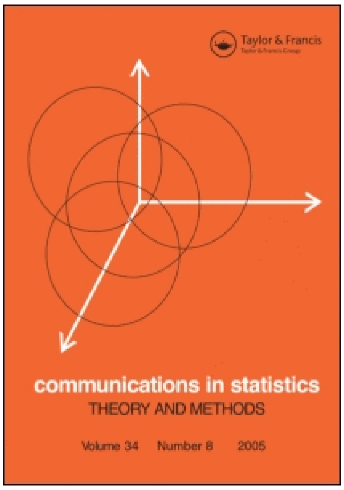

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Asymptotic Expansion for Functionals of a Marked Point Process

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We consider a functional of a marked point process and derive asymptotic expansion of the distribution. Also, we apply this result to obtain expansion for the M-estimator. A moving average process sampled by a point process and a point process marked by a diffusion process are discussed.

Keywords Asymptotic expansion; Malliavin calculus; Marked point process; Mixing; Point process; Sampling.

Mathematics Subject Classification Primary 62M05, 62M09, 60G55, 60F99, 60H07; Secondary 60F05.

1. Introduction

Given a probability space $(\Omega, \mathcal{F}, P)$ and a sub σ-field $\widehat{\mathcal{F}} \subset \mathcal{F}$, let $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ and $\lambda : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ are $\widehat{\mathcal{F}} \times \mathbb{B}(\mathbb{R}_+)$-measurable mappings. $N = (N_t)_{t\in\mathbb{R}_+}$ is assumed to be an $\widehat{\mathcal{F}}$-conditional inhomogeneous Poisson process with intensity $\lambda(t)$, namely, $N = (N_t)_{t\in\mathbb{R}_+}$ is a right-continuous simple point process, in that $N_0 = 0$ and $\Delta N_t \in \{0, 1\}$, N is a process with $\widehat{\mathcal{F}}$-conditionally independent increments, $\int_0^t \lambda(s)ds < \infty$ a.s. for all $t \in \mathbb{R}_+$, and

$$E\big[N_t - N_s \,|\, \check{\mathcal{F}}_s\big] = \int_s^t \lambda(r)dr$$

for $0 \le s < t$, where $\check{\mathcal{F}}_s = \sigma[N_r; r \le s] \vee \widehat{\mathcal{F}}$. By definition, N possesses a doubly stochastic structure.

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We consider an n-dimensional functional

$$Z_T = \int_0^T f(t)dN_t, \tag{1.1}$$

and the centered functional

$$\overline{Z}_T = \frac{1}{\sqrt{T}}\int_0^T f(t)dN_t - E\left[\frac{1}{\sqrt{T}}\int_0^T f(t)\lambda(t)dt\right] \tag{1.2}$$

to investigate the distribution of Z_T.

In introducing the functional (1.1), we have motives for statistical inference for stochastic processes sampled discretely at times that are determined by a stochastic mechanism described by N. Since the asymptotic normality of an estimator under a mixing property is rather obvious, we focus our mind on the higher-order asymptotics in this article.

In Sec. 2 we prepare formulas for the cumulants of the functional $\overline{Z}_T$. Section 3 starts with reconstruction of the random elements to apply the Malliavin calculus to validate the asymptotic expansion. A basic result (Theorem 3.1) is presented there. Section 4 treats an M-estimation based on the data sampled at occurrence times of the point process N. Since N has a random intensity process $\lambda(t)$, it is possible to describe various random sampling schemes in practice. Sections 3 and 4 validate the asymptotic expansion scheme in a general setting; however, it is a somewhat abstract one. As an illustrative example, we will discuss a moment estimator for a sampled moving average process in Sec. 5. Finally, in Sec. 6, we treat a point process marked by a diffusion process and give coefficients of the expansion, taking advantage of the nature of the diffusion process. We also discuss a parameter estimation problem for a randomly sampled diffusion process.

2. Cumulants of $\overline{Z}_T$

This section gives expressions to the cumulants of $\overline{Z}_T$. We will assume the following conditions.

[A1] There exists a positive constant a such that

$$\sup_{A\in\mathscr{B}_0^t,B\in\mathscr{B}_{t+h}^\infty}\left|P[A\cap B]-P[A]P[B]\right| \le \mathsf{a}^{-1}e^{-\mathsf{a}h} \quad (t, h\in\mathbb{R}_+)$$

where $\mathscr{B}_0^t = \sigma[\lambda(s), f(s); s\in[0,t]]$ and $\mathscr{B}_t^\infty = \sigma[\lambda(s), f(s); s\in[t,\infty)]$.

[A2] $(\lambda(t), f(t))$ is stationary and $\lambda(0), f(0)\in\bigcap_{p>1}L^p$.

Remark 2.1. Condition [A1] is the exponential mixing of the process $(\lambda(t), f(t))$. It is possible to relax it to a polynomial type mixing condition to obtain the results in this article. Moreover, the stationarity assumption in [A2] is not essential, just for simplicity of exposition.

By stationarity,

$$E\left[\frac{1}{\sqrt{T}}\int_0^T f(t)\lambda(t)dt\right] = T^{\frac{1}{2}}\mu$$

for $\mu = (\mu_a) = E[\lambda(0)f(0)] \in \mathbb{R}^n$. The characteristic function $\varphi(u)$ of $\overline{Z}_T$ is given by

$$\varphi(u) = e^{-iu\cdot T^{\frac{1}{2}}\mu} E\left[\exp\left\{\int_0^T \lambda(t)(e^{iu\cdot T^{-1/2}f(t)} - 1)dt\right\}\right]. \tag{2.1}$$

Since the absolute value of the integrant of the above expectation is not greater than 1, the differentiations under the expectation sign are valid because of [A2].

We write $\tilde{h}(t) = h(t) - E[h(t)]$ for a measurable process $h : \Omega \times \mathbb{R}_+ \to \mathbb{R}$. Moreover, define $\mathbb{J}_T(h)$ and $\widetilde{\mathbb{J}}_T(h)$ as

$$\mathbb{J}_T(h) = \frac{1}{T}\int_0^T h(t)dt$$

and

$$\widetilde{\mathbb{J}}_T(h) = \frac{1}{\sqrt{T}}\int_0^T \tilde{h}(t)dt$$

if exist.

Denote by κ_{ab} and κ_{abc} the second and third cumulants of $\overline{Z}_T$ (depending on T), respectively.

Lemma 2.1. *Under conditions* [A1] *and* [A2],

(a) κ_{ab} *admits the representations*

$$\begin{aligned}\kappa_{ab} &= E\left[\widetilde{\mathbb{J}}_T(\lambda f_a)\widetilde{\mathbb{J}}_T(\lambda f_b)\right] + E\left[\mathbb{J}_T(\lambda f_a f_b)\right] \\ &= \int_0^\infty \left(Cov[\lambda(t)f(t)_a, \lambda(0)f(0)_b] + Cov[\lambda(t)f(t)_b, \lambda(0)f(0)_a]\right)dt \\ &\quad + E[\lambda(0)f(0)_a f(0)_b] + O\left(\frac{1}{T}\right)\end{aligned}$$

as $T \to \infty$.

(b) κ_{abc} *admits the representations*

$$\begin{aligned}\kappa_{abc} &= E\left[\widetilde{\mathbb{J}}_T(\lambda f_a)\widetilde{\mathbb{J}}_T(\lambda f_b)\widetilde{\mathbb{J}}_T(\lambda f_c)\right] + T^{-1/2}\sum{}^{*}E\left[\widetilde{\mathbb{J}}_T(\lambda f_a f_b)\widetilde{\mathbb{J}}_T(\lambda f_c)\right] \\ &\quad + T^{-1/2}E\left[\mathbb{J}_T(\lambda f_a f_b f_c)\right]\end{aligned}$$

and

$$\begin{aligned}T^{\frac{1}{2}}\kappa_{abc} &= \sum{}^{**}\int_0^\infty\int_0^\infty E\left[\widetilde{\lambda f}_a(s+t)\widetilde{\lambda f}_b(s)\widetilde{\lambda f}_c(0)\right]dt\,ds \\ &\quad + \sum{}^{*}\int_0^\infty E\left[\widetilde{\lambda f_a f}_b(t)\widetilde{\lambda f}_c(0) + \widetilde{\lambda f_a f}_b(0)\widetilde{\lambda f}_c(t)\right]dt \\ &\quad + E[\lambda(0)f_a(0)f_b(0)f_c(0)] + o(1)\end{aligned}$$

as $T \to \infty$. *Here* $\sum{}^{*}m_{a,b,c} = m_{a,b,c} + m_{b,c,a} + m_{c,a,b}$ *and* $\sum{}^{**}m_{a,b,c} = m_{a,b,c} + m_{a,c,b} + m_{b,a,c} + m_{b,c,a} + m_{c,a,b} + m_{c,b,a}$ *for any* $m_{a,b,c}$ *with subscripts* a, b, c.

Proof. From (2.1),

$$
\begin{aligned}
\kappa_{ab} &= E\left[\left(T^{-1/2}\int_0^T \lambda(t)f(t)_a\,dt\right)\left(T^{-1/2}\int_0^T \lambda(t)f(t)_b\,dt\right)\right] \\
&\quad + E\left[T^{-1}\int_0^T \lambda(t)f(t)_a f(t)_b\,dt\right] - T\mu_a\mu_b \\
&= E\left[\widetilde{\mathbb{J}}_T(\lambda f_a)\widetilde{\mathbb{J}}_T(\lambda f_b)\right] + E\left[\mathbb{J}_T(\lambda f_a f_b)\right].
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\kappa_{ab} &= \int_0^T \frac{T-t}{T}\left(E[\widetilde{\lambda f}_a(t)\widetilde{\lambda f}_b(0) + \widetilde{\lambda f}_a(0)\widetilde{\lambda f}_b(t)]\right)dt + E[\lambda(0)f(0)_a f(0)_b] \\
&= \int_0^\infty \left(E[\widetilde{\lambda f}_a(t)\widetilde{\lambda f}_b(0) + \widetilde{\lambda f}_a(0)\widetilde{\lambda f}_b(t)]\right)dt \\
&\quad + E[\lambda(0)f(0)_a f(0)_b] + O\left(\frac{1}{T}\right).
\end{aligned}
$$

The last equation was due to the covariance inequality associated to the mixing property.

Next, for the third-order cumulant,

$$
\begin{aligned}
\kappa_{abc} &= E\left[\left(T^{-1/2}\int_0^T \lambda(t)f(t)_a dt\right)\left(T^{-1/2}\int_0^T \lambda(t)f(t)_b dt\right)\left(T^{-1/2}\int_0^T \lambda(t)f(t)_c\,dt\right)\right] \\
&\quad + \sum{}^{*} E\left[\left(T^{-1}\int_0^T \lambda(t)f(t)_a f(t)_b dt\right)\left(T^{-1/2}\int_0^T \lambda(t)f(t)_c dt\right)\right] \\
&\quad + E\left[T^{-3/2}\int_0^T \lambda(t)f(t)_a f(t)_b f(t)_c dt\right] - T^{\frac{1}{2}}\sum{}^{*}\kappa_{ab}\mu_c - T^{3/2}\mu_a\mu_b\mu_c \\
&= E\Bigg[\left(T^{-1/2}\int_0^T \lambda(t)f(t)_a dt - T^{\frac{1}{2}}\mu_a\right)\left(T^{-1/2}\int_0^T \lambda(t)f(t)_b\,dt - T^{\frac{1}{2}}\mu_b\right) \\
&\quad \times \left(T^{-1/2}\int_0^T \lambda(t)f(t)_c dt - T^{\frac{1}{2}}\mu_c\right)\Bigg] \\
&\quad + \sum{}^{*} E\left[T^{-1}\int_0^T \widetilde{\lambda f_a f_b}(t)dt \times T^{-1/2}\int_0^T \widetilde{\lambda f}_c(t)dt\right] \\
&\quad + E\left[T^{-3/2}\int_0^T \lambda(t)f(t)_a f(t)_b f(t)_c dt\right],
\end{aligned}
$$

which is the first assertion of (b). Moreover, we have

$$
\begin{aligned}
&T^{\frac{1}{2}}E\left[\widetilde{\mathbb{J}}_T(\lambda f_a)\widetilde{\mathbb{J}}_T(\lambda f_b)\widetilde{\mathbb{J}}_T(\lambda f_c)\right] \\
&\quad = \sum{}^{**}\iint 1_{\{0<s<t<T\}}\frac{T-t}{T}E\left[\widetilde{\lambda f}_a(t)\widetilde{\lambda f}_b(s)\widetilde{\lambda f}_c(0)\right]ds\,dt \\
&\quad \to \sum{}^{**}\int_0^\infty\int_0^\infty E\left[\widetilde{\lambda f}_a(s+t)\widetilde{\lambda f}_b(s)\widetilde{\lambda f}_c(0)\right]dt\,ds.
\end{aligned}
$$

Here we used the integrability of $E[\widetilde{\lambda f}_a(s+t)\widetilde{\lambda f}_b(s)\widetilde{\lambda f}_c(0)]$ in $(t,s) \in (0,\infty)^2$ but it is clear if one applies the covariance inequality twice together with the estimate

$$\int_0^\infty \int_0^\infty e^{-\mathsf{q}(s\vee u)} ds\, du < \int_0^\infty \int_0^\infty e^{-\frac{\mathsf{q}}{2}(s+u)} ds\, du < \infty.$$

We also have

$$E\left[\widetilde{\mathbb{J}}_T(\lambda f_a f_b)\widetilde{\mathbb{J}}_T(\lambda f_c)\right] \to \int_0^\infty E\left[\widetilde{\lambda f_a f_b}(t)\widetilde{\lambda f}_c(0) + \widetilde{\lambda f_a f_b}(0)\widetilde{\lambda f}_c(t)\right] dt.$$

Thus, we have the second assertion of (b). □

3. Asymptotic Expansion of $\mathcal{L}\{\overline{Z}_T\}$

In order to validate a formal Edgeworth expansion of $\mathcal{L}\{\overline{Z}_T\}$, we need more structure of random variables. Processes $w = (w_t)_{t\in\mathbb{R}_+}$ and $Y^\dagger = (Y_t^\dagger)_{t\in\mathbb{R}_+}$ are abstract separable stochastic processes defined on a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ taking values in $\mathbb{R}^{d_0}$ and $\mathbb{R}^{d_1}$ for some d_0 and $d_1 \in \{1, 2, \ldots, \infty\}$, respectively. Let p be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_+$ defined on another probability space $(\Omega', \mathcal{F}', P')$ with compensator $dt \times dx$. Let $(\Omega, \mathcal{F}, P) = (\widehat{\Omega} \times \Omega', \widehat{\mathcal{F}} \times \mathcal{F}', \widehat{P} \times P')$. Stochastic processes defined on $\widehat{\Omega}$ or Ω' are extended to Ω in a natural way. Denote by $\mathcal{B}_I^{dU}$ the σ-field generated by the increments of a process U over the set $I \subset \mathbb{R}_+$. Similarly, $\mathcal{B}_I^U$ denotes the σ-field generated by U_t, $t \in I$.

By definition, $(Y^\dagger, w)$ is independent of p. A basic structure we will assume is described as follows.

(i) w is a process with independent increments in the sense that $\mathcal{B}_{[0,r]}^{dw} \vee \mathcal{B}_{[0,r]}^{Y^\dagger}$ is independent of $\mathcal{B}_{[r,\infty)}^{dw}$ for all $r \in \mathbb{R}_+$.
(ii) $Y^\dagger$ is an ϵ-Markov process driven by w in that $Y_t^\dagger$ is $\mathcal{B}_{[s-\epsilon,s]}^{Y^\dagger} \vee \mathcal{B}_{[s,t]}^{dw}$-measurable for any $s, t \in \mathbb{R}_+$ with $\epsilon \le s \le t$. Here ϵ is a nonnegative constant.

Put $X_t^\dagger = (w_t, \mathsf{p}([0,t] \times B); B \in \mathbb{B}_0)$, where $\mathbb{B}_0$ is a countable family of generators of the Borel sets $\mathbb{B}(\mathbb{R}_+)$. Then $X^\dagger$ is a process with independent increments; that is, $\mathcal{B}_{[0,r]} := \mathcal{B}_{[0,r]}^{dX^\dagger} \vee \mathcal{B}_{[0,r]}^{Y^\dagger}$ is independent of $\mathcal{B}_{[r,\infty)}^{dX^\dagger}$ for all $r \in \mathbb{R}_+$. Moreover, $Y^\dagger$ is an ϵ-Markov process driven by $X^\dagger$, namely, $Y_t^\dagger$ is $\mathcal{B}_{[s-\epsilon,s]}^{Y^\dagger} \vee \mathcal{B}_{[s,t]}^{dX^\dagger}$-measurable for any $s, t \in \mathbb{R}_+$ with $\epsilon \le s \le t$.

Let $\lambda = (\lambda(t))_{t\in\mathbb{R}_+}$ and $f = (f(t))_{t\in\mathbb{R}_+}$ be respectively $\mathbb{R}_+$ and $\mathbb{R}^n$-valued stochastic processes defined on $\widehat{\Omega}$ (therefore extended to Ω) satisfying that $(\lambda, f)|_{\Omega\times[s,t]}$ is $\mathcal{B}_{[s,t]} \times \mathbb{B}([s,t])$-measurable for every $s < t$, where $\mathcal{B}_{[s,t]} = \mathcal{B}_{[s,t]}^{dX^\dagger} \vee \mathcal{B}_{[s,t]}^{Y^\dagger}$, and that $t \mapsto \lambda(t)$ is locally bounded a.s. The process N_t is now defined by

$$N_t = \iint 1_{[0,t]\times[0,\lambda(s)]}(s,x)\mathsf{p}(ds, dx).$$

Then the process Z_t defined by (1.1) satisfies that $Z_t - Z_s$ is $\mathcal{B}_{[s,t]}$-measurable for every $s, t \in \mathbb{R}_+$ with $s \le t$.

Let $I(j) = [u(j), v(j)]$ $(j = 1, \ldots, n(T))$ be a sequence of intervals with $\epsilon \le u(j) < v(j) \le u(j+1)$, $0 < \inf_j |I(j)| \le \sup_j |I(j)| < \infty$, and $\liminf_{T\to\infty} n(T)/T > 0$.

Suppose that for some sub σ-field $\mathscr{B}'_{[v(j)-\epsilon,v(j)]}$ of $\mathscr{B}_{[v(j)-\epsilon,v(j)]}$,

$$E[\cdot \mid \mathscr{B}_{[v(j)-\epsilon,v(j)]}] = E[\cdot \mid \mathscr{B}'_{[v(j)-\epsilon,v(j)]}] \quad \text{on } b\mathscr{B}_{[v(j),\infty)}.$$

Set $\widehat{\mathscr{C}}(j) = \mathscr{B}_{[u(j)-\epsilon,u(j)]} \vee \mathscr{B}'_{[v(j)-\epsilon,v(j)]}$. Then $(I(j))$ forms a set of *dense reduction intervals* associated with $\widehat{\mathscr{C}}(j)$; see Yoshida (2004).

In order to validate the formal Edgeworth expansion, besides [A1] and [A2], we will assume the following condition, which can be relaxed but simple to state.

[A3] There exist truncation functionals $\psi_j : \Omega \to [0, 1]$ and constants $b, b' \in (0, 1)$ and $B > 0$ such that $4b' < (1-b)^2$ and

(i) $\frac{1}{n(T)} \sum_j E\big[\sup_{u:|u|\geq B} \big|E\big[e^{iu\cdot(Z_{v(j)}-Z_{u(j)})}\psi_j \mid \widehat{\mathscr{C}}(j)\big]\big|\big] < b'$ for large T.
(ii) $\frac{1}{n(T)} \sum_j E[\psi_j] > 1 - b$ for large T.

It is usually possible to simplify the above condition if one may take advantage of the stationarity. On the other hand, it is not difficult to verify [A3] for a stationary model specified more concretely.

The cumulant functions $\chi_{T,r}(u)$ of $\overline{Z}_T$ given by (1.2) are defined by

$$\chi_{T,r}(u) = \left(\frac{d}{d\epsilon}\right)^r \Big|_{\epsilon=0} \log P[\exp(i\epsilon u \cdot \overline{Z}_T)].$$

By the partial sum of the formal expansion

$$\exp\left(\sum_{r=2}^{\infty} \frac{1}{r!}\epsilon^{r-2}\chi_{T,r}(u)\right) = \exp\left(\frac{1}{2}\chi_{T,2}(u)\right) + \sum_{r=1}^{\infty} \epsilon^r T^{-\frac{r}{2}} \widetilde{P}_{T,r}(u),$$

we define the function $\widehat{\Psi}_{T,p}(u)$ as

$$\widehat{\Psi}_{T,p}(u) = \exp\left(\frac{1}{2}\chi_{T,2}(u)\right) + \sum_{r=1}^{p-2} T^{-\frac{r}{2}} \widetilde{P}_{T,r}(u).$$

Let $\Psi_{T,s} = \mathscr{F}^{-1}[\widehat{\Psi}_{T,s}]$, the Fourier inversion of $\widehat{\Psi}_{T,s}$.

Let $p \in \mathbb{N}$ with $p \geq 3$, $p_0 = 2[p/2]$ and set $\mathscr{E}(M, \gamma) = \{h : \mathbb{R}^d \to \mathbb{R},\ \text{measurable}, |h(x)| \leq M(1 + |x|^\gamma) \quad (x \in \mathbb{R}^d)\}$. For measurable function $h : \mathbb{R}^d \to \mathbb{R}$, $r > 0$ and a Borel measure ν on $\mathbb{R}^d$, we set $\omega_h(x; r) = \sup\{|h(x+h) - h(x)|; |h| \leq r\}$ and write $\omega(h; r, \nu) = \int \omega_h(x; r) d\nu$. Let

$$\Delta_T(h) := |E[h(\overline{Z}_T)] - \Psi_{T,p}[h]|.$$

We write $r_T(h) = \bar{o}(T^{-a})$ when $r_T(h) = o(T^{-a})$ uniformly in a class $\mathscr{E}(M, \gamma)$.

Theorem 3.1. *Suppose that a positive-definite limit $\Sigma = \lim_{T\to\infty} \mathrm{Cov}[\overline{Z}_T]$ exists and let Σ^o denote a symmetric matrix satisfying $\Sigma < \Sigma^o$. Let $M, K > 0$. Then under* [A1]–[A3], *there exist positive constants M^* and δ^* such that*

$$\Delta_T(h) \leq M^* \omega(h; T^{-K}, \phi(x; \Sigma^o)dx) + \bar{o}(T^{-(p-2+\delta^*)/2}) \tag{3.1}$$

uniformly in $\mathscr{E}(M, p_0)$ *as* $T \to \infty$. *Here* $\phi(z; \Sigma^o)$ *is the normal density with positive definite covariance matrix* Σ^o.

This result follows from Theorem 1 and Sec. 3.3 of Yoshida (2004).

Remark 3.1. In the present exposition of the result, we assumed the stationarity of the process (λ, f). However, it is possible to replace the stationarity in [A2] by the uniform boundedness of the moments of the increments of the functional.

4. *M*-Estimator

In this section, we will consider an application of Theorem 3.1 to an M-estimator. Let $\Theta \subset \mathbb{R}^p$ be an open bounded convex set, fix $\theta_0 \in \Theta$ arbitrarily as the target value to be estimated. We use p for the dimension of Θ in this section. Let $\check{f} : \Omega \times \mathbb{R}_+ \times \Theta \to \mathbb{R}^p$ be an $\widehat{\mathscr{F}} \otimes \mathbb{B}(\mathbb{R}_+) \otimes \mathbb{B}(\Theta)$-measurable mapping. Suppose that for each $\theta \in \Theta$, $(\check{f}(t, \theta))_{t \in \mathbb{R}_+}$ is a stationary process. In order to estimate θ_0, let us consider an estimating function defined by

$$\psi(T, \theta) = \int_0^T \check{f}(t, \theta) dN_t - T\mu(T, \theta),$$

where N_t is given in Sec. 3 and μ is an auxiliary function. Concrete examples of this estimation function ψ will be shown in Secs. 5 and 6.

Denote the ath elements of ψ, $\check{f}$, and μ by $\psi_{a;}$, $\check{f}_{a;}$, and $\mu_{a;}$ and put $\psi_{a;a_1\cdots a_k}(\cdot, \theta) = \delta_{a_1} \cdots \delta_{a_k} \psi_{a;}(\cdot, \theta)$, $\check{f}_{a;a_1\cdots a_k}(\cdot, \theta) = \delta_{a_1} \cdots \delta_{a_k} \check{f}_{a;}(\cdot, \theta)$, $\mu_{a;a_1\cdots a_k}(\cdot, \theta) = \delta_{a_1} \cdots \delta_{a_k} \mu_{a;}(\cdot, \theta)$, and $\bar{\nu}_{a;a_1\cdots a_k}(T, \theta) = T^{-1} E[\psi_{a;a_1\cdots a_k}(T, \theta)]$, where $\delta_a = \partial / \partial \theta^a$. From the stationarity,

$$\bar{\nu}_{a;a_1\cdots a_k}(T, \theta) = E[\check{f}_{a;a_1\cdots a_k}(0, \theta)\lambda(0)] - \mu_{a;a_1\cdots a_k}(T, \theta).$$

We will assume the following conditions for $K \in \mathbb{N}$, $q > 1$ and $\gamma > 0$:

$[\text{C0}]^K$ $\psi(T, \cdot) \in C^K(\Theta)$ a.s.;
$[\text{C1}]_q$ $\sup_{T > T_0} \left\| T^{-1/2} \psi_{a;}(T, \theta_0) \right\|_q < \infty$ for $a = 1, \ldots, p$.
$[\text{C2}]^K_{q,\gamma}$ $\sup_{T > T_0, \theta \in \Theta} \left\| T^{\gamma/2} (T^{-1} \psi_{a;a_1\cdots a_K}(T, \theta) - \bar{\nu}_{a;a_1\cdots a_K}(T, \theta)) \right\|_q < \infty$;
[C3] There exists an open set $\widetilde{\Theta}$ including θ_0 such that

$$\inf_{T > T_0, \theta_1, \theta_2 \in \widetilde{\Theta}, |x| = 1} \left| x' \left(\int_0^1 \bar{\nu}_{a;b}(T, \theta_1 + s(\theta_2 - \theta_1)) ds \right) \right| > 0;$$

$[\text{C4}]^K_q$ $\sup_{T > T_0} \left\| \sup_{\theta \in \Theta} |T^{-1} \psi_{a;a_1\cdots a_K}(T, \theta)| \right\|_q < \infty$ for $a, a_j = 1, \ldots, p$, $j = 1, \ldots, K$.

Under some of these conditions, we can show the existence of M-estimator $\hat{\theta}_T$ defined by $\psi(T, \hat{\theta}_T) = 0$. For the details, see Sakamoto and Yoshida (2004). For some bounded function β, define a bias-corrected M-estimator $\hat{\theta}^*_T$ by

$$\hat{\theta}^*_T = \hat{\theta}_T - r_T^2 \beta(\hat{\theta}_T). \tag{4.1}$$

Let

$$\overline{Z}_{a;}(T) = \frac{1}{\sqrt{T}}\int_0^T \check{f}_{a;}(t,\theta_0)dN_t - \sqrt{T}E[\check{f}_{a;}(0,\theta_0)\lambda(0)]$$

and

$$\overline{Z}_{a;b}(T) = \frac{1}{\sqrt{T}}\int_0^T \check{f}_{a;b}(t,\theta_0)dN_t - \sqrt{T}E[\check{f}_{a;b}(0,\theta_0)\lambda(0)],$$

then $\check{\kappa}_{a;,b;} := \mathrm{Cov}[\overline{Z}_{a;}, \overline{Z}_{b;}]$ and $\check{\kappa}_{a;,b;,c;} := \mathrm{Cum}[\overline{Z}_{a;}, \overline{Z}_{b;}, \overline{Z}_{c;}]$ take the same forms as κ_{ab} and κ_{abc} in Lemma 2.1 with $\check{f}_{a;}(\cdot,\theta_0)$, $\check{f}_{b;}(\cdot,\theta_0)$ and $\check{f}_{c;}(\cdot,\theta_0)$ in place of f_a, f_b and f_c, respectively. Also $\check{\kappa}_{a;b,c;} := \mathrm{Cov}[\overline{Z}_{a;b}, \overline{Z}_{c;}]$ can be represented as κ_{ab} in Lemma 2.1 with $\check{f}_{a;b}$ and $\check{f}_{c;}$ in place of f_a and f_b.

Under condition [C3], there exists the inverse matrix of $(\bar{\nu}_{a;b}(T,\theta_0))_{a,b=1}^p$, which is denoted by $(\bar{\nu}^{a;b})_{a,b=1}^p$. Let $\bar{\lambda}^{abc} = -T^{1/2}\bar{\nu}^{a;a'}\bar{\nu}^{b;b'}\bar{\nu}^{b;b'}\check{\kappa}_{a';,b';,c';}$ and $g^{ab} = \bar{\nu}^{a;a'}\bar{\nu}^{b;b'}\check{\kappa}_{a';,b';}$. Suppose that $(g^{ab})_{a,b=1}^p$ is nonsingular, denote its inverse matrix by $(g_{ab})_{a,b=1}^p$, and put

$$\tilde{\mu}^{a;}{}_{bc} = \frac{1}{2}(V^{a;}{}_{b,c} + V^{a;}{}_{b,c} + \bar{\nu}^{a;}{}_{b,c})$$

where $V^{a;}{}_{b,c} = \bar{\nu}^{a;a'}\bar{\nu}^{c';c''}\check{\kappa}_{a';b,c'';}g_{c'c}$ and $\bar{\nu}^{a;}{}_{bc} = -\bar{\nu}^{a;a'}\bar{\nu}_{a';bc}(T,\theta_0)$. Assume $\sup_{T,\theta}\Delta_{a;}(T,\theta) < \infty$, where $\Delta_{a;}(T,\theta) = T\bar{\nu}_{a;}(T,\theta)$, put

$$\Delta^{a;}(T) = -\bar{\nu}^{a;a'}\Delta_{a';}(T,\theta_0) \quad \text{and} \quad \tilde{\beta}^{a;} = \beta^a(\theta_0) - \Delta^{a;}(T).$$

Finally, suppose that there exists a random vector $\dot{Z}_T$ consisting of a subset of $\{T^{1/2}\overline{Z}_{a;b}(T)\}_{a,b=1,\ldots,p}$ such that

(i) $\ddot{Z}_T = L\dot{Z}_T$ for some matrix L, where $\ddot{Z}_T$ is a random vector consisting of the other elements of $\{T^{1/2}\overline{Z}_{a;b}(T)\}_{a,b=1,\ldots,p}$ than those of $\dot{Z}_T$,
(ii) $\mathrm{Cov}(T^{-1/2}Z_T^*)$ converges to a positive matrix, where $Z_T^* = (T^{1/2}\overline{Z}_{1;}(T), \ldots, T^{1/2}\overline{Z}_{p;}(T), \dot{Z}_T)$.

We then obtain an asymptotic expansion of the distributions of the M-estimator $\hat{\theta}_T^*$ given by (4.1).

Theorem 4.1. *Let $\hat{g} > \lim_{T\to\infty} g$, M and γ' be positive constants, γ a constant $\in (0,1)$, and m a positive constant satisfying $m > \gamma' + 2$. Suppose that* $[\mathrm{C0}]^3$, $[\mathrm{C1}]_{p_1}$, $[\mathrm{C2}]^k_{p_2,\gamma}$, $k = 1, 2$, [C3], *and* $[\mathrm{C4}]^3_{p_3}$ *hold true for some $p_1 > 3m$, $p_2 > \max(p, 3m)$, $p_3 > m$ with $2/3 + \max(m/p_2, m/(3p_3)) < \gamma < 1 - m/p_1$, and that for the tensors $\bar{\nu}_{a;b}$ and $\bar{\nu}_{a;bc}$ in* $[\mathrm{C2}]^1_{p_2,\gamma}$ *and* $[\mathrm{C2}]^2_{p_2,\gamma}$, *$\delta_c\bar{\nu}_{a;b}(\theta) = \bar{\nu}_{a;bc}(\theta)$. Moreover, suppose that the conditions* [A1]–[A3] *in Theorem* 3.1 *for Z_T^* in place of Z_T hold true. Then there exist constants $c > 0$, $\widetilde{C} > 0$, and $\tilde{\epsilon} > 0$ and such that for any function $f \in \mathscr{E}(M,\gamma')$,*

$$\left|E\big[f\big[\sqrt{T}(\hat{\theta}_T^* - \theta)\big]\big] - \int dy^{(0)}f(y^{(0)})q_{T,1}(y^{(0)})\right| \le c\omega(f, \widetilde{C}T^{-(\tilde{\epsilon}+1)/2}, \hat{g}) + o(T^{-1/2}), \tag{4.2}$$

where

$$q_{T,1}(y^{(0)}) = \phi\big(y^{(0)}; (g^{ab})\big)\bigg(1 + \frac{1}{6\sqrt{T}} c^{abc} h_{abc}(y^{(0)}; g^{ab}) + \frac{1}{\sqrt{T}} (\tilde{\mu}^{a;}{}_{cd} g^{cd} - \tilde{\beta}^a) h_a(y^{(0)}; g^{ab})\bigg),$$

$$c^{abc} = \bar{\lambda}^{abc} + 6\tilde{\mu}^{c;}{}_{a'b'} g^{a'a} g^{b'b}.$$

Here $h_{a_1\ldots a_k}(z; \sigma^{ab})$ *is the Hermite polynomial defined by*

$$h_{a_1\ldots a_k}(z; \sigma^{ab}) = \frac{(-1)^k}{\phi(z; (\sigma^{ab}))} \frac{\partial^k}{\partial z_{a_1} \cdots \partial z_{a_k}} \phi(z; (\sigma^{ab})).$$

For notational simplicity, T is omitted from the coefficients.

Proof. In the same way as the proof of Theorem 6.2 in Sakamoto and Yoshida (2004), it is easy to show that the second-order stochastic expansion of $\sqrt{T}(\hat{\theta}^*_T - \theta_0)$ is given by

$$\sqrt{T}(\hat{\theta}^*_T - \theta_0)^a = \overline{Z}^{a;} + \frac{1}{\sqrt{T}}\bigg(\Delta^{a;} + \overline{Z}^{a;}{}_b \overline{Z}^{b;} + \frac{1}{2}\bar{\nu}^{a;}{}_{cd} \overline{Z}^{c;} \overline{Z}^{d;}\bigg) + \frac{1}{T} R^a_2$$

for some remainder term R^a_2, where $\overline{Z}^{a;}$ and $\overline{Z}^{a;}{}_b$ are random variables defined by $\overline{Z}^{a;} = -\bar{\nu}^{a;a'}(T, \theta_0)\overline{Z}_{a';}(T)$ and $\overline{Z}^{a;}{}_b = -\bar{\nu}^{a;a'}(T, \theta_0)\overline{Z}_{a';b}(T)$, and $\bar{\nu}^{a;}{}_{bc}$ is a constant defined by $\bar{\nu}^{a;}{}_{bc} = -\bar{\nu}^{a;a'}(T, \theta_0)\bar{\nu}_{a';bc}(T, \theta_0)$. As in the proof of Theorem 6.2 in Sakamoto and Yoshida (2004), putting $\epsilon = \min(3\gamma' - 2, 3\gamma - 2 - \alpha)$ for $\gamma' \in (2/3, \gamma - m/p_2)$ and $\alpha \in (m/p_3, 3\gamma - 2)$, one can show that

$$P[r_T|R^a_2| \leq Cr^\epsilon_T, a = 1, \ldots, p] = 1 - o(r^m_T).$$

By using the Delta-method as Lemma 8.5 in Sakamoto and Yoshida (2004) with Theorem 3.1, the assertion can be proved. The rigorous validation of this step is not straightforward; see the proof of Theorem 6.4 of Sakamoto and Yoshida (2004) for details. □

Remark 4.1. Theorem 5.2 and other results in Sakamoto and Yoshida (2004) gave a general asymptotic expansion formula for the distribution of a random variable defined as a transform of additive functionals like $(\sqrt{T}\,\overline{Z}_{a;}(T), \sqrt{T}\,\overline{Z}_{a;b}(T))$ above or $\sqrt{T}\,\overline{Z}_T$ in the previous section. Combining such a theorem with the additive functional's expansion and the Delta-method, one can derive the distributional asymptotic expansion for many statistics in iid models, linear and nonlinear time series models, and continuous time stochastic processes. As typical applications, expansion formulas for M-estimators, minimum contrast estimators, and MLE were obtained in Sakamoto and Yoshida (2004). Their results cover the case where a stochastic expansion consists of linearly dependent random variables. Since we have freedom to choose the basic additive functionals as well as the functions defining the transform, this method works in various situations; for example, in the likelihood analysis even when the score function and its derivatives are linearly dependent. For an extreme example, even if the $\dot{Z}$-part vanishes, one can make a new element for

$\dot{Z}$ so that it is independent of the originally given functionals. In an analogous way, the third-order expansions for the test statistics, e.g., Wald, Rao, likelihood ratio statistics, were obtained by Sakamoto (2000).

5. Estimation of a Sampled Moving Average Process

In order to illustrate the asymptotic expansion of Theorem 4.1, let us consider an M-estimator of a sampled moving average process. Let $w = (w_t)_{t\in\mathbb{R}}$ be a Wiener process on $\mathbb{R}$ with $w_0 = 0$. We define a stochastic process $X = (X_t)_{t\in\mathbb{R}}$ by

$$X_t = \int_{\mathbb{R}} r(t-s, \theta) dw_s,$$

with a non-zero element $r(\cdot, \theta) \in L^2(\mathbb{R})$, the diameter of whose support is less than ϵ; θ is an unknown parameter in an open set $\Theta \subset \mathbb{R}$. Without loss of generality, we may assume that $\operatorname{supp} r(\cdot, \theta) \subset [0, \epsilon]$. Let $N = (N_t)_{t\mathbb{R}_+}$ be a Poisson process with a positive rate λ independent of w. With

$$v(\theta) = E_\theta[X_t^2],$$

the estimating function we will discuss is

$$\psi(\theta) = \int_0^T \left(X_t^4 - 3v(\theta)X_t^2\right)dN_t.$$

We assume that $v \in C^2(\Theta)$. $Y_t^\dagger := (X_t, w_t)$ can be regarded as an ϵ-Markov process driven by $X^\dagger := (w, N)$ since $Y_t^\dagger$ is measurable with respect to

$$\sigma\left[Y_r^\dagger; r \in (s-\epsilon, s]\right] \vee \sigma\left[w_u - w_s; u \in [s, t]\right]$$

for $s \le t$. Let $I(j) = [(8j+1)\epsilon, (8j+7)\epsilon]$, $I_j = [8j\epsilon, (8j+1)\epsilon]$, and $J_j = [(8j+6)\epsilon, (8j+7)\epsilon]$. Further, let $\mathscr{B}_I = \mathscr{B}_I^{Y^\dagger} \vee \mathscr{B}_I^{dX^\dagger}$ and $\widehat{\mathscr{C}}(j) = \mathscr{B}_{I_j} \vee \mathscr{B}_{J_j}$. Then the ϵ-Markov property yields that the intervals $I(j)$ form dense reduction intervals with $(\widehat{\mathscr{C}}(j))$; see Sec. 3.2 and Yoshida (2004).

In order to verify condition [A3] in Sec. 3, we realize random variables on a Wiener–Poisson space (the product of a two-sided Wiener space and a Poisson space) and will apply the partial Malliavin calculus that uses shifts in w only over $\bigcup_j[(8j+1)\epsilon, (8j+5)\epsilon]$. Then for any bounded $\widehat{\mathscr{C}}(j)$-measurable function F, the jth Malliavin operator L_j corresponding to the shifts of w over the interval $[(8j+1)\epsilon, (8j+5)\epsilon]$ satisfies $L_jF = 0$. It is sufficient to verify condition $[A3^M]$ of Section 4.1 of Yoshida (2004) in order to apply Theorem 2 in it. In the present situation, the functional Z_T in question is $Z_T = (\psi(\theta_0), \partial_\theta\psi(\theta))$. It should be noted that we do not need to attach the information of the right end $(X_t^\dagger)_{t\in[(8j+6)\epsilon,(8j+7)\epsilon]}$ thanks to the above property of L_j, and that $\partial_\theta^2\psi(\theta)$ is suppressed since it is linearly dependent on $\partial_\theta\psi(\theta)$. We will show the nondegeneracy of

$$Z_{I(j)} = \left(\int_{I(j)} (X_t^4 - 3v(\theta_0)X_t^2)dN_t, -3\partial_\theta v(\theta_0)\int_{I(j)} X_t^2 dN_t\right).$$

This is reduced to show the nondegeneracy of

$$H = \left(\int_{I(0)} X_t^4 dN_t, \int_{I(0)} X_t^2 dN_t \right)$$

by stationarity and a linear transform. Let $I_1^\delta = [2.5\epsilon - \delta, 2.5\epsilon]$ and $I_2^\delta = [4.5\epsilon - \delta, 4.5\epsilon]$ for $\delta \in (0, 0.5\epsilon)$. We consider the event

$$A^\delta = \{N(I_1^\delta) = 1, N(I_2^\delta) = 1, N([0, 8\epsilon]) = 2\}.$$

In a point of the event A^δ, we denote by $t_1 \in I_1^\delta$ and $t_2 \in I_2^\delta$ the first and second jump times, respectively. Then H is expressed as

$$H_{t_1,t_2} = \sum_{k=1}^{2} (X_{t_k}^4, X_{t_k}^2).$$

Thus, the Malliavin covariance matrix of H is given by

$$\sigma_{H_{t_1,t_2}} = v(\theta_0) \sum_{k=1}^{2} \begin{bmatrix} 4X_{t_k}^3 \\ 2X_{t_k} \end{bmatrix}^{\otimes 2}.$$

It is possible to find two intervals $\mathbb{I}_k = (c_k - \epsilon', c_k + \epsilon')$ $(k = 1, 2)$ in $\mathbb{R}$ such that

$$\sum_{k=1}^{2} \begin{bmatrix} 4x_k^3 \\ 2x_k \end{bmatrix}^{\otimes 2}$$

is elliptic uniformly in $(x_1, x_2) \in \mathbb{I}_1 \times \mathbb{I}_2$. The support of X_t is $\mathbb{R}$, X_t is continuous, and the family $(X_{t_1})_{t_1 \in I_1^\delta}$ is independent of $(X_{t_2})_{t_2 \in I_2^\delta}$; therefore,

$$P\big[\det \sigma_{H_{t_1,t_2}} > c_0 \ (t_1 \in I_1^\delta, t_2 \in I_2^\delta)\big] =: c_* > 0$$

for some small positive constants δ and c_0. Let $\varphi \in C^\infty(\mathbb{R}; [0, 1])$ such that $\varphi(x) = 0$ if $x \le c_0/2$ and $\varphi(x) = 1$ if $x \ge c_0$. We define a truncation functional ψ by

$$\psi = \varphi\big(\det \sigma_{H_{T_1,T_2}}\big) 1_{A^\delta}.$$

Here T_k is the (first) jump time in I_k^δ. Note that ψ depends on (T_1, T_2) as well as w. Then by stationarity, we can check $[A3^M]$ of Yoshida (2004, p. 571), $\widehat{Z}_j$ there is for $Z_{I(j)}$ here, and $\mathscr{C}$ is the trivial σ-field in the present case. Indeed,

$$\begin{aligned} E[\psi] &= E\Big[E\big[\varphi\big(\det \sigma_{H_{t_1,t_2}}\big)\big]\big|_{t_1=T_1, t_2=T_2} 1_{A^\delta}\Big] \\ &\ge c_* P[A^\delta] > 0. \end{aligned}$$

After all, the expansion of Z_T, or its linear extension, turns out to be valid.

Let us derive the second-order expansion of the M-estimator $\hat{\theta}_T$ under suitable regularity conditions. Let $\rho_t(\theta) = \mathrm{Cov}[X_t, X_0]$ and $\check{f}_t(\theta) = X_t^4 - 3v(\theta)X_t^2$. We see that $v(\theta) = \rho_0(\theta)$ and

$$\begin{aligned}\rho_t(\theta) &= E\left[\int_{t-\epsilon}^{t} r(t-s,\theta)dw_s \int_{-\epsilon}^{0} r(-u,\theta)dw_u\right]\\ &= 1_{[0,\epsilon]}(t)\int_{t-\epsilon}^{0} r(t-s,\theta)r(-s,\theta)ds \quad \text{for } t>0.\end{aligned}$$

The normality and the stationarity of X_t yield that

$$\begin{aligned}\mathrm{Cov}[\check{f}_t(\theta),\check{f}_0(\theta)] &= 6\rho_t^2(\theta)\left(4\rho_t^2(\theta)+3v^2(\theta)\right),\\ \mathrm{Cov}[\delta\check{f}_t(\theta),\check{f}_0(\theta)] &= \mathrm{Cov}[\check{f}_t(\theta),\delta\check{f}_0(\theta)] = -18\delta v(\theta)v(\theta)\rho_t^2(\theta),\\ E[\check{f}_{s+t}(\theta)\check{f}_s(\theta)\check{f}_0(\theta)] &= 72\big[3v^2(\theta)\big\{\rho_s^2(\theta)\rho_t^2(\theta)+\rho_s^2(\theta)\rho_{s+t}^2(\theta)+\rho_t^2(\theta)\rho_{s+t}^2(\theta)\big\}\\ &\quad + \rho_s(\theta)\rho_t(\theta)\rho_{s+t}(\theta)\big\{3v^3(\theta)+8v(\theta)\rho_s^2(\theta)\\ &\quad + 8v(\theta)\rho_t^2(\theta)+8v(\theta)\rho_{s+t}^2(\theta)+24\rho_s(\theta)\rho_t(\theta)\rho_{s+t}(\theta)\big\}\big],\\ E[\check{f}_t^2(\theta)\check{f}_0(\theta)] &= 72v^2(\theta)\left(17\rho_t^2(\theta)v^2(\theta)+43\rho_t^4(\theta)\right).\end{aligned}$$

Put $\bar{\rho}[i;\theta] = \int_0^\infty \rho_t^i(\theta)dt$, $\bar{\rho}[i,j;\theta] = \int_0^\infty\int_0^\infty \rho_s^i(\theta)\rho_{s+t}^j(\theta)ds\,dt$, and $\bar{\rho}[i,j,k;\theta] = \int_0^\infty\int_0^\infty \rho_s^i(\theta)\rho_t^j(\theta)\rho_{s+t}^k(\theta)ds\,dt$. By using these notations, the constants for the asymptotic expansion in Sec. 4 are represented as

$$\begin{aligned}\check{\kappa}_{1;,1;} &= 12\lambda^2(4\bar{\rho}[4;\theta]+3v^2(\theta)\bar{\rho}[2;\theta])+42\lambda v^4(\theta)+O\left(\frac{1}{T}\right),\\ \check{\kappa}_{1;1,1;} &= -36\lambda^2\delta v(\theta)v(\theta)\bar{\rho}[2;\theta]-18\lambda\delta v(\theta)v^3(\theta)+O\left(\frac{1}{T}\right),\\ T^{\frac{1}{2}}\check{\kappa}_{1;,1;,1;} &= 432\lambda\big\{3\lambda^2v^2(\theta)\big((\bar{\rho}[2;\theta])^2+2\bar{\rho}[2,2;\theta]+v(\theta)\bar{\rho}[1,1,1;\theta]\big)\\ &\quad + 8\lambda^2v(\theta)\big(2\bar{\rho}[1,3,1;\theta]+\bar{\rho}[1,1,3;\theta]\big)+24\lambda^2\bar{\rho}[2,2,2;\theta]\\ &\quad + \lambda v^2(\theta)\big(17\bar{\rho}[2;\theta]v^2(\theta)+43\bar{\rho}[4;\theta]\big)+10v^6(\theta)\big\}+o(1),\\ \bar{v}_{1;1}(T,\theta) &= -3\delta v(\theta)v(\theta)\lambda, \quad \bar{v}_{1;11}(T,\theta) = -3\delta^2 v(\theta)v(\theta)\lambda.\end{aligned}$$

6. Diffusion-Point Process

In this section, we will discuss the case where processes f and λ are functions of a diffusion process and will obtain asymptotic expansions of $\overline{Z}_T$ and $\hat{\theta}_T^*$ as applications of Theorems 3.1 and 4.1, respectively.

Let $X = (X_t)_{t\in\mathbb{R}_+}$ be a d-dimensional stationary diffusion process satisfying a stochastic differential equation

$$dX_t = V_0(X_t)dt + V(X_t)dw_t, \tag{6.1}$$

where $V_0 = (V_0^i)_{i=1,\dots,d} \in C_b^\infty(\mathbb{R}^d;\mathbb{R}^d)$,[1] $V = (V_\alpha^i)_{i=1,\dots,d,\alpha=1,\dots,r} \in C_b^\infty(\mathbb{R}^d;\mathbb{R}^d \otimes \mathbb{R}^r)$. We realize X on $\widehat{\Omega} \times \Omega'$ in Sec. 3, where $\widehat{\Omega}$ is a product space of an r-dimensional Wiener space and a probability space corresponding to the initial value X_0, and the partial Malliavin calculus that uses shifts only in w will be applied.

6.1. *Functional* $\overline{Z}_T$

Let $\lambda(t) = \lambda(X_t)$ and $f(t) = f(X_t)$ for some measurable functions $\lambda : \mathbb{R}^d \to \mathbb{R}_+$ and $f : \mathbb{R}^d \to \mathbb{R}^n$. For this process λ, we take a point process N_t as in Sec. 3.

Suppose that (i) $E[|X_0|^p] < \infty$ for any $p > 0$ and that (ii)[2] there exists a positive constant a such that

$$\|E[h \mid \mathscr{B}_{[s]}^X] - E[h]\|_1 \le \mathsf{a}^{-1} e^{-\mathsf{a}(t-s)} \|h\|_\infty$$

for any $s, t \in \mathbb{R}_+$, $s \le t$ and for any bounded $\mathscr{B}_{[t,\infty)}^X$-measurable function h.

The generator of the diffusion process X is given by

$$\mathscr{A} = \sum_{i=1}^d V_0^i(x) \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j}^d \sum_{k=1}^r V_k^i(x) V_k^j(x) \frac{\partial^2}{\partial x^i \partial x^j}.$$

For a measurable function $h : \mathbb{R}^d \to \mathbb{R}$, $G\langle h \rangle$ denotes a function such that $\mathscr{A}G\langle h \rangle = h - \nu(h)$, and let $[h] = -V'\nabla G\langle h \rangle$, where $\nu(h)$ is the expectation of h w.r.t. the stationary distribution ν of X : $\nu(h) = \int_{\mathbb{R}^d} h(x)\nu(dx)$. If the Green functions exist and satisfy certain regularity conditions, then we have

$$\kappa_{ab} = \nu([\lambda f_a] \cdot [\lambda f_b]) + \nu(\lambda f_a f_b) + o(1) \tag{6.2}$$

and

$$\kappa_{abc} = \frac{1}{\sqrt{T}} \Big\{ \sum\nolimits^* \big\{ \nu([[\lambda f_a] \cdot [\lambda f_b]] \cdot [\lambda f_c]) + \nu([\lambda f_a f_b] \cdot [\lambda f_c]) \big\} + \nu(\lambda f_a f_b f_c) \Big\} + o(T^{-1/2}). \tag{6.3}$$

In the one-dimensional case ($d = 1$), under some regularity conditions, the invariant probability measure ν has the support $\mathbb{R}$, and it is given by the density

$$\frac{d\nu}{dx} = \frac{n(x)}{\int_{-\infty}^\infty n(y)dy}, \tag{6.4}$$

where

$$n(x) = \frac{1}{V^2(x)p(x)}, \quad p(x) = \exp\left\{-2\int_0^x \frac{V_0(u)}{V(u)^2}\,\mathrm{d}u\right\}, \quad \int_{-\infty}^\infty n(x)\mathrm{d}x < \infty.$$

[1] C_b^∞ is the space of smooth functions with bounded derivatives of positive order.
[2] Veretennikov (1987, 1997) and Kusuoka and Yoshida (2000) provided sufficient conditions for (ii).

The Green function $G\langle h\rangle$ can be expressed explicitly as

$$G\langle h\rangle(x) = \int_{-\infty}^{x} p(y)\left(\int_{-\infty}^{y} 2(h(v) - \nu(h))n(v)\mathrm{d}v\right)\mathrm{d}y$$

whenever it exists.

Example 6.1. Let $V_0(x) = -\theta x$, $V(x) = \sigma > 0$, $\lambda(x) = ax^2 + b > 0$, $f(x) = cx + d$, then the conditions in Theorem 3.1 are easily verified and the coefficients of the expansion can be derived from the following asymptotic representation of cumulants:

$$\begin{aligned}
\lim_{T\to\infty} \kappa_{11} &= \frac{11a^2c^2\sigma^6}{4\theta^4} + \frac{a\left(6bc^2 + ad^2\right)\sigma^4}{2\theta^3} + \frac{c^2\left(4b^2 + 3a\sigma^2\right)\sigma^2}{4\theta^2} \\
&\quad + \frac{\left(bc^2 + ad^2\right)\sigma^2}{2\theta} + bd^2, \\
\lim_{T\to\infty} \sqrt{T}\kappa_{111} &= \frac{33a^3c^2d\sigma^8}{\theta^6} + \frac{3a^2d\left(18bc^2 + ad^2\right)\sigma^6}{2\theta^5} + \frac{3ac^2d\left(2b^2 + 7a\sigma^2\right)\sigma^4}{\theta^4} \\
&\quad + \frac{3ad\left(13bc^2 + ad^2\right)\sigma^4}{2\theta^3} + \frac{3c^2d\left(8b^2 + 3a\sigma^2\right)\sigma^2}{4\theta^2} \\
&\quad + \frac{d\left(3bc^2 + ad^2\right)\sigma^2}{2\theta} + bd^3.
\end{aligned}$$

Example 6.2. Let $V_0(x) = -\theta x/(1 + x^2)$, $V(x) = \sqrt{\theta}/\sqrt{1 + x^2}$, $\lambda(x) = ax^2 + b > 0$, $f(x) = cx + d$, and $u(x) = \int \sqrt{1 + x^2}/\sqrt{\theta}dx$. Then $Y_t = u(X_t)$ satisfies $dY_t = \tilde{\beta}(Y_t)dt + dw_t$ where $\tilde{\beta}(y) = \frac{z\sqrt{\theta}}{2\left(1+z^2\right)^{3/2}} - \frac{z\sqrt{\theta}}{\sqrt{1+z^2}}$, $z = u^{-1}(y)$. Therefore, according to Theorem 1 in Veretennikov (1997), it can be shown that X_t as well as Y_t is polynomial mixing with sufficiently small order. As mentioned in Remark 2.1, this takes the place of [A1] for Theorem 3.1. One can verify other conditions and obtain the representation of the cumulants as follows:

$$\begin{aligned}
\lim_{T\to\infty} \kappa_{11} &= \frac{7ac^2}{4} + \frac{5bc^2}{6} + \frac{5ad^2}{6} + bd^2 \\
&\quad + \frac{\left(6777c^2 + 830d^2\right)a^2 + 4644bc^2a + 972b^2c^2}{216\theta}, \\
\lim_{T\to\infty} \sqrt{T}\kappa_{111} &= \frac{d\left(\left(8661a^2 + 5059ba + 972b^2\right)c^2 + 415a^2d^2\right)}{36\theta} \\
&\quad + \frac{1}{12}d\left(3(21a + 10b)c^2 + 2(5a + 6b)d^2\right) \\
&\quad + \frac{d\left(103900d^2a^3 + 27\left(116689a^2 + 55600ba + 7324b^2\right)c^2a\right)}{1296\theta^2}.
\end{aligned}$$

We use the partial Malliavin calculus in which the shifts on the probability space are only in w. Let $0 < t_1 < t_0$. We shall confine our attention to the events $\{\Delta N_{t_1} = 1, N_{t_0} = 1\}$ to deduce the local nondegeneracy of the functional Z_{t_0}. Rewrite

(6.1) in the Stratonovich form:

$$dX_t = \widetilde{V}_0(X_t)dt + V(X_t) \circ dw_t, \tag{6.5}$$

where

$$\widetilde{V}_0 = V_0 - \frac{1}{2}\sum_{\alpha=1}^{r}(\partial_x V_\alpha)[V_\alpha].$$

Here $L[v] = (\sum_j L^i_j v^j)$ for matrix $L = (L^i_j)$ and vector $v = (v^j)$.

The Malliavin covariance matrix of $(X_{t_0}, f(X_{t_1}))$ has the expression

$$\sigma_{t_1,t_0} = \int_0^{t_0} \overline{Y}_{t_0}\overline{Y}_t^{-1} \begin{pmatrix} v(X_t) & 1_{[0,t_1]}(t)v(X_t)(\partial_x f(X_t))' \\ \text{sym.} & 1_{[0,t_1]}(t)\partial_x f(X_t)v(X_t)(\partial_x f(X_t))' \end{pmatrix} (\overline{Y}_t^{-1})'\overline{Y}_{t_0}' dt,$$

where

$$v(x) = \sum_{\alpha=1}^{r} V_\alpha(x)V_\alpha(x)'$$

and $\overline{Y}_t$ is defined by

$$\begin{cases} d\overline{Y}_t = \begin{bmatrix} \partial_x\widetilde{V}_0(X_t) & 0 \\ 1_{[0,t_1]}(t)\partial_x((\partial_x f)[\widetilde{V}_0])(X_t) & 0 \end{bmatrix}\overline{Y}_t dt \\ \qquad + \displaystyle\sum_{\alpha=1}^{r}\begin{bmatrix} \partial_x V_\alpha(X_t) & 0 \\ 1_{[0,t_1]}(t)\partial_x((\partial_x f)[V_\alpha])(X_t) & 0 \end{bmatrix}\overline{Y}_t \circ dw_t^\alpha \\ \overline{Y}_0 = I_{d+n}. \end{cases}$$

Sakamoto and Yoshida (2008) discussed the nondegeneracy of σ_{t_1,t_0} by taking advantage of the nondegeneracy of v and the function f. Here we will take another approach by the support theorem. Let us consider a system of ordinary differential equations

$$\begin{cases} dx_t = \widetilde{V}_0(x_t)dt + V(x_t)u_t dt \\ d\bar{y}_t = \begin{bmatrix} \partial_x\widetilde{V}_0(x_t) & 0 \\ \partial_x((\partial_x f)[\widetilde{V}_0])(x_t) & 0 \end{bmatrix}\bar{y}_t dt + \displaystyle\sum_{\alpha=1}^{r}\begin{bmatrix} \partial_x V_\alpha(x_t) & 0 \\ \partial_x((\partial_x f)[V_\alpha])(x_t) & 0 \end{bmatrix}\bar{y}_t u_t^\alpha dt \\ x_0 = x_* \\ \bar{y}_0 = I_{d+n}. \end{cases} \tag{6.6}$$

(S) There exist a point x_* in $\operatorname{supp} v$, positive constants t_0, c_0 and a piecewise smooth mapping $u_t = (u_t^\alpha) : [0, t_0] \to \mathbb{R}^r$ such that for the solution of the ordinary differential equation (6.6), it holds that

$$\int_0^{t_0} \bar{y}_t^{-1}\begin{bmatrix} v(x_t) & v(x_t)(\partial_x f(x_t))' \\ \text{sym.} & \partial_x f(x_t)v(x_t)(\partial_x f(x_t))' \end{bmatrix}(\bar{y}_t^{-1})' dt \geq c_0 I_{d+n}.$$

Under condition (S), if we choose a sufficiently small positive number ϵ, then

$$\sigma_{t_1,t_0} \geq \epsilon I_{d+n} \quad (\forall t_1 \in [t_0 - \epsilon, t_0])$$

on the event

$$\left\{ \sup_{t\in[0,t_0]} |X_t - x_t| + |\overline{Y}_t - \bar{y}| < \epsilon,\ \mathsf{p}([t_0 - \epsilon, t_0] \times (0, \epsilon]) = N_{t_0} = 1 \right\}$$

having positive probability, suppose that λ is continuous and $\lambda(x_{t_0}) > 0$. Then we can apply Theorem 6 of Yoshida (2004).

6.2. *M-Estimator*

As in Sec. 4, for an open and bounded subspace $\Theta \subset \mathbb{R}^p$, $\theta_0 \in \Theta$ denotes the target value to be estimated. Let $(X_t)_{t\in[0,\infty)}$ be a d-dimensional diffusion process of Sec. 6.1, and processes $(\check{f}(t, \theta))_{t\in[0,\infty)}$ for each $\theta \in \Theta$ and $(\lambda(t))_{t\in[0,\infty)}$ given in Sec. 4 are here redefined by $\check{f}(t, \theta) = \check{f}(X_t, \theta)$ and $\lambda(t) = \lambda(X_t)$. Note that in this section $\check{f} : \mathbb{R}^d \times \Theta \to \mathbb{R}^p$ and $\lambda : \mathbb{R}^d \to \mathbb{R}_+$. According to this modification, the estimating function ψ becomes

$$\psi(T, \theta) = \int_0^T \check{f}(X_t, \theta) dN_t - T\mu(T, \theta)$$

where $(\mu(t, \theta))$ is an auxiliary process.

For this estimating function, we can apply Theorem 4.1 to derive an asymptotic expansion formula. Note that the cumulants $\check{\kappa}_{a;,b;}$, $\check{\kappa}_{a;,b;,c;}$, and $\check{\kappa}_{a;b,c}$, determining the coefficients of the expansion admit asymptotic representation with Green function as in (6.2) and (6.3).

In the case where $(X_t)_{t\in\mathbb{R}_+}$ satisfies

$$dX_t = V_0(X_t, \theta)dt + V(X_t, \theta)dw_t, \tag{6.7}$$

instead of (6.1), and λ depends on θ, i.e., $\lambda(\cdot) = \lambda(\cdot, \theta)$, it is usual to choose the auxiliary process μ as

$$\mu(T, \theta) = \nu_\theta[\check{f}(\cdot, \theta)\lambda(\cdot, \theta)],$$

where ν_θ is the stationary distribution of (X_t) and $\nu_\theta(g) = \int_{\mathbb{R}^d} g(x) d\nu_\theta(x)$ if g is integrable w.r.t $d\nu_\theta$. Note that the target value θ_0 is the true value in this parametric model.

Example 6.3. Let $d = 1$ and $p = 2$. Let $\Theta = \{\theta = (\theta_1, \theta_2) \mid \theta_2 > 0\}$, $V_0(x, \theta) = -c_0(x - \theta_1)$ for some known constant $c_0 > 0$, $V(x, \theta) = \theta_2$, and $\lambda(x, \theta) = c_1(x - \theta_1)^2 + c_2$ for some known positive constant c_1 and c_2. Put $\check{f}(x, \theta) = (2c_0, x - \theta_1)$. Then ν_θ is the probability measure of the normal distribution with mean θ_1 and variance $\theta_2/2c_0$, and

$$\mu(T, \theta) := \nu_\theta(\check{f}(\cdot, \theta)\lambda(\cdot, \theta)) = (c_1\theta_2^2 + 2c_0c_2, 0).$$

Moreover, the constants representing the coefficients in the asymptotic expansion can be straightforwardly computed;

$$\bar{v}_{1;1} = 0, \quad \bar{v}_{1;2} = -2c_1\theta_2, \quad \bar{v}_{2;1} = -\frac{c_1\theta_2^2}{2c_0} - c_2, \quad \bar{v}_{2;2} = 0,$$

$$\bar{v}_{a;bc} = -2c_1 \quad (a = 1, b = c = 2), \quad 0 \text{ (otherwise)},$$

$$\check{\kappa}_{1;,1;} = \frac{2c_1^2\theta_2^4}{c_0} + 2c_0c_1\theta_2^2 + 4c_0^2c_2, \quad \check{\kappa}_{1;,2;} = 0,$$

$$\check{\kappa}_{2;,1;} = 0, \quad \check{\kappa}_{2;,2;} = \frac{11c_1^2\theta_2^6}{4c_0^4} + \frac{3c_1(c_0+4c_2)\theta_2^4}{4c_0^3} + \frac{c_2(c_0+2c_2)\theta_2^2}{2c_0^2},$$

$$\check{\kappa}_{1;,1;,1;} = \frac{12c_1^3\theta_2^6}{c_0^2} + 12c_1^2\theta_2^4 + 4c_0^2c_1\theta_2^2 + 8c_0^3c_2,$$

$$\check{\kappa}_{1;,1;,2;} = \check{\kappa}_{1;,2;,1;} = \check{\kappa}_{2;,1;,1;} = 0,$$

$$\check{\kappa}_{1;,2;,2;} = \check{\kappa}_{2;,1;,2;} = \check{\kappa}_{2;,2;,1;}$$

$$= \frac{22c_1^3\theta_2^8}{c_0^5} + \frac{2c_1^2(7c_0+9c_2)\theta_2^6}{c_0^4} + \frac{c_1\left(3c_0^2+26c_2c_0+8c_2^2\right)\theta_2^4}{2c_0^3} + \left(\frac{4c_2^2}{c_0} + c_2\right)\theta_2^2,$$

$$\check{\kappa}_{2;,2;,2;} = 0,$$

$$\check{\kappa}_{a;b,c} = -\frac{c_1^2\theta_2^4}{c_0^2} - c_1\theta_2^2 - 2c_0c_2 \quad (a = 2, b = c = 1), \quad 0 \text{ (otherwise)}.$$

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