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Asymptotic Expansion for Small Diffusions Applied to Option Pricing

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Abstract. Using the Malliavin calculus, we derive asymptotic expansion of the distribution of statistics related to small diffusions. Applications to option pricing in economics are presented.

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1. Introduction

Consider a family of d-dimensional diffusion processes defined by the stochastic differential equations (statistical models)

$$\begin{aligned} \mathrm{d}X_t^\varepsilon(\theta) &= V_0(X_t^\varepsilon(\theta),\theta)\,\mathrm{d}t + \varepsilon V(X_t^\varepsilon(\theta),\theta)\,\mathrm{d}w_t, \\ &\qquad t\in[T_0,T],\ \ \varepsilon\in(0,1], \\ X_{T_0}^\varepsilon(\theta) &= x_{T_0}, \end{aligned} \tag{1}$$

where T_0 and T are fixed values, x_{T_0} is a constant, a p-dimensional unknown parameter $\theta\in\Theta$: a bounded convex domain of $\mathbf{R}^p$, V_0 is an $\mathbf{R}^d$-valued smooth function defined on $\mathbf{R}^d\times\Theta$, V is an $\mathbf{R}^d\otimes\mathbf{R}^r$-valued smooth function defined on $\mathbf{R}^d\times\Theta$ with bounded x-derivative and w is an r-dimensional standard Wiener process.

Under an equivalent martingale measure $\tilde{P}$, we consider d-dimensional diffusion processes defined by the stochastic differential equations

$$\begin{aligned} \mathrm{d}X_t^\varepsilon(\theta) &= \tilde{V}_0(X_t^\varepsilon(\theta),\theta)\,\mathrm{d}t + \varepsilon V(X_t^\varepsilon(\theta),\theta)\,\mathrm{d}\tilde{w}_t, \\ &\qquad t\in[T_0,T],\ \ \varepsilon\in(0,1], \\ X_{T_0}^\varepsilon(\theta) &= x_{T_0}, \end{aligned} \tag{2}$$

where $\tilde{V}_0$ is an $\mathbf{R}^d$-valued smooth function defined on $\mathbf{R}^d\times\Theta$, $\tilde{w}$ is an r-dimensional standard Wiener process under $\tilde{P}$ and θ, V, T_0, T and x_{T_0} are the same as (1).

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There are many applications of small diffusion models to finance, especially, option pricing, see [7, 8, 21, 22, 31] and references therein. Many option pricing problems associated with small diffusion are related to functionals of the form

$$F_T^\varepsilon(\theta, \tilde{w}) = F_{T_0} + \int_{T_0}^T f_0(X_t^\varepsilon(\theta), \theta)\,\mathrm{d}t + \varepsilon \int_{T_0}^T f(X_t^\varepsilon(\theta), \theta)\,\mathrm{d}\tilde{w}_t, \tag{3}$$

where $F_{T_0} \in \mathbf{R}^k$, $f_0(x, \theta) \in \bar{C}_\uparrow^\infty(\mathbf{R}^d \times \Theta \to \mathbf{R}^k)$, that is, $f_0(x, \theta)$ is an $\mathbf{R}^k$-valued smooth function defined on $\mathbf{R}^d \times \Theta$ and for any $i, j \in \mathbf{N}_0$ there exist $m_1, C_1 > 0$ such that $\sup_{\theta\in\Theta} |\delta^i \partial^j f_0(x, \theta)| \leqslant C_1(1 + |x|)^{m_1}$, and $f(x, \theta) \in \bar{C}_\uparrow^\infty(\mathbf{R}^d \times \Theta \to \mathbf{R}^k \otimes \mathbf{R}^r)$. From the viewpoint of the stochastic control theory, it may be useful that instead of the form (3), $F_T^\varepsilon(\theta, \tilde{w})$ is defined by

$$F_T^\varepsilon(\theta, \tilde{w}) = F_{T_0} + \int_{T_0}^T f_0(X_t^\varepsilon(\theta), \theta)\,\mathrm{d}t + \varepsilon \int_{T_0}^T f(X_t^\varepsilon(\theta), \theta)\,\mathrm{d}\tilde{w}_t + F(X_T^\varepsilon(\theta)), \tag{4}$$

where $F(x)$ is an $\mathbf{R}^k$-valued smooth function. For example, if $d = k$ and $F_T^\varepsilon(\theta, \tilde{w}) = X_T^\varepsilon(\theta)$, then we put $F_{T_0} = 0$, $f_0(x, \theta) = 0$, $f^i(x, \theta) = 0$ and $F(x) = x$. However, by using Itô's formula, $F(X_T^\varepsilon(\theta))$ can be represented by the right-hand side of (3). Consequently, we will take the form (3) in place of the form (4) for prices at time T.

Let

$$\begin{aligned} G_T^\varepsilon(\theta, \tilde{w}) &= \exp\left\{-\int_{T_0}^T r_s^\varepsilon(\theta, \tilde{w})\,\mathrm{d}s\right\}, \\ r_T^\varepsilon(\theta, \tilde{w}) &= r_{T_0} + \int_{T_0}^T h_0(X_t^\varepsilon(\theta), \theta)\,\mathrm{d}t + \varepsilon \int_{T_0}^T h(X_t^\varepsilon(\theta), \theta)\,\mathrm{d}\tilde{w}_t, \end{aligned}$$

where $r_{T_0} \in \mathbf{R}$, $h_0(x, \theta) \in \bar{C}_\uparrow^\infty(\mathbf{R}^d \times \Theta \to \mathbf{R})$, $h(x, \theta) \in \bar{C}_\uparrow^\infty(\mathbf{R}^d \times \Theta \to \mathbf{R} \otimes \mathbf{R}^r)$ and $r_s^\varepsilon(\theta, \tilde{w}) \geqslant 0$ for $s \in [T_0, T]$. To price call-options at time $t = T_0$ we want to calculate the expectation

$$E_{T_0,\theta_0}[G_T^\varepsilon(\theta_0, \tilde{w})\,\mathrm{Max}\{F_T^\varepsilon(\theta_0, \tilde{w}) - K, 0\}], \tag{5}$$

where $\mathrm{Max}\{x, 0\} = (\mathrm{Max}\{x_1, 0\}, \ldots, \mathrm{Max}\{x_k, 0\})'$ for $x = (x_1, \ldots, x_k)' \in \mathbf{R}^k$, $E_{T_0,\theta_0}[\cdot]$ stands for the conditional expectation operator under $\tilde{P}$ given $\theta_0 \in \Theta$ at time $t = T_0$ and K is an $\mathbf{R}^k$-valued striking price (see [4, 12]).

Unfortunately, we cannot obtain this expectation (5) because θ_0 is an unknown parameter. Therefore, we estimate θ_0 and predict (5) by means of

$$E_{T_0,\hat{\theta}_\varepsilon(w)}[G_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})\,\mathrm{Max}\{F_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) - K, 0\}], \tag{6}$$

where $\hat{\theta}_\varepsilon(w)$ is an estimator of θ_0 independent of $\{\tilde{w}_t; t \in [T_0, T]\}$. It is difficult to obtain this expectation explicitly, so we will derive the asymptotic expansion of

(6). For the first order asymptotic theory for small diffusion models, see [3, 9–11, 19, 20, 25, 26]. For details of asymptotic expansions for small diffusion models, see [2, 6, 30–33, 35].

As $\varepsilon \downarrow 0$, the process $X_t^\varepsilon(\theta)$ defined by (2) converges to $X_t^0(\theta)$ satisfying the ordinary differential equation

$$\begin{aligned} \frac{\mathrm{d}X_t^0(\theta)}{\mathrm{d}t} &= \tilde{V}_0(X_t^0(\theta), \theta), \quad t \in [T_0, T], \\ X_{T_0}^0(\theta) &= x_{T_0}. \end{aligned}$$

For example, it is known that $\sup_{T_0 \leqslant t \leqslant T} |X_t^\varepsilon(\theta_0) - X_t^0(\theta_0)| \to 0$ a.s. as $\varepsilon \downarrow 0$. Hence, under some regularity conditions, it follows that $F_T^\varepsilon(\theta_0, \tilde{w}) \to f_{-1}(\theta_0)$ a.s. as $\varepsilon \downarrow 0$, where

$$f_{-1}(\theta) = F_{T_0} + \int_{T_0}^T f_0(X_t^0(\theta), \theta)\,\mathrm{d}t.$$

Therefore it is more convenient to treat $\tilde{F}_T^\varepsilon(\theta, \tilde{w}) = [F_T^\varepsilon(\theta, \tilde{w}) - f_{-1}(\theta)]/\varepsilon$ instead of $F_T^\varepsilon(\theta, \tilde{w})$ itself. From (6), we consider the functional $H_T^\varepsilon(\theta, \tilde{w})$ defined by

$$H_T^\varepsilon(\theta, \tilde{w}) = G_T^\varepsilon(\theta, \tilde{w})(F_T^\varepsilon(\theta, \tilde{w}) - K)1_{A_\varepsilon(\theta)}(\tilde{F}_T^\varepsilon(\theta, \tilde{w})), \tag{7}$$

where $A_\varepsilon(\theta) = \{x; x \geqslant (K - f_{-1}(\theta))/\varepsilon\}$ and derive the asymptotic expansion of the expectation

$$E_{T_0, \hat{\theta}_\varepsilon(w)}[H_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})]. \tag{8}$$

In order to obtain the asymptotic expansion of (8), we must consider the stochastic expansion of (7). Let us discuss this from a mathematical point of view. Assume that families of random variables F_T^ε and G_T^ε have the asymptotic expansions:

$$F_T^\varepsilon \sim f_{-1} + \varepsilon f_0 + \varepsilon^2 f_1 + \cdots, \tag{9}$$
$$G_T^\varepsilon \sim g_{-1} + \varepsilon g_0 + \varepsilon^2 g_1 + \cdots \tag{10}$$

as $\varepsilon \downarrow 0$ in some sense. If function $T(x)$ satisfies a certain regularity, we then have the stochastic expansion of $T(\tilde{F}_T^\varepsilon)$ defined by

$$T(\tilde{F}_T^\varepsilon) \sim \Phi_0 + \varepsilon \Phi_1 + \cdots \tag{11}$$

as $\varepsilon \downarrow 0$ in some sense, where $\Phi_0, \Phi_1, \ldots$ are determined by formal Taylor expansion, and in particular $\Phi_0 = T(f_0)$ and $\Phi_1 = f_1(\partial T/\partial x)(f_0)$. If we can take $T(x) = 1_A(x)$, the indicator function of the Borel set A, the stochastic expansion of (11) is formally given by

$$1_A(\tilde{F}_T^\varepsilon) \sim 1_A(f_0) + \varepsilon f_1 \frac{\partial 1_A}{\partial x}(f_0) + \cdots \tag{12}$$

as $\varepsilon \downarrow 0$ in some sense. From (9), (10) and (12), we obtain the expansion of $G_T^\varepsilon(F_T^\varepsilon - K)1_A(\tilde{F}_T^\varepsilon)$:

$$\begin{aligned} G_T^\varepsilon(F_T^\varepsilon - K)1_A(\tilde{F}_T^\varepsilon) \sim\ & g_{-1}(f_{-1} - K)1_A(f_0) + \\ & + \varepsilon\bigg[\{g_0(f_{-1} - K) + g_{-1}f_0\}1_A(f_0) + \\ & + g_{-1}(f_{-1} - K)f_1\frac{\partial 1_A}{\partial x}(f_0)\bigg] + \cdots \end{aligned}$$

as $\varepsilon \downarrow 0$ in some sense.

However, there are three difficulties in this situation. We note that the second term of the right-hand side of (12) is a composite function of the random variable f_0 and the Schwartz distribution $\partial 1_A/\partial x$. We have the first difficulty that there is no usual meaning of a random variable as a measurable function on a probability space. Therefore, we are faced with the problem of how to define or interpret such composite functionals. Next, we meet the second question of how to justify the formal expansion (12) after removing the first difficulty. The above two difficulties have been solved by Watanabe [27, 29] in the Malliavin calculus. Watanabe [27] presented the concept of the generalized Wiener functional (i.e. the Schwartz distribution on the probability space) and the pull-back of Schwartz distribution under Wiener mappings. Moreover, he formulated the asymptotic expansion of the generalized Wiener functionals in some Sobolev space in his renowned work [29]. For more details of the Malliavin calculus and Watanabe's theory, see [5, 13–18, 23, 24, 28–30, 34]. To use Watanabe's theory, the crucial step is to show the nondegeneracy of the Malliavin covariance of functionals. However, we here have the third problem that it is not easy to check this even for a simple statistical estimator, whose Malliavin covariance is given by an integration of some nonadaptive process. In addition, as for estimators such as maximum likelihood estimators, we cannot ensure their existence on the whole sample space in general. This difficulty has been solved by Yashida [31] in the modification of Watanabe's theory with truncation. We call this the theory of Malliavin–Watanabe with truncation. The present paper will prove that this theory is very useful to derive quite directly the asymptotic expansion of estimator for option price.

The organization of the article is as follows. In Section 2, the asymptotic expansion of the estimator for a price of a call-option is obtained. In Section 3, we present our main result. The first and second order asymptotically expectation-unbiased estimators for a price of a call-option are derived. Section 4 presents proofs of the results.

2. Asymptotic Expansion and Option Pricing

Let $(W, \mathcal{F}, P)$ be a probability space. Let $(\tilde{W}, \tilde{H}, \tilde{P})$ be an r-dimensional Wiener space with the Ornstein–Uhlenbeck operator $\tilde{L}$. It is then possible to extend $\tilde{L}$ over the product space $\bar{W} = W \times \tilde{W}$ (cf. [1]). We will in the sequel use the same $\tilde{L}$ for this extended Ornstein–Uhlenbeck operator over $\bar{W}$. $D_p^s(\bar{W})$ is regarded as a Sobolev space over $\bar{W}$. Moreover, we define D^{∞}, $D^{-\infty}$, $\tilde{D}^{-\infty}$ as usual.

First, we prepare several notations. Let $X_t^0(\theta)$ be the solution of the ordinary differential equation

$$\begin{aligned}\frac{\mathrm{d}X_t^0(\theta)}{\mathrm{d}t} &= \tilde{V}_0(X_t^0(\theta),\theta), \quad t \in [T_0, T],\\ X_{T_0}^0(\theta) &= x_{T_0}.\end{aligned}$$

Let an $\mathbf{R}^d \otimes \mathbf{R}^d$-valued process $Y_t^{\varepsilon}(\theta)$ be the solution of the stochastic differential equation

$$\begin{aligned}&\mathrm{d}Y_t^{\varepsilon}(\theta) = \partial\tilde{V}_0(X_t^{\varepsilon}(\theta),\theta)Y_t^{\varepsilon}(\theta)\,\mathrm{d}t + \varepsilon\sum_{\alpha=1}^{r}\partial V_\alpha(X_t^{\varepsilon}(\theta),\theta)Y_t^{\varepsilon}(\theta)\,\mathrm{d}\tilde{w}_t^{\alpha},\\ &\qquad\qquad t \in [T_0, T],\\ &Y_{T_0}^{\varepsilon}(\theta) = I_d,\end{aligned}$$

where $[\partial\tilde{V}_0]^{i,j} = \partial_j\tilde{V}_0^i$, $[\partial V_\alpha]^{i,j} = \partial_j V_\alpha^i$, $\partial_j = \partial/\partial x^j$, $i, j = 1, \ldots, d$, $\alpha = 1, \ldots, r$. Then, $Y_t(\theta) := Y_t^0(\theta)$ is a deterministic $\mathbf{R}^d \otimes \mathbf{R}^d$-valued process. It is known that $\varepsilon \to X_t^{\varepsilon}$ is smooth. In particular, $D_t(\theta) := \partial X_t^{\varepsilon}(\theta)/\partial\varepsilon|_{\varepsilon=0}$ satisfies the stochastic differential equation

$$\begin{aligned}\mathrm{d}D_t(\theta) &= \partial\tilde{V}_0(X_t^0(\theta),\theta)D_t(\theta)\,\mathrm{d}t + V(X_t^0(\theta),\theta)\,\mathrm{d}\tilde{w}_t, \quad t \in [T_0, T],\\ D_{T_0}(\theta) &= 0.\end{aligned}$$

Then, $D_t(\theta)$ is represented by

$$D_t(\theta) = \int_{T_0}^{t} Y_t(\theta)Y_s^{-1}(\theta)V(X_s^0(\theta),\theta)\,\mathrm{d}\tilde{w}_s, \quad t \in [T_0, T].$$

$E_t(\theta) := \partial^2 X_t^{\varepsilon}(\theta)/\partial\varepsilon^2|_{\varepsilon=0}$ satisfies the stochastic differential equation

$$\begin{aligned}\mathrm{d}E_t(\theta) &= \sum_{i,j=1}^{d}\partial_i\partial_j\tilde{V}_0(X_t^0(\theta),\theta)D_t^i(\theta)D_t^j(\theta)\,\mathrm{d}t +\\ &\quad + \sum_{i=1}^{d}\partial_i\tilde{V}_0(X_t^0(\theta),\theta)E_t^i(\theta)\,\mathrm{d}t +\\ &\quad + 2\sum_{i=1}^{d}\partial_i V(X_t^0(\theta),\theta)D_t^i(\theta)\,\mathrm{d}\tilde{w}_t, \quad t \in [T_0, T],\\ E_{T_0}(\theta) &= 0.\end{aligned}$$

Then, $E_t(\theta)$ is represented by

$$E_t(\theta) = \int_{T_0}^{t} Y_t(\theta) Y_s^{-1}(\theta) \sum_{i,j=1}^{d} \partial_i \partial_j \tilde{V}_0(X_s^0(\theta), \theta) D_s^i(\theta) D_s^j(\theta)\, \mathrm{d}s + \\ + 2 \int_{T_0}^{t} Y_t(\theta) Y_s^{-1}(\theta) \sum_{i=1}^{d} \partial_i V(X_s^0(\theta), \theta) D_s^i(\theta)\, \mathrm{d}\tilde{w}_s, \quad t \in [T_0, T].$$

LEMMA 1. *$F_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \in D^\infty(\bar{W}; \mathbf{R}^k)$ and it has the asymptotic expansion*

$$F_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \sim f_{-1}(\hat{\theta}_\varepsilon(w)) + \varepsilon f_0(\hat{\theta}_\varepsilon(w), \tilde{w}) + \varepsilon^2 f_1(\hat{\theta}_\varepsilon(w), \tilde{w}) + \cdots$$

in $D^\infty(\bar{W}; \mathbf{R}^k)$ as $\varepsilon \downarrow 0$ with $f_{-1}(\hat{\theta}_\varepsilon(w)), f_0(\hat{\theta}_\varepsilon(w), \tilde{w}), f_1(\hat{\theta}_\varepsilon(w), \tilde{w}), \ldots \in D^\infty(\bar{W}; \mathbf{R}^k)$. In particular,

$$f_{-1}(\theta) = F_{T_0} + \int_{T_0}^{T} f_0(X_t^0(\theta), \theta)\, \mathrm{d}t, \tag{13}$$

$$f_0(\theta, \tilde{w}) = \int_{T_0}^{T} \partial_i f_0(X_t^0(\theta), \theta) D_t^i(\theta, \tilde{w})\, \mathrm{d}t + \int_{T_0}^{T} f(X_t^0(\theta), \theta)\, \mathrm{d}\tilde{w}_t, \tag{14}$$

$$f_1(\theta, \tilde{w}) = \frac{1}{2} \int_{T_0}^{T} \partial_i \partial_j f_0(X_t^0(\theta), \theta) D_t^i(\theta, \tilde{w}) D_t^j(\theta, \tilde{w})\, \mathrm{d}t + \\ + \frac{1}{2} \int_{T_0}^{T} \partial_i f_0(X_t^0(\theta), \theta) E_t^i(\theta, \tilde{w})\, \mathrm{d}t + \\ + \int_{T_0}^{T} \partial_i f(X_t^0(\theta), \theta) D_t^i(\theta, \tilde{w})\, \mathrm{d}\tilde{w}_t, \tag{15}$$

where we use Einstein's rule for repeated indices.

LEMMA 2. *$G_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \in D^\infty(\bar{W})$ and it has the asymptotic expansion*

$$G_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \sim g_{-1}(\hat{\theta}_\varepsilon(w)) + \varepsilon g_0(\hat{\theta}_\varepsilon(w), \tilde{w}) + \cdots$$

in $D^\infty(\bar{W})$ as $\varepsilon \downarrow 0$ with $g_{-1}(\hat{\theta}_\varepsilon(w)), g_0(\hat{\theta}_\varepsilon(w), \tilde{w}), \ldots \in D^\infty(\bar{W})$. In particular,

$$g_{-1}(\theta) = \exp\left(- \int_{T_0}^{T} \left\{ r_{T_0} + \int_{T_0}^{t} h_0(X_s^0(\theta), \theta)\, \mathrm{d}s \right\} \mathrm{d}t \right), \tag{16}$$

$$g_0(\theta, \tilde{w}) = -g_{-1}(\theta) \left[\int_{T_0}^{T} \left\{ \int_{T_0}^{t} \partial_i h_0(X_s^0(\theta), \theta) D_s^i(\theta, \tilde{w})\, \mathrm{d}s + \\ + \int_{T_0}^{t} h(X_s^0(\theta), \theta)\, \mathrm{d}\tilde{w}_s \right\} \mathrm{d}t \right]. \tag{17}$$

ASSUMPTION 1. *For any* $p \in (1, \infty)$, *the Malliavin covariance* $\sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)}$ *of* $f_0(\hat{\theta}_\varepsilon(w), \cdot)$ *in Lemma 1 satisfies*

$$\sup_{\varepsilon \in (0,1]} E[(\det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)})^{-p}] < \infty.$$

LEMMA 3. *Let* $\tilde{F}_T^\varepsilon(\theta, \tilde{w}) = [F_T^\varepsilon(\theta, \tilde{w}) - f_{-1}(\theta)]/\varepsilon$ *and* $\psi : \mathbf{R} \to \mathbf{R}$ *be a smooth function such that* $0 \leqslant \psi(x) \leqslant 1$ for $x \in \mathbf{R}$, $\psi(x) = 1$ for $|x| \leqslant 1/2$ *and* $\psi(x) = 0$ *for* $|x| \geqslant 1$. *Let* $\xi_\varepsilon(\theta, \tilde{w}) = 2(\det \sigma_{\tilde{F}_T^\varepsilon(\theta,\cdot)} - \det \sigma_{f_0(\theta,\cdot)})/\det \sigma_{f_0(\theta,\cdot)}$ *and* $\psi_\varepsilon^*(\theta, \tilde{w}) = \psi(\xi_\varepsilon(\theta, \tilde{w}))$. *Suppose that Assumption* 1 *holds true. Then for any* $T(x) \in S'(\mathbf{R}^k)$ (*the space of Schwartz tempered distributions*), $\psi_\varepsilon^*(\hat{\theta}_\varepsilon(w), \tilde{w}) T(\tilde{F}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})) \in \tilde{D}^{-\infty}(\bar{W})$ *has the asymptotic expansion*

$$\psi_\varepsilon^*(\hat{\theta}_\varepsilon(w), \tilde{w}) T(\tilde{F}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})) \sim \Phi_0(\hat{\theta}_\varepsilon(w), \tilde{w}) + \varepsilon \Phi_1(\hat{\theta}_\varepsilon(w), \tilde{w}) + \cdots$$

in $\tilde{D}^{-\infty}(\bar{W})$ *as* $\varepsilon \downarrow 0$ *uniformly in every class* $\{T\}$ *satisfying the condition* (5) *of Theorem* 4.1 *in* [31], *and* $\Phi_0(\hat{\theta}_\varepsilon(w), \tilde{w}), \Phi_1(\hat{\theta}_\varepsilon(w), \tilde{w}), \ldots$ *in* $\tilde{D}^{-\infty}(\bar{W})$ *are determined by the formal Taylor expansion. In particular*,

$$\begin{aligned} \Phi_0(\theta, \tilde{w}) &= T(f_0(\theta, \tilde{w})), \\ \Phi_1(\theta, \tilde{w}) &= \sum_i f_1^i(\theta, \tilde{w}) \partial_i T(f_0(\theta, \tilde{w})). \end{aligned}$$

From Lemmas 2.1 and 2.2 in [30] and Lemmas 1 and 3, we see that the composite functional

$$\begin{aligned} &(F_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) - K)\psi_\varepsilon^*(\hat{\theta}_\varepsilon(w), \tilde{w}) 1_{A_\varepsilon(\hat{\theta}_\varepsilon(w))}(\tilde{F}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})) \\ &\quad =: \bar{F}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \quad \text{(say)} \end{aligned}$$

is well-defined, where $A_\varepsilon(\theta) = \{x; x \geqslant (K - f_{-1}(\theta))/\varepsilon\}$, and we define

$$\bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) = G_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \bar{F}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}).$$

LEMMA 4. *Suppose that Assumption* 1 *holds true. Then* $\bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \in \tilde{D}^{-\infty}(\bar{W}; \mathbf{R}^k)$ *has the asymptotic expansion*

$$\bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \sim \Xi_0(\hat{\theta}_\varepsilon(w), \tilde{w}) + \varepsilon \Xi_1(\hat{\theta}_\varepsilon(w), \tilde{w}) + \cdots$$

in $\tilde{D}^{-\infty}(\bar{W}; \mathbf{R}^k)$ *as* $\varepsilon \downarrow 0$, *and* $\Xi_0(\hat{\theta}_\varepsilon(w), \tilde{w}), \Xi_1(\hat{\theta}_\varepsilon(w), \tilde{w}), \ldots$ *in* $\tilde{D}^{-\infty}(\bar{W}; \mathbf{R}^k)$ *are determined by the formal Taylor expansion. In particular*,

$$\begin{aligned} \Xi_0(\theta, \tilde{w}) &= g_{-1}(\theta)(f_{-1}(\theta) - K) 1_{A_\varepsilon(\theta)}(f_0(\theta, \tilde{w})), \\ \Xi_1(\theta, \tilde{w}) &= g_{-1}(\theta) f_0(\theta, \tilde{w}) 1_{A_\varepsilon(\theta)}(f_0(\theta, \tilde{w})) + \\ &\quad + g_{-1}(\theta)(f_{-1}(\theta) - K) f_1^i(\theta, \tilde{w}) \partial_i 1_{A_\varepsilon(\theta)}(f_0(\theta, \tilde{w})) + \\ &\quad + g_0(\theta, \tilde{w})(f_{-1}(\theta) - K) 1_{A_\varepsilon(\theta)}(f_0(\theta, \tilde{w})). \end{aligned}$$

Remark 1. $\Xi_i(\hat{\theta}_\varepsilon(w), \tilde{w})$ depends on ε for $i = 0, 1, \ldots$. However, the same proof as Watanabe [29] and Yoshida [30] works in such a situation. Moreover, note that we may replace $\tilde{F}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})$ by $(\tilde{F}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) + [f_{-1}(\hat{\theta}_\varepsilon(w)) - f_{-1}(\theta_0)]/\varepsilon)$ and $A_\varepsilon(\hat{\theta}_\varepsilon(w))$ by $A_\varepsilon(\theta_0)$, respectively, if necessary. The Malliavin covariance is invariant and there is no problem in applying the partial version of Theorem 2.2 in [30]. See the second part of the proof of Theorem 1.

We will use the Einstein's rule for repeated indices. For matrix A, $[A]^{ij}$ denotes its (i, j)-element. Moreover, $[A]^{i\cdot}$ and $[A]^{\cdot j}$ are the ith row vector and the jth column vector of A, respectively. For vector a, a^i is its ith element. Define several functions as follows:

$$\lambda_{t,s}^i(\theta) = [Y_t(\theta)Y_s^{-1}(\theta)V(X_s^0(\theta), \theta)]^{i\cdot}, \quad i = 1, \ldots, d,$$

$$\mu_{i,t,s}^j(\theta) = [Y_t(\theta)Y_s^{-1}(\theta)\partial_i V(X_s^0(\theta), \theta)]^{j\cdot}, \quad i, j = 1, \ldots, d,$$

$$\nu_{i,j,t,s}^l(\theta) = [Y_t(\theta)Y_s^{-1}(\theta)\partial_i\partial_j \tilde{V}_0(X_s^0(\theta), \theta)]^l, \quad i, j, l = 1, \ldots, d,$$

$$a_t(\theta) = \int_t^T \partial_\alpha f_0(X_s^0(\theta), \theta)\lambda_{s,t}^\alpha(\theta)\,\mathrm{d}s + f(X_t^0(\theta), \theta),$$

$$b_t(\theta) = -\int_t^T \left(\int_t^s \partial_\alpha h_0(X_u^0(\theta), \theta)\lambda_{u,t}^\alpha(\theta)\,\mathrm{d}u + h(X_s^0(\theta), \theta)\right)\mathrm{d}s,$$

$$\Sigma(\theta) = \int_{T_0}^T a_t(\theta)a_t(\theta)'\,\mathrm{d}t,$$

$$B(\theta) = \int_{T_0}^T a_t(\theta)b_t(\theta)'\,\mathrm{d}t.$$

From (14), (15) and (17),

$$f_0(\theta, \tilde{w}) = \int_{T_0}^T a_t(\theta)\,\mathrm{d}\tilde{w}_t, \tag{18}$$

$$\begin{aligned} f_1^\alpha(\theta, \tilde{w}) \;=\; & \frac{1}{2}\int_{T_0}^T \partial_i\partial_j f_0^\alpha(X_t^0(\theta), \theta)D_t^i(\theta, \tilde{w})D_t^j(\theta, \tilde{w})\,\mathrm{d}t + \\ & + \frac{1}{2}\int_{T_0}^T \partial_i f_0^\alpha(X_t^0(\theta), \theta)E_t^i(\theta, \tilde{w})\,\mathrm{d}t + \\ & + \int_{T_0}^T \partial_i f^\alpha(X_t^0(\theta), \theta)D_t^i(\theta, \tilde{w})\,\mathrm{d}\tilde{w}_t, \end{aligned} \tag{19}$$

$$g_0(\theta, \tilde{w}) = g_{-1}(\theta)\int_{T_0}^T b_t(\theta)\,\mathrm{d}\tilde{w}_t. \tag{20}$$

For $\mathbf{R}$-valued function h_t,

$$C_1(h, \theta)_T = \int_{T_0}^T \int_{T_0}^t h_t\lambda_{t,s}(\theta)a_s'(\theta)\,\mathrm{d}s\,\mathrm{d}t.$$

For $\mathbf{R}^1 \otimes \mathbf{R}^r$-valued function h_t,

$$C_2^i(h,\theta)_T = \int_{T_0}^T \int_{T_0}^t a_t(\theta) h_t' \lambda_{t,s}^i(\theta) a_s'(\theta)\,\mathrm{d}s\,\mathrm{d}t, \quad i = 1,\dots,d.$$

For $\mathbf{R}^1 \otimes \mathbf{R}^r$-valued functions b_t and c_t, put

$$C_2(b,c,\theta)_T = \frac{1}{2}\int_{T_0}^T \int_{T_0}^T a_t(\theta)[b_t' c_s + c_t' b_s] a_s'(\theta)\,\mathrm{d}s\,\mathrm{d}t.$$

Let $C_2^{i,j}(t,\theta) = C_2(\lambda_{t,\cdot}^i(\theta) I_{\{\cdot \leqslant t\}}, \lambda_{t,\cdot}^j(\theta) I_{\{\cdot \leqslant t\}}, \theta)_T$. Put $\sigma^{ij}(\theta) = [\Sigma(\theta)]^{ij}$ and $\sigma_{ij}(\theta) = [(\Sigma(\theta))^{-1}]^{ij}$.

Define $A^{0,\alpha}(\theta)$ and $A_{p,q}^{2,\alpha}(\theta)$ by

$$\begin{aligned}
A^{0,\alpha}(\theta) &= \frac{1}{2}\int_{T_0}^T \int_{T_0}^t \partial_i \partial_j f_0^\alpha(X_t^0(\theta),\theta) \lambda_{t,s}^i(\theta)(\lambda_{t,s}^j(\theta))'\,\mathrm{d}s\,\mathrm{d}t + {} \\
&\quad + \frac{1}{2}\int_{T_0}^T \partial_\beta f_0^\alpha(X_t^0(\theta),\theta) \int_{T_0}^t \int_{T_0}^s \nu_{i,j,t,s}^\beta(\theta) \lambda_{s,u}^i(\theta) \times {} \\
&\quad \times (\lambda_{s,u}^j(\theta))'\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t, \\
A_{p,q}^{2,\alpha}(\theta) &= \frac{1}{2}\int_{T_0}^T \partial_i \partial_j f_0^\alpha(X_t^0(\theta),\theta)[C_2^{i,j}(t,\theta)]^{mn}\,\mathrm{d}t\sigma_{pm}(\theta)\sigma_{qn}(\theta) + {} \\
&\quad + [C_2^i(\partial_i f^\alpha(X_\cdot^0(\theta),\theta),\theta)_T]^{mn}\sigma_{pm}(\theta)\sigma_{qn}(\theta) + {} \\
&\quad + \frac{1}{2}\int_{T_0}^T \partial_\beta f_0^\alpha(X_t^0(\theta),\theta) \int_{T_0}^t \nu_{i,j,t,s}^\beta(\theta) \times {} \\
&\quad \times [C_2^{i,j}(t,\theta)]^{mn}\,\mathrm{d}s\,\mathrm{d}t\sigma_{pm}(\theta)\sigma_{qn}(\theta) + {} \\
&\quad + \int_{T_0}^T \partial_\beta f_0^\alpha(X_t^0(\theta),\theta)[C_2^i(\mu_{i,t,\cdot}^\beta(\theta),\theta)_t]^{mn}\,\mathrm{d}t\sigma_{pm}(\theta)\sigma_{qn}(\theta).
\end{aligned}$$

Let $\phi(x;\mu,\Sigma)$ be the probability density function of the k-dimensional normal distribution with mean vector μ and covariance matrix Σ.

THEOREM 1. Let $p > 1$. *Suppose that Assumption* 1 *holds true. Then* $E_{T_0,\hat\theta_\varepsilon(w)}[H_T^\varepsilon(\hat\theta_\varepsilon(w),\tilde w)] \in L_p(\mathbf{R}^k)$ *has the asymptotic expansion*

$$\begin{aligned}
E_{T_0,\hat\theta_\varepsilon(w)}[H_T^\varepsilon(\hat\theta_\varepsilon(w),\tilde w)] &\sim \int_{A_\varepsilon(\hat\theta_\varepsilon(w))} p_0(x,\hat\theta_\varepsilon(w))\,\mathrm{d}x + {} \\
&\quad + \varepsilon \int_{A_\varepsilon(\hat\theta_\varepsilon(w))} p_1(x,\hat\theta_\varepsilon(w))\,\mathrm{d}x + \cdots
\end{aligned}$$

in $L_p(\mathbf{R}^k)$ *as* $\varepsilon \downarrow 0$ *with* $\int_{A_\varepsilon(\hat{\theta}_\varepsilon(w))} p_0(x, \hat{\theta}_\varepsilon(w))\,\mathrm{d}x, \int_{A_\varepsilon(\hat{\theta}_\varepsilon(w))} p_1(x, \hat{\theta}_\varepsilon(w))\,\mathrm{d}x, \ldots$ *in* $L_p(\mathbf{R}^k)$. *In particular*,

$$\begin{aligned} p_0(x,\theta) &= g_{-1}(\theta)(f_{-1}(\theta)-K)\phi(x;0,\Sigma(\theta)),\\ p_1(x,\theta) &= g_{-1}(\theta)x\phi(x;0,\Sigma(\theta)) - g_{-1}(\theta)(f_{-1}(\theta)-K)\big[\big\{A^{2,\alpha}_{\alpha,p}(\theta) +\\ &\quad + A^{2,\alpha}_{p,\alpha}(\theta) - A^{0,\alpha}(\theta)\sigma_{\alpha p}(\theta) + A^{2,\alpha}_{q,l}(\theta)\sigma^{ql}(\theta)\sigma_{\alpha p}(\theta) -\\ &\quad - B^{\alpha}(\theta)\sigma_{\alpha p}(\theta)\big\}x^p - A^{2,\alpha}_{p,q}(\theta)\sigma_{\alpha l}(\theta)x^p x^q x^l\big]\phi(x;0,\Sigma(\theta)). \end{aligned}$$

Remark 2. $\hat{\theta}_\varepsilon(w)$ may be degenerate because we have used the Malliavin calculus on $\tilde{W}$. In particular, we have the same result as in [31] by replacing $\hat{\theta}_\varepsilon(w)$ by θ_0.

ASSUMPTION 2. $f_0(\theta_0, \cdot)$ *in Lemma* 1 *is nondegenerate in the sense of Malliavin, that is*,

$$\det\left(\int_{T_0}^{T} a_t(\theta_0)a_t(\theta_0)'\,\mathrm{d}t\right) > 0.$$

COROLLARY 1. *Suppose that Assumption* 2 *holds true. Then* $E_{T_0,\theta_0}[H^\varepsilon_T(\theta_0,\tilde{w})]$ *has the asymptotic expansion*

$$E_{T_0,\theta_0}[H^\varepsilon_T(\theta_0,\tilde{w})] \sim \int_{A_\varepsilon(\theta_0)} p_0(x,\theta_0)\,\mathrm{d}x + \varepsilon\int_{A_\varepsilon(\theta_0)} p_1(x,\theta_0)\,\mathrm{d}x + \cdots$$

as $\varepsilon \downarrow 0$.

3. Unbiased Estimator

ASSUMPTION 3. *There exists* $R_\varepsilon(w) = \mathrm{O}(1)$ *in* $\cap_{p>1}L_p(W)$ *such that for every* $\tilde{K} > 0$ *and* $c > 0$,

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-\tilde{K}}P[|R_\varepsilon(w)| > c] = 0$$

and for $\psi_\varepsilon(w) = \psi(3R_\varepsilon(w))$ *and* $i = 1, 2, \ldots, p$,

$$\psi_\varepsilon(w)\varepsilon^{-1}(\hat{\theta}_\varepsilon(w)-\theta_0)^i \sim \zeta_i^{(0)}(w) + \tfrac{1}{2}\varepsilon\zeta_i^{(1)}(w) + \cdots$$

in $\cap_{p>1}L_p(W)$ *as* $\varepsilon \downarrow 0$ *with* $\zeta_i^{(0)}(w), \zeta_i^{(1)}(w), \ldots$ *in* $\cap_{p>1}L_p(W)$, *where* $\zeta_i^{(0)}(w) = \int_0^{T_0} g_i(t)\,\mathrm{d}w_t$ *for some function* $g_i \in L^2([0,T_0],\mathrm{d}t)$.

Define $\hat{x}_0(w)$, $\hat{x}_1(w)$, $\bar{f}_{1,i}(\tilde{w})$ and $C_i^{l\alpha}(\theta_0)$ for $i = 1,\ldots,p$ and $l,\alpha = 1,\ldots,k$ by

$$\begin{aligned} \hat{x}_0(w) &= \int_{T_0}^{T}\delta_i^{(2)} f_0(X_t^0(\theta_0),\theta_0)\,\mathrm{d}t\,\zeta_i^{(0)}(w) +\\ &\quad + \int_{T_0}^{T}\partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\delta_i X_t^{0,i_1}(\theta_0)\,\mathrm{d}t\,\zeta_i^{(0)}(w), \end{aligned}$$

$$
\begin{aligned}
\hat{x}_1(w) &= \frac{1}{2}\int_{T_0}^{T} \delta_i^{(2)} f_0(X_t^0(\theta_0),\theta_0)\,\mathrm{d}t\,\zeta_i^{(1)}(w) + \\
&\quad + \frac{1}{2}\int_{T_0}^{T} \partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\delta_i X_t^{0,i_1}(\theta_0)\,\mathrm{d}t\,\zeta_i^{(1)}(w) + \\
&\quad + \frac{1}{2}\int_{T_0}^{T} \delta_i^{(2)}\delta_j^{(2)} f_0(X_t^0(\theta_0),\theta_0)\,\mathrm{d}t\,\zeta_i^{(0)}(w)\zeta_j^{(0)}(w) + \\
&\quad + \frac{1}{2}\int_{T_0}^{T} \partial_{i_1}\partial_{i_2} f_0(X_t^0(\theta_0),\theta_0) \times \\
&\qquad \times \delta_i X_t^{0,i_1}(\theta_0)\delta_j X_t^{0,i_2}(\theta_0)\,\mathrm{d}t\,\zeta_i^{(0)}(w)\zeta_j^{(0)}(w) + \\
&\quad + \frac{1}{2}\int_{T_0}^{T} \partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\delta_i\delta_j X_t^{0,i_1}(\theta_0)\,\mathrm{d}t\,\zeta_i^{(0)}(w)\zeta_j^{(0)}(w) + \\
&\quad + \int_{T_0}^{T} \partial_{i_1}\delta_j^{(2)} f_0(X_t^0(\theta_0),\theta_0)\delta_i X_t^{0,i_1}(\theta_0)\,\mathrm{d}t\,\zeta_i^{(0)}(w)\zeta_j^{(0)}(w), \\
\hat{y}_0(w) &= -\int_{T_0}^{T}\int_{T_0}^{s} \big\{\partial_{i_1} h_0(X_s^0(\theta_0),\theta_0)\delta_i X_t^{0,i_1}(\theta_0) + \\
&\quad + \delta_i^{(2)} h_0(X_s^0(\theta_0),\theta_0)\big\}\,\mathrm{d}t\,\mathrm{d}s\,\zeta_i^{(0)}(w), \\
\bar{f}_{1,i}(\tilde{w}) &= \int_{T_0}^{T} \partial_{i_1}\delta_i^{(2)} f_0(X_t^0(\theta_0),\theta_0) D_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}t + \\
&\quad + \int_{T_0}^{T} \partial_{i_1}\partial_{i_2} f_0(X_t^0(\theta_0),\theta_0) D_t^{i_1}(\theta_0,\tilde{w})\delta_i X_t^{0,i_2}(\theta_0)\,\mathrm{d}t + \\
&\quad + \int_{T_0}^{T} \partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\delta_i D_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}t + \\
&\quad + \int_{T_0}^{T} \delta_i^{(2)} f(X_t^0(\theta_0),\theta_0)\,\mathrm{d}\tilde{w}_t + \\
&\quad + \int_{T_0}^{T} \partial_{i_1} f(X_t^0(\theta_0),\theta_0)\delta_i X_t^{0,i_1}(\theta_0)\,\mathrm{d}\tilde{w}_t \\
&:= \int_{T_0}^{T} c_{t,i}(\theta_0)\,\mathrm{d}\tilde{w}_t, \\
C_i^{l\alpha}(\theta_0) &= \int_{T_0}^{T} \left[a_t(\theta_0)c'_{t,i}(\theta_0)\right]^{l\alpha}\,\mathrm{d}t,
\end{aligned}
$$

where $\delta_i = \partial/\partial\theta^i$ and $\delta_i^{(2)} f_0(X_t^0(\theta_0),\theta_0) = \delta_i f_0(X_t^0(\theta_0),\theta)|_{\theta=\theta_0}$.

LEMMA 5. *Let $\bar{\psi}_\varepsilon(w) = \psi(9R_\varepsilon(w))$. Suppose that Assumption* 3 *holds true. Then $\bar{\psi}_\varepsilon(w)F_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}) \in D^\infty(\bar{W};\mathbf{R}^k)$ and it has the asymptotic expansion*

$$
\bar{\psi}_\varepsilon(w)F_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}) \sim f_{-1}(\theta_0) + \varepsilon\hat{f}_0(w,\tilde{w}) + \varepsilon^2\hat{f}_1(w,\tilde{w}) + \cdots
$$

in $D^{\infty}(\bar{W}; \mathbf{R}^k)$ *as* $\varepsilon \downarrow 0$ *with* $f_{-1}(\theta_0)$, $\hat{f}_0(w, \tilde{w})$, $\hat{f}_1(w, \tilde{w}), \ldots \in D^{\infty}(\bar{W}; \mathbf{R}^k)$. *In particular*,

$$\hat{f}_0(w, \tilde{w}) = f_0(\theta_0, \tilde{w}) + \hat{x}_0(w), \tag{21}$$
$$\hat{f}_1(w, \tilde{w}) = f_1(\theta_0, \tilde{w}) + \bar{f}_{1,i}(\tilde{w})\zeta_i^{(0)}(w) + \hat{x}_1(w). \tag{22}$$

LEMMA 6. *Suppose that Assumption* 3 *holds true. Then* $\bar{\psi}_{\varepsilon}(w)G_T^{\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w}) \in D^{\infty}(\bar{W})$ *and it has the asymptotic expansion*

$$\bar{\psi}_{\varepsilon}(w)G_T^{\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w}) \sim g_{-1}(\theta_0) + \varepsilon \hat{g}_0(w, \tilde{w}) + \cdots$$

in $D^{\infty}(\bar{W})$ *as* $\varepsilon \downarrow 0$ *with* $g_{-1}(\theta_0)$, $\hat{g}_0(w, \tilde{w}), \ldots \in D^{\infty}(\bar{W})$. *In particular*,

$$\hat{g}_0(w, \tilde{w}) = g_{-1}(\theta_0)\int_{T_0}^{T} b_t(\theta_0)\, \mathrm{d}\tilde{w}_t + g_{-1}(\theta_0)\hat{y}_0(w). \tag{23}$$

LEMMA 7. *Let* $\tilde{F}_T^{*\varepsilon}(\theta, \tilde{w}) = [\bar{\psi}_{\varepsilon}(w)F_T^{\varepsilon}(\theta, \tilde{w}) - f_{-1}(\theta_0)]/\varepsilon$ *and* $\hat{\psi}_{\varepsilon}^{*}(w, \tilde{w}) = \psi(\hat{\xi}_{\varepsilon}(w, \tilde{w}))$, *where* $\hat{\xi}_{\varepsilon}(w, \tilde{w}) = 2(\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w), \cdot)} - \det \sigma_{\hat{f}_0(w, \cdot)})/\det \sigma_{\hat{f}_0(w, \cdot)}$. *Suppose that Assumptions* 2 *and* 3 *hold true. Then for any* $T(x) \in S'(\mathbf{R}^k)$, $\hat{\psi}_{\varepsilon}^{*}(w, \tilde{w})T(\tilde{F}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w})) \in \tilde{D}^{-\infty}(\bar{W})$ *has the asymptotic expansion*

$$\hat{\psi}_{\varepsilon}^{*}(w, \tilde{w})T(\tilde{F}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w})) \sim \hat{\Phi}_0(w, \tilde{w}) + \varepsilon\hat{\Phi}_1(w, \tilde{w}) + \cdots$$

in $\tilde{D}^{-\infty}(\bar{W})$ *as* $\varepsilon \downarrow 0$ *uniformly in every class* $\{T\}$ *satisfying the condition* (5) *of Theorem* 4.1 *in* [31], *and* $\hat{\Phi}_0(w, \tilde{w})$, $\hat{\Phi}_1(w, \tilde{w}), \ldots$ *in* $\tilde{D}^{-\infty}(\bar{W})$ *are determined by the formal Taylor expansion. In particular*,

$$\hat{\Phi}_0(w, \tilde{w}) = T(\hat{f}_0(w, \tilde{w})),$$
$$\hat{\Phi}_1(w, \tilde{w}) = \sum_i \hat{f}_1^i(w, \tilde{w})\partial_i T(\hat{f}_0(w, \tilde{w})).$$

From Lemmas 2.1 and 2.2 in [30] and Lemmas 5 and 7, we see that the composite functional

$$\bar{\psi}_{\varepsilon}(w)(F_T^{\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w}) - K)\hat{\psi}_{\varepsilon}^{*}(w, \tilde{w})1_{A_{\varepsilon}(\theta_0)}(\tilde{F}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w}))$$
$$=: \bar{F}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w}) \quad \text{(say)}$$

is well-defined and we define $\bar{H}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w}) = \bar{\psi}_{\varepsilon}(w)G_T^{\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w})\bar{F}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w})$.

LEMMA 8. *Suppose that Assumptions* 2 *and* 3 *hold true. Then* $\bar{H}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w}) \in \tilde{D}^{-\infty}(\bar{W}; \mathbf{R}^k)$ *has the asymptotic expansion*

$$\bar{H}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w), \tilde{w}) \sim \hat{\Xi}_0(w, \tilde{w}) + \varepsilon\hat{\Xi}_1(w, \tilde{w}) + \cdots$$

in $\tilde{D}^{-\infty}(\bar{W};\mathbf{R}^k)$ *as* $\varepsilon \downarrow 0$, *and* $\hat{\Xi}_0(w,\tilde{w}), \hat{\Xi}_1(w,\tilde{w}),\dots$ *in* $\tilde{D}^{-\infty}(\bar{W};\mathbf{R}^k)$ *are determined by the formal Taylor expansion. In particular,*

$$\begin{aligned}
\hat{\Xi}_0(w,\tilde{w}) &= g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w,\tilde{w})),\\
\hat{\Xi}_1(w,\tilde{w}) &= g_{-1}(\theta_0)\hat{f}_0(w,\tilde{w})1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w,\tilde{w}))+\\
&\quad + g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)\hat{f}_1^i(w,\tilde{w})\partial_i 1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w,\tilde{w}))+\\
&\quad + (f_{-1}(\theta_0)-K)\hat{g}_0(w,\tilde{w})1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w,\tilde{w})).
\end{aligned}$$

THEOREM 2. *Let* $p>1$. *Suppose that Assumptions* 2 *and* 3 *hold true. Then* $E_{T_0,\hat{\theta}_\varepsilon(w)}[\bar{H}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\tilde{w})] \in L_p(\mathbf{R}^k)$ *has the asymptotic expansion*

$$E_{T_0,\hat{\theta}_\varepsilon(w)}[\bar{H}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\tilde{w})] \sim \int_{A_\varepsilon(\theta_0)} \hat{p}_0(x,w)\,\mathrm{d}x + \varepsilon\int_{A_\varepsilon(\theta_0)} \hat{p}_1(x,w)\,\mathrm{d}x + \cdots$$

in $L_p(\mathbf{R}^k)$ *as* $\varepsilon\downarrow 0$ *with* $\int_{A_\varepsilon(\theta_0)}\hat{p}_0(x,w)\,\mathrm{d}x, \int_{A_\varepsilon(\theta_0)}\hat{p}_1(x,w)\,\mathrm{d}x,\dots$ *in* $L_p(\mathbf{R}^k)$. *In particular,*

$$\begin{aligned}
\hat{p}_0(x,w) &= g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)\phi(x;\hat{x}_0(w),\Sigma(\theta_0)),\\
\hat{p}_1(x,w) &= g_{-1}(\theta_0)x\phi(x;\hat{x}_0(w),\Sigma(\theta_0)) - g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)\times\\
&\quad\times\big[C_i^{l\alpha}(\theta_0)\sigma_{l\alpha}(\theta_0)\zeta_i^{(0)}(w) - \hat{y}_0(w) + \big\{A^{2,\alpha}_{\alpha,p}(\theta_0) + A^{2,\alpha}_{p,\alpha}(\theta_0) -\\
&\quad - \big(A^{0,\alpha}(\theta_0) - A^{2,\alpha}_{l,q}(\theta_0)\sigma^{lq}(\theta_0) + \hat{x}_1^\alpha(w) + B^\alpha(\theta_0)\big)\sigma_{\alpha p}(\theta_0)\big\}\times\\
&\quad\times(x-\hat{x}_0(w))^p - C_i^{l\alpha}(\theta_0)\sigma_{lp}(\theta_0)\sigma_{\alpha q}(\theta_0)\zeta_i^{(0)}(w)\times\\
&\quad\times(x-\hat{x}_0(w))^p(x-\hat{x}_0(w))^q - A^{2,\alpha}_{p,q}(\theta_0)\sigma_{\alpha l}(\theta_0)\times\\
&\quad\times(x-\hat{x}_0(w))^p(x-\hat{x}_0(w))^q(x-\hat{x}_0(w))^l\big]\times\\
&\quad\times\phi(x;\hat{x}_0(w),\Sigma(\theta_0)).
\end{aligned}$$

We here consider the first-order asymptotically expectation-unbiased (AEU) estimator of $E_{T_0,\theta_0}[H_T^\varepsilon(\theta_0,\tilde{w})]$. Let $\Sigma_1(\theta_0) = \Sigma(\theta_0) + \mathrm{Var}[\hat{x}_0(w)]$, where $\mathrm{Var}[\hat{x}_0(w)] = \int_0^{T_0} d_t(\theta_0)d_t(\theta_0)'\,\mathrm{d}t$ and

$$d_s(\theta_0) = \left(\int_{T_0}^T \left\{\delta_i^{(2)} f_0(X_t^0(\theta_0),\theta_0) + \partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\delta_i X_t^{0,i_1}(\theta_0)\right\}\mathrm{d}t\right) g_i(s).$$

THEOREM 3. *Suppose that Assumptions* 2 *and* 3 *hold true. Let*

$$b_1(\theta_0) = g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)\int_{A_\varepsilon(\theta_0)}\{\phi(x;0,\Sigma_1(\theta_0)) - \phi(x;0,\Sigma(\theta_0))\}\,\mathrm{d}x.$$

Then $E_{T_0,\hat{\theta}_\varepsilon(w)}[H_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w})] - b_1(\hat{\theta}_\varepsilon(w))$ *is the first-order AEU estimator of* $E_{T_0,\theta_0}[H_T^\varepsilon(\theta_0,\tilde{w})]$.

Next, we obtain the second-order AEU estimator of $E_{T_0,\theta_0}[H_T^\varepsilon(\theta_0,\tilde{w})]$.

THEOREM 4. *Suppose that Assumptions* 2 *and* 3 *hold true. Let*

$$b_2(\theta_0) = \int_W \int_{A_\varepsilon(\theta_0)} \hat{p}_1(x, w)\,\mathrm{d}x\,P(dw) - \int_{A_\varepsilon(\theta_0)} p_1(x, \theta_0)\,\mathrm{d}x.$$

Then $E_{T_0,\hat{\theta}_\varepsilon(w)}[H_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})] - b_1(\hat{\theta}_\varepsilon(w)) - \varepsilon b_2(\hat{\theta}_\varepsilon(w))$ *is the second-order AEU estimator of* $E_{T_0,\theta_0}[H_T^\varepsilon(\theta_0, \tilde{w})]$.

Suppose that a one-dimensional small diffusion F satisfies (3), that is, $k = 1$. From Theorems 3 and 4, we obtain the second-order AEU estimator of $E_{T_0,\theta_0}[H_T^\varepsilon(\theta_0, \tilde{w})]$.

ASSUMPTION 4. *There exists* $t \in [T_0, T]$ *such that* $a_t \neq 0$.

THEOREM 5. *Suppose that* F_T^ε *satisfies* (3) for $k = 1$. *Moreover, suppose that Assumptions* 3 *and* 4 *hold true. Let*

$$b_2^{(1)}(\theta) = g_{-1}(\theta) \int_{A_\varepsilon(\theta)} x\,\{\phi(x; 0, \Sigma_1(\theta) - \phi(x; 0, \Sigma(\theta))\}\,\mathrm{d}x.$$

Then $E_{T_0,\hat{\theta}_\varepsilon(w)}[H_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})] - b_1(\hat{\theta}_\varepsilon(w)) - \varepsilon b_2^{(1)}(\hat{\theta}_\varepsilon(w))$ *is the second-order AEU estimator of* $E_{T_0,\theta_0}[H_T^\varepsilon(\theta_0, \tilde{w})]$.

Remark 3. To derive the results in Theorems 3–5, we can relax the conditions of C^∞ smoothness about the drift and the diffusion coefficient functions appearing in the stochastic differential equations concerned with these theorems.

4. Proofs

Proof of Lemma 1. Expanding X_t^ε in a Taylor series around X_t^0, we have

$$\begin{aligned}
&\int_{T_0}^T f_0(X_t^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}), \hat{\theta}_\varepsilon(w))\,\mathrm{d}t \\
&\quad = \int_{T_0}^T f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w))\,\mathrm{d}t + \\
&\qquad + \varepsilon \int_{T_0}^T \partial_i f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) D_t^i(\hat{\theta}_\varepsilon(w), \tilde{w})\,\mathrm{d}t + \\
&\qquad + \varepsilon^2 \left\{ \frac{1}{2} \int_{T_0}^T \partial_i \partial_j f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) D_t^i(\hat{\theta}_\varepsilon(w), \tilde{w}) D_t^j(\hat{\theta}_\varepsilon(w), \tilde{w})\,\mathrm{d}t + \right. \\
&\qquad \left. + \frac{1}{2} \int_{T_0}^T \partial_i f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) E_t^i(\hat{\theta}_\varepsilon(w), \tilde{w})\,\mathrm{d}t \right\} + \\
&\qquad + \varepsilon^3 f_2^{(2)}(X_t^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}), \hat{\theta}_\varepsilon(w)),
\end{aligned}$$

where

$$
\begin{aligned}
& f_2^{(2)}(X_t^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}), \hat{\theta}_\varepsilon(w)) \\
&\quad = \frac{1}{2} \int_0^1 \int_{T_0}^T \frac{\partial^3}{\partial \varepsilon^3} f_0(X_t^\varepsilon(\theta, \tilde{w}), \theta) \, \mathrm{d}t \bigg|_{\varepsilon \leftarrow u\varepsilon} (1-u)^2 \, \mathrm{d}u \bigg|_{\theta \leftarrow \hat{\theta}_\varepsilon(w)} .
\end{aligned}
$$

From $f_2^{(2)}(X_t^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}), \hat{\theta}_\varepsilon(w)) = \mathrm{O}_M(1)$, where $\mathrm{O}_M(1)$ means $\mathrm{O}(1)$ in $D^\infty(\bar{W}; \mathbf{R}^k)$, we have

$$
\begin{aligned}
& \int_{T_0}^T f_0(X_t^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}), \hat{\theta}_\varepsilon(w)) \, \mathrm{d}t \\
&\quad = \int_{T_0}^T f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) \, \mathrm{d}t + \\
&\qquad + \varepsilon \int_{T_0}^T \partial_i f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) D_t^i(\hat{\theta}_\varepsilon(w), \tilde{w}) \, \mathrm{d}t + \\
&\qquad + \varepsilon^2 \left\{ \frac{1}{2} \int_{T_0}^T \partial_i \partial_j f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) D_t^i(\hat{\theta}_\varepsilon(w), \tilde{w}) D_t^j(\hat{\theta}_\varepsilon(w), \tilde{w}) \, \mathrm{d}t + \right. \\
&\qquad \left. + \frac{1}{2} \int_{T_0}^T \partial_i f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) E_t^i(\hat{\theta}_\varepsilon(w), \tilde{w}) \, \mathrm{d}t \right\} + \mathrm{O}_M(\varepsilon^3).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \varepsilon \int_{T_0}^T f(X_t^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}), \hat{\theta}_\varepsilon(w)) \, \mathrm{d}\tilde{w}_t \\
&\quad = \varepsilon \int_{T_0}^T f(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) \, \mathrm{d}\tilde{w}_t + \\
&\qquad + \varepsilon^2 \int_{T_0}^T \partial_i f(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) D_t^i(\hat{\theta}_\varepsilon(w), \tilde{w}) \, \mathrm{d}\tilde{w}_t + \mathrm{O}_M(\varepsilon^3).
\end{aligned}
$$

We then have

$$
\begin{aligned}
& F_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \\
&\quad = F_{T_0} + \int_{T_0}^T f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) \, \mathrm{d}t + \\
&\qquad + \varepsilon \left\{ \int_{T_0}^T \partial_i f_0(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) D_t^i(\hat{\theta}_\varepsilon(w), \tilde{w}) \, \mathrm{d}t + \right. \\
&\qquad \left. + \int_{T_0}^T f(X_t^0(\hat{\theta}_\varepsilon(w)), \hat{\theta}_\varepsilon(w)) \, \mathrm{d}\tilde{w}_t \right\} +
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon^2\left\{\frac{1}{2}\int_{T_0}^{T}\partial_i\partial_j f_0(X_t^0(\hat\theta_\varepsilon(w)),\hat\theta_\varepsilon(w))D_t^i(\hat\theta_\varepsilon(w),\tilde w)D_t^j(\hat\theta_\varepsilon(w),\tilde w)\,\mathrm{d}t+\right.\\
&+\frac{1}{2}\int_{T_0}^{T}\partial_i f_0(X_t^0(\hat\theta_\varepsilon(w)),\hat\theta_\varepsilon(w))E_t^i(\hat\theta_\varepsilon(w),\tilde w)\,\mathrm{d}t+\\
&\left.+\int_{T_0}^{T}\partial_i f(X_t^0(\hat\theta_\varepsilon(w)),\hat\theta_\varepsilon(w))D_t^i(\hat\theta_\varepsilon(w),\tilde w)\,\mathrm{d}\tilde w_t\right\}+\\
&+\mathrm{O}_M(\varepsilon^3).
\end{aligned}
$$

From the second-order stochastic expansion of $F_T^\varepsilon(\hat\theta_\varepsilon(w),\tilde w)$ as above, f_0 and f_1 are determined.

In the same way as the second-order stochastic expansion, we can see that for any $m\in\mathbf{N}$,

$$
\begin{aligned}
F_T^\varepsilon(\hat\theta_\varepsilon(w),\tilde w) \;=\; & f_{-1}(\hat\theta_\varepsilon(w))+\varepsilon f_0(\hat\theta_\varepsilon(w),\tilde w)+\varepsilon^2 f_1(\hat\theta_\varepsilon(w),\tilde w)+\cdots+\\
&+\varepsilon^{m-1}f_{m-1}(\hat\theta_\varepsilon(w),\tilde w)+\varepsilon^m f_\varepsilon^{(m)}(\hat\theta_\varepsilon(w),\tilde w),
\end{aligned}
$$

where $f_\varepsilon^{(m)}(\hat\theta_\varepsilon(w),\tilde w)=\mathrm{O}_M(1)$. This completes the proof.

Proof of Lemma 2. In the same way as the proof of Lemma 1, we can show the result.

Proof of Lemma 3. From the definition of ξ_ε, $|\xi_\varepsilon(\hat\theta_\varepsilon(w),\tilde w)|>1$ if $\det\sigma_{\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\cdot)}<(1/2)\det\sigma_{f_0(\hat\theta_\varepsilon(w),\cdot)}$ or $\det\sigma_{\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\cdot)}>(3/2)\det\sigma_{f_0(\hat\theta_\varepsilon(w),\cdot)}$. In case that $(3/2)\det\sigma_{f_0(\hat\theta_\varepsilon(w),\cdot)}\geqslant\det\sigma_{\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\cdot)}\geqslant(1/2)\det\sigma_{f_0(\hat\theta_\varepsilon(w),\cdot)}$, we obtain $(2/3)\ (\det\sigma_{f_0(\hat\theta_\varepsilon(w),\cdot)})^{-1}\leqslant(\det\sigma_{\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\cdot)})^{-1}\leqslant 2(\det\sigma_{f_0(\hat\theta_\varepsilon(w),\cdot)})^{-1}$. Therefore, it follows from Assumption 1 that for any $p\in(1,\infty)$,

$$
\sup_{\varepsilon\in(0,1]}E[1_{\{|\xi_\varepsilon(\hat\theta_\varepsilon(w),\tilde w)|\leqslant 1\}}(\det\sigma_{\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\cdot)})^{-p}]<\infty.
$$

From Lemma 1, $\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\tilde w)\in D^\infty(\bar W;\mathbf{R}^k)$ has the asymptotic expansion

$$
\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\tilde w)\sim f_0(\hat\theta_\varepsilon(w),\tilde w)+\varepsilon f_1(\hat\theta_\varepsilon(w),\tilde w)+\cdots
$$

in $D^\infty(\bar W;\mathbf{R}^k)$ as $\varepsilon\downarrow 0$ with $f_0(\hat\theta_\varepsilon(w),\tilde w),\ f_1(\hat\theta_\varepsilon(w),\tilde w),\ldots\in D^\infty(\bar W;\mathbf{R}^k)$. We then have $\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\tilde w)=f_0(\hat\theta_\varepsilon(w),\tilde w)+\varepsilon f_\varepsilon^{(1)}(\hat\theta_\varepsilon(w),\tilde w)$, where $f_0(\hat\theta_\varepsilon(w),\tilde w)\in D^\infty(\bar W;\mathbf{R}^k)$ and $f_\varepsilon^{(1)}(\hat\theta_\varepsilon(w),\tilde w)=\mathrm{O}_M(1)$. Since

$$
\begin{aligned}
&|(\det\sigma_{\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\cdot)})^{1/2}-(\det\sigma_{f_0(\hat\theta_\varepsilon(w),\cdot)})^{1/2}|\\
&\quad\leqslant(\det\sigma_{\tilde F_T^\varepsilon(\hat\theta_\varepsilon(w),\cdot)-f_0(\hat\theta_\varepsilon(w),\cdot)})^{1/2}=(\det\sigma_{\varepsilon f_\varepsilon^{(1)}(\hat\theta_\varepsilon(w),\cdot)})^{1/2}\\
&\quad=\varepsilon(\det\sigma_{f_\varepsilon^{(1)}(\hat\theta_\varepsilon(w),\cdot)})^{1/2},
\end{aligned}
$$

we have

$$
\begin{aligned}
&|\det \sigma_{\tilde{F}^{\varepsilon}_T(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)}| \\
&\quad = |(\det \sigma_{\tilde{F}^{\varepsilon}_T(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2} - (\det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2}||(\det \sigma_{\tilde{F}^{\varepsilon}_T(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2} + \\
&\qquad + (\det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2}| \\
&\quad \leqslant \varepsilon (\det \sigma_{f^{(1)}_\varepsilon(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2} \left\{ (\det \sigma_{\tilde{F}^{\varepsilon}_T(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2} + (\det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2} \right\}.
\end{aligned}
$$

Hence, it follows that there exists $C > 0$ such that for any $p > 1$,

$$
\| \det \sigma_{\tilde{F}^{\varepsilon}_T(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)} \|_p \leqslant C\varepsilon. \tag{24}
$$

From the definition of ξ_ε, $\xi_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) = \mathrm{O}(1)$ in $D^\infty(\bar{W})$ as $\varepsilon \downarrow 0$. It follows from Chebyshev's inequality that for any $a > 0$, $K > 0$,

$$
\begin{aligned}
\bar{P}[\xi_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) > a] \;&\leqslant\; \bar{P}\left[\left| \frac{2(\det \sigma_{\tilde{F}^{\varepsilon}_T(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)})}{\det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)}} \right| > a \right] \\
&\leqslant\; \frac{1}{a^K} \left\| \frac{2(\det \sigma_{\tilde{F}^{\varepsilon}_T(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)})}{\det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)}} \right\|_K^K .
\end{aligned}
$$

It follows from (24), Hölder's inequality and Assumption 1 that for any $K > 0$,

$$
\left\| \frac{2(\det \sigma_{\tilde{F}^{\varepsilon}_T(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)})}{\det \sigma_{f_0(\hat{\theta}_\varepsilon(w),\cdot)}} \right\|_K^K = \mathrm{O}(\varepsilon^K).
$$

We then see that for any $a > 0$, $K > 0$,

$$
\bar{P}[\xi_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) > a] = \mathrm{O}(\varepsilon^K). \tag{25}
$$

From (25), we see that for any $p > 1$,

$$
\begin{aligned}
\|1 - \psi^*_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})\|_p \;&=\; \|1 - \psi(\xi_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}))\|_p \leqslant \|1_{\left\{|\xi_\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w})| > \frac{1}{2}\right\}}\|_p \\
&=\; \mathrm{O}(\varepsilon^K).
\end{aligned}
$$

In view of the chain rule for H-derivatives,

$$
\begin{aligned}
D(1 - \psi^*_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})) \;&=\; -D\psi^*_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) \\
&=\; -\psi'(\xi_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}))D\xi_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}).
\end{aligned}
$$

Since $\|\psi'(\xi_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}))\|_p = \mathrm{O}(\varepsilon^K)$, we see that for any $p > 1$,

$$
\|D(1 - \psi^*_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}))\|_p = \mathrm{O}(\varepsilon^K).
$$

Similarly, it follows that for any $p > 1$ and $j > 0$,

$$
\|D^j(1 - \psi^*_\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}))\|_p = \mathrm{O}(\varepsilon^K).
$$

Therefore, we see that for any $K > 0$, $\psi_\varepsilon^*(\hat{\theta}_\varepsilon(w), \tilde{w}) = 1 - \mathrm{O}(\varepsilon^K)$ in $D^\infty(\bar{W})$. By using Theorem 4.1 in [31],

$$\psi_\varepsilon^*(\hat{\theta}_\varepsilon(w), \tilde{w}) T(\tilde{F}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w})) \sim \Phi_0(\hat{\theta}_\varepsilon(w), \tilde{w}) + \varepsilon \Phi_1(\hat{\theta}_\varepsilon(w), \tilde{w}) + \cdots$$

in $\tilde{D}^{-\infty}(\bar{W})$ as $\varepsilon \downarrow 0$ with $\Phi_0(\hat{\theta}_\varepsilon(w), \tilde{w}), \Phi_1(\hat{\theta}_\varepsilon(w), \tilde{w}), \ldots$ in $\tilde{D}^{-\infty}(\bar{W})$. This completes the proof.

Proof of Lemma 4. As in the proof of Lemma 3, using Theorem 4.1 in [31], we obtain the result.

Proof of Theorem 1. For every $G \in \tilde{D}^{-\infty}(\bar{W})$, there exists $m \in Z_+$ such that

$$(I - \tilde{L})^{-m} G \in \bigcap_{p>1} L_p(\bar{P}).$$

Denote by $J(w)$ and $\tilde{J}(\tilde{w})$ Wiener functionals (over $\bar{W}$) depending only on W and $\tilde{W}$, respectively. Put

$$I(G, \tilde{J})(w) = \int_{\tilde{W}} (I - \tilde{L})^{-m} G(w, \tilde{w})(I - \tilde{L})^m \tilde{J}(\tilde{w}) \tilde{P}(\mathrm{d}\tilde{w}).$$

For any $J(w) \in L_p(W, P)$ and $\tilde{J}(\tilde{w}) \in D^\infty(\tilde{W}, \tilde{P})$,

$$\begin{aligned} \int_W I(G, \tilde{J})(w) J(w) P(\mathrm{d}w) \ &= \ \int_{\bar{W}} (I - \tilde{L})^{-m} G(w, \tilde{w})(I - \tilde{L})^m \times \\ &\qquad \times (\tilde{J}(\tilde{w}) J(w)) \bar{P}(\mathrm{d}w, \mathrm{d}\tilde{w}) \\ &= \ {}_{D^{-\infty}}\langle G(w, \tilde{w}), \tilde{J}(\tilde{w}) J(w) \rangle_{D^\infty}. \end{aligned}$$

In particular, for $p_1, p_2, p' > 1$ such that $1/p_1 + 1/p_2 + 1/p' = 1$,

$$\left| \int_W I(G, \tilde{J})(w) J(w) P(\mathrm{d}w) \right| \leqslant \|G(w, \tilde{w})\|_{p_1, -2m} \cdot \left\| \tilde{J}(\tilde{w}) \right\|_{p_2, 2m} \cdot \|J(w)\|_{p'}.$$

Thus,

$$\left\| I(G, \tilde{J})(w) \right\|_p \leqslant \|G(w, \tilde{w})\|_{p_1, -2m} \cdot \left\| \tilde{J}(\tilde{w}) \right\|_{p_2, 2m},$$

where $p = p'/(p' - 1)$. Put

$$\begin{aligned} G_\varepsilon(w, \tilde{w}) \ &= \ \bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}) - \big(\Xi_0(\hat{\theta}_\varepsilon(w), \tilde{w}) + \varepsilon \Xi_1(\hat{\theta}_\varepsilon(w), \tilde{w}) + \cdots + \\ &\qquad + \varepsilon^{k-1} \Xi_{k-1}(\hat{\theta}_\varepsilon(w), \tilde{w}) \big). \end{aligned}$$

We then have

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{-k} \left\| I(\bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w), \tilde{w}), \tilde{J}) - \sum_{\gamma=0}^{k-1} \varepsilon^\gamma I(\Xi_\gamma(\hat{\theta}_\varepsilon(w), \tilde{w}), \tilde{J}) \right\|_p < \infty. \tag{26}$$

Since

$$\begin{aligned}\int_W I(\bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}),\tilde{J})J(w)P(\mathrm{d}w) &= {}_{D^{-\infty}}\langle \bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}),\tilde{J}(\tilde{w})J(w)\rangle_{D^\infty}\\ &= \int_{\bar{W}} \bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w})\tilde{J}(\tilde{w})J(w)\times\\ &\quad\times \bar{P}(\mathrm{d}w,\mathrm{d}\tilde{w})\\ &= \int_W E^{\tilde{P}}\left[\bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w})\tilde{J}(\tilde{w})\right]\times\\ &\quad\times J(w)P(\mathrm{d}w),\end{aligned}$$

in particular, we know

$$I(\bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}),1)(w) = E^{\tilde{P}}\left[\bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w})\right]$$

P-a.s. It follows from $\psi_\varepsilon^*(\hat{\theta}_\varepsilon(w),\tilde{w}) = 1 - \mathrm{O}(\varepsilon^K)$ in $D^\infty(\bar{W})$ for any $K > 0$ that

$$\limsup_{\varepsilon\downarrow 0}\varepsilon^{-k}\left\|E^{\tilde{P}}\left[H_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w})\right] - E^{\tilde{P}}\left[\bar{H}_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w})\right]\right\|_p < \infty. \tag{27}$$

From (26) and (27), we have

$$\limsup_{\varepsilon\downarrow 0}\varepsilon^{-k}\left\|E^{\tilde{P}}\left[H_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w})\right] - \sum_{\gamma=0}^{k-1}\varepsilon^\gamma I(\Xi_\gamma(\hat{\theta}_\varepsilon(w),\tilde{w}),1)\right\|_p < \infty.$$

We can easily show that

$$\begin{aligned}I(\Xi_0(\hat{\theta}_\varepsilon(w),\tilde{w}),1)(w) &= \int_{\tilde{W}} \Xi_0(\hat{\theta}_\varepsilon(w),\tilde{w})\tilde{P}(\mathrm{d}\tilde{w})\\ &= \int_{\tilde{W}} g_{-1}(\hat{\theta}_\varepsilon(w))(f_{-1}(\hat{\theta}_\varepsilon(w)) - K)1_{A_\varepsilon(\hat{\theta}_\varepsilon(w))}\times\\ &\quad\times(f_0(\hat{\theta}_\varepsilon(w),\tilde{w}))\tilde{P}(\mathrm{d}\tilde{w})\\ &= \int_{A_\varepsilon(\hat{\theta}_\varepsilon(w))} p_0(x,\hat{\theta}_\varepsilon(w))\mathrm{d}x\end{aligned}$$

P-a.s., where $p_0(x,\theta) = g_{-1}(\theta)(f_{-1}(\theta) - K)p_{f_0}(x,\theta)$ and $p_{f_0}(x,\theta) = \phi(x; 0,\Sigma(\theta))$.

Next, we obtain the second-order term $p_1(x,\theta)$. It is easy to obtain that

$$\begin{aligned}&I(g_{-1}(\hat{\theta}_\varepsilon(w))f_0(\hat{\theta}_\varepsilon(w),\tilde{w})1_{A_\varepsilon(\hat{\theta}_\varepsilon(w))}(f_0(\hat{\theta}_\varepsilon(w),\tilde{w})),1)(w)\\ &\quad= \int_{A_\varepsilon(\hat{\theta}_\varepsilon(w))} q_0(x,\hat{\theta}_\varepsilon(w))\,\mathrm{d}x\end{aligned}$$

P-a.s., where $q_0(x,\theta) = g_{-1}(\theta) x p_{f_0}(x,\theta)$. For any $J(w) \in L_p(W,P)$,

$$
\begin{aligned}
&\int_W I(f_1^i(\hat{\theta}_\varepsilon(w),\tilde{w})\partial_i 1_{A_\varepsilon(\hat{\theta}_\varepsilon(w))}(f_0(\hat{\theta}_\varepsilon(w),\tilde{w})),1) J(w) P(\mathrm{d}w) \\
&\quad = {}_{D^{-\infty}}\langle f_1^i(\hat{\theta}_\varepsilon(w),\tilde{w})\partial_i 1_{A_\varepsilon(\hat{\theta}_\varepsilon(w))}(f_0(\hat{\theta}_\varepsilon(w),\tilde{w})), J(w)\rangle_{D^\infty} \\
&\quad = {}_{D^{-\infty}}\Bigg\langle f_1^i(\hat{\theta}_\varepsilon(w),\tilde{w})\partial_i 1_{A_\varepsilon(\theta_0)} \times \\
&\qquad \times \left(f_0(\hat{\theta}_\varepsilon(w),\tilde{w}) + \frac{f_{-1}(\hat{\theta}_\varepsilon(w)) - f_{-1}(\theta_0)}{\varepsilon} \right), J(w) \Bigg\rangle_{D^\infty} \\
&\quad = {}_{D^{-\infty}}\langle f_1^i(\hat{\theta}_\varepsilon(w),\tilde{w})\partial_i 1_{A_\varepsilon(\theta_0)}(g(w,\tilde{w})), J(w)\rangle_{D^\infty},
\end{aligned}
$$

where

$$
g(w,\tilde{w}) = f_0(\hat{\theta}_\varepsilon(w),\tilde{w}) + \frac{f_{-1}(\hat{\theta}_\varepsilon(w)) - f_{-1}(\theta_0)}{\varepsilon}.
$$

Moreover, we have

$$
\begin{aligned}
&{}_{D^{-\infty}}\langle f_1^i(\hat{\theta}_\varepsilon(w),\tilde{w})\partial_i 1_{A_\varepsilon(\theta_0)}(g(w,\tilde{w})), J(w)\rangle_{D^\infty} \\
&\quad = \int_{\bar{W}} 1_{A_\varepsilon(\theta_0)}(g(w,\tilde{w})) G_1^i(w,\tilde{w}) J(w) \bar{P}(\mathrm{d}w,\mathrm{d}\tilde{w}) \\
&\quad = \int_{\bar{W}} 1_{A_\varepsilon(\hat{\theta}_\varepsilon(w))}(f_0(\hat{\theta}_\varepsilon(w),\tilde{w})) G_1^i(w,\tilde{w}) J(w) \bar{P}(\mathrm{d}w,\mathrm{d}\tilde{w}) \\
&\quad = \int_W \left\{ \int_{\tilde{W}} 1_{A_\varepsilon(\hat{\theta}_\varepsilon(w))}(f_0(\hat{\theta}_\varepsilon(w),\tilde{w})) G_1^i(w,\tilde{w}) P(\mathrm{d}\tilde{w}) \right\} J(w) P(\mathrm{d}w) \\
&\quad = \int_W \left\{ \int_{\tilde{W}} 1_A(f_0(\hat{\theta}_\varepsilon(w),\tilde{w})) G_1^i(w,\tilde{w}) P(\mathrm{d}\tilde{w}) \right\} \Bigg|_{A = A_\varepsilon(\hat{\theta}_\varepsilon(w))} J(w) P(\mathrm{d}w),
\end{aligned}
$$

where

$$
G_1^i(w,\tilde{w}) = D^* \left[\sum_{j=1}^k \left(\sigma^{-1}_{f_0(\hat{\theta}_\varepsilon(w),\tilde{w})} \right)^{ij} f_1^i(\hat{\theta}_\varepsilon(w),\tilde{w}) D f_0^j(\hat{\theta}_\varepsilon(w),\tilde{w}) \right].
$$

It follows that for a Borel set $A \in \mathbf{B}^k$,

$$
\begin{aligned}
&\int_W \int_{\tilde{W}} 1_A(f_0(\hat{\theta}_\varepsilon(w),\tilde{w})) G_1^i(w,\tilde{w}) P(\mathrm{d}\tilde{w}) J(w) P(\mathrm{d}w) \\
&\quad = {}_{D^{-\infty}}\langle 1_A(f_0(\hat{\theta}_\varepsilon(w),\tilde{w})), G_1^i(w,\tilde{w}) J(w)\rangle_{D^\infty} \\
&\quad = {}_{D^{-\infty}}\langle \partial_i 1_A(f_0(\hat{\theta}_\varepsilon(w),\tilde{w})), f_1^i(\hat{\theta}_\varepsilon(w),\tilde{w}) J(w)\rangle_{D^\infty} \\
&\quad = \int_A q(z)\,\mathrm{d}z.
\end{aligned}
$$

Thus, by the very routine work,

$$\begin{aligned} q(z) &= -\partial_i \Big(E^{\tilde{P}}[f_1^i(\hat{\theta}_\varepsilon(w), \cdot) J(w) | f_0(\hat{\theta}_\varepsilon(w), \tilde{w}) = z] p_{f_0}(z, \hat{\theta}_\varepsilon(w)) \Big) \\ &= -\partial_i \bigg(\int_W E^{\tilde{P}}[f_1^i(\hat{\theta}_\varepsilon(w), \cdot) | f_0(\hat{\theta}_\varepsilon(w), \tilde{w}) = z] p_{f_0}(z, \hat{\theta}_\varepsilon(w)) \times \\ &\qquad \times J(w) P(\mathrm{d}w) \bigg) \\ &= \int_W -\partial_i \Big(E^{\tilde{P}}[f_1^i(\hat{\theta}_\varepsilon(w), \cdot) | f_0(\hat{\theta}_\varepsilon(w), \tilde{w}) = z] p_{f_0}(z, \hat{\theta}_\varepsilon(w)) \Big) \times \\ &\qquad \times J(w) P(\mathrm{d}w). \end{aligned}$$

We then have

$$\begin{aligned} &{}_{D^{-\infty}}\langle f_1^i(\hat{\theta}_\varepsilon(w), \tilde{w}) \partial_i 1_{A_\varepsilon(\theta_0)}(g(w, \tilde{w})), J(w) \rangle_{D^\infty} \\ &\quad = \int_W \left\{ \int_{A_\varepsilon(\hat{\theta}_\varepsilon(w))} q_1(z, \hat{\theta}_\varepsilon(w)) \, \mathrm{d}z \right\} J(w) P(\mathrm{d}w), \end{aligned}$$

where $q_1(z, \hat{\theta}_\varepsilon(w)) = -\partial_i \Big(E^{\tilde{P}}[f_1^i(\hat{\theta}_\varepsilon(w), \cdot) | f_0(\hat{\theta}_\varepsilon(w), \tilde{w}) = z] p_{f_0}(z, \hat{\theta}_\varepsilon(w)) \Big)$. Therefore, we obtain

$$I(f_1^i(\hat{\theta}_\varepsilon(w), \tilde{w}) \partial_i 1_{A_\varepsilon(\hat{\theta}_\varepsilon(w))}(f_0(\hat{\theta}_\varepsilon(w), \tilde{w})), 1)(w) = \int_{A_\varepsilon(\hat{\theta}_\varepsilon(w))} q_1(z, \hat{\theta}_\varepsilon(w)) \, \mathrm{d}z$$

P-a.s. Similarly, we have

$$I(g_0(\hat{\theta}_\varepsilon(w), \tilde{w}) 1_{A_\varepsilon(\hat{\theta}_\varepsilon(w))}(f_0(\hat{\theta}_\varepsilon(w), \tilde{w})), 1)(w) = \int_{A_\varepsilon(\hat{\theta}_\varepsilon(w))} q_2(z, \hat{\theta}_\varepsilon(w)) \, \mathrm{d}z$$

P-a.s., where $q_2(z, \hat{\theta}_\varepsilon(w)) = E^{\tilde{P}}[g_0(\hat{\theta}_\varepsilon(w), \cdot) | f_0(\hat{\theta}_\varepsilon(w), \tilde{w}) = z] p_{f_0}(z, \hat{\theta}_\varepsilon(w))$.

From (18)–(20) and Lemmas 5.7, 5.8 in [31], we have

$$\begin{aligned} &E^{\tilde{P}} \left[f_1^\alpha(\hat{\theta}_\varepsilon(w), \cdot) \middle| \int_{T_0}^T a_t(\hat{\theta}_\varepsilon(w)) \, \mathrm{d}\tilde{w}_t = x \right] \\ &\quad = A^{0,\alpha}(\hat{\theta}_\varepsilon(w)) - A^{2,\alpha}_{p,q}(\hat{\theta}_\varepsilon(w)) \sigma^{pq}(\hat{\theta}_\varepsilon(w)) + A^{2,\alpha}_{p,q}(\hat{\theta}_\varepsilon(w)) x^p x^q, \\ &E^{\tilde{P}} \left[g_0(\hat{\theta}_\varepsilon(w), \cdot) \middle| \int_{T_0}^T a_t(\hat{\theta}_\varepsilon(w)) \, \mathrm{d}\tilde{w}_t = x \right] \\ &\quad = g_{-1}(\hat{\theta}_\varepsilon(w)) B^\alpha(\hat{\theta}_\varepsilon(w)) \sigma_{\alpha p}(\hat{\theta}_\varepsilon(w)) x^p. \end{aligned}$$

Since it follows that

$$
\begin{aligned}
q_1(x,\hat{\theta}_\varepsilon(w)) &= -\partial_\alpha\{E^{\tilde{P}}[f_1^\alpha(\hat{\theta}_\varepsilon(w),\cdot)|f_0(\hat{\theta}_\varepsilon(w),\tilde{w})=x]\phi(x;0,\Sigma(\hat{\theta}_\varepsilon(w)))\} \\
&= -\Big[A^{2,\alpha}_{\alpha,p}(\hat{\theta}_\varepsilon(w)) + A^{2,\alpha}_{p,\alpha}(\hat{\theta}_\varepsilon(w)) - A^{0,\alpha}(\hat{\theta}_\varepsilon(w))\sigma_{\alpha p}(\hat{\theta}_\varepsilon(w)) + \\
&\quad + A^{2,\alpha}_{q,l}(\hat{\theta}_\varepsilon(w))\sigma^{ql}(\hat{\theta}_\varepsilon(w))\sigma_{\alpha p}(\hat{\theta}_\varepsilon(w))\Big] \times \\
&\quad \times x^p\phi(x;0,\Sigma(\hat{\theta}_\varepsilon(w))) + A^{2,\alpha}_{p,q}(\hat{\theta}_\varepsilon(w))\sigma_{\alpha l}(\hat{\theta}_\varepsilon(w)) \times \\
&\quad \times x^p x^q x^l\phi(x;0,\Sigma(\hat{\theta}_\varepsilon(w))),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
p_1(x,\hat{\theta}_\varepsilon(w)) &= g_{-1}(\hat{\theta}_\varepsilon(w))x\phi(x;0,\Sigma(\hat{\theta}_\varepsilon(w))) + \\
&\quad + g_{-1}(\hat{\theta}_\varepsilon(w))(f_{-1}(\hat{\theta}_\varepsilon(w)) - K)q_1(x,\hat{\theta}_\varepsilon(w)) + \\
&\quad + g_{-1}(\hat{\theta}_\varepsilon(w))(f_{-1}(\hat{\theta}_\varepsilon(w)) - K)B^\alpha(\hat{\theta}_\varepsilon(w))\sigma_{\alpha p} \times \\
&\quad \times (\hat{\theta}_\varepsilon(w))x^p\phi(x;0,\Sigma(\hat{\theta}_\varepsilon(w))).
\end{aligned}
$$

In the same fashion, we obtain $p_i(x,w)$ for $i \geqslant 2$. This completes the proof.

Proof of Lemma 5. From Assumption 3, we see

$$
\begin{aligned}
\psi_\varepsilon(w)\varepsilon^{-1}(\hat{\theta}_\varepsilon(w)-\theta_0)^i &= \zeta_i^{(0)}(w) + \frac{1}{2}\varepsilon\zeta_i^{(1)}(w) + \cdots + \frac{1}{m!}\varepsilon^{m-1} \times \\
&\quad \times \zeta_i^{(m-1)}(w) + \varepsilon^m\zeta_{\varepsilon,i}^{(m)}(w),
\end{aligned}
$$

where $\zeta_{\varepsilon,i}^{(m)}(w) = \mathrm{O}(1)$ in $\cap_{p>1}L_p(W)$. Moreover, in case that $|R_\varepsilon| < 1/9$, it follows from $\psi_\varepsilon(w) = \psi(3R_\varepsilon(w)) = 1$ that

$$
\begin{aligned}
1\cdot\varepsilon^{-1}(\hat{\theta}_\varepsilon(w)-\theta_0)^i &= \psi_\varepsilon(w)\varepsilon^{-1}(\hat{\theta}_\varepsilon(w)-\theta_0)^i \\
&= \zeta_i^{(0)}(w) + \frac{1}{2}\varepsilon\zeta_i^{(1)}(w) + \cdots + \frac{1}{m!}\varepsilon^{m-1}\zeta_i^{(m-1)}(w) + \\
&\quad + \varepsilon^m\zeta_{\varepsilon,i}^{(m)}(w).
\end{aligned} \tag{28}
$$

First, expanding $\hat{\theta}_\varepsilon(w)$ in a Taylor series around θ_0 and substituting (28) in the resulting expansion, we obtain the second-order stochastic expansion as follows:

$$
\begin{aligned}
&\bar{\psi}_\varepsilon(w)\int_{T_0}^T f_0(X_t^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}),\hat{\theta}_\varepsilon(w))\,\mathrm{d}t \\
&\quad = \bar{\psi}_\varepsilon(w)\bigg[\int_{T_0}^T f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t + \\
&\qquad + \int_{T_0}^T \delta_i f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\,(\hat{\theta}_\varepsilon(w)-\theta_0)^i + \\
&\qquad + \frac{1}{2}\int_{T_0}^T \delta_i\delta_j f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\,(\hat{\theta}_\varepsilon(w)-\theta_0)^i(\hat{\theta}_\varepsilon(w)-\theta_0)^j +
\end{aligned}
$$

$$+\frac{1}{2}\int_0^1\int_{T_0}^T \delta_i\delta_j\delta_k f_0(X_t^\varepsilon(\theta,\tilde{w}),\theta)\,\mathrm{d}t\bigg|_{\theta=\theta_0+u(\hat{\theta}_\varepsilon(w)-\theta_0)}(1-u)^2\,\mathrm{d}u\ \times$$
$$\times\ (\hat{\theta}_\varepsilon(w)-\theta_0)^i(\hat{\theta}_\varepsilon(w)-\theta_0)^j(\hat{\theta}_\varepsilon(w)-\theta_0)^k\bigg]$$
$$=\ \bar{\psi}_\varepsilon(w)\bigg[\int_{T_0}^T f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t+\varepsilon\int_{T_0}^T \delta_i f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\ \times$$
$$\times\ \mathrm{d}t\,\zeta_i^{(0)}(w)+\varepsilon^2\bigg\{\frac{1}{2}\int_{T_0}^T \delta_i f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\,\zeta_i^{(1)}(w)\ +$$
$$+\ \frac{1}{2}\int_{T_0}^T \delta_i\delta_j f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\,\zeta_i^{(0)}(w)\zeta_j^{(0)}(w)\bigg\}\ +$$
$$+\ \varepsilon^3 f_1^{(2)}(w,\tilde{w})\bigg],$$

where

$$f_1^{(2)}(w,\tilde{w})$$
$$=\int_{T_0}^T \delta_i f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\,\zeta_{\varepsilon,i}^{(2)}(w)\ +$$
$$+\ \frac{1}{2}\left(\int_{T_0}^T \delta_i\delta_j f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\right)(\zeta_i^{(0)}(w)\zeta_{\varepsilon,j}^{(1)}(w)+\zeta_{\varepsilon,i}^{(1)}(w)\ \times$$
$$\times\ \zeta_j^{(0)}(w))+\varepsilon\frac{1}{2}\int_{T_0}^T \delta_i\delta_j f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\,\zeta_{\varepsilon,i}^{(1)}(w)\zeta_{\varepsilon,j}^{(1)}(w)\ +$$
$$+\ \frac{1}{2}\int_0^1\int_{T_0}^T \delta_i\delta_j\delta_k f_0(X_t^\varepsilon(\theta,\tilde{w}),\theta)\,\mathrm{d}t\bigg|_{\theta=\theta_0+u(\hat{\theta}_\varepsilon(w)-\theta_0)}\ \times$$
$$\times\ (1-u)^2\,\mathrm{d}u\,\zeta_{\varepsilon,i}^{(0)}(w)\zeta_{\varepsilon,j}^{(0)}(w)\zeta_{\varepsilon,k}^{(0)}(w).$$

Since it follows from

$$\bar{\psi}_\varepsilon(w)=1-\mathrm{O}(\varepsilon^K)\quad\text{in }\cap_{p>1}L_p(W)\text{ for any }K>0 \tag{29}$$

that

$$\bar{\psi}_\varepsilon(w)\int_0^1\int_{T_0}^T \delta_i\delta_j\delta_k f_0(X_t^\varepsilon(\theta,\tilde{w}),\theta)\,\mathrm{d}t\bigg|_{\theta=\theta_0+u(\hat{\theta}_\varepsilon(w)-\theta_0)}\ \times$$
$$\times\ (1-u)^2\,\mathrm{d}u=\mathrm{O}_M(1),$$

we obtain $\bar{\psi}_\varepsilon(w)f_1^{(2)}(w,\tilde{w})=\mathrm{O}_M(1)$ and

$$\bar{\psi}_\varepsilon(w)\int_{T_0}^T f_0(X_t^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}),\hat{\theta}_\varepsilon(w))\,\mathrm{d}t$$
$$=\bar{\psi}_\varepsilon(w)\bigg[\int_{T_0}^T f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\ +$$

$$
\begin{aligned}
&+ \varepsilon\biggl\{ \int_{T_0}^{T} \partial_{i_1} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\delta_i X_t^{\varepsilon,i_1}(\theta_0,\tilde{w})\,\mathrm{d}t + \\
&+ \int_{T_0}^{T} \delta_i^{(2)} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\biggr\}\zeta_i^{(0)}(w) + \\
&+ \varepsilon^2\biggl\{\frac{1}{2}\biggl(\int_{T_0}^{T} \partial_{i_1} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\delta_i X_t^{\varepsilon,i_1}(\theta_0,\tilde{w})\,\mathrm{d}t + \\
&+ \int_{T_0}^{T} \delta_i^{(2)} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\biggr)\zeta_i^{(1)}(w) + \\
&+ \frac{1}{2}\biggl(\int_{T_0}^{T} \partial_{i_1}\partial_{i_2} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\delta_i X_t^{\varepsilon,i_1}(\theta_0,\tilde{w})\delta_j X_t^{\varepsilon,i_2}(\theta_0,\tilde{w})\,\mathrm{d}t + \\
&+ \int_{T_0}^{T} \delta_i^{(2)}\partial_{i_1} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\delta_j X_t^{\varepsilon,i_1}(\theta_0,\tilde{w})\,\mathrm{d}t + \\
&+ \int_{T_0}^{T} \partial_{i_1} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\delta_i\delta_j X_t^{\varepsilon,i_1}(\theta_0,\tilde{w})\,\mathrm{d}t + \\
&+ \int_{T_0}^{T} \partial_{i_1}\delta_i^{(2)} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\delta_j X_t^{\varepsilon,i_1}(\theta_0,\tilde{w})\,\mathrm{d}t + \\
&+ \int_{T_0}^{T} \delta_i^{(2)}\delta_j^{(2)} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\biggr)\zeta_i^{(0)}(w)\zeta_j^{(0)}(w)\biggr\}\biggr] + \mathrm{O}_M(\varepsilon^3).
\end{aligned}
$$

Moreover, expanding X_t^ε in a Taylor series around X_t^0, we have

$$
\begin{aligned}
&\bar{\psi}_\varepsilon(w)\int_{T_0}^{T} f_0(X_t^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}),\hat{\theta}_\varepsilon(w))\,\mathrm{d}t \\
&\quad = \bar{\psi}_\varepsilon(w)\biggl[\int_{T_0}^{T} f_0(X_t^0(\theta_0),\theta_0)\,\mathrm{d}t + \varepsilon\biggl\{\int_{T_0}^{T} \partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\times \\
&\qquad \times D_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}t + \biggl(\int_{T_0}^{T} \partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\delta_i X_t^{0,i_1}(\theta_0)\,\mathrm{d}t + \\
&\quad + \int_{T_0}^{T} \delta_i^{(2)} f_0(X_t^0(\theta_0),\theta_0)\,\mathrm{d}t\biggr)\zeta_i^{(0)}(w)\biggr\} + \\
&\quad + \varepsilon^2\biggl\{\frac{1}{2}\biggl(\int_{T_0}^{T} \partial_{i_1}\partial_{i_2} f_0(X_t^0(\theta_0),\theta_0)D_t^{i_1}(\theta_0,\tilde{w})D_t^{i_2}(\theta_0,\tilde{w})\,\mathrm{d}t + \\
&\quad + \int_{T_0}^{T} \partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)E_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}t\biggr) + \\
&\quad + \biggl(\int_{T_0}^{T} \partial_{i_1}\partial_{i_2} f_0(X_t^0(\theta_0),\theta_0)D_t^{i_1}(\theta_0,\tilde{w})\delta_i X_t^{0,i_2}(\theta_0)\,\mathrm{d}t + \\
&\quad + \int_{T_0}^{T} \partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\delta_i D_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}t +
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{T_0}^{T} \partial_{i_1}\delta_i^{(2)} f_0(X_t^0(\theta_0), \theta_0) D_t^{i_1}(\theta_0, \tilde{w})\,\mathrm{d}t\bigg)\zeta_i^{(0)}(w) + \\
&+ \frac{1}{2}\bigg(\int_{T_0}^{T} \partial_{i_1} f_0(X_t^0(\theta_0), \theta_0)\delta_i X_t^{0,i_1}(\theta_0)\,\mathrm{d}t + \\
&+ \int_{T_0}^{T} \delta_i^{(2)} f_0(X_t^0(\theta_0), \theta_0)\,\mathrm{d}t\bigg)\zeta_i^{(1)}(w) + \\
&+ \frac{1}{2}\bigg(\int_{T_0}^{T} \partial_{i_1}\partial_{i_2} f_0(X_t^0(\theta_0), \theta_0)\delta_i X_t^{0,i_1}(\theta_0)\delta_j X_t^{0,i_2}(\theta_0)\,\mathrm{d}t + \\
&+ \int_{T_0}^{T} \partial_{i_1} f_0(X_t^0(\theta_0), \theta_0)\delta_i\delta_j X_t^{0,i_1}(\theta_0)\,\mathrm{d}t + \int_{T_0}^{T} \delta_i^{(2)}\delta_j^{(2)} f_0 \times \\
&\times (X_t^0(\theta_0), \theta_0)\,\mathrm{d}t + 2\int_{T_0}^{T} \partial_{i_1}\delta_i^{(2)} f_0(X_t^0(\theta_0), \theta_0)\delta_j X_t^{0,i_1}(\theta_0)\,\mathrm{d}t\bigg) \times \\
&\times \zeta_i^{(0)}(w)\zeta_j^{(0)}(w)\bigg\} + \varepsilon^3 f_2^{(2)}(w, \tilde{w})\bigg] + \mathrm{O}_M(\varepsilon^3),
\end{aligned}
$$

where

$$
\begin{aligned}
&f_2^{(2)}(w, \tilde{w}) \\
&\quad = \frac{1}{2}\int_0^1 \int_{T_0}^{T} \frac{\partial^3}{\partial\varepsilon^3} f_0(X_t^\varepsilon(\theta_0, \tilde{w}), \theta_0)\,\mathrm{d}t\bigg|_{\varepsilon\leftarrow u\varepsilon} (1-u)^2\,\mathrm{d}u + \\
&\quad + \int_0^1 \int_{T_0}^{T} \frac{\partial^2}{\partial\varepsilon^2}\partial_{i_1} f_0(X_t^\varepsilon(\theta_0, \tilde{w}), \theta_0)\delta_i X_t^{\varepsilon,i_1}(\theta_0, \tilde{w})\,\mathrm{d}t\bigg|_{\varepsilon\leftarrow u\varepsilon} \times \\
&\quad \times (1-u)\,\mathrm{d}u\zeta_i^{(0)}(w) + \\
&\quad + \int_0^1 \int_{T_0}^{T} \frac{\partial^2}{\partial\varepsilon^2}\delta_i^{(2)} f_0(X_t^\varepsilon(\theta_0, \tilde{w}), \theta_0)\,\mathrm{d}t\bigg|_{\varepsilon\leftarrow u\varepsilon} (1-u)\,\mathrm{d}u\zeta_i^{(0)}(w) + \\
&\quad + \frac{1}{2}\int_0^1 \int_{T_0}^{T} \frac{\partial}{\partial\varepsilon}\partial_{i_1} f_0(X_t^\varepsilon(\theta_0, \tilde{w}), \theta_0)\delta_i X_t^{\varepsilon,i_1}(\theta_0, \tilde{w})\,\mathrm{d}t\bigg|_{\varepsilon\leftarrow u\varepsilon} \mathrm{d}u\zeta_i^{(1)}(w) + \\
&\quad + \frac{1}{2}\int_0^1 \int_{T_0}^{T} \frac{\partial}{\partial\varepsilon}\delta_i^{(2)} f_0(X_t^\varepsilon(\theta_0, \tilde{w}), \theta_0)\,\mathrm{d}t\bigg|_{\varepsilon\leftarrow u\varepsilon} \mathrm{d}u\zeta_i^{(1)}(w) + \\
&\quad + \frac{1}{2}\int_0^1 \int_{T_0}^{T} \frac{\partial}{\partial\varepsilon}\partial_{i_1}\partial_{i_2} f_0(X_t^\varepsilon(\theta_0, \tilde{w}), \theta_0)\delta_i X_t^{\varepsilon,i_1}(\theta_0, \tilde{w})\delta_j X_t^{\varepsilon,i_2} \times \\
&\quad \times (\theta_0, \tilde{w})\,\mathrm{d}t\bigg|_{\varepsilon\leftarrow u\varepsilon} \mathrm{d}u\zeta_i^{(0)}(w)\zeta_j^{(0)}(w) + \\
&\quad + \frac{1}{2}\int_0^1 \int_{T_0}^{T} \frac{\partial}{\partial\varepsilon}\partial_{i_1} f_0(X_t^\varepsilon(\theta_0, \tilde{w}), \theta_0)\delta_i\delta_j X_t^{\varepsilon,i_1}(\theta_0, \tilde{w})\,\mathrm{d}t\bigg|_{\varepsilon\leftarrow u\varepsilon} \times \\
&\quad \times \mathrm{d}u\zeta_i^{(0)}(w)\zeta_j^{(0)}(w) +
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^1 \int_{T_0}^{T} \frac{\partial}{\partial\varepsilon}\delta_i^{(2)}\delta_j^{(2)} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\,\mathrm{d}t\Big|_{\varepsilon\leftarrow u\varepsilon} \mathrm{d}u\,\zeta_i^{(0)}(w)\zeta_j^{(0)}(w) + \\
&+\int_0^1 \int_{T_0}^{T} \frac{\partial}{\partial\varepsilon}\partial_{i_1}\delta_i^{(2)} f_0(X_t^\varepsilon(\theta_0,\tilde{w}),\theta_0)\delta_j X_t^{\varepsilon,i_1}(\theta_0,\tilde{w})\,\mathrm{d}t\Big|_{\varepsilon\leftarrow u\varepsilon} \times \\
&\times \mathrm{d}u\,\zeta_i^{(0)}(w)\zeta_j^{(0)}(w).
\end{aligned}
$$

It follows from (29) that $\bar{\psi}_\varepsilon(w) f_2^{(2)}(w,\tilde{w}) = \mathrm{O}_M(1)$. Similarly, we have

$$
\begin{aligned}
&\bar{\psi}_\varepsilon(w)\left(\varepsilon\int_{T_0}^{T} f(X_t^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}),\hat{\theta}_\varepsilon(w))\,\mathrm{d}\tilde{w}_t\right) \\
&\quad = \varepsilon\int_{T_0}^{T} f(X_t^0(\theta_0),\theta_0)\,\mathrm{d}\tilde{w}_t + \varepsilon^2\Bigg[\int_{T_0}^{T}\partial_{i_1} f(X_t^0(\theta_0),\theta_0) D_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}\tilde{w}_t + \\
&\qquad + \left(\int_{T_0}^{T}\partial_{i_1} f(X_t^0(\theta_0),\theta_0)\delta_i X_t^{0,i_1}(\theta_0)\,\mathrm{d}\tilde{w}_t + \right.\\
&\qquad \left. + \int_{T_0}^{T}\delta_i^{(2)} f(X_t^0(\theta_0),\theta_0)\,\mathrm{d}\tilde{w}_t\right)\zeta_i^{(0)}(w)\Bigg] + \mathrm{O}_M(\varepsilon^3).
\end{aligned}
$$

We then have

$$
\begin{aligned}
&\bar{\psi}_\varepsilon(w) F_T^\varepsilon(\hat{\theta}_\varepsilon(w),\tilde{w}) \\
&\quad = F_{T_0} + \int_{T_0}^{T} f_0(X_t^0(\theta_0),\theta_0)\,\mathrm{d}t + \\
&\qquad + \varepsilon\Bigg\{\int_{T_0}^{T}\partial_{i_1} f_0(X_t^0(\theta_0),\theta_0) D_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}t + \int_{T_0}^{T} f(X_t^0(\theta_0),\theta_0)\,\mathrm{d}\tilde{w}_t + \\
&\qquad + \left(\int_{T_0}^{T}\partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\delta_i X_t^{0,i_1}(\theta_0)\,\mathrm{d}t + \right.\\
&\qquad \left. + \int_{T_0}^{T}\delta_i^{(2)} f_0(X_t^0(\theta_0),\theta_0)\,\mathrm{d}t\right)\zeta_i^{(0)}(w)\Bigg\} + \\
&\qquad + \varepsilon^2\Bigg\{\frac{1}{2}\left(\int_{T_0}^{T}\partial_{i_1}\partial_{i_2} f_0(X_t^0(\theta_0),\theta_0) D_t^{i_1}(\theta_0,\tilde{w}) D_t^{i_2}(\theta_0,\tilde{w})\,\mathrm{d}t + \right.\\
&\qquad \left. + \int_{T_0}^{T}\partial_{i_1} f_0(X_t^0(\theta_0),\theta_0) E_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}t\right) + \\
&\qquad + \int_{T_0}^{T}\partial_{i_1} f(X_t^0(\theta_0),\theta_0) D_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}\tilde{w}_t + \\
&\qquad + \left(\int_{T_0}^{T}\partial_{i_1}\partial_{i_2} f_0(X_t^0(\theta_0),\theta_0) D_t^{i_1}(\theta_0,\tilde{w})\delta_i X_t^{0,i_2}(\theta_0)\,\mathrm{d}t + \right.\\
&\qquad + \int_{T_0}^{T}\partial_{i_1} f_0(X_t^0(\theta_0),\theta_0)\delta_i D_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}t +
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{T_0}^{T}\partial_{i_1}\delta_i^{(2)}f_0(X_t^0(\theta_0),\theta_0)D_t^{i_1}(\theta_0,\tilde{w})\,\mathrm{d}t+\\
&+\int_{T_0}^{T}\partial_{i_1}f(X_t^0(\theta_0),\theta_0)\delta_iX_t^{0,i_1}(\theta_0)\,\mathrm{d}\tilde{w}_t+\\
&+\left.\int_{T_0}^{T}\delta_i^{(2)}f(X_t^0(\theta_0),\theta_0)\,\mathrm{d}\tilde{w}_t\right)\zeta_i^{(0)}(w)+\\
&+\frac{1}{2}\left(\int_{T_0}^{T}\partial_{i_1}f_0(X_t^0(\theta_0),\theta_0)\delta_iX_t^{0,i_1}(\theta_0)\,\mathrm{d}t+\right.\\
&+\left.\int_{T_0}^{T}\delta_i^{(2)}f_0(X_t^0(\theta_0),\theta_0)\,\mathrm{d}t\right)\zeta_i^{(1)}(w)+\\
&+\frac{1}{2}\left(\int_{T_0}^{T}\partial_{i_1}\partial_{i_2}f_0(X_t^0(\theta_0),\theta_0)\delta_iX_t^{0,i_1}(\theta_0)\delta_jX_t^{0,i_2}(\theta_0)\,\mathrm{d}t+\right.\\
&+\int_{T_0}^{T}\partial_{i_1}f_0(X_t^0(\theta_0),\theta_0)\delta_i\delta_jX_t^{0,i_1}(\theta_0)\,\mathrm{d}t+\\
&+\int_{T_0}^{T}\delta_i^{(2)}\delta_j^{(2)}f_0(X_t^0(\theta_0),\theta_0)\,\mathrm{d}t+\\
&+\left.\left.2\int_{T_0}^{T}\partial_{i_1}\delta_i^{(2)}f_0(X_t^0(\theta_0),\theta_0)\delta_jX_t^{0,i_1}(\theta_0)\,\mathrm{d}t\right)\zeta_i^{(0)}(w)\zeta_j^{(0)}(w)\right\}+\mathrm{O}_M(\varepsilon^3).
\end{aligned}
$$

From the second-order asymptotic expansion of $F_T^{\varepsilon}(\hat{\theta}_{\varepsilon}(w),\tilde{w})$ as above, $\hat{f}_0$ and $\hat{f}_1$ are determined.

In the same way as the second-order asymptotic expansion, we can see that for any $m\in\mathbf{N}$,

$$
\begin{aligned}
&\bar{\psi}_{\varepsilon}(w)F_T^{\varepsilon}(\hat{\theta}_{\varepsilon}(w),\tilde{w})\\
&\quad=\bar{\psi}_{\varepsilon}(w)(f_{-1}(\theta_0)+\varepsilon\hat{f}_0(w,\tilde{w})+\varepsilon^2\hat{f}_1(w,\tilde{w})+\cdots+\\
&\quad+\varepsilon^{m-1}\hat{f}_{m-1}(w,\tilde{w})+\varepsilon^m\hat{f}_{\varepsilon}^{(m)}(w,\tilde{w})),
\end{aligned}
$$

where $\hat{f}_{\varepsilon}^{(m)}(w,\tilde{w})=\mathrm{O}_M(1)$. Moreover, it follows from (29) that we obtain

$$
\bar{\psi}_{\varepsilon}(w)F_T^{\varepsilon}(\hat{\theta}_{\varepsilon}(w),\tilde{w})\sim f_{-1}(\theta_0)+\varepsilon\hat{f}_0(w,\tilde{w})+\varepsilon^2\hat{f}_1(w,\tilde{w})+\cdots
$$

in $D^{\infty}(\bar{W};\mathbf{R}^k)$ as $\varepsilon\downarrow 0$. This completes the proof.

Proof of Lemma 6. Following the same way as the proof of Lemma 5, we have the result.

Proof of Lemma 7. From the definition of $\hat{\xi}_{\varepsilon}$, $|\hat{\xi}_{\varepsilon}(w,\tilde{w})|>1$ if $\det\sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w),\cdot)}<(1/2)\det\sigma_{\hat{f}_0(w,\cdot)}$ or $\det\sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w),\cdot)}>(3/2)\det\sigma_{\hat{f}_0(w,\cdot)}$. In case that $(3/2)\det\sigma_{\hat{f}_0(w,\cdot)}\geqslant\det\sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_{\varepsilon}(w),\cdot)}\geqslant(1/2)\det\sigma_{\hat{f}_0(w,\cdot)}$, we obtain $(2/3)(\det\sigma_{\hat{f}_0(w,\cdot)})^{-1}\leqslant$

$(\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)})^{-1} \leqslant 2(\det \sigma_{\hat{f}_0(w,\cdot)})^{-1}$. Therefore, it follows from Assumption 2 that for any $p \in (1, \infty)$,

$$\sup_{\varepsilon \in (0,1]} E[1_{\{|\hat{\xi}_\varepsilon(w,\tilde{w})| \leqslant 1\}} (\det \sigma_{\tilde{F}_T^{\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)})^{-p}] < \infty.$$

From Lemma 5, $\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w), \tilde{w}) \in D^\infty(\bar{W}; \mathbf{R}^k)$ has the asymptotic expansion

$$\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w), \tilde{w}) \sim \hat{f}_0(w, \tilde{w}) + \varepsilon \hat{f}_1(w, \tilde{w}) + \cdots$$

in $D^\infty(\bar{W}; \mathbf{R}^k)$ as $\varepsilon \downarrow 0$ with $\hat{f}_0(w, \tilde{w}), \hat{f}_1(w, \tilde{w}), \ldots \in D^\infty(\bar{W}; \mathbf{R}^k)$. Hence, $\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w), \tilde{w}) = \hat{f}_0(w, \tilde{w}) + \varepsilon f_\varepsilon^{(1)}(w, \tilde{w})$, where $\hat{f}_0(w, \tilde{w}) \in D^\infty(\bar{W}; \mathbf{R}^k)$ and $f_\varepsilon^{(1)}(w, \tilde{w}) = \mathrm{O}_M(1)$. Since

$$\begin{aligned}
&|(\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2} - (\det \sigma_{\hat{f}_0(w,\cdot)})^{1/2}| \\
&\quad \leqslant (\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot) - \hat{f}_0(w,\cdot)})^{1/2} = (\det \sigma_{\varepsilon \hat{f}_\varepsilon^{(1)}(w,\cdot)})^{1/2} = \varepsilon (\det \sigma_{\hat{f}_\varepsilon^{(1)}(w,\cdot)})^{1/2},
\end{aligned}$$

we have

$$\begin{aligned}
&|\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{\hat{f}_0(w,\cdot)}| \\
&\quad = |(\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2} - (\det \sigma_{\hat{f}_0(w,\cdot)})^{1/2}||(\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2} + \\
&\qquad + (\det \sigma_{\hat{f}_0(w,\cdot)})^{1/2}| \\
&\quad \leqslant \varepsilon (\det \sigma_{\hat{f}_\varepsilon^{(1)}(w,\cdot)})^{1/2} \left\{ (\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)})^{1/2} + (\det \sigma_{\hat{f}_0(w,\cdot)})^{1/2} \right\}.
\end{aligned}$$

Hence, it follows that there exists $C > 0$ such that for any $p > 1$,

$$\| \det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{\hat{f}_0(w,\cdot)} \|_p \leqslant C\varepsilon. \tag{30}$$

From definition, $\hat{\xi}_\varepsilon(w, \tilde{w}) = \mathrm{O}(1)$ in $D^\infty(\bar{W})$ as $\varepsilon \downarrow 0$. It follows from Chebyshev's inequality that for any $a > 0$, $K > 0$,

$$\begin{aligned}
\bar{P}[\hat{\xi}_\varepsilon(w, \tilde{w}) > a] &\leqslant \bar{P}\left[\left| \frac{2(\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{\hat{f}_0(w,\cdot)})}{\det \sigma_{\hat{f}_0(w,\cdot)}} \right| > a \right] \\
&\leqslant \frac{1}{a^K} \left\| \frac{2(\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{\hat{f}_0(w,\cdot)})}{\det \sigma_{\hat{f}_0(w,\cdot)}} \right\|_K^K.
\end{aligned}$$

It follows from (30), Hölder's inequality and Assumption 2 that for any $K > 0$,

$$\left\| \frac{2(\det \sigma_{\tilde{F}_T^{*\varepsilon}(\hat{\theta}_\varepsilon(w),\cdot)} - \det \sigma_{\hat{f}_0(w,\cdot)})}{\det \sigma_{\hat{f}_0(w,\cdot)}} \right\|_K^K = \mathrm{O}(\varepsilon^K).$$

We then see that for any $a > 0$, $K > 0$,

$$\bar{P}[\hat{\xi}_\varepsilon(w, \tilde{w}) > a] = \mathrm{O}(\varepsilon^K). \tag{31}$$

From (31), we see that for any $p > 1$,

$$\|1 - \hat{\psi}^*_\varepsilon(w, \tilde{w})\|_p = \|1 - \psi(\hat{\xi}_\varepsilon(w, \tilde{w}))\|_p \leqslant \|1_{\left\{|\hat{\xi}_\varepsilon(w,\tilde{w})|>\frac{1}{2}\right\}}\|_p = \mathrm{O}(\varepsilon^K).$$

In view of the chain rule for H-derivatives,

$$D(1 - \hat{\psi}^*_\varepsilon(w, \tilde{w})) = -D\hat{\psi}^*_\varepsilon(w, \tilde{w}) = -\psi'(\hat{\xi}_\varepsilon(w, \tilde{w}))D\hat{\xi}_\varepsilon(w, \tilde{w}).$$

Since $\|\psi'(\hat{\xi}_\varepsilon(w, \tilde{w}))\|_p = \mathrm{O}(\varepsilon^K)$, we see that for any $p > 1$,

$$\|D(1 - \hat{\psi}^*_\varepsilon(w, \tilde{w}))\|_p = \mathrm{O}(\varepsilon^K).$$

Similarly, it follows that for any $p > 1$ and $j > 0$,

$$\|D^j(1 - \hat{\psi}^*_\varepsilon(w, \tilde{w}))\|_p = \mathrm{O}(\varepsilon^K).$$

Therefore, we see that

$$\hat{\psi}^*_\varepsilon(w, \tilde{w}) = 1 - \mathrm{O}(\varepsilon^K) \quad \text{in } D^\infty(\bar{W}) \text{ for any } K > 0. \tag{32}$$

By using Theorem 4.1 in [31],

$$\hat{\psi}^*_\varepsilon(w, \tilde{w})T(\tilde{F}^{*\varepsilon}_T(\hat{\theta}_\varepsilon(w), \tilde{w})) \sim \hat{\Phi}_0(w, \tilde{w}) + \varepsilon\hat{\Phi}_1(w, \tilde{w}) + \cdots$$

in $\tilde{D}^{-\infty}(\bar{W})$ as $\varepsilon \downarrow 0$ with $\hat{\Phi}_0(w, \tilde{w}), \hat{\Phi}_1(w, \tilde{w}), \ldots$ in $\tilde{D}^{-\infty}(\bar{W})$. This completes the proof.

Proof of Lemma 8. In the same way as the proof of Lemma 7, it follows from Theorem 4.1 of [31] that we obtain the result.

Proof of Theorem 2. We use the same notation as the proof of Theorem 1. Let

$$\begin{aligned} G_\varepsilon(w, \tilde{w}) \ &= \ \bar{H}^{*\varepsilon}_T(\hat{\theta}_\varepsilon(w), \tilde{w}) - \big(\hat{\Xi}_0(w, \tilde{w}) + \varepsilon\,\hat{\Xi}_1(w, \tilde{w}) + \cdots + \\ &\quad + \varepsilon^{k-1}\,\hat{\Xi}_{k-1}(w, \tilde{w})\big). \end{aligned}$$

We then have

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{-k} \left\| I(\bar{H}^{*\varepsilon}_T(\hat{\theta}_\varepsilon(w), \tilde{w}), \tilde{J}) - \sum_{\gamma=0}^{k-1} \varepsilon^\gamma I(\hat{\Xi}_\gamma(w, \tilde{w}), \tilde{J}) \right\|_p < \infty.$$

Since $I(\bar{H}^{*\varepsilon}_T(\hat{\theta}_\varepsilon(w), \tilde{w}), 1)(w) = E^{\tilde{P}}[\bar{H}^{*\varepsilon}_T(\hat{\theta}_\varepsilon(w), \tilde{w})]$ P-a.s., it follows that $I(\hat{\Xi}_0(w, \tilde{w}), 1)(w) = \int_{A_\varepsilon(\theta_0)} \hat{p}_0(x, w)\,\mathrm{d}x$ P-a.s., where $\hat{p}_0(x, w) = g_{-1}(\theta_0)(f_{-1}(\theta_0) - K)\phi(x; \hat{x}_0(w), \Sigma(\theta_0))$.

Next, we obtain the second-order term $\hat{p}_1(x, \theta)$. In the same way as the proof of Theorem 1,

$$I(g_{-1}(\theta_0)\hat{f}_0(w, \tilde{w})1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w, \tilde{w})), 1)(w) = \int_{A_\varepsilon(\theta_0)} q_0(x, w)\,\mathrm{d}x$$

P-a.s., where $q_0(x, w) = g_{-1}(\theta_0) x \phi(x; \hat{x}_0(w), \Sigma(\theta_0))$. For any $J(w) \in L_p(W, P)$,

$$
\begin{aligned}
&\int_W I(\hat{f}_1^i(w, \tilde{w}) \partial_i 1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w, \tilde{w})), 1) J(w) P(\mathrm{d}w) \\
&= {}_{D^{-\infty}}\langle \hat{f}_1^i(w, \tilde{w}) \partial_i 1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w, \tilde{w})), J(w) \rangle_{D^\infty} \\
&= \int_{\bar{W}} 1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w, \tilde{w})) G_1^i(w, \tilde{w}) J(w) \bar{P}(\mathrm{d}w, \mathrm{d}\tilde{w}) \\
&= \int_{\bar{W}} 1_{A_\varepsilon^*(w)}(g_0(\tilde{w})) G_1^i(w, \tilde{w}) J(w) \bar{P}(\mathrm{d}w, \mathrm{d}\tilde{w}) \\
&= \int_W \left\{ \int_{\tilde{W}} 1_{A_\varepsilon^*(w)}(g_0(\tilde{w})) G_1^i(w, \tilde{w}) P(\mathrm{d}\tilde{w}) \right\} J(w) P(\mathrm{d}w) \\
&= \int_W \left\{ \int_{\tilde{W}} 1_A(g_0(\tilde{w})) G_1^i(w, \tilde{w}) P(\mathrm{d}\tilde{w}) \right\} \Bigg|_{A = A_\varepsilon^*(w)} J(w) P(\mathrm{d}w),
\end{aligned}
$$

where

$$
\begin{aligned}
G_1^i(w, \tilde{w}) &= D^* \left[\sum_{j=1}^k \left(\sigma_{g_0(\tilde{w})}^{-1} \right)^{ij} \hat{f}_1^i(w, \tilde{w}) D g_0^j(\tilde{w}) \right], \\
g_0(\tilde{w}) &= \hat{f}_0(w, \tilde{w}) - \hat{x}_0(w) = f_0(\theta_0, \tilde{w}), \\
A_\varepsilon^*(w) &= \left\{ x; x \geqslant -\hat{x}_0(w) + \frac{K - f_{-1}(\theta_0)}{\varepsilon} \right\}.
\end{aligned}
$$

For a Borel set $A \in \mathbf{B}^k$,

$$
\begin{aligned}
&\int_W \int_{\tilde{W}} 1_A(g_0(\tilde{w})) G_1^i(w, \tilde{w}) P(\mathrm{d}\tilde{w}) J(w) P(\mathrm{d}w) \\
&= {}_{D^{-\infty}}\langle 1_A(g_0(\tilde{w})), G_1^i(w, \tilde{w}) J(w) \rangle_{D^\infty} \\
&= {}_{D^{-\infty}}\langle \partial_i 1_A(g_0(\tilde{w})), \hat{f}_1^i(w, \tilde{w}) J(w) \rangle_{D^\infty} \\
&= \int_A q(z)\, \mathrm{d}z.
\end{aligned}
$$

Since it is easy to show that

$$
\begin{aligned}
q(x) &= -\partial_i \left(E^{\bar{P}}[\hat{f}_1^i(w, \cdot) J(w) | g_0(\tilde{w}) = z] p_{g_0}(z) \right) \\
&= -\partial_i \left(\int_W E^{\tilde{P}}[\hat{f}_1^i(w, \cdot) | g_0(\tilde{w}) = z] p_{g_0}(z) J(w) P(\mathrm{d}w) \right) \\
&= \int_W -\partial_i \left(E^{\tilde{P}}[\hat{f}_1^i(w, \cdot) | g_0(\tilde{w}) = z] p_{g_0}(z) \right) J(w) P(\mathrm{d}w),
\end{aligned}
$$

where $p_{g_0}(z) = \phi(z; 0, \Sigma(\theta_0))$, it follows that

$$
\begin{aligned}
&{}_{D^{-\infty}}\langle \hat{f}_1^i(w, \tilde{w})\partial_i 1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w, \tilde{w})), J(w)\rangle_{D^\infty} \\
&\quad = \int_W \left\{ \int_{A_\varepsilon^*(w)} -\partial_i \left(E^{\tilde{P}}[\hat{f}_1^i(w, \cdot)|g_0(\tilde{w}) = z] p_{g_0}(z) \right) \mathrm{d}z \right\} J(w)P(\mathrm{d}w) \\
&\quad = \int_W \left\{ \int_{A_\varepsilon(\theta_0)} q_1(z - \hat{x}_0(w), w)\, \mathrm{d}z \right\} J(w)P(\mathrm{d}w),
\end{aligned}
$$

where $q_1(z, w) = -\partial_i \left(E^{\tilde{P}}[\hat{f}_1^i(w, \cdot)|g_0(\tilde{w}) = z] p_{g_0}(z) \right)$. Therefore, we obtain

$$
\begin{aligned}
I(\hat{f}_1^i(w, \tilde{w})\partial_i 1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w, \tilde{w})), 1)(w) &= \int_{A_\varepsilon(\theta_0)} q_1(z - \hat{x}_0(w), w)\, \mathrm{d}z \\
&= \int_{A_\varepsilon(\theta_0)} p_1'(x, w)\, \mathrm{d}x
\end{aligned}
$$

P-a.s. Similarly, we have

$$
I(\hat{g}_0(w, \tilde{w}) 1_{A_\varepsilon(\theta_0)}(\hat{f}_0(w, \tilde{w})), 1)(w) = \int_{A_\varepsilon(\theta_0)} q_2(z - \hat{x}_0(w), w)\, \mathrm{d}z
$$

P-a.s., where $q_2(z, w) = E^{\tilde{P}} \left[\hat{g}_0(w, \cdot) \middle| \int_{T_0}^T a_t(\theta_0)\, \mathrm{d}\tilde{w}_t = z \right] p_{g_0}(z)$.

From (21)–(23) and Lemmas 5.7, 5.8 in [31], we have

$$
\begin{aligned}
E^{\tilde{P}} \left[\hat{f}_1^\alpha(w, \cdot) \middle| \int_{T_0}^T a_t(\theta_0)\, \mathrm{d}\tilde{w}_t = x \right] &= A^{0,\alpha}(\theta_0) - A_{p,q}^{2,\alpha}(\theta_0)\sigma^{pq}(\theta_0) + \\
&\quad + \hat{x}_1^\alpha(w) + C_i^{l\alpha}(\theta_0)\sigma_{lp}(\theta_0)x^p\zeta_i^{(0)}(w) \\
&\quad + A_{p,q}^{2,\alpha}(\theta_0)x^p x^q, \\
E^{\tilde{P}} \left[\hat{g}_0(w, \cdot) \middle| \int_{T_0}^T a_t(\theta_0)\, \mathrm{d}\tilde{w}_t = x \right] &= g_{-1}(\theta_0) \left(B^\alpha(\theta_0)\sigma_{\alpha p}(\theta_0)x^p + \hat{y}_0(w) \right).
\end{aligned}
$$

Since it follows that

$$
\begin{aligned}
p_1'(x, w) &= -\partial_\alpha \{ E^{\tilde{P}}[\hat{f}_1^\alpha(w, \cdot)|g_0(\tilde{w}) = x - \hat{x}_0(w)]\phi(x; \hat{x}_0(w), \Sigma(\theta_0)) \} \\
&= - \Big[C_i^{l\alpha}(\theta_0)\sigma_{l\alpha}(\theta_0)\zeta_i^{(0)}(w) \\
&\quad + \Big\{ A_{\alpha,p}^{2,\alpha}(\theta_0) + A_{p,\alpha}^{2,\alpha}(\theta_0) - \\
&\quad - \left(A^{0,\alpha}(\theta_0) - A_{l,q}^{2,\alpha}(\theta_0)\sigma^{lq}(\theta_0) + \hat{x}_\alpha(w) \right) \sigma_{\alpha p}(\theta_0) \Big\} \times \\
&\quad \times (x - \hat{x}_0(w))^p - C_i^{l\alpha}(\theta_0)\sigma_{lp}(\theta_0)\sigma_{\alpha q}(\theta_0)\zeta_i^{(0)}(w) \times \\
&\quad \times (x - \hat{x}_0(w))^p (x - \hat{x}_0(w))^q - A_{p,q}^{2,\alpha}(\theta_0)\sigma_{\alpha l} \times \\
&\quad \times (\theta_0)\ (x - \hat{x}_0(w))^p (x - \hat{x}_0(w))^q (x - \hat{x}_0(w))^l \Big] \times \\
&\quad \times \phi(x; \hat{x}_0(w), \Sigma(\theta_0)),
\end{aligned}
$$

we obtain

$$\begin{aligned}\hat{p}_1(x,w) &= g_{-1}(\theta_0)x\phi(x;\hat{x}_0(w),\Sigma(\theta_0)) + g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)p_1'(x,w)\\ &\quad + g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)[\hat{y}_0(w) +\\ &\quad + B^{\alpha}\sigma_{\alpha p}(\theta_0)(x-\hat{x}_0(w))^p]\phi(x;\hat{x}_0(w),\Sigma(\theta_0)).\end{aligned}$$

In the same fashion, we obtain $\hat{p}_i(x,w)$ for $i \geqslant 2$. This completes the proof.

Proof of Theorem 3. Let $\bar{S}^*_\varepsilon(w) = E_{T_0,\hat{\theta}_\varepsilon(w)}[\bar{H}^{*\varepsilon}_T(\hat{\theta}_\varepsilon(w),\tilde{w})] - E_{T_0,\theta_0}[H^\varepsilon_T(\theta_0,\tilde{w})]$. It follows from Corollary 1 and Theorem 2 that $\bar{S}^*_\varepsilon(w) \in L_p(\mathbf{R}^k)$ has the asymptotic expansion

$$\begin{aligned}\bar{S}^*_\varepsilon(w) &\sim \int_{A_\varepsilon(\theta_0)} g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)[\phi(x;\hat{x}_0(w),\Sigma(\theta_0)) -\\ &\quad - \phi(x;0,\Sigma(\theta_0))]\,\mathrm{d}x + \mathrm{O}_p(\varepsilon)\end{aligned}$$

in $L_p(\mathbf{R}^k)$ as $\varepsilon \downarrow 0$. Set $S_\varepsilon(w) = E_{T_0,\hat{\theta}_\varepsilon(w)}[H^\varepsilon_T(\hat{\theta}_\varepsilon(w),\tilde{w})] - E_{T_0,\theta_0}[H^\varepsilon_T(\theta_0,\tilde{w})]$. Let

$$\begin{aligned}b(w,\theta_0) &= \bar{S}^*_\varepsilon(w) - \int_{A_\varepsilon(\theta_0)} g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)\times\\ &\quad \times \left[\phi(x;\hat{x}_0(w),\Sigma(\theta_0)) - \phi(x;0,\Sigma(\theta_0))\right]\mathrm{d}x.\end{aligned}$$

It follows from (29) and (32) that

$$\begin{aligned}E[S_\varepsilon(w) - b_1(\hat{\theta}_\varepsilon(w))] &\sim E[\bar{S}^*_\varepsilon(w) - \bar{\psi}_\varepsilon b_1(\hat{\theta}_\varepsilon(w))]\\ &\sim E[\bar{S}^*_\varepsilon(w) - b_1(\theta_0)] + \mathrm{O}(\varepsilon)\\ &\sim E[b(w,\theta_0)] + \mathrm{O}(\varepsilon)\\ &\sim \mathrm{O}(\varepsilon)\end{aligned}$$

as $\varepsilon \downarrow 0$. This completes the proof.

Proof of Theorem 4. In the same way as Theorem 3, $\bar{S}^*_\varepsilon(w) \in L_p(\mathbf{R}^k)$ has the second-order asymptotic expansion

$$\begin{aligned}\bar{S}^*_\varepsilon(w) &\sim \int_{A_\varepsilon(\theta_0)} g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)\times\\ &\quad \times [\phi(x;\hat{x}_0(w),\Sigma(\theta_0)) - \phi(x;0,\Sigma(\theta_0))]\,\mathrm{d}x +\\ &\quad + \varepsilon\int_{A_\varepsilon(\theta_0)} \{\hat{p}_1(x,w) - p_1(x,\theta_0)\}\,\mathrm{d}x + \mathrm{O}_p(\varepsilon^2)\end{aligned}$$

in $L_p(\mathbf{R}^k)$ as $\varepsilon \downarrow 0$. Let

$$\begin{aligned}b(w,\theta_0) &= \bar{S}^*_\varepsilon(w) - \int_{A_\varepsilon(\theta_0)} g_{-1}(\theta_0)(f_{-1}(\theta_0)-K)\times\\ &\quad \times [\phi(x;\hat{x}_0(w),\Sigma(\theta_0)) - \phi(x;0,\Sigma(\theta_0))]\,\mathrm{d}x -\\ &\quad - \varepsilon\int_{A_\varepsilon(\theta_0)} \{\hat{p}_1(x,w) - p_1(x,\theta_0)\}\,\mathrm{d}x.\end{aligned}$$

In the same way as Theorem 3,

$$\begin{aligned}
&E[S_\varepsilon(w) - b_1(\hat{\theta}_\varepsilon(w)) - \varepsilon b_2(\hat{\theta}_\varepsilon(w))] \\
&\quad \sim E[\bar{S}^*_\varepsilon(w) - \bar{\psi}_\varepsilon b_1(\hat{\theta}_\varepsilon(w)) - \varepsilon\bar{\psi}_\varepsilon b_2(\hat{\theta}_\varepsilon(w))] \\
&\quad \sim E[\bar{S}^*_\varepsilon(w) - b_1(\theta_0) - \varepsilon b_2(\theta_0)] + \mathrm{O}(\varepsilon^2) \\
&\quad \sim E[b(w,\theta_0)] + \mathrm{O}(\varepsilon^2) \\
&\quad \sim \mathrm{O}(\varepsilon^2)
\end{aligned}$$

as $\varepsilon \downarrow 0$. This completes the proof.

Proof of Theorem 5. Let

$$\begin{aligned}
b(w,\theta_0) \;=\; & \bar{S}^*_\varepsilon(w) - g_{-1}(\theta_0)(f_{-1}(\theta_0) - K) \times \\
& \times \int_{A_\varepsilon(\theta_0)} [\phi(x;\hat{x}_0(w),\Sigma(\theta_0)) - \phi(x;0,\Sigma(\theta_0))]\,\mathrm{d}x - \\
& - \varepsilon g_{-1}(\theta_0) \int_{A_\varepsilon(\theta_0)} \big[x\phi(x;\hat{x}_0(w),\Sigma(\theta_0)) - \\
& - \; x\phi(x;0,\Sigma(\theta_0))\big]\,\mathrm{d}x.
\end{aligned}$$

We have the expansion of $b(w,\theta_0)$ in ε-power as follows.

(i) when $K - f_{-1}(\theta_0) < 0$, $b(w,\theta_0) \in L_p(\mathbf{R}^k)$ has the asymptotic expansion
$$b(w,\theta_0) \sim \varepsilon g_{-1}(\theta_0)(f_{-1}(\theta_0) - K)\hat{y}_0(w) + \mathrm{O}_p(\varepsilon^2)$$
in $L_p(\mathbf{R}^k)$ as $\varepsilon \downarrow 0$.

(ii) when $K^i - f_{-1}(\theta_0) > 0$, $b(w,\theta_0) \in L_p(\mathbf{R}^k)$ has the asymptotic expansion
$$b(w,\theta_0) \sim \mathrm{O}_p(\varepsilon^n)$$
for $n = 2, 3, \ldots$ in $L_p(\mathbf{R}^k)$ as $\varepsilon \downarrow 0$.

(iii) when $K^i - f_{-1}(\theta_0) = 0$, $b(w,\theta_0) \in L_p(\mathbf{R}^k)$ has the asymptotic expansion
$$b(w,\theta_0) \sim \mathrm{O}_p(\varepsilon^2)$$
in $L_p(\mathbf{R}^k)$ as $\varepsilon \downarrow 0$.

In the same way as Theorem 4,

$$\begin{aligned}
&E[S_\varepsilon(w) - b_1(\hat{\theta}_\varepsilon(w)) - \varepsilon b_2(\hat{\theta}_\varepsilon(w))] \\
&\quad \sim E[\bar{S}^*_\varepsilon(w) - \bar{\psi}_\varepsilon b_1(\hat{\theta}_\varepsilon(w)) - \varepsilon\bar{\psi}_\varepsilon b_2(\hat{\theta}_\varepsilon(w))] \\
&\quad \sim E[\bar{S}^*_\varepsilon(w) - b_1(\theta_0) - \varepsilon b_2(\theta_0)] + \mathrm{O}(\varepsilon^2) \\
&\quad \sim E[b(w,\theta_0)] + \mathrm{O}(\varepsilon^2) \\
&\quad \sim \mathrm{O}(\varepsilon^2)
\end{aligned}$$

as $\varepsilon \downarrow 0$. This completes the proof.

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References

1. Bichteler, K., Gravereaux, J.-B. and Jacod, J.: *Malliavin Calculus for Processes with Jumps*, Gordon and Breach Science Publishers, New York, London, Paris, Montreux, Tokyo 1987.
2. Dermoune, A. and Kutoyants, Yu. A.: Expansion of distribution function of maximum likelihood estimate for misspecified diffusion type observations. *Stochastics and Stochastic Reports* **52** (1995), 121–145.
3. Genon-Catalot, V.: Maximum contrast estimation for diffusion processes from discrete observations. *Statistics* **21** (1990), 99–116.
4. Hull, J.: *Options, Futures, and Other Derivative Securities*, 2nd edn, Prentice-Hall, New Jersey, 1993.
5. Ikeda, N. and Watanabe, S.: *Stochastic Differential Equations and Diffusion Processes*, 2nd edn, North-Holland/Kodansha, Tokyo, 1989.
6. Kashiwakura, K., Sørensen, M. and Yoshida, N.: Random limit expansion for small diffusion processes (2002) (in preparation).
7. Kim, Y. J. and Kunitomo, N.: Pricing options under stochastic interest rates. *Asia Pacific Financial Markets* **6** (1999), 49–70.
8. Kunitomo, N. and Takahashi, A.: The asymptotic expansion approach to the valuation of interest rate contingent claims. *Mathematical Finance* **11** (2001), 117–151.
9. Kutoyants, Yu. A.: In: *Parameter Estimation for Stochastic Processes*. B. L. S. Prakasa Rao (ed.), Heldermann, Berlin, 1984.
10. Kutoyants, Yu. A.: *Identification of Dynamical Systems with Small Noise*, Kluwer, Dordrecht, 1994.
11. Laredo, C. F.: A sufficient condition for asymptotic sufficiency of incomplete observations of a diffusion process, *Ann. Statist*. **18** (1990), 1158–1171.
12. Lamberton, D. and Lapeyre, B.: *Introduction to Stochastic Calculus Applied to Finance*, Chapman and Hall, London, 1996.
13. Malliavin, P.: Stochastic calculus of variation and hypoelliptic operators. In: K. Itô (ed.), *Proceedings of the International Symposium SDE Kyoto*, Kinokuniya, Tokyo, 1976, pp. 195–263.
14. Malliavin, P.: C^k-hypoellipticity with degeneracy. In: A. Friedman and M. Oinsky (eds), *Stochastic Analysis*, Academic Press, New York, 1978, pp. 199–214, 327–340.
15. Malliavin, P.: *Stochastic Analysis*, Springer, Berlin, Heidelberg, New York, 1997.
16. Nualart, D.: The Malliavin Calculus and Related Topics, Springer, Berlin, Heidelberg, New York, 1995.
17. Sakamoto, Y. and Yoshida, N.: Expansion of perturbed random variables based on generalized Wiener functionals, *J. Multivariate Anal.* **59** (1996), 34–59.
18. Sakamoto, Y. and Yoshida, N.: Asymptotic expansions of M-estimator over Wiener space, *Statist. Infer. Stochastic Process.* **1** (1998), 85–103.
19. Sørensen, M.: *Small Dispersion Asymptotics for Diffusion Martingale Estimating Functions*, Preprint No. 2000-2, Department of Statistics and Operations Research, University of Copenhagen, 2000.
20. Sørensen, M. and Uchida, M.: Small diffusion asymptotics for discretely sampled stochastic differential equations, *Bernoulli* (2002) (to appear).
21. Takahashi, A.: An asymptotic expansion approach to pricing contingent claims, *Asia Pacific Financial Markets* **6** (1999), 115–151.
22. Takahashi, A. and Yoshida, N.: Asymptotic expansion scheme for the optimal portfolio for investment, *Statist. Infer. Stochastic Process.* (2001) (to appear).

23. Takanobu, S. and Watanabe, S.: Asymptotic expansion formula of the Schilder type for a class of conditional Wiener functional integrations. In: K. D. Elworthy and N. Ikeda (eds), Asymptotic Problems in Probability Theory: Wiener Functionals and Asymptotics, *Proceedings*, *the Taniguchi International Symposium*, Sanda and Kyoto, Longman, Harlow, UK, 1993, pp. 194–241.
24. Uchida, M. and Yoshida, N.: Information criteria for small diffusions via the theory of Malliavin–Watanabe, *Statist. Infer. Stochastic Process.* (2002) (to appear).
25. Uchida, M.: Estimation for dynamical systems with small noise from discrete observations, (2002) (revised).
26. Uchida, M.: Estimation for discretely observed small diffusions based on approximate martingale estimating functions, (2003) (submitted).
27. Watanabe, S.: Malliavin's calculus in terms of generalized Wiener functionals. In: *Lecture Notes Control Inf. Sci.*, Vol. 49, Springer, Berlin, Heidelberg, New York, 1983.
28. Watanabe, S.: *Lectures on Stochastic Differential Equations and Malliavin Calculus*, Tata Institute of Fundamental Research, Springer, Berlin, Heidelberg, New York, 1984.
29. Watanabe, S.: Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels, *Ann. Probab*. **15** (1987), 1–39.
30. Yoshida, N.: Asymptotic expansion of maximum likelihood estimators for small diffusions via the theory of Malliavin–Watanabe. *Probab. Theory Relat. Fields* **92** (1992), 275–311.
31. Yoshida, N.: Asymptotic expansion for statistics related to small diffusions, *J. Japan Statist. Soc*. **22** (1992), 139–159.
32. Yoshida, N.: Asymptotic expansion of Bayes estimators for small diffusions, *Probab. Theory Relat. Fields* **95** (1993), 429–450.
33. Yoshida, N.: Asymptotic expansions for perturbed systems on Wiener space: maximum likelihood estimators, *J. Multivariate Anal*. **57** (1996), 1–36.
34. Yoshida, N.: Malliavin calculus and statistics. In: S. Kotz et al. (eds), *Encyclopedia of Statistical Sciences*, Update Vol. 3, Wiley, New York, 1999.
35. Yoshida, N.: Conditional expansions and their applications, *Stoch. Proc. Appl*. (2001) (to appear).