n,T_n}, N_n), \sigma_{N_n}, M_{n,T_n}, \xi_n, N_n, \kappa_n, \lambda_n\}.$$

The set $\mathcal{S}^n_1$ is defined only when $\mathcal{S}^n_0 \subset D^{n,2}_{\infty-}$ by $\mathcal{S}^n_1 = \{\mathcal{L}^n X_n\} \cup \mathcal{S}^n_0 \cup \Gamma^n(\mathcal{S}^n_0, X_n)$. For $k \in \mathbf{N}$, $k \geqslant 2$, $\mathcal{S}^n_k$ is defined only when $\mathcal{S}^n_{k-1} \subset D^{n,2}_{\infty-}$ by $\mathcal{S}^n_k = \mathcal{S}^n_{k-1} \cup \Gamma^n(\mathcal{S}^n_{k-1}, X_n)$.

We will assume that

[A1] For a random vector (Z, ξ, η) taking values in $\mathbf{R}^d \times (\mathbf{R}^d \otimes \mathbf{R}^d) \times \mathbf{R}^d$,

$$(M_{n,T_n}, \xi_n, N_n) \to^d (Z, \xi, \eta) \quad \text{as } n \to \infty.$$

[A2] The set $\mathcal{S}_l^n$ is bounded with respect to $\|\cdot\|_{L_p(P^n)}$ uniformly in $n \in \mathbf{N}$ for any $p > 1$.

We denote by ϕ the d-dimensional standard normal density. Define $A : \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^d$ and $B : \mathbf{R}^d \to \mathbf{R}^d$ by $A(x) = (A^{jk}(x))_{j,k=1}^d = P[\xi | Z = x]$ and $B(x) = (B^j(x))_{j=1}^d = P[\eta | Z = x]$; moreover, let

$$p_n(x) = \phi(x) + \frac{1}{2} r_n \sum_{j,k=1}^d \partial_j \partial_k \big(A^{jk}(x)\phi(x)\big) - r_n \sum_{j=1}^d \partial_j \big(B^j(x)\phi(x)\big).$$

Remark 1. – Under the assumptions, it is easy to show the existence of the derivatives of A^{jk} and B^j, and the integrability of the each term on the right-hand side of the above definition of p_n.

Let $\mathcal{E}(M, \gamma)$ be the set of measurable functions $f : \mathbf{R}^d \to \mathbf{R}$ satisfying that $|f(x)| \leqslant M(1 + |x|^\gamma)$. The following is our main result:

THEOREM 1. – *Let Y_n denote either X_n or M_{n,T_n}. Suppose that conditions* [A1] *and* [A2] *for some $l > 2d + 4$. Moreover, suppose that* $\sup_{n\in\mathbf{N}} P^n[s_n^{-p}] < \infty$ *for any $p > 1$, and that* $\lim_{n\to\infty} P^n[\det \sigma_{Y_n} \geqslant s_n] = 1$. *Then, for any $M > 0$, $\gamma > 0$, $a_1 > (3 + d)/(l - d)$ and $a_2 < 1$, there exist a constant C and a sequence $\varepsilon_n = \mathrm{o}(r_n)$ such that*

$$P^n\big[f(X_n)\big] = \int_{\mathbf{R}^d} f(x) p_n(x)\, \mathrm{d}x + C r_n^{-a_1} P^n[\det \sigma_{Y_n} < s_n]^{a_2} + \varepsilon_n$$

for any $n \in \mathbf{N}$ and any $f \in \mathcal{E}(M, \gamma)$.

Remark 2. – The differentiability index in Theorem 1 is greater than the one given in [27]. It would be natural because what we treat here is essentially the expansion of the (local) density of X_n differently from the distribution function as in [27].

In [10,18,19,21], we treated functionals of a continuous-time parameter ε-Markov process with geometrical mixing property (*local approach*). The result in the present article can also apply to a class of such models to derive the second order expansions. Here, we give a simple application of our result. Our previous result by (unconditional) local approach cannot apply to this example since it is based on a long-memory process breaking the geometric mixing condition.

Example 1. – Consider a regression model defined by:

$$X_t = \theta Z_t + e_t,$$

where $Z = (Z_t)_{t\in\mathbf{N}}$ and $e = (e_t)_{t\in\mathbf{N}}$ are independent and e is an i.i.d. sequence with $E[e_t] = 0$ and $\mathrm{Var}[e_t] = 1$. Given observations $(X_t, Z_t)_{t=1,\dots,T}$, the least square estimator $\hat{\theta}_T$ for θ is given by

$$\sqrt{T}(\hat{\theta}_T - \theta) = \frac{\sum_{i=1}^T Z_t e_t/\sqrt{T}}{\sum_{t=1}^T Z_t^2/T}.$$

In the present situation, we can naturally embed the processes $(X_t, Z_t)_{t=1,\dots,T}$ into the continuous-time processes $(X_{[t]}, Z_{[t]})_{t\in\mathbf{R}_+}$, and to simplify our argument, for a while, let us confine our attention to the martingale which forms the principal part: the martingale $M^T = (M_t^T)_{t\in\mathbf{R}_+}$ defined by

$$M_t^T = \frac{1}{\sqrt{T}} \int_0^t Z_{[s+1]}\,\mathrm{d}w_s,$$

where $w = (w_t)_{t\in\mathbf{R}_+}$ is a Wiener process independent of Z, and the reference filtration $(\mathcal{F}_t)_{t\in\mathbf{R}_+}$ is given by $\mathcal{F}_t = \sigma[Z_n\ (n \in \mathbf{Z}_+), w_s\ (s \leqslant t)]$. Let $(Z_n)_{n\in\mathbf{Z}_+}$ be a stationary Gaussian sequence satisfying $E[Z_n] = 0$ and $\mathrm{Var}[Z_n] = 1$. Then $\langle M^T\rangle = \int_0^\cdot Z_{[s+1]}^2\,\mathrm{d}s/T$ and hence, for the Hermite polynomial H_2 of degree 2,

$$\xi_T = r_T^{-1} T^{-1} \sum_{t=1}^T H_2(Z_t)$$

for $T \in \mathbf{Z}_+$. Suppose that the autocovariance function

$$r(n) := E[Z_0 Z_n] \sim |n|^{-q} L(n)$$

as $n \to \infty$ for some constant q $(0 < q < 1)$ and some slowly varying function L. By Herglotz' theorem, there exists a measure ν such that $r(n) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}nx}\nu(\mathrm{d}x)$. Moreover, it is known that there exists a weak limit ν_∞ of the sequence of measures $\nu_n(\mathrm{d}x) = n^q L(n)^{-1}\nu(\frac{\mathrm{d}x}{n} \cup [-\pi, \pi))$ on $\mathbf{B}_1$. Let $X(t)$ be the second order Hermite process corresponding to ν_∞. It is then known that for $A_T = T^{1-q} L(T)$ $(T \in \mathbf{N})$,

$$A_T^{-1}\sum_{t=1}^{T} H_2(Z_t) \to^d X(1)$$

as $T \to \infty$, $T \in \mathbf{N}$, if $q < 1/2$. Take $r_T = A_T/T$. Then we see that

$$\begin{aligned}
&E\left[\exp\left(\mathrm{i}uM_T^T + \mathrm{i}v\xi_T\right)\right] \\
&\quad = E\left[\exp\left(-\frac{1}{2}u^2T^{-1}\int_0^T Z_{[t+1]}^2\,\mathrm{d}t + \mathrm{i}vA_T^{-1}\sum_{t=1}^{T} H_2(Z_t)\right)\right] \\
&\quad \to E\left[\exp\left(-\frac{1}{2}u^2 + \mathrm{i}vX(1)\right)\right];
\end{aligned}$$

thus $(M_T^T, \xi_T) \to^d (Z', X(1))$, where Z' is independent of $X(1)$. From our theorem, it is possible to obtain the asymptotic expansion of the distribution of M_T^T. The order of the second term is $r_T \sim T^{-q}L(T)$; since the conditional expectation of ξ given Z' vanishes, what we obtained is a Berry–Esseen type bound, i.e., for any $q' < q$, there exists a constant C independent of n such that

$$\sup_{x\in\mathbf{R}}\left|P\left[M_T^T \leqslant x\right] - \Phi(x)\right| = \mathrm{o}(r_n) \leqslant CT^{-q'}.$$

In a similar way, we can also obtain a uniform bound for the distribution of the least square estimator $\hat{\theta}_T$ if we use our present result and the Delta-method. Note that this example does not follow from our previous result on the asymptotic expansion for geometric strong mixing processes.

It is also possible to generalize the above argument to non-gaussian models. In the general case, we do not use the embedding of the martingale to a continuous martingale but use our result for jump martingales directly. A longer discussion will be necessary to describe the strong dependency and limit theorems to compute the higher order term. Available is the result by Ho and Tsing [8] on a moving average with slowly decaying coefficients.

Robinson and Hidalgo [16] more precisely studied the first-order asymptotics of the stochastic linear regression model.

Another treatment of the asymptotic expansion by the partial mixing is in [28].

4. Preliminaries and proofs

Suppose that we are given a one-dimensional random variable (truncation functional) $\psi_n : \Omega^n \to [0,1]$ for each $n \in \mathbf{N}$. We will use the multi-index: for $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}_+^d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$; moreover, $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ with $\partial_i = \partial/\partial_{x_i}$, $i = 1, \ldots, d$.

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}_+^d$, let

$$\hat{g}_n^\alpha(u) = P^n\big[\mathrm{e}^{\mathrm{i}u\cdot X_n}\psi_n X_n^\alpha\big].$$

We define $g_n^\alpha : \mathbf{R}^d \to \mathbf{R}$ by

$$g_n^\alpha(x) = \left(\frac{1}{2\pi}\right)^d \int\limits_{\mathbf{R}^d} \mathrm{e}^{-\mathrm{i}u\cdot x}\hat{g}_n^\alpha(u)\,\mathrm{d}u, \qquad x \in \mathbf{R}^d,$$

if $\hat{g}_n^\alpha$ is integrable. The function g_n^0 is referred to as the local density of X_n. It is easy to see that under regularity conditions,

$$g_n^\alpha(x) = x^\alpha g_n^0(x)$$

for $x \in \mathbf{R}^d$ and $\alpha \in \mathbf{Z}_+^d$.

Define $c_n \in \mathbf{R}$ by $c_n = P^n[\psi_n]$; moreover, for $\alpha \in \mathbf{Z}_+^d$ and $x \in \mathbf{R}^d$, let $h_n^\alpha(x) = x^\alpha h_n^0(x)$, where

$$h_n^0(x) = c_n\phi(x) + \frac{1}{2}r_n\sum_{j,k=1}^d \partial_j\partial_k\big(A^{jk}(x)\phi(x)\big) - r_n\sum_{j=1}^d \partial_j\big(B^j(x)\phi(x)\big).$$

Define $A_\alpha(u,x)$ and $B_\alpha(u,x)$ by:

$$A_\alpha(u,x) = \sum_{j,k=1}^d A^{jk}(x)\big((\mathrm{i}u)^j(\mathrm{i}u)^k x^\alpha + (\mathrm{i}u)^j\partial_k x^\alpha + (\mathrm{i}u)^k\partial_j x^\alpha + \partial_j\partial_k x^\alpha\big),$$

and

$$B_\alpha(u,x) = \sum_{j=1}^d B^j(x)\big((\mathrm{i}u)^j x^\alpha + \partial_j x^\alpha\big),$$

respectively. Set

$$\hat{h}_n^\alpha(u) = \int\limits_{\mathbf{R}^d} \mathrm{e}^{\mathrm{i}u\cdot x}\left[c_n x^\alpha + \frac{1}{2}r_n A_\alpha(u,x) + r_n B_\alpha(u,x)\right]\phi(x)\,\mathrm{d}x.$$

Denote by $\mathcal{S}^d$ the set of all $d \times d$-symmetric matrices. For $\alpha \in \mathbf{Z}_+^d$, $u, z \in \mathbf{R}^d$ and $r \in \mathcal{S}^d$, define $P_\alpha(u,z,r)$ and $Q_\alpha(u,z,r)$ by

$$P_\alpha(u,z,r) = \exp\left(-\mathrm{i}u \cdot z - \frac{1}{2} r(u,u)\right)(-\mathrm{i}\partial_u)^\alpha \times \exp\left(\mathrm{i}u \cdot z + \frac{1}{2} r(u,u)\right),$$

and

$$Q_\alpha(u,z,r) = \mathrm{e}^{\frac{1}{2}r(u,u)} P_\alpha(u,z,r),$$

respectively. Clearly,

$$(-\mathrm{i}\partial_u)^\alpha \exp\left(\mathrm{i}u \cdot z + \frac{1}{2} r(u,u)\right) = \mathrm{e}^{\mathrm{i}u\cdot z} Q_\alpha(u,z,r).$$

In the sequel, we often denote M_{n,T_n} simply by M_n. The integral $\hat{g}_n^\alpha(u)$ of Fourier type will be approximated by $\hat{h}_n^\alpha(u)$. The gap between them is decomposed into three parts as follows:

$$\hat{g}_n^\alpha(u) - \hat{h}_n^\alpha(u) = J_n^\alpha(u) + K_n^\alpha(u) + L_n^\alpha(u),$$

where

$$\begin{aligned} J_n^\alpha(u) &= P^n\left[\psi_n X^\alpha \mathrm{e}^{\mathrm{i}u\cdot X_n}\right] \\ &\quad - P^n\left[\psi_n \sum_{0\leqslant\beta\leqslant\alpha} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} Q_\beta\left(u, M_n, \widetilde{\Xi}_{n,T_n}\right) \mathrm{e}^{\mathrm{i}u\cdot X_n}\right] \\ &\quad - \frac{1}{2} r_n E\left[\mathrm{e}^{\mathrm{i}u\cdot Z} A_\alpha(u,Z)\right], \\ K_n^\alpha(u) &= P^n\left[\psi_n \sum_{0\leqslant\beta\leqslant\alpha} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} Q_\beta\left(u, M_n, \widetilde{\Xi}_{n,T_n}\right) \mathrm{e}^{\mathrm{i}u\cdot X_n}\right] \\ &\quad - P^n\left[\psi_n Q_\alpha\left(u, M_n, \widetilde{\Xi}_{n,T_n}\right) \mathrm{e}^{iu\cdot M_n}\right] - r_n E\left[\mathrm{e}^{\mathrm{i}u\cdot Z} B_\alpha(u,Z)\right] \end{aligned}$$

and

$$\begin{aligned} L_n^\alpha(u) &= P^n\left[\psi_n Q_\alpha\left(u, M_n, \widetilde{\Xi}_{n,T_n}\right) \mathrm{e}^{\mathrm{i}u\cdot M_n}\right] \\ &\quad - P^n\left[\psi_n (-\mathrm{i}\partial_u)^\alpha \mathrm{e}^{-\frac{1}{2}|u|^2}\right]. \end{aligned}$$

For $\beta \in \mathbf{Z}_+^d$, $u \in \mathbf{R}^d$ and $r \in \mathcal{S}^d$, let

$$S_\beta(u,r) = \mathrm{e}^{-\frac{1}{2}r(u,u)} \partial_u^\beta \mathrm{e}^{\frac{1}{2}r(u,u)} = \mathrm{i}^{|\beta|} P_\beta(u,0,r).$$

LEMMA 1. –
(1) *There exist constants* $c_j^{|\beta|}$ (*independent of* u, r) *such that*

$$|S_\beta(u,r)| \leqslant \sum_{j=0}^{|\beta|} c_j^{|\beta|} |u|^j |r|^{(j+|\beta|)/2}$$

and that $c_{\text{odd}}^{\text{even}} = 0$ *and* $c_{\text{even}}^{\text{odd}} = 0$.
(2) *For* $u, z \in \mathbf{R}^d$, $r \in \mathcal{S}^d$,

$$P_\alpha(u,z,r) = \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} (-\mathrm{i})^{|\beta|} z^{\alpha-\beta} S_\beta(u,r).$$

Proof. – It is easy to show (1). For (2), we may use the multi-index Leibniz rule:

$$\begin{aligned}(-\mathrm{i}\partial_u)^\alpha & \exp\left(\mathrm{i}u \cdot z + \frac{1}{2} r(u,u)\right) \\ &= (-\mathrm{i})^{|\alpha|} \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} (\partial_u)^{\alpha-\beta} \mathrm{e}^{\mathrm{i}u \cdot z} \cdot (\partial_u)^\beta \mathrm{e}^{\frac{1}{2} r(u,u)}. \qquad \square\end{aligned}$$

Let $q > 1$ and $\rho > 0$. Before applying the Malliavin calculus, we will clarify what kind conditions should be verified to obtain the asymptotic expansion.

[C1]

$$\sup_{n \in \mathbf{N}} \left\| r_n^{-2} \left(|x|^4 * \mu_{n,T_n} + |x|^4 * \nu_{n,T_n} \right) \right\|_q < \infty.$$

[C2] (i) $\psi_n \to^p 1$ as $n \to \infty$.

(ii) For any $p_1, p_2, p_3 \in \mathbf{Z}_+$,

$$\sup_{n \in \mathbf{N}} P^n \left[\psi_n |M_n|^{p_1} |\xi_n|^{p_2} |N_n|^{p_3} \right] < \infty.$$

(iii) Assumption [A1] is satisfied.

[C3] For $b_n = r_n^{2\rho}$, $|\widetilde{\Xi}_{n,T_n}| \leqslant b_n$ a.s. on the set $\{\psi_n > 0\}$.

[C4] There exists a constant A independent of $n \in \mathbf{N}$ such that on $\{\psi_n > 0\}$,

$$r_n^{-\rho} \sup_{s \leqslant T_n} \left| \Delta M_{n,s}^d \right| \leqslant A \quad \text{a.s.}$$

and

$$r_n^{-2\rho} \sup_{s \leqslant T_n} \Delta \langle M_n^{d,i}, M_n^{d,i} \rangle_s \leqslant A^2 \quad \text{a.s.}$$

for all $i = 1, \ldots, d$.

$[\mathrm{C5}]_l$ For any $\alpha \in \mathbf{Z}_+^d$,

$$\sup_{n \in \mathbf{N}} \sup_{u:\ |u| \geqslant r_n^{-\rho}} \big| |u|^l P^n \big[\mathrm{e}^{\mathrm{i}u \cdot X_n} \psi_n X_n^\alpha \big] \big| < \infty.$$

$[\mathrm{C6}]_j$ For any $\alpha \in \mathbf{Z}_+^d$,

$$\sup_{n \in \mathbf{N}} \sup_{u:\ 1 \leqslant |u| \leqslant r_n^{-\rho}} |u|^j \big| P^n \big[\mathrm{e}^{\mathrm{i}u \cdot X_n} \psi_n A_{n,u}^\alpha \big] \big| < \infty,$$

where

$$A_{n,u}^\alpha = \frac{1}{(1+|u|^2) r_n} \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} \times \big\{ Q_\beta(u, M_n, 0) - Q_\beta(u, M_n, r_n \xi_n) \big\}.$$

$[\mathrm{C7}]_k$ For any $\alpha \in \mathbf{Z}_+^d$,

$$\sup_{n \in \mathbf{N}} \sup_{u:\ 1 \leqslant |u| \leqslant r_n^{-\rho}} |u|^k \big| E \big[\mathrm{e}^{\mathrm{i}u \cdot X_n} \psi_n B_{n,u}^\alpha \big] \big| < \infty,$$

where

$$B_{n,u}^\alpha = \frac{1}{(1+|u|) r_n} \bigg\{ \sum_{0 \leqslant \beta \leqslant \alpha} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} Q_\beta(u, M_n, r_n \xi_n) - Q_\alpha(u, M_n, r_n \xi_n)\, \mathrm{e}^{-\mathrm{i}u \cdot r_n N_n} \bigg\}.$$

We will denote by C a generic constant independent of n and u. Also we write $F(n,u) \lesssim G(n,u)$ if there exists a constant C independent of n and u such that $F(n,u) \leqslant C G(n,u)$ for any n and u. For a fixed positive number ρ, let $\Lambda_n^1 = \{u \in \mathbf{R}^d; 1 \leqslant |u| \leqslant r_n^{-\rho}\}$ and $\Lambda_n^0 = \{u \in \mathbf{R}^d; |u| \leqslant r_n^{-\rho}\}$. Put $e_k = (\delta_{k,i})_{i=1}^d \in \mathbf{R}^d$: the kth vector of the standard basis of $\mathbf{R}^d$. We assume [C3].

LEMMA 2. – *If* [C2] *is satisfied, then* $J_n^\alpha(u) = \mathrm{o}(r_n)$ $(n \to \infty)$ *for each* $u \in \mathbf{R}^d$. *Moreover, if* $[\mathrm{C6}]_j$ *holds, then*

$$r_n^{-1} \big| J_n^\alpha(u) \big| 1_{\Lambda_n^1}(u) \leqslant C (1+|u|)^{-(j-2)}$$

for any $u \in \mathbf{R}^d$*. Under* [C2] *and* $[\mathrm{C6}]_j$*, if* $j > d + 2$*, then*

$$r_n^{-1} \int\limits_{\mathbf{R}^d} \bigl| J_n^\alpha(u) \bigr| 1_{\Lambda_n^0}(u)\, \mathrm{d}u \to 0$$

as $n \to \infty$*.*

Proof. – Put

$$j_{1,n}^\alpha(u) = \psi_n \sum_{\substack{\beta:\ 0 \leqslant \beta \leqslant \alpha \\ |\beta| \leqslant |\alpha|-1}} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} \times \bigl\{ Q_\beta(u, M_n, 0) - Q_\beta(u, M_n, r_n \xi_n) \bigr\} \mathrm{e}^{\mathrm{i}u \cdot X_n}$$

and

$$j_{2,n}^\alpha(u) = \psi_n \bigl\{ Q_\alpha(u, M_n, 0) - Q_\alpha(u, M_n, r_n \xi_n) \bigr\} \mathrm{e}^{\mathrm{i}u \cdot X_n}.$$

Clearly, [C2] implies that $r_n^{-1} P^n[|j_{1,n}^\alpha(u)|] = \mathrm{o}(1)$ for each $u \in \mathbf{R}^d$. On the other hand, since $Q_\alpha(u, z, 0) = z^\alpha$ (multi-index),

$$\begin{aligned}
r_n^{-1} j_{2,n}^\alpha(u) &= \psi_n r_n^{-1} \Biggl\{ M_n^\alpha - \sum_{j:\ 0 \leqslant j \leqslant \alpha} \binom{\alpha}{j} (-\mathrm{i})^{|j|} M_n^{\alpha-j} \\
&\qquad \times S_j(u, r_n \xi_n)\, \mathrm{e}^{\frac{1}{2} r_n \xi_n(u,u)} \Biggr\} \mathrm{e}^{\mathrm{i}u \cdot X_n} \\
&= \mathrm{e}^{\mathrm{i}u \cdot X_n} \psi_n r_n^{-1} \Biggl\{ M_n^\alpha - M_n^\alpha\, \mathrm{e}^{\frac{1}{2} r_n \xi_n(u,u)} \\
&\quad - \sum_{k=1}^d \binom{\alpha}{e_k} (-\mathrm{i}) M_n^{\alpha - e_k} r_n \xi_n^{k\cdot} \cdot u\, \mathrm{e}^{\frac{1}{2} r_n \xi_n(u,u)} \\
&\quad - \sum_{k_1, k_2:\ 1 \leqslant k_1 \leqslant k_2 \leqslant d} \binom{\alpha}{e_{k_1} + e_{k_2}} (-1) M_n^{\alpha - e_{k_1} - e_{k_2}} \\
&\quad \times \Biggl(r_n \xi_n^{k_1 k_2} + \sum_{l_1=1}^d \sum_{l_2=1}^d r_n^2 \xi_n^{k_1 l_1} \xi_n^{k_2 l_2} u^{l_1} u^{l_2} \Biggr) \mathrm{e}^{\frac{1}{2} r_n \xi_n(u,u)} \\
&\quad - \sum_{\substack{j:\ j \leqslant \alpha \\ |j| \geqslant 3}} \binom{\alpha}{j} (-\mathrm{i})^{|j|} M_n^{\alpha-j} S_j(u, r_n \xi_n)\, \mathrm{e}^{\frac{1}{2} r_n \xi_n(u,u)} \Biggr\}.
\end{aligned}$$

Write $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}_+^d$. It follows from [C2] and the implied uniform integrability that

$$
\begin{aligned}
&r_n^{-1} P^n\left[j_{2,n}^{\alpha}(u)\right] \\
&\quad \to E\left[\mathrm{e}^{\mathrm{i}u\cdot Z}\left\{-\frac{1}{2} Z^{\alpha} E[\xi|Z](u,u) + \sum_{k=1}^{d} \mathrm{i}\alpha_k Z^{\alpha-l_k} E\left[\xi^{k\cdot}|Z\right]\cdot u\right.\right. \\
&\qquad \left.\left. + \sum_{k_1,k_2:\ 1\leqslant k_1\leqslant k_2\leqslant d} \binom{\alpha}{e_{k_1}+e_{k_2}} Z^{\alpha-l_1-l_2} E\left[\xi^{k_1k_2}|Z\right]\right\}\right] \\
&\quad = \frac{1}{2} E\left[\mathrm{e}^{\mathrm{i}u\cdot Z}\left\{\sum_{j,k=1}^{d} \left(\mathrm{i}u^j\right)\left(\mathrm{i}u^k\right) A^{jk}(Z) Z^{\alpha}\right.\right. \\
&\qquad \left.\left. + 2\sum_{j,k=1}^{d} \left(\mathrm{i}u^j\right) A^{kj}(Z)\partial_k(x^{\alpha})|_{x=Z} + \sum_{k_1,k_2=1}^{d} \partial_{k_1}\partial_{k_2}\left(x^{\alpha}\right)|_{x=Z} A^{k_1k_2}(Z)\right\}\right] \\
&\quad = \frac{1}{2} E\left[\mathrm{e}^{\mathrm{i}u\cdot Z} A_{\alpha}(u,Z)\right].
\end{aligned}
$$

Thus we have obtained the first assertion. With $[\mathrm{C6}]_j$, taking limit, we have

$$
\left|E\left[\mathrm{e}^{\mathrm{i}u\cdot Z} A_{\alpha}(u,Z)\right]\right| \lesssim (1+|u|)^{-(j-2)},
$$

and then the second and the third assertions are easy consequences. □

LEMMA 3. – *If* [C2] *is satisfied, then* $K_n^{\alpha}(u) = \mathrm{o}(r_n)$ $(n\to\infty)$ *for each* $u\in\mathbf{R}^d$. *Moreover, if* $[\mathrm{C7}]_k$ *holds, then*

$$
r_n^{-1}\left|K_n^{\alpha}(u)\right| 1_{\Lambda_n^1}(u) \leqslant C(1+|u|)^{-(k-1)}
$$

for any $u\in\mathbf{R}^d$. *Under* [C2] *and* $[\mathrm{C7}]_k$, *if* $k>d+1$, *then*

$$
r_n^{-1}\int_{\mathbf{R}^d} \left|K_n^{\alpha}(u)\right| 1_{\Lambda_n^0}(u)\,\mathrm{d}u \to 0
$$

as $n\to\infty$.

Proof. – Let

$$
\begin{aligned}
k_{1,n}^{\alpha} &= \psi_n Q_{\alpha}(u, M_n, r_n\xi_n)\left(\mathrm{e}^{\mathrm{i}u\cdot X_n} - \mathrm{e}^{\mathrm{i}u\cdot M_n}\right) \\
&\quad + \psi_n \sum_{\substack{\beta:\ 0\leqslant\beta\leqslant\alpha \\ |\beta|=|\alpha|-1}} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} Q_{\beta}(u, M_n, r_n\xi_n)\,\mathrm{e}^{\mathrm{i}u\cdot X_n}
\end{aligned}
$$

and

$$k_{2,n}^{\alpha} = \psi_n \sum_{\substack{\beta:\ 0\leqslant\beta\leqslant\alpha \\ |\beta|\leqslant|\alpha|-2}} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} Q_\beta(u, M_n, r_n\xi_n)\, \mathrm{e}^{\mathrm{i}u\cdot X_n}.$$

Clearly, [C2] implies that $r_n^{-1} P^n[k_{2,n}^{\alpha}(u)] \to 0$ and that

$$r_n^{-1} P^n\left[k_{1,n}^{\alpha}(u)\right] \to E\left[Z^\alpha \mathrm{e}^{\mathrm{i}u\cdot Z}\, \mathrm{i}u\cdot\eta + \sum_{\substack{\beta:\ 0\leqslant\beta\leqslant\alpha \\ |\beta|=|\alpha|-1}} \binom{\alpha}{\beta} \eta^{\alpha-\beta} Z^\beta \mathrm{e}^{\mathrm{i}u\cdot Z}\right]$$

as $n\to\infty$. Since

$$\begin{aligned}
&E\left[\mathrm{e}^{\mathrm{i}u\cdot Z}\left\{Z^\alpha(\mathrm{i}u)\cdot\eta + \sum_{\substack{\beta:\ 0\leqslant\beta\leqslant\alpha \\ |\beta|=|\alpha|-1}} \binom{\alpha}{\beta} Z^\beta \eta^{\alpha-\beta}\right\}\right] \\
&\quad = E\left[\mathrm{e}^{\mathrm{i}u\cdot Z}\left\{Z^\alpha(\mathrm{i}u)\cdot B(Z) + \sum_{\substack{\beta:\ 0\leqslant\beta\leqslant\alpha \\ |\beta|=|\alpha|-1}} \binom{\alpha}{\beta} Z^\beta B(Z)^{\alpha-\beta}\right\}\right] \\
&\quad = E\left[\mathrm{e}^{\mathrm{i}u\cdot Z}\left\{Z^\alpha(\mathrm{i}u)\cdot B(Z) + \sum_{j=1}^{d} \left(\partial_j x^\alpha\right)\big|_{x=Z} B(Z)^j\right\}\right] \\
&\quad = E\left[\mathrm{e}^{\mathrm{i}u\cdot Z} B_\alpha(u, Z)\right],
\end{aligned}$$

it holds that

$$r_n^{-1} K_n^\alpha(u) = r_n^{-1} P^n\left[k_{1,n}^\alpha + k_{2,n}^\alpha\right] - E\left[\mathrm{e}^{\mathrm{i}u\cdot Z} B_\alpha(u, Z)\right] \to 0$$

as $n\to\infty$. This shows the first assertion. What rest easily follow from $[\mathrm{C7}]_k$. □

Though the following lemma is more or less well known, we will state it here for later convenience.

LEMMA 4. – *Suppose that*

$$\left\| |x|^2 * \mu_{n,T_n} \right\|_1 < \infty.$$

Then the following inequalities hold true:

(1) $0 \leqslant \Delta\langle M_n^d, M_n^d\rangle_s = \int_{\mathbf{R}^d} x^{\otimes 2} \nu_n(\{s\}, \mathrm{d}x) - (\hat{x}_{n,s})^{\otimes 2} = \int_{\mathbf{R}^d} x^{\otimes 2} \times \nu_n(\{s\}, \mathrm{d}x)$, *where* $\hat{x}_{n,s} = \int_{\mathbf{R}^d} x\, \nu_n(\cdot; \{s\}, \mathrm{d}x) = 0$.

(2) $|\Delta\langle M_n^d, M_n^d\rangle_s|^2 = |\int_{\mathbf{R}^d} x^{\otimes 2} \nu_n(\{s\}, \mathrm{d}x)|^2 \leqslant \int_{\mathbf{R}^d} |x|^4 \nu_n(\{s\}, \mathrm{d}x)$.

LEMMA 5. – *Let* $\alpha^* = \frac{1}{3}$ *and* $\beta^* = \frac{2}{3}$. *Suppose that conditions* [C1], [C2], [C3] *and* [C4] *are satisfied. If* $p > 2q(2q-1)^{-1}$, *then for some constants* C_p^α *and* C_α (*independent of* p), *it holds that*

$$\left|L_n^\alpha(u)\right| 1_{\Lambda_n^0}(u) \leqslant C_p^\alpha\left(|u|^3+1\right)\|\psi_n - 1\|_p + C_\alpha r_n^2\left(|u|^4+1\right)$$

for any $n \in \mathbf{N}$ *and* $u \in \mathbf{R}^d$. *Moreover,*

$$\int_{\mathbf{R}^d} \left|L_n^\alpha(u)\right| 1_{\Lambda_n^0}(u)\,\mathrm{d}u \leqslant C_p^{\alpha,d} r_n^{-\rho(3+d)}\|\psi_n - 1\|_p + C_\alpha^d r_n^{2-\rho(4+d)}$$

for any $n \in \mathbf{N}$.

Proof. – *Step* 1. Let

$$Y_{n,t} = \mathrm{i}u \cdot M_{n,t} + \frac{1}{2}\,\widetilde{\Xi}_{n,t}(u,u).$$

Define a stopping time τ_n by

$$\tau_n = \inf\left\{t;\ \sup_{u:\ |u|=1} \widetilde{\Xi}_{n,t}(u,u) \geqslant b_n\right\},$$

where $b_n = r_n^{2\rho}$. Define Φ_i, $i = 1, \ldots, 5$, as follows:

$$\Phi_1 = P^n\left[\psi_n\, \mathrm{e}^{Y_{n,-}} \cdot (\mathrm{i}u \cdot M_n)_{\tau_n}\right],$$

(note that on the right-hand side, the first dot stands for the stochastic integration and the second one for the inner product)

$$\Phi_2 = \frac{1}{2}\left(\alpha^* - 1\right) P^n\left[\psi_n \sum_{s \leqslant \tau_n} \mathrm{e}^{Y_{n,s-}}\left(\Delta M_{n,s}^d\right)^{\otimes 2}(u,u)\right]$$
$$+ \frac{1}{2}\,\beta^* P^n\left[\psi_n\, \mathrm{e}^{Y_{n,s-}} \cdot \left\langle M_n^d\right\rangle_{\tau_n}(u,u)\right],$$

$$\Phi_3 = -\frac{1}{2}\left(\alpha^* - \frac{1}{3}\right)\mathrm{i}^3 P^n\left[\psi_n \sum_{s \leqslant \tau_n} \mathrm{e}^{Y_{n,s-}}\left(\Delta M_{n,s}^d\right)^{\otimes 3}(u,u,u)\right],$$

$$\Phi_4 = -\frac{1}{2}\,\mathrm{i}^3 \beta^* P^n\left[\psi_n \sum_{s \leqslant \tau_n} \mathrm{e}^{Y_{n,s-}}\,\Delta M_{n,s}^d(u)\,\Delta\left\langle M_n^d\right\rangle_s(u,u)\right],$$

and

$$\Phi_5 = P^n\left[\psi_n\left\{R_{2,n}(u) + R_{3,n}(u) + R_{4,n}(u)\right\}\right].$$

Here $R_{i,n}(u)$, $i = 2, 3, 4$, are defined by:

$$R_{2,n}(u)=\frac{1}{2}\sum_{s\leqslant T_n}\mathrm{e}^{Y_{n,s-}}(\Delta Y_{n,s})^2-P_{2,n}(u),$$
$$R_{3,n}(u)=\frac{1}{6}\sum_{s\leqslant T_n}\mathrm{e}^{Y_{n,s-}}(\Delta Y_{n,s})^3-\frac{1}{6}\sum_{s\leqslant T_n}\mathrm{e}^{Y_{n,s-}}\,\mathrm{i}^3\big(\Delta M^d_{n,s}\big)^{\otimes 3}(u,u,u)$$

and

$$R_{4,n}(u)=\frac{1}{6}\sum_{s\leqslant T_n}\mathrm{e}^{Y_{n,s-}}\int_0^1(\Delta Y_{n,s})^4(1-v)^3\,\mathrm{e}^{v\Delta Y_{n,s}}\,\mathrm{d}s,$$

where

$$P_{2,n}(u)=\sum_{s\leqslant T_n}\mathrm{e}^{Y_{n,s-}}\bigg\{\frac{1}{2}\big(\Delta M^d_{n,s}\big)^{\otimes 2}(\mathrm{i}u,\mathrm{i}u)-\frac{1}{2}\alpha^*\big(\Delta M^d_{n,s}\big)^{\otimes 3}(\mathrm{i}u,\mathrm{i}u,\mathrm{i}u)$$
$$-\frac{1}{2}\,\mathrm{i}^3\beta^*\Delta M^d_{n,s}\otimes\Delta\big\langle M^d_n\big\rangle_s(u;u,u)\bigg\}.$$

Note that if $\alpha^*=1/3$ and $\beta^*=2/3$, we have $\Phi_3=0$ and

$$\Phi_2=-\frac{1}{3}P^n\bigg[\psi_n\bigg\{\sum_{s\leqslant\tau_n}\mathrm{e}^{Y_{n,s-}}\big(\Delta M^d_{n,s}\big)^{\otimes 2}(u,u)-\mathrm{e}^{Y_{n,-}}\cdot\big\langle M^d_n\big\rangle_{\tau_n}(u,u)\bigg\}\bigg].$$

We then have a decomposition of $L^0_n(u)$:

$$L^0_n(u)=\sum_{i=1}^{5}\Phi_i. \tag{2}$$

In fact, by using Itô's formula, we see that a.s. on $\{\psi_n>0\}$,

$$\begin{aligned}(3)\ \mathrm{e}^{Y_{n,T_n}}-\mathrm{e}^{-\frac{1}{2}|u|^2}&\\ =\mathrm{e}^{Y_{n,-}}\cdot M_{n,\tau_n}(\mathrm{i}u)&\\ +\frac{1}{2}\mathrm{e}^{Y_{n,-}}\cdot\bigg(\alpha^*\sum_{s\leqslant\cdot}\big(\Delta M^d_{n,s}\big)^{\otimes 2}_{\tau_n}(u,u)+\beta^*\big\langle M^d_n,M^d_n\big\rangle_{\tau_n}(u,u)\bigg)&\\ +\sum_{s\leqslant\tau_n}\mathrm{e}^{Y_{n,s-}}\big[\mathrm{e}^{\Delta Y_{n,s}}-1-\mathrm{i}u^{\cdot}\Delta Y_{n,s}\big],&\end{aligned}$$

where $M_n(u)$ denotes the 1-form $M_n\cdot u$. In view of the choice of numbers α^* and β^*, we see

$$
\begin{aligned}
L_n^0(u) &= \Phi_1 + \Phi_2 + \frac{1}{2} P^n\left[\psi_n \sum_{s \leqslant \tau_n} \mathrm{e}^{Y_{n,s-}} \left(\Delta M_{n,s}^d\right)^{\otimes 2}(u,u)\right] \\
&\quad + P^n\left[\psi_n \sum_{s \leqslant \tau_n} \left[\mathrm{e}^{\Delta Y_{n,s}} - 1 - \mathrm{i}u^{\cdot} \Delta Y_{n,s}\right]\right] \\
&= \Phi_1 + \Phi_2 + \frac{1}{2} P^n\left[\psi_n \sum_{s \leqslant \tau_n} \mathrm{e}^{Y_{n,s-}} \left(\Delta M_{n,s}^d\right)^{\otimes 2}(u,u)\right] \\
&\quad + P^n\left[\psi_n \sum_{s \leqslant \tau_n} \mathrm{e}^{Y_{n,s-}} (\Delta Y_{n,s})^2/2!\right] \\
&\quad + P^n\left[\psi_n \sum_{s \leqslant \tau_n} \mathrm{e}^{Y_{n,s-}} (\Delta Y_{n,s})^3/3!\right] + P^n\left[\psi_n R_{4,n}(u)\right] \\
&= \Phi_1 + \Phi_2 + P^n\left[\psi_n \left(-\mathrm{i}^3/2\right) \alpha^* \sum_{s \leqslant \tau_n} \left(\Delta M_{n,s}^d\right)^{\otimes 3}(u,u,u)\right] \\
&\quad + P^n\left[\psi_n \left(-\frac{1}{2}\right) \mathrm{i}^3 \beta^* \sum_{s \leqslant \tau_n} \Delta M_{n,s}^d \otimes \Delta\left\langle M_n^d, M_n^d\right\rangle_s (u; u, u)\right] \\
&\quad + P^n\left[\psi_n R_{2,n}(u)\right] + P^n\left[\psi_n \sum_{s \leqslant \tau_n} \mathrm{e}^{Y_{n,s-}} (\Delta Y_{n,s})^3/3!\right] + P^n[\psi_n R_{4,n}] \\
&= \Phi_1 + \Phi_2 + P^n\left[\psi_n R_{3,n}(u)\right] \\
&\quad + \Phi_4 + P^n\left[\psi_n R_{2,n}(u)\right] + P^n\left[\psi_n R_{4,n}(u)\right] \\
&= \sum_{i=1}^5 \Phi_i,
\end{aligned}
$$

which is the desired decomposition (2).

Differentiating L_n^0 by u, we obtain

$$
L_n^\alpha(u) = \sum_{i=1}^5 (-\mathrm{i}\partial_u)^\alpha \Phi_i.
$$

Step 2: By Burkholder–Davis–Gundy's inequality, Hölder's inequality and Doob's inequality, we have

$$
\left\|(b \cdot M_n)_{\tau_n}^*\right\|_r \leqslant C \left\| |b|_{T_n}^* \right\|_p \| \, |M_{n,T_n}| \, \|_q
$$

for $\mathbf{R}^d$-valued (bounded) predictable processes b and $p,q,r>1$ with $1/p+1/q=1/r$. Here dot means both the inner product and the stochastic integration. Use the above inequality for $b=U(u,M_n,\widetilde{\Xi}_n)$, where

$$U(u,z,r)=(-\mathrm{i}\partial_u)^\alpha\big[\mathrm{e}^{\mathrm{i}u\cdot z+\frac{1}{2}r(u,u)}\,\mathrm{i}u\big],$$

we see that $U(u,M_n,\widetilde{\Xi}_n)_-\cdot M_{n,\tau_n\wedge\cdot}$ is a uniformly integrable martingale, and hence, the marginal mean vanishes and for $p,p'>1$ $(1/p+1/p'=1)$,

$$\begin{aligned}\big|(-\mathrm{i}\partial_u)^\alpha\Phi_1\big|&=\big|P^n\big[\psi_nU(u,M_n,\widetilde{\Xi}_n)_-\cdot M_{n,\tau_n}\big]\big|\\&=\big|P^n\big[(\psi_n-1)U(u,M_n,\widetilde{\Xi}_n)_-\cdot M_{n,\tau_n}\big]\big|\\&\leqslant C\|\psi_n-1\|_p\big\|U(u,M_n,\widetilde{\Xi}_n)_-\cdot M_{n,\tau_n}\big\|_{p'}\\&\leqslant\|\psi_n-1\|_pC_p^\alpha(|u|+1)\,\mathrm{e}^{\frac{1}{2}b_n|u|^2}.\end{aligned}$$

Next we will consider Φ_2. For $k=1,\dots,d$, let

$$\mathcal{M}_1=\big[\big(M_n^k\big)^d,\big(M_n^k\big)^d\big]-\big\langle\big(M_n^k\big)^d,\big(M_n^k\big)^d\big\rangle.$$

Then $\mathcal{M}_1^{\tau_n}\in\mathcal{M}^n$, the set of (element-wise) uniformly integrable martingales. Indeed, from [9] I.4.50(b), $(M_n^{d,\tau_n})^2-[M_n^{d,\tau_n},M_n^{d,\tau_n}]\in\mathcal{M}^n$; similarly from [9] I.4.2., $(M_n^{d,\tau_n})^2-\langle M_n^{d,\tau_n}\rangle\in\mathcal{M}^n$. For $j,k=1,\dots,d$ and $l\in\mathbf{Z}_+$, we have the estimate:

$$\begin{aligned}&\big\|\big(\big(M_{n,-}^j\big)^l\cdot\mathcal{M}_1\big)_{\tau_n}^*\big\|_{p''}\\&\quad\lesssim\big\|\big[\big(M_{n,-}^j\big)^l\cdot\mathcal{M}_1,\big(M_{n,-}^j\big)^l\cdot\mathcal{M}_1\big]_{\tau_n}^{1/2}\big\|_{p''}\\&\quad\lesssim\big\|\big(\big(M_{n,-}^j\big)^l\big)_{T_n}^*\big\|_{p_1}\big\|[\mathcal{M}_1,\mathcal{M}_1]_{T_n}^{1/2}\big\|_{p_2}\\&\quad\lesssim\big\|\big(\big(M_{n,-}^j\big)^l\big)_{T_n}^*\big\|_{p_1}\big(\big\||x|^4*\mu_{n,T_n}\big\|_{p_2/2}^{1/2}+\big\||x|^4*\nu_{n,T_n}\big\|_{p_2/2}^{1/2}\big).\end{aligned}$$

Here $p'',p_1>1$, $2<p_2\leqslant 2q$ $(1/p_1+1/p_2=1/p'')$, and we used the Burkholder–Davis–Gundy inequality, the Hölder inequality, and the estimate:

$$\begin{aligned}[\mathcal{M}_1,\mathcal{M}_1]_{T_n}&=\sum_{s\leqslant T_n}(\Delta\mathcal{M}_{1,s})^2\\&\leqslant 2\bigg[\sum_{s\leqslant T_n}\big(\Delta\big(M_n^k\big)^d\big)^4+\sum_{s\leqslant T_n}\big(\Delta\big\langle\big(M_n^k\big)^d,\big(M_n^k\big)^d\big\rangle\big)^2\bigg]\\&\leqslant 2\big[|x|^4*\mu_{n,T_n}+|x|^4*\nu_{n,T_n}\big]\end{aligned}$$

by Lemma 4 in the last inequality. Therefore, taking the derivatives of $Y_{n,s-}$ into account, we see that for p ($p'' > p/(p-1)$, in particular, $p > 2q/(2q-1)$),

$$\left|(-\mathrm{i}\partial)^\alpha \Phi_2\right| \lesssim \|\psi_n - 1\|_p \left(|u|^2+1\right) \mathrm{e}^{\frac{1}{2}b_n|u|^2}$$

for $u \in \Lambda_n^0$.

Put

$$\mathcal{M}_2 = \left(\Delta\left\langle M_n^d\right\rangle\right) \cdot M_n^{d,\tau_n}.$$

Here the integration on the right-hand side reads as a tri-linear form valued process. Then

$$(-\partial_u)^\alpha \Phi_4 = -\frac{1}{2}\,\mathrm{i}^3 \beta^* P^n\left[\psi_n (-\partial_u)^\alpha \left(\mathrm{e}^{Y_{n,-}} \cdot \mathcal{M}_{2,\tau_n}(u;u,u)\right)\right].$$

Since

$$\left\| |\mathcal{M}_{2,\tau_n}|^* \right\|_{p'} \lesssim \left\| \left(|x|^2 * \mu_{n,\tau_n}\right)^{\frac{1}{2}} \right\|_{p'}$$

for $u \in \Lambda_n^0$, and $\Delta\langle M_n^d, M_n^d\rangle \leqslant (3/2)\Xi_n$ is bounded on $\{\psi_n > 0\}$ a.s. because of [C4], we obtain for $p, p_1 > 1$ $(1/p + 1/p_1 < 1)$,

$$\left|(-\partial_u)^\alpha \Phi_4\right| \lesssim \|\psi_n - 1\|_p \left(1+|u|^3\right) \mathrm{e}^{\frac{1}{2}b_n|u|^2} \left(1 + \left\| \left(|x|^2 * \mu_{n,\tau_n}\right)^{\frac{1}{2}} \right\|_{p_1}\right)$$

for $u \in \Lambda_n^0$.

Step 3: It follows from definition that

$$\begin{aligned}
(4) \qquad R_{2,n} = \frac{1}{2} \sum_{s \leqslant T_n} \mathrm{e}^{Y_{n,s-}} \Bigg\{ &\left(\frac{1}{2}\alpha^*\right)^2 \left(\Delta M_{n,s}^d\right)^{\otimes 4}(u,u,u,u) \\
&+ \left(\frac{1}{2}\beta^*\right)^2 \left(\Delta\left\langle M_n^d, M_n^d\right\rangle_s\right)^{\otimes 2}(u,u;u,u) \\
&+ \frac{1}{2}\alpha^*\beta^* \left(\Delta M_{n,s}^d\right)^{\otimes 2} \otimes \Delta\left\langle M_n^d, M_n^d\right\rangle_s(u,u;u,u) \Bigg\}.
\end{aligned}$$

Differentiating (4) and using [C1] and [C2] (moment), it is not difficult to obtain

$$(5) \qquad \left|P^n\left[\psi_n (-\partial_u)^\alpha R_{2,n}(u)\right]\right| \leqslant r_n^2 \left(1+|u|^4\right) \mathrm{e}^{\frac{1}{2}b_n|u|^2} P_{1,\alpha}\left(r_n^\rho |u|\right),$$

where $P_{1,\alpha}$ is a polynomial.

With $\mathcal{X}^j := \mathrm{i}u^j \Delta M_{n,s}^{d,j}$, $\mathcal{Y}^{jk} := \frac{1}{2}\alpha^* u^j u^k \Delta M_{n,s}^{d,j} \Delta M_{n,s}^{d,k}$ and $\mathcal{Z}^{jk} := \frac{1}{2}\beta^* u^j u^k \Delta\langle M_n^{d,j}, M_n^{d,k}\rangle_s$, $R_{3,n}(u)$ is expressed as

$$R_{3,n}(u)=\frac{1}{6}\sum_{s\leqslant T_n}\mathrm{e}^{Y_{n,s-}}\biggl\{\biggl(\sum_j\mathcal{X}^j+\sum_{j,k}\mathcal{Y}^{jk}+\sum_{j,k}\mathcal{Z}^{jk}\biggr)^3 -\sum_j\sum_k\sum_l\mathcal{X}^j\mathcal{X}^k\mathcal{X}^l\biggr\}.$$

This is a third order homogenous polynomial of $\mathcal{X}^j$, $\mathcal{Y}^{jk}$, $\mathcal{Z}^{jk}$ without the terms $\mathcal{X}^j\mathcal{X}^k\mathcal{X}^l$. Differentiating this expression and taking [C1], [C4] and Lemma 4 into account, we obtain the estimate:

$$(6)\qquad \bigl|P^n\bigl[\psi_n(-\partial_u)^\alpha R_{3,n}(u)\bigr]\bigr|\leqslant \mathrm{e}^{\frac{1}{2}b_n|u|^2}\bigl(|u|^4+1\bigr)r_n^2P_{2,\alpha}\bigl(r_n^\rho|u|\bigr),$$

where $P_{2,\alpha}$ is a polynomial independent of n and u.

In a similar way, it is easy to show

$$(7)\ \bigl|P^n\bigl[\psi_n(-\partial_u)^\alpha R_{4,n}(u)\bigr]\bigr| \leqslant r_n^2\bigl(1+|u|^4\bigr)P_{3,\alpha}\bigl(r_n^\rho|u|\bigr)\exp\biggl(\frac{1}{2}\bigl(\alpha^*+\beta^*\bigr)A^2r_n^{2\rho}|u|^2+\frac{1}{2}b_n|u|^2\biggr)$$

for some polynomial $P_{3,\alpha}$.

Thus from (5), (6) and (7), we see that for $\alpha\in\mathbf{Z}_+^d$, there exists a polynomial P_α independent of n and u such that

$$\bigl|(-\partial_u)^\alpha\Phi_5\bigr|\leqslant r_n^2\bigl(|u|^4+1\bigr)P_\alpha\bigl(r_n^\rho|u|\bigr)\exp\biggl(\frac{1}{2}A^2|u|^2r_n^{2\rho}+\frac{1}{2}b_n|u|^2\biggr).$$

Step 4: Finally, we will estimate L_n^α. From above Steps, it follows that for some constant C_p^α and C^α,

$$\begin{aligned}\bigl|L_n^\alpha(u)\bigr|1_{\Lambda_n^0}(u)&\leqslant\sum_{i=1}^5\bigl|(-\mathrm{i}\partial_u)^\alpha\Phi_i\bigr|1_{\Lambda_n^0}(u)\\&\leqslant C_p^\alpha\bigl(|u|^3+1\bigr)\|\psi_n-1\|_p+C^\alpha r_n^2\bigl(|u|^4+1\bigr)\end{aligned}$$

for any $n\in\mathbf{N}$ and $u\in\mathbf{R}^d$. Consequently,

$$\begin{aligned}&\int_{\mathbf{R}^d}\bigl|L_n^\alpha(u)\bigr|1_{\Lambda_n^0}(u)\,\mathrm{d}u\\&\quad\leqslant\int_{\Lambda_n^0}C_p^\alpha\bigl(|u|^3+1\bigr)\,\mathrm{d}u\,\|\psi_n-1\|_p+r_n^2\int_{\Lambda_n^0}\bigl(|u|^4+1\bigr)\,\mathrm{d}u\\&\quad\sim C_p^\alpha(d)r_n^{-\rho(3+d)}\|\psi_n-1\|_p+C^\alpha(d)r_n^{-\rho(4+d)+2},\end{aligned}$$

which completes the proof of Lemma 5. □

LEMMA 6. – *Suppose conditions* $[\mathrm{C1}]$–$[\mathrm{C7}]_k$ *are satisfied. Suppose that* $j > d+2$, $k > d+1$, $l > 2d+4$ *and* $p > 2q(2q-1)^{-1}$. *Then, for* ρ *satisfying* $(l-d)^{-1} < \rho < (4+d)^{-1}$ *and for each* $\alpha \in \mathbf{Z}_+^d$, *there exists a constant* $C_p^{\alpha,d}$ *such that*

$$\sup_{x\in\mathbf{R}^d}\left|g_n^\alpha(x) - h_n^\alpha(x)\right| \leqslant C_p^{\alpha,d} r_n^{-\rho(3+d)}\|\psi_n - 1\|_p + \mathrm{o}(r_n).$$

Proof. – It follows from the Fourier inversion formula that

$$\begin{aligned}
&\sup_{x\in\mathbf{R}^d}\left|g_n^\alpha(x) - h_n^\alpha(x)\right| \\
&\quad = \sup_{x\in\mathbf{R}^d}\left|\left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d} \mathrm{e}^{-\mathrm{i}u\cdot x}\left(\hat{g}_n^\alpha(u) - \hat{h}_n^\alpha(u)\right)\mathrm{d}u\right| \\
&\quad \leqslant \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{R}^d - \Lambda_n^0} \left(\left|\hat{g}_n^\alpha(u)\right| + \left|\hat{h}_n^\alpha(u)\right|\right)\mathrm{d}u \\
&\qquad + \left(\frac{1}{2\pi}\right)^d \int_{\Lambda_n^0} \left|\hat{g}_n^\alpha(u) - \hat{h}_n^\alpha(u)\right|\mathrm{d}u.
\end{aligned} \tag{8}$$

By Lemmas 2, 3 and 5, one has

$$\begin{aligned}
\int_{\Lambda_n^0}\left|\hat{g}_n^\alpha(u) - \hat{h}_n^\alpha(u)\right|\mathrm{d}u &\leqslant \int_{\Lambda_n^0}\left(\left|J_n^\alpha(u)\right| + \left|K_n^\alpha(u)\right| + \left|L_n^\alpha(u)\right|\right)\mathrm{d}u \\
&\leqslant \mathrm{o}(r_n) + C_p^{\alpha,d} r_n^{-\rho(3+d)}\|\psi_n - 1\|_p + C^{\alpha,d} r_n^{2-\rho(4+d)} \\
&= \mathrm{o}(r_n) + C_p^{\alpha,d} r_n^{-\rho(3+d)}\|\psi_n - 1\|_p.
\end{aligned}$$

On the other hand, condition $[\mathrm{C5}]_l$ yields

$$\int_{\mathbf{R}^d - \Lambda_n^0} \left|\hat{g}_n^\alpha(u)\right|\mathrm{d}u \leqslant \int_{r_n^{-\rho}}^{\infty} \frac{C}{(1+r)^l} r^{d-1}\,\mathrm{d}r \sim r_n^{\rho(l-d)} = \mathrm{o}(r_n). \tag{9}$$

We then obtain the result from (8), (9), and a similar estimate for the integration of $\hat{h}_n^\alpha$. This completes the proof. □

PROPOSITION 2. – *Let* $M, \gamma > 0$. *Under the same assumptions as Lemma* 6, *there exist constants* $C_1 = C_1(M,\gamma,d,p)$, $C_2 = C_2(M,\gamma,d)$ *and a sequence* ε_n *with* $\varepsilon_n = \mathrm{o}(r_n)$ *such that*

$$\left|P^n\left[f(X_n)\right]-P_n[f]\right|\leqslant\left[M\sup_{n\in\mathbf{N}}\left\|1+|X_n|^\gamma\right\|_{p'}+C_1r_n^{-\rho(3+d)}\right]\|\psi_n-1\|_p \\ +C_2\|\psi_n-1\|_1+\varepsilon_n$$

for any $n\in\mathbf{N}$ *and any* $f\in\mathcal{E}(M,\gamma)$, *where* $p'=p/(p-1)$ *and* $P_n[f]=\int_{\mathbf{R}^d}f(x)p_n(x)\,\mathrm{d}x$.

Proof. – Define a measure Q^n on Ω^n by $\mathrm{d}Q^n=c_n^{-1}\psi_n\,\mathrm{d}P^n$. By condition [C5], we see that the characteristic function

$$Ch^{Q^n}[X_n](u)=c_n^{-1}P^n\left[\mathrm{e}^{\mathrm{i}u\cdot X_n}\psi_n\right]$$

of the distribution $\mathcal{L}\{X_n\mid Q^n\}$ is $\mathrm{d}u$-integrable, and hence, $\mathcal{L}\{X_n\mid Q^n\}$ has a density which is nothing but $c_n^{-1}g_n^0$. In particular, $P^n[f(X_n)\psi_n]=\int_{\mathbf{R}^d}f(x)g_n^0(x)\,\mathrm{d}x$. Let d' be the minimum even number greater than d, γ' the minimum even number greater or equal to γ, and $k'=d'+\gamma'$. Clearly, for $f\in\mathcal{E}(M,\gamma)$,

$$\left|\int_{\mathbf{R}^d}f(x)\left(g_n^0(x)-h_n^0(x)\right)\mathrm{d}x\right| \\ \leqslant M\int_{\mathbf{R}^d}\left(1+|x|^{d'}\right)^{-1}\mathrm{d}x\cdot C_{d',\gamma'}\sum_{\substack{\alpha:\ |\alpha|\leqslant k'\\ |\alpha|:\ \text{even}}}\sup_{x\in\mathbf{R}^d}\left|g_n^\alpha(x)-h_n^\alpha(x)\right|.$$

By using this inequality and applying Lemma 6, we have

$$\begin{aligned}
&\left|P^n\left[f(X_n)\right]-P_n[f]\right| \\
&\quad\leqslant\left|P^n\left[f(X_n)\psi_n\right]-P^n\left[f(X_n)\right]\right| \\
&\qquad+\left|\int_{\mathbf{R}^d}f(x)g_n^0(x)\,\mathrm{d}x-\int_{\mathbf{R}^d}f(x)h_n^0(x)\,\mathrm{d}x\right| \\
&\qquad+\left|\int_{\mathbf{R}^d}f(x)h_n^0(x)\,\mathrm{d}x-\int_{\mathbf{R}^d}f(x)p_n(x)\,\mathrm{d}x\right| \\
&\quad\leqslant M\sup_{n\in\mathbf{N}}\left\|1+|X_n|^\gamma\right\|_{p'}\|\psi_n-1\|_p \\
&\qquad+C(M,\gamma,d,p)r_n^{-\rho(3+d)}\|\psi_n-1\|_p+\mathrm{o}(r_n) \\
&\qquad+|1-c_n|\int_{\mathbf{R}^d}M\left(1+|x|^\gamma\right)\phi(x)\,\mathrm{d}x,
\end{aligned}$$

from which one has the result. □

Proof of Theorem 1. – Since $a_1 > (d+3)/(l-d)$ and $l > 2d+4$, we can take ρ so that $1/(l-d) < \rho \leqslant \min\{a_1/(d+3), 1/(d+4)\}$. Let $a = 1-2\rho$ and $b = 2a$. Take a smooth function $\varphi : \mathbf{R} \to [0,1]$ satisfying that $\varphi(x) = 1$ if $|x| \leqslant \frac{1}{2}$, and $\varphi(x) = 0$ if $|x| \geqslant 1$. Define truncation functional ψ_n on Ω^n by

$$\psi_n = \varphi\left(\frac{s_n}{2\det\sigma_{Y_n}}\right)\varphi\left(\left|\frac{4\sigma_{r_nN_n}}{s_n}\right|^2\right)\varphi\left(\left|r_n^a\xi_n\right|^2\right)\varphi\left(r_n^b(\kappa_n+\lambda_n)\right).$$

In order to apply Proposition 2, we shall verify Conditions [C1]–$[\mathrm{C7}]_k$. By definition of ψ_n, if $\psi_n > 0$, then $r_n^a|\xi_n| < 1$ and hence $|\widetilde{\Xi}_{n,T_n}| \leqslant r_n^{2\rho}$, i.e., [C3] holds.

The definition of ψ_n also implies that if $\psi_n > 0$, then $r_n^b\kappa_n < 1$ and $r_n^b\lambda_n < 1$ a.s. Remembering $\kappa_n = r_n^{-2}|x|^4 * \mu_{n,T_n}$, we see that if $\psi_n > 0$,

$$\sup_{s\leqslant T_n}|\Delta M_s| \leqslant \left(\sum_{s\leqslant T_n}|\Delta M_s|^4\right)^{1/4} = \left(r_n^2\kappa_n\right)^{1/4} < r_n^\rho$$

since $(2-b)/4 = \rho$. Similarly, using Lemma 4, we have

$$\begin{aligned}\sup_{s\leqslant T_n}\left|\Delta\langle M_n^d, M_n^d\rangle_s\right| &\leqslant \left(\sum_{s\leqslant T_n}\left|\Delta\langle M_n^d, M_n^d\rangle_s\right|^2\right)^{1/2} \leqslant \left(|x|^4 * \nu_{n,T_n}\right)^{1/2}\\ &= \left(r_n^2\lambda_n\right)^{1/2} < r_n^{(2-b)/2} = r_n^{2\rho}\end{aligned}$$

if $\psi_n > 0$. Thus we obtained [C4] with $A = 1$. Note that $\||x|^2 * \mu_{n,T_n}\|_q < \infty$ for any $q > 1$; indeed, from Doob's inequality and the Burkholder–Davis–Gundy inequality, we see that $|M_n|^*_{T_n} \in L^{2p}$ and that $|[M_n, M_n]| \in L^p$, and hence that $|x|^2 * \mu_{n,T_n} \in L^p$.

Condition [C1] holds since $\mathcal{S}_0^n \subset \bigcap_{p>1} L^p(P^n)$ by [A2]. Condition [C2] (iii) is nothing but [A1]. It is not difficult to verify condition [C2] (i) by definition of ψ_n and by L^2-boundedness of s_n^{-1} and σ_{N_n}, and L^1-boundedness of ξ_n, κ_n, λ_n. Since $\mathcal{S}_0^n$ is L^p-bounded uniformly in $n \in \mathbf{N}$ for every $p > 1$, condition [C2] (ii) holds true.

Using the chain rule for Γ^n, we see that Proposition 1 can apply to the case where $F = X_n$, $\psi = \psi_n$, $\psi_n A^\alpha_{n,u}$, $\psi_n B^\alpha_{n,u}$, and $f(x) = \mathrm{e}^{\mathrm{i}u\cdot x}$. Noticing that $|u^2 r_n| \leqslant 1$ if $|u| \leqslant r_n^{-\rho}$, it is possible to verify Condition $[\mathrm{C5}]_l$, $[\mathrm{C6}]_l$ and $[\mathrm{C7}]_l$.

Moreover, it is easy to show that $\sup_{u\in\mathbf{R}^d}|u|^l|E[\mathrm{e}^{\mathrm{i}u\cdot Z}Z^\alpha\chi]| < \infty$ for $\chi = \xi^j, \eta^j$ and any $\alpha \in \mathbf{Z}_+^d$. Therefore, a version of the function $y \mapsto$

$y^\beta \partial_y^i(E[Z^\alpha \chi | Z = y]\phi(y))$, $|i| \leqslant d+4$, is continuous, tending to zero as $|y| \to \infty$, and integrable with respect to the Lebesgue measure for any $\alpha, \beta \in \mathbf{Z}_+^d$. This completes the proof. $\square$

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