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Expansion of Perturbed Random Variables Based on Generalized Wiener Functionals

Yuji Sakamoto*

Nagoya University, Nagoya, Japan

and

Nakahiro Yoshida

The Institute of Statistical Mathematics, Tokyo, Japan

By means of the Malliavin calculus, we present an expansion formula for the distribution of a random variable F having a stochastic expansion $F = F_0 + R$, where F_0 is an easily tractable random variable and R is the remainder term. From this result, we derive an expansion of the distribution of the scale mixture sZ of a normal random variable Z by a scale random variable s. Applications to shrinkage estimators of the Stein type are mentioned

1. INTRODUCTION

In this article, we will treat a perturbed random variable F taking the form of $F = F_0 + R$, where F_0 is the principal part of F which is easily tractable, and R is the remainder term which one expects to be small in a certain sense. In some cases, it is small in some Banach space, e.g., L^p-space; in other cases, R involves parameters and it is small with respect to a Banach norm uniformly in the parameter space.

We will present an expansion of the distribution of F, estimating the error term. One typical example is the expansion of the t-distribution as the degree of freedom is sufficiently large. Including this problem, Fujikoshi [2], Fujikoshi and Shimizu [3, 4] obtained asymptotic expansions for scale mixture sZ, where Z is a normal random variable and s is a positive scale random variable. They treated the case where s and Z are mutually

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* Fax: 81-52-789-3724. E-mail: yuji@nuap.nagoya-u.ac.jp.

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independent. If we consider a shrinkage type estimator such as $(1-b/(|Z|^2+a))Z$, it is in the form of sZ again, while in this case Z and s are no longer independent. Following Watanabe [14] essentially, we take an approach to such problems by using the notion of the asymptotic expansion of generalized Wiener functionals. The method adopted here is well applicable to dependent cases, since we use neither an explicit representation of density of F nor that of characteristic function. Watanabe's theory was intensively applied to derive asymptotic expansions for heat kernels in Watanabe [14], Uemura [13], Takanobu [11], Takanobu and Watanabe [12]. In a different way, Kusuoka and Stroock [8] also presented expansions for certain Wiener functionals in the light of the Malliavin calculus. In statistics, the Malliavin calculus was used for statistical estimators by [15–18], and recently by Dermoune and Kutoyants [1].

The expansions presented here are not asymptotic ones: in some statistical applications, such as the problem of inadmissibility, we inevitably encounter error terms involving parameters, and it is then necessary to express the error bounds explicitly as a function of the parameters.

The organization of this article is as follows: in Section 2, an expansion formula will be presented in terms of the generalized Wiener functionals (Theorem 1). Theorem 2 provides an expansion of the distribution of F, while the proof is deferred until Section 4 with preliminary lemmas. In Section 3, we mention several applications of Theorem 2. In the last part of this paper, the integration-by-parts formulas under truncation are presented. Some of them are more or less well-known; however, we restate them explicitly not only for the convenience of reference, but also for the reason that we want to clarify how the regularity conditions appearing in theorems are rooted in those basic results. The elementary and detailed proofs for the results in this paper are presented in Sakamoto and Yoshida [9].

2. MAIN RESULTS

Let (W, H, P) be an r-dimensional Wiener space. For a separable Hilbert space E, define a norm $\|F\|_p$ of E-valued Wiener functionals F by

$$\|F\|_p=\left(\int_W |F|_E^p\, P(dw)\right)^{1/p},$$

where $|F|_E=\langle F, F\rangle_E^{1/2}$ and $\langle\ ,\ \rangle_E$ is the inner product of E. As usual, we denote the E-valued L_p-space by $\mathbf{L}_p(E)$, eliminating E when $E=\mathbf{R}$. For

$p>1$ and $s\in\mathbf{R}$, the norm $\|\cdot\|_{p,s}$ on the totality of E-valued polynomial functionals F is defined by

$$\|F\|_{p,s}=\|(I-L)^{s/2}F\|_p,$$

where L is the Ornstein–Uhlenbeck operator and I is the identity operator. The Banach space $\mathbf{D}_p^s(E)$ is the completion of the totality of E-valued polynomial functionals with respect to the norm $\|\cdot\|_{p,s}$. The set of Wiener test functionals is denoted by

$$\mathbf{D}^{\infty}(E)=\bigcap_{s>0}\ \bigcap_{1<p<\infty}\mathbf{D}_p^s(E).$$

The couple of $F\in\mathbf{D}_p^s(E)$ and $G\in\mathbf{D}_q^{-s}$, where $1/p+1/q=1$, $p>1$ and $s>0$, is denoted by ${}_{D_q^{-s}}\langle G,F\rangle_{D_p^s}$, which is often called the generalized expectation. If $F=(F^1,...,F^d)\in\mathbf{D}_2^1(\mathbf{R}^d)$, we denote the Malliavin covariance of F by σ_F. For these definitions, see Ikeda and Watanabe [6].

Let $\mathscr{S}(\mathbf{R}^d)$ be the Schwartz space on $\mathbf{R}^d$ and $\mathscr{S}'(\mathbf{R}^d)$ be the space of Schwartz tempered distributions. For $m\in\mathbf{Z}^+$ and $T\in\mathscr{S}'(\mathbf{R}^d)$, if $t^{m-1}e^{-t}\langle T(y),p(t,x,y)\rangle$ is absolutely integrable in t for any x, an integral operator $A^{-m}T(x)$ is defined by

$$A^{-m}T(x)=\int_0^{\infty}\frac{t^{m-1}e^{-t}}{\Gamma(m)}\langle T(y),p(t,x,y)\rangle\,dt,$$

where

$$p(t,x,y)=\prod_{i=1}^{d}\frac{\exp\left\{-\frac{\sqrt{2}}{2}(\coth\sqrt{2}\,t)[(x^i)^2-2x^iy^i\operatorname{sech}\sqrt{2}\,t+(y^i)^2]\right\}}{\sqrt{2\pi(\sinh\sqrt{2}\,t)\,2^{-1/2}}}.$$

For $\mathbf{n}=(n_1,..,n_d)\in\mathbf{Z}^{+d}$, let $|\mathbf{n}|=n_1+\cdots+n_d$ and $\partial^{\mathbf{n}}=(\partial/\partial x^1)^{n_1}\cdots(\partial/\partial x^d)^{n_d}$. Moreover, for any $k\in\mathbf{Z}^+$ and any $m\in\mathbf{Z}^+$, the subspaces $\tilde{C}^{k,-2m}(\mathbf{R}^d)$ of $\mathscr{S}'(\mathbf{R}^d)$ is defined by

$$\tilde{C}^{k,-2m}(\mathbf{R}^d)=\Big\{T\in\mathscr{S}'(\mathbf{R}^d);\text{ for any }\mathbf{n}\text{ such that }|\mathbf{n}|\leqslant k,$$
$$A^{-m}\partial^{\mathbf{n}}T\in\hat{C}(\mathbf{R}^d),\text{ there exist }T_n\in\mathscr{S}(\mathbf{R}^d)\text{ such that}$$
$$\lim_{n\to\infty}\sum_{|\mathbf{n}|\leqslant k}\|A^{-m}(\partial^{\mathbf{n}}T_n-\partial^{\mathbf{n}}T)\|_{\infty}=0\Big\},$$

where $\hat{C}(\mathbf{R}^d)$ is the totality of continuous functions $f(x)$ on $\mathbf{R}^d$ such that $f(x)\to 0$ as $|x|\to\infty$. For simplicity, we denote $\tilde{C}^{0,-2m}$ by $\tilde{C}^{-2m}$.

Using Proposition 3 in the Appendix, we define composite functionals $(\psi \cdot T) \circ F$. If $F \in \bigcap_{p>1} \mathbf{D}_p^{2m+1}(\mathbf{R}^d)$ and $\psi \in \bigcup_{p>1} \mathbf{D}_p^{2m}$ satisfy (5.10) for some $m \in \mathbf{N}$ and $q>1$, then, for any $T \in \tilde{C}^{-2m}(\mathbf{R}^d)$ and any p with $q>p>1$, the composite functional $(\psi \cdot T) \circ F$ is defined by

$$(\psi \cdot T) \circ F = \lim_{n \to \infty} \psi \cdot T_n(F) \qquad \text{in} \quad \mathbf{D}_p^{-2m},$$

where $T_n \in \mathscr{S}(\mathbf{R}^d)$ is any sequence satisfying $\lim_{n \to \infty} \|A^{-m}T - A^{-m}T_n\|_\infty = 0$. From this definition, we see

$$_{D_p^{-2m}}\langle (\psi \cdot T) \circ F, G \rangle_{D_{p'}^{2m}} = \lim_{n \to \infty} \int_W \psi \cdot T_n(F)\, GP(dw) \tag{2.1}$$

for $G \in \mathbf{D}_{p'}^{2m}$ and $p'>1$ such that $1/p + 1/p' = 1$.

In the following, we will present expansion formulas for a perturbed Wiener functional $F = F_0 + R$, and error bounds with the functional Ψ_{2m}^F defined just before Proposition 2 of the Appendix. Their applications will be given in the next section.

THEOREM 1. *Suppose the following conditions are satisfied*:

(1) $T \in \tilde{C}^{K,\,-2m}(\mathbf{R}^d)$;

(2) $F = F_0 + R$ *for some* $F_0,\ R \in \bigcap_{p>1} \mathbf{D}_p^{2m+1}(\mathbf{R}^d)$;

(3) $\psi \in \bigcup_{p>1} \mathbf{D}_p^{2m}$;

(4) *there exists* $q>1$ *such that*

$$i \leqslant 2m,\ i+j \leqslant 4m,\ i,j \in \mathbf{Z}^+ \Rightarrow \sup_{0 \leqslant u \leqslant 1} \|(\det \sigma_{F_u})^{-j} D^i \psi\|_q < \infty, \tag{2.2}$$

where $F_u = F_0 + uR$. *Then, for any* p *with* $q>p>1$, *the composite functionals* $(\psi \cdot T) \circ F$ *and* $(\psi R^{\mathbf{n}} \cdot \partial^{\mathbf{n}} T) \circ F_0$ $(|\mathbf{n}| \leqslant K)$ *are well-defined in* $\mathbf{D}_p^{-2m}$, *and the generalized expectation* $_{D_p^{-2m}}\langle (\psi \cdot T) \circ F, G \rangle_{D_{p'}^{2m}}$ *admits the following expansion for any* $G \in \mathbf{D}_{p'}^{2m}$, $1/p + 1/p' = 1$:

$$_{D_p^{-2m}}\langle (\psi \cdot T) \circ F, G \rangle_{D_{p'}^{2m}} = \sum_{|\mathbf{n}| \leqslant K-1} \frac{1}{\mathbf{n}!}\, {}_{D_p^{-2m}}\langle (R^{\mathbf{n}}\psi \cdot \partial^{\mathbf{n}} T) \circ F_0, G \rangle_{D_{p'}^{2m}} + r_K^G(T),$$

where $\mathbf{n}! = n_1! \cdots n_d!$ *and*

$$|r_K^G(T)| \leqslant \sum_{|\mathbf{n}| = K} \frac{K}{\mathbf{n}!} \|A^{-m} \partial^{\mathbf{n}} T\|_\infty \int_0^1 (1-u)^{K-1} \|\Psi_{2m}^{F_u}(\cdot\,; R^{\mathbf{n}}\psi G)\|_1\, du.$$

Proof. It is easy to see that $(\psi \cdot T) \circ F$ and $(\psi R^{\mathbf{n}} \cdot \partial^{\mathbf{n}} T) \circ F_0 \in \mathbf{D}_p^{-2m}$ are well-defined in $\mathbf{D}_p^{-2m}$. Therefore choosing $T_n \in \mathscr{S}(\mathbf{R}^d)$ such that $\|A^{-m}(\partial^{\mathbf{n}} T_n - \partial^{\mathbf{n}} T)\|_\infty$ tends to zero as $n \to \infty$ for any $\mathbf{n} \in \mathbf{Z}^{+d}$ with $|\mathbf{n}| \leqslant K$ and expanding $T_n(F)$, we obtain

$$\begin{aligned}
{}_{D_p^{-2m}}\langle (\psi \cdot T)\circ F, G\rangle_{D_{p'}^{2m}} &= \lim_{n\to\infty} \int_W T_n(F)\,\psi G P(dw) \\
&= \sum_{|\mathbf{n}|\leqslant K-1} \frac{1}{\mathbf{n}!}\, {}_{D_p^{-2m}}\langle (\psi R^{\mathbf{n}} \cdot \partial^{\mathbf{n}} T)\circ F_0, G\rangle_{D_{p'}^{2m}} \\
&\quad + \lim_{n\to\infty} \int_W \rho_{n,K}\psi G P(dw),
\end{aligned}$$

where

$$\rho_{n,K} = \sum_{|\mathbf{n}|=K} \frac{K}{\mathbf{n}!} \int_0^1 (1-u)^{K-1} (\partial^{\mathbf{n}} T_n)(F_u)\, du\, R^{\mathbf{n}}.$$

Due to (2.2), Proposition 2 can be applied to F_u and $R^{\mathbf{n}}\psi G$, and hence we obtain

$$\begin{aligned}
\left| \int_W \rho_{n,K}\psi G P(dw) \right| &\leqslant \sum_{|\mathbf{n}|=K} \frac{K}{\mathbf{n}!} \int_0^1 (1-u)^{K-1}\, du \\
&\quad \times \int_W |(A^{-m}\partial^{\mathbf{n}} T_n)(F_u)\, \Psi_{2m}^{F_u}(w; R^{\mathbf{n}}\psi G)|\, P(dw) \\
&\leqslant \sum_{|\mathbf{n}|=K} \frac{K}{\mathbf{n}!} \|A^{-m}\partial^{\mathbf{n}} T_n\|_\infty \int_0^1 (1-u)^{K-1} \\
&\quad \times \|\Psi_{2m}^{F_u}(\cdot; R^{\mathbf{n}}\psi G)\|_1\, du.
\end{aligned}$$

Thus this theorem has been shown. ∎

For any Borel set $B\subset \mathbf{R}^d$, let the indicator functions 1_B be defined by

$$1_B(x) = \begin{cases} 1 & (x\in B) \\ 0 & (x\notin B) \end{cases}.$$

If $T=1_B$ in Theorem 1, we have the following results.

THEOREM 2. *Let K and d be positive integers, and let m be a positive integer such that $m>(d+K)/2$. Assume that the functionals F and ψ satisfy Conditions* (2), (3), (4) *in Theorem* 1 *and another condition that*

$$j\leqslant 4m, \qquad j\in \mathbf{Z}^+ \Rightarrow (\det \sigma_{F_0})^{-j} \in \bigcup_{q>1} \mathbf{L}_q.$$

Then F_0 has a density $p_{F_0}(x)$, for every $B \in \mathbf{B}^d$, and

$$\Pr\{F \in B\} = \sum_{|\mathbf{n}| \leqslant K-1} \frac{1}{\mathbf{n}!} \int_B (-\partial_x)^{\mathbf{n}} \{E[R^{\mathbf{n}} \mid F_0 = x]\, p_{F_0}(x)\}\, dx + \tilde{r}_K(1_B), \tag{2.3}$$

where

$$|\tilde{r}_K(1_B)| \leqslant |r_K^1(1_B)| + E[|\psi - 1|] + \sum_{|\mathbf{n}| \leqslant K-1} \frac{1}{\mathbf{n}!} \|A^{-m} \partial^{\mathbf{n}} 1_B\|_\infty \, \|\Psi_{2m}^{F_0}(\cdot; (\psi - 1)\, R^{\mathbf{n}})\|_1. \tag{2.4}$$

The expansion (2.3) implicitly means the differentiability of $E[R^{\mathbf{n}} \mid F_0 = x]\, p_{F_0}(x)$ and the integrability of the derivatives. The proof of Theorem 2 will be presented in Section 4.

In the multivariate analysis based on normal populations, each statistic F can be regarded as a smooth function of $([h_1](w), ..., [h_d](w))$, where $\{h_i\}$ is any orthonormal system of H and $[h_i](w) = \sum_{\alpha=1}^d \int_0^\infty \dot{h}_i^\alpha(t)\, dw^\alpha(t)$. If there exists a suitable truncation functional ψ for F, then Theorem 2 can be applied to the distribution of F. In the following theorem, it is shown that there exists a truncation functional ψ for the scale mixture sZ of a normal random variable Z with a scale factor s. In the next section, similar truncation functionals will appear in other situations.

THEOREM 3. *Let K, m, and d be positive integers satisfying $m > (d+K)/2$. Assume that Wiener functionals Z and s satisfy the following conditions*: (a) *$Z(w) = \tau^{1/2}([h_1](w), ..., [h_d](w))' + \theta$, where $\{h_i\}$ is an orthonormal system of H, $\tau \in \mathbf{R}^d \otimes \mathbf{R}^d$ is a positive definite matrix, and $\theta \in \mathbf{R}^d$*; (b) *$s \in \bigcap_{p>1} \mathbf{D}_p^{2m+1}(\mathbf{R}^d \otimes \mathbf{R}^d)$. Then* (1) *there exists $\psi \in \bigcap_{q>1} \mathbf{D}_q^{2m}$ such that*

$$i \leqslant 2m, \qquad i, j \in \mathbf{Z}^+, \qquad q > 1 \Rightarrow \sup_{0 \leqslant u \leqslant 1} \|(\det \sigma_{F_u})^{-j} D^i \psi\|_q < \infty,$$

where $F_u = ((1-u)\, I_d + us) Z$; (2) *for every Borel set $B \subset \mathbf{R}^d$, $\Pr\{sZ \in B\}$ has the expansion*

$$\Pr\{sZ \in B\} = \sum_{|\mathbf{n}| \leqslant K-1} \frac{1}{\mathbf{n}!} \int_B (-\partial_x)^{\mathbf{n}} \{E[((s - I_d) x)^{\mathbf{n}} \mid Z = x]\, p_Z(x)\}\, dx + \tilde{r}_K(1_B), \tag{2.5}$$

where $p_Z(x)$ is the density of the normal distribution $N_d(\theta, \tau)$ and $\tilde{r}_K(1_B)$ is estimated by (2.4) *with $R = (s - I_d) Z$.*

Proof. We see that $\sigma_{F_u}^{i,j} = \tau^{i,j} + S^{i,j}$, where

$$S^{i,j} = \langle DZ^i, D(uR^j) \rangle_H + \langle D(uR^i), DZ^j \rangle_H + \langle D(uR^i), D(uR^j) \rangle_H.$$

Since $|S^{i,j}| \leqslant |\tau^{i,i}|^{1/2} |\sigma_R^{j,j}|^{1/2} + |\tau^{j,j}|^{1/2} |\sigma_R^{i,i}|^{1/2} + |\sigma_R^{i,j}|$, there exists $c > 0$ such that

$$c\, |\sigma_R|^2 < 1 \Rightarrow \det \sigma_{F_u} > \tfrac{1}{2} \det \tau.$$

Let $\psi = \eta(c\, |\sigma_R|^2)$ for $c > 0$, where η is a C^∞-function: $\mathbf{R} \to [0, 1]$ such that $\eta(x) = 1$ for $|x| \leqslant 1/2$ and $\eta(x) = 0$ for $|x| \geqslant 1$. By the chain rule, we obtain

$$D^i \psi = \begin{cases} \sum_{k=0}^{i} \eta^{(k)}(c\, |\sigma_R|^2)\, P_k & (c\, |\sigma_R|^2 < 1) \\ 0 & (c\, |\sigma_R|^2 \geqslant 1) \end{cases},$$

where P_k is a polynomial in H-derivatives of $c\, |\sigma_R|^2$. Thus, if $q > 1$ and $0 \leqslant u \leqslant 1$, then

$$\begin{aligned} \|(\det \sigma_{F_u})^{-j} D^i \psi\|_q &= \|1_{\{c\, |\sigma_R|^2 < 1\}} (\det \sigma_{F_u})^{-j} D^i \psi\|_q \\ &\leqslant \left(\frac{2}{\det \tau}\right)^j \left\| \sum_{k=0}^{i} \eta^{(k)}(c\, |\sigma_R|^2)\, P_k \right\|_q < \infty \end{aligned}$$

for $i \leqslant 2m$, $j \in \mathbf{Z}^+$, because $R \in \bigcap_{p>1} \mathbf{D}_p^{2m+1}(\mathbf{R}^d)$; hence (1) has been proved. Moreover we see F_u and ψ satisfy (2.2). By virtue of Theorem 2, (2.5) follows. ▮

In the case that s is independent of Z, Fujikoshi [2] and Fujikoshi and Shimizu [3, 4] obtained asymptotic expansions of the distribution of sZ with some error bound. As the method used here relies neither on the characteristic function nor on the density function, our result given above does not need the condition that s is independent of Z.

3. APPLICATIONS

3.1. *t-Distribution*

Suppose that Z is distributed according to $N_1(0, 1)$ and that s is distributed as $1/\sqrt{\chi_n^2/n}$ independently of Z. Then sZ is distributed as the t-distribution with degree n. Since s does not have higher order moments, $s \notin \mathbf{D}^\infty$. Therefore we modify s as follows. Fix any a satisfying $0 < a < 1$. Let g be a monotone increasing C^∞-function such that $g(0) = a/2$ and $g(x) = x$ for $x \geqslant a$. Let $\tilde{s} = 1/\sqrt{g(s^{-2})}$. Since $\tilde{s} \in \mathbf{D}^\infty$, we get the asymptotic expansion of $\Pr\{\tilde{s}Z \leqslant x\}$ from Theorem 3. By large-deviation argument, we have

$\Pr\{s \neq \tilde{s}\} = O(\delta^n)$ for some $0 < \delta < 1$. Note that $\limsup_{n\to\infty} E\,|s-1|^k < \infty$ for any $k \in \mathbf{Z}^+$. Hence, we have

$$|\Pr\{sZ \leqslant x\} - \Pr\{\tilde{s}Z \leqslant x\}| = O(\delta^n)$$

and

$$|E(s-1)^k - E(\tilde{s}-1)^k| = O(\delta^{n/2})$$

for any $k \in \mathbf{Z}^+$. We have to calculate $E(s-1)^k$ to get the asymptotic expansion. Using Stirling's formula, we have

$$E(s-1)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \left(\frac{1}{n} T_1(j) + \frac{1}{n^2} T_2(j)\right) + O\left(\frac{1}{n^3}\right),$$

where $T_1(k) = k^2/4 + k/2$ and $T_2(k) = k^4/32 + 5k^3/24 + 3k^2/8 + k/6$. Consequently, we get the well-known Gram–Charlier expansion

$$\Pr\{sZ \leqslant x\} = \Phi(x) - \phi(x) \left\{\frac{1}{4n} x(x^2+1) + \frac{1}{96n^2} x(3x^6 - 7x^4 - 5x^2 - 3)\right\} + O\left(\frac{1}{n^3}\right),$$

where $\Phi(x)$ is the distribution function of the standard normal distribution and $\phi(x)$ is its density.

3.2. *Scale Mixture of Multidimensional Normal Distributions*

Consider independent Wiener functionals $Z \in \mathbf{D}^\infty(\mathbf{R}^d)$, $Q \in \mathbf{D}^\infty(\mathbf{R}^d \otimes \mathbf{R}^q)$, and $U \in \mathbf{D}^\infty(\mathbf{R}^q)$. Assume that Z, U have the normal distributions $N_d(0, I_d)$, $N_q(0, I_q)$, respectively. Let $s = (I_d + QQ')^{1/2}$, then the distribution of sZ coincides with that of $Z - QU$.

If we put $F_0 = Z$, $R = -QU$ in Theorem 2 and choose ψ as in Theorem 3, then we obtain the expansion of the distribution of $F = Z - QU$ as follows. Noting that $E[X^{2m}] = v^m(2m)!/2^m m!$ and $E[X^{2m-1}] = 0$, if X is distributed according to $N(0, v)$, we have

$$\sum_{|\mathbf{n}| \leqslant 2K-2} \frac{1}{\mathbf{n}!} E[(QU)^{\mathbf{n}}]\, \partial_x^{\mathbf{n}} = \sum_{j=0}^{K-1} \frac{1}{2^j j!} E[\{\nabla_x Q(\nabla_x Q)'\}^j],$$

where $\nabla_x = (\partial/\partial x^1, ..., \partial/\partial x^d)$. Thus we obtain the following asymptotic expansion of sZ:

$$\Pr\{sZ \in B\} = \sum_{j=0}^{K-1} \frac{1}{2^j j!} \int_B E_Q\{(\nabla_x Q(\nabla_x Q)')^j \phi(x)\}\, dx + \tilde{r}_{2K-1}(1_B);$$

here E_Q stands for the expectation with respect to Q. This formula coincides with that in Theorem 3.1 of Fujikoshi and Shimizu [4] except for the estimation of the remainder terms.

3.3. *Shrinkage-Type Estimator*

Let X be distributed according to $N_d(\theta, \sigma^2 I_d/n)$ and let ξ_n be distributed according to $\sigma^2\chi^2_{c_n}/c_n$ independently of X, where $c_n = O(n)$ as n tends to ∞, e.g., $c_n = d(n-1)$. For a constant $a > 0$, let us consider a shrinkage-type estimator JS_a defined by

$$JS_a = \left(1 - \frac{c_n(d-2)}{n(c_n+2)} \frac{\xi_n}{|X|^2 + a}\right) X.$$

Note that JS_a is a better estimator than X with respect to mean square error.

Set $F_0 = \sqrt{n}\,(X - \theta)$ and

$$R = -\frac{c_n(d-2)}{\sqrt{n}\,(c_n+2)} \frac{\xi_n}{|X|^2 + a} X$$

in Theorem 2. If we choose a truncation functional ψ in the same fashion as in Theorem 3, we see easily that for any $c > 0$, there exist some positive constants C_1, C_2 such that

$$\Pr\{|\sigma_R| > c\} \leqslant C_1 e^{-C_2 n}$$

for any n. Thus we can obtain the following asymptotic expansion of $\Pr\{\sqrt{n}\,(JS_a - \theta) \in B\}$:

$$\begin{aligned}\Pr\{\sqrt{n}\,(JS_a - \theta) \in B\} = &\sum_{|\mathbf{n}| \leqslant K-1} \frac{1}{\mathbf{n}!} \left(\frac{c_n(d-2)}{\sqrt{n}\,(c_n+2)}\right)^{|\mathbf{n}|} E[\xi_n^{|\mathbf{n}|}] \\ &\times \frac{1}{\sigma^d} \int_B \partial_x^{\mathbf{n}} \left\{(g_a(x/\sqrt{n} + \theta))^{\mathbf{n}}\, \phi\left(\frac{x}{\sigma}\right)\right\} dx \\ &+ O(n^{-K/2});\end{aligned}$$

here $E[\xi_n^{|\mathbf{n}|}] = c_n(c_n+2) \cdots (c_n + 2(|\mathbf{n}| - 1))(\sigma^2/c_n)^{|\mathbf{n}|}$ and

$$g_a(x) = \frac{1}{|x|^2 + a} x. \tag{3.1}$$

As in Fig. 1, we give the results of the numerical studies for $\Pr\{\sqrt{n}\,((JS_a)^1 - \theta^1) \leqslant x^1\}$ in the cases where $d = 6$, $n = 5, 10$, $c_n = d(n-1)$, $a = 1$, $\theta = (0, 0, ..., 0)$, $(1, 0, ..., 0)$, $(1, 1, ..., 1)$, and $\sigma = 1$. Figure 1 compares the true distribution functions obtained by Monte-Carlo simulation (100,000 repetitions) with the distribution functions obtained by our asymptotic expansion formulas for $K = 0, 1, 2$.

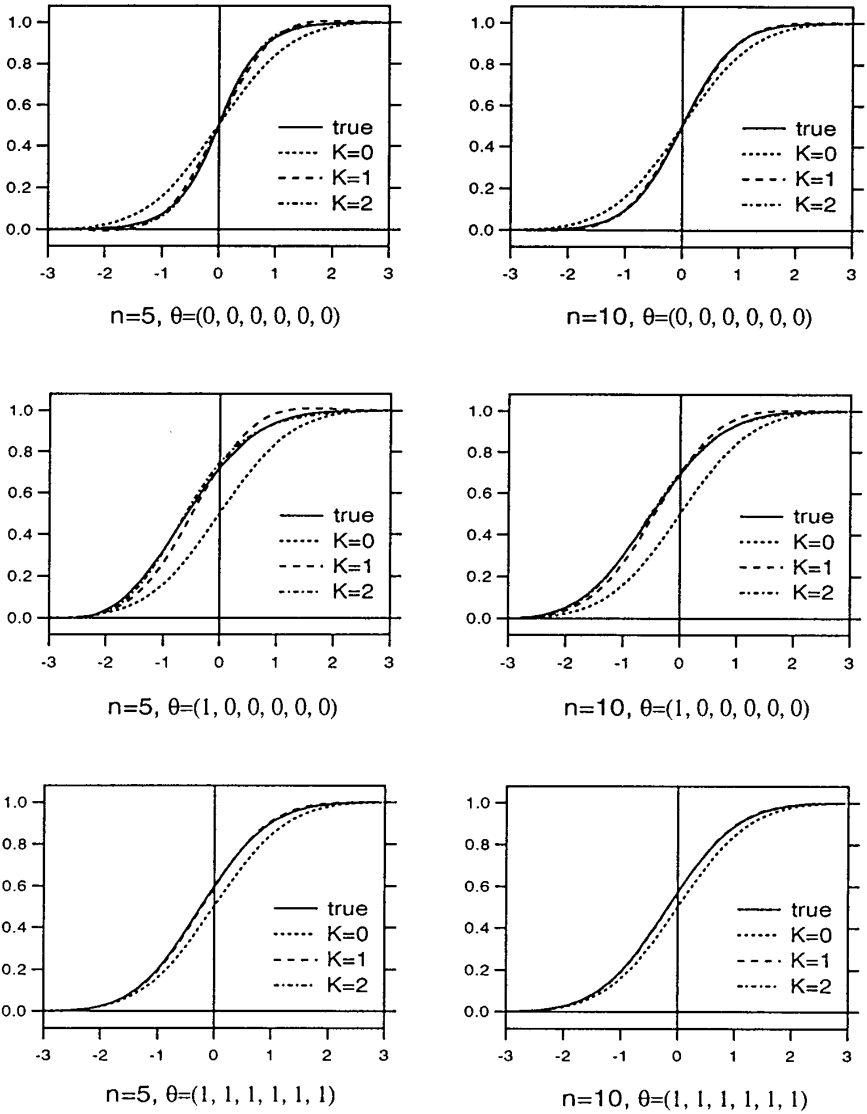

FIG. 1. Numerical studies for shrinkage type estimator; $d=6$, $c_n=n(d-1)$, $a=1$, $\sigma=1$.

3.4. *Confidence Region and Prediction Region*

Let X be distributed according to $N_d(\theta, I_d)$. We consider the shrinkage-type estimator

$$\delta_{a,b}(X)=\left(1-\frac{b}{a+b+|X|^2}\right)X,$$

where a, b are positive constants. Let $Q(B)=\Pr\{\delta_{a,b}(X)\in B\}$ for every Borel set $B\subset\mathbf{R}^d$. Define the function $\psi_{a,b}(z)$ by $\psi_{a,b}(z)=\eta_1(2bd^2/(a+b+|z+\theta|^2))$, where η_1 is a C^∞-function $\mathbf{R}\to[0,1]$ such that $\eta_1(x)=1$ for $|x|\leqslant 1/2$ and $\eta_1(x)=0$ for $|x|\geqslant 3/4$. Applying Theorem 2 with truncation functional $\psi=\psi_{a,b}(X-\theta)$, we obtain

$$\begin{aligned}Q(B)=&\sum_{|\mathbf{n}|\leqslant K-1}\frac{b^{|\mathbf{n}|}}{\mathbf{n}!}\int_B(\partial_x)^{\mathbf{n}}\{(g_{a+b}(x))^{\mathbf{n}}\,\phi(x-\theta)\}\,dx\\&+\tilde{r}_0(a,b,B,d,K,\theta),\end{aligned}\tag{3.2}$$

where $\phi(z)$ is the d-dimensional standard normal density and $g_a(x)$ is defined by (3.1).

If θ is unknown, a usual confidence region $C_0(X)$ for θ based on the observed value of X is given by

$$C_0(X)=\{\theta;\,|\theta-X|<c\},$$

where c is chosen so that $\Pr\{\theta\in C_0(X)\}=1-\alpha$. Joshi [7] considered the confidence region

$$C(X)=\{\theta;\,|\theta-\delta_{a,b}(X)|<c\},$$

and showed that if $d\geqslant 3$, $C(X)$ improves the coverage probability of $C_0(X)$ for sufficiently large a and sufficiently small b. Hwang and Casella [5] presented another practical confidence region. Let

$$I(w)=\int_{\mathbf{R}^d}1_{B_w}(z-bg_{a+b}(z+\theta))\,\phi(z)\,dz,$$

where $B_w=\{z;\,|z-w|<c\}$. Then, as a special case of (3.2), one has

$$\begin{aligned}I(w)&=Q(B_w+\theta)\\&=\sum_{|\mathbf{n}|\leqslant K-1}\frac{b^{|\mathbf{n}|}}{\mathbf{n}!}\int_{B_w}(\partial_z)^{\mathbf{n}}\{(g_{a+b}(z+\theta))^{\mathbf{n}}\,\phi(z)\}\,dz\\&\quad+\tilde{r}_0(a,b,B_w+\theta,d,K,\theta).\end{aligned}$$

Moreover, the remainder term $\tilde{r}_0(a,b,B_w+\theta,d,K,\theta)$ can be estimated as

$$\begin{aligned}|\tilde{r}_0(a,b,B_w+\theta,d,K,\theta)|\leqslant P_{d,K}(a^{-1/2},b)\Bigg\{&\left(\frac{b}{\sqrt{a+b+|\theta|^2}}\right)^K\\&+\sqrt{\Pr\left\{\frac{2bd^2}{a+b+|X|^2}\geqslant\frac{1}{2}\right\}}\Bigg\}\end{aligned}\tag{3.3}$$

for some polynomial $P_{d,K}(x, y)$ with positive coefficients depending only on p and K; for details, see Sakamoto *et al.* [10]. Note that the bound of $\tilde{r}_0(a, b, B_w+\theta, d, K, \theta)$ on the right-hand side of (3.3) is independent of B_w. Since $I(0)=\Pr\{\theta\in C(X)\}$, by using a similar method as in Sakamoto *et al.* [10], we obtain the expansion

$$\Pr\{\theta\in C(X)\}=1-\alpha+\frac{b(1-\alpha-h(\alpha))}{a+b+|\theta|^2}\left(d-2-\frac{b}{2}+\frac{(b+4)(a+b)}{2(a+b+|\theta|^2)}\right)$$
$$+\bar{r}(a, b, c, d, \theta),$$

where $h(\alpha)=(1/d)\int_{B_0}|z|^2\,\phi(z)\,dz$. For any $a_0, b_0>0$, there exists a positive constant c_1 such that

$$|\bar{r}(a, b, c, d, \theta)|\leqslant c_1(a+b+|\theta|^2)^{-3/2},$$

for any $a\geqslant a_0$, $b\leqslant b_0$, $\theta\in\mathbf{R}^d$. By this expansion, one can prove the inadmissibility of the usual confidence region $C_0(X)$ again.

Theorem 2 can also be applied to a prediction problem as follows. Suppose that Y is distributed according to $N_d(\theta, I_d)$ independently of X. A natural prediction region for the value of Y based on the observed value of X is given by

$$S_0(X)=\{y;\,|y-X|<c\},$$

where c is chosen so that $\Pr\{Y\in S_0(X)\}=1-\alpha$. In the same fashion as the confidence region, we consider the prediction region $S(X)$ defined by

$$S(X)=\{y;\,|y-\delta_{a,b}(X)|<c\}$$

for $a>0$ and $b>0$. Since

$$\Pr_\theta\{Y\in S(X)\}=\int_{\mathbf{R}^d}I(w)\,\phi(w)\,dw,$$

we can obtain an expansion of $\Pr_\theta\{Y\in S(X)\}$ with the remainder term having the same bound as $\tilde{r}_0(a, b, B_w+\theta, d, K, \theta)$. With this expansion, we may show that $S(X)$ improves the usual prediction region $S_0(X)$, cf. Sakamoto *et al.* [10].

4. PROOF OF THEOREM 2

4.1. *Preliminary Lemmas*

First we rewrite the kernel function $p(t, x, y)$ as

$$p(t, x, y)=\frac{g(x, t)}{\sigma^{d}(t)}\, \phi\left(\frac{y-x \operatorname{sech} \sqrt{2}\, t}{\sigma(t)}\right)=\frac{g(y, t)}{\sigma^{d}(t)}\, \phi\left(\frac{x-y \operatorname{sech} \sqrt{2}\, t}{\sigma(t)}\right), \tag{4.1}$$

where

$$\sigma(t)=\left(\frac{\tanh \sqrt{2}\, t}{\sqrt{2}}\right)^{1/2}, \qquad g(x, t)=\frac{\exp(-\sigma^{2}(t)|x|^{2})}{\sqrt{\cosh \sqrt{2}\, t}^{\,d}}$$

and $\phi(x)$ is d-dimensional standard normal density. For $\mathbf{n}=(n_1, ..., n_d) \in \mathbf{Z}^{+d}$, let $|\mathbf{n}|=n_1+\cdots+n_d$ and $H_{\mathbf{n}}(x)$ be defined by $H_{\mathbf{n}}(x)=\{(-\partial)^{\mathbf{n}}\, \phi(x)\}/\phi(x)$. Then we see

$$\partial_x^{\mathbf{n}} p(t, x, y)=\frac{1}{(-\sigma(t))^{|\mathbf{n}|}}\, H_{\mathbf{n}}\left(\frac{x-y \operatorname{sech} \sqrt{2}\, t}{\sigma(t)}\right) p(t, x, y) \tag{4.2}$$

and

$$\partial_x^{\mathbf{n}} g(x, t)=(-\sqrt{2}\, \sigma(t))^{|\mathbf{n}|}\, H_{\mathbf{n}}(\sqrt{2}\, \sigma(t) x)\, g(x, t). \tag{4.3}$$

LEMMA 1. *Suppose that $T \in \mathscr{S}'(\mathbf{R}^d)$ can be expressed as*

$$T(x)=(1+|x|^{2})^{s}\, \partial^{\boldsymbol{\alpha}} f(x),$$

where $\boldsymbol{\alpha}=(\alpha_1, ..., \alpha_d) \in \mathbf{Z}^{+d}$, $s \geqslant 0$ and $f(x)$ is a bounded measurable function. Suppose that $m>|\boldsymbol{\alpha}|/2+s$. Then $T \in \tilde{C}^{-2m}(\mathbf{R}^d)$. Moreover there exists a positive constant $C_{m, s, \boldsymbol{\alpha}}$ independent of f such that

$$\|A^{-m} T\|_{\infty} \leqslant C_{m, s, \boldsymbol{\alpha}}\, \|f\|_{\infty}.$$

Proof. First we show the continuity of $A^{-m} T(x)$ in x. Let $Q_{\boldsymbol{\beta}}(y)=\partial_y^{\boldsymbol{\beta}}(1+|y|^{2})^{s}$. By (4.1) and (4.2), it is obtained that

$$\begin{aligned}
&\langle T(y), p(t, x, y)\rangle \\
&\quad=\langle f(y), (-\partial_y)^{\boldsymbol{\alpha}}\, (1+|y|^{2})^{s}\, p(t, x, y)\rangle \\
&\quad=(-1)^{|\boldsymbol{\alpha}|} \sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \int_{\mathbf{R}^d} f(y)\, \partial_y^{\boldsymbol{\alpha}-\boldsymbol{\beta}}(1+|y|^{2})^{s}\, \partial_y^{\boldsymbol{\beta}} p(t, x, y)\, dy \\
&\quad=\sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \frac{(-1)^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}\, g(x, t)}{\sigma(t)^{|\boldsymbol{\beta}|+d}} \int_{\mathbf{R}^d} f(y)\, Q_{\boldsymbol{\alpha}-\boldsymbol{\beta}}(y) \\
&\qquad \times H_{\boldsymbol{\beta}}\left(\frac{y-x \operatorname{sech} \sqrt{2}\, t}{\sigma(t)}\right) \phi\left(\frac{y-x \operatorname{sech} \sqrt{2}\, t}{\sigma(t)}\right) dy
\end{aligned} \tag{4.4}$$

where $\boldsymbol{\beta} = (\beta_1, ..., \beta_d) \in \mathbf{Z}^{+d}$ such that $\beta_k \leqslant \alpha_k$, $k = 1, ..., d$ and

$$\sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} = \sum_{\beta_1 = 0}^{\alpha_1} \cdots \sum_{\beta_d = 0}^{\alpha_d}, \qquad \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d}.$$

Obviously if we fix t and y, then the integrand of (4.4) is continuous in x; for any $t > 0$ and $x \in \mathbf{R}^d$, it is integrable with respect to y uniformly on any compact set of x, from which it follows that if t is fixed, then $\langle T(y), p(t, x, y) \rangle$ is continuous in x. Moreover setting $c_1 = \sup_y |f(y)|$, we have

$$|\langle T(y), p(t, x, y) \rangle| \leqslant c_1 \sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \frac{g(x, t)}{(\sigma(t))^{|\boldsymbol{\beta}|}} \times \int_{\mathbf{R}^d} |Q_{\boldsymbol{\alpha} - \boldsymbol{\beta}}(z\sigma(t) + x \operatorname{sech} \sqrt{2}\, t)\, H_{\boldsymbol{\beta}}(z)\, \phi(z)|\, dz.$$

Since $Q_{\boldsymbol{\beta}}(y) = O(|y|^{2s})$ as $y \to \infty$ and $|\sigma(t)| \leqslant 1$, there exists a function $R(x) = O(|x|^{2s})$ as $x \to \infty$ such that

$$\int_{\mathbf{R}^d} |Q_{\boldsymbol{\alpha} - \boldsymbol{\beta}}(z\sigma(t) + x \operatorname{sech} \sqrt{2}\, t)\, H_{\boldsymbol{\beta}}(z)\, \phi(z)|\, dz \leqslant R(|x|).$$

From this, we see

$$|\langle T(y), p(t, x, y) \rangle| \leqslant c_1 \sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \frac{g(x, t)}{(\sigma(t))^{|\boldsymbol{\beta}|}} R(|x|). \tag{4.5}$$

Note that $|g(x, t)| \leqslant 1$ and $\sigma(t) = O(t^{1/2})$ as $t \to 0$. If $m > |\boldsymbol{\alpha}|/2$, then the right-hand side of (4.5) multiplied by $t^{m-1} e^{-t}$ is integrable with respect to t uniformly on a compact set of x. Therefore, we can show the continuity of $A^{-m} T(x)$.

Next we show that $\lim_{|x| \to \infty} A^{-m} T(x) = 0$. If $m > |\boldsymbol{\alpha}|/2 + s$, then we can choose r such that $0 < r/2 < m - (|\boldsymbol{\alpha}|/2 + s)$. For such r, we see $c_2 = \sup_{x, t} ||\sigma(t) x|^{2s + r} g(x, t)| < \infty$. Combining this and (4.5), we have

$$\begin{aligned} |A^{-m} T(x)| &\leqslant c_1 \sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \frac{R(|x|)}{\Gamma(m)} \int_0^\infty \frac{t^{m-1} e^{-t} g(x, t)}{(\sigma(t))^{|\boldsymbol{\beta}|}}\, dt \\ &\leqslant c_1 c_2 \sum_{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \frac{R(|x|)}{\Gamma(m) |x|^{2s + r}} \int_0^\infty \frac{t^{m-1} e^{-t}}{(\sigma(t))^{|\boldsymbol{\beta}| + 2s + r}}\, dt. \end{aligned}$$

Since $t^{m-1} e^{-t} / (\sigma(t))^{|\boldsymbol{\beta}| + 2s + r}$ is integrable and $R(|x|) = O(|x|^{2s})$ as $x \to \infty$, it follows that $\lim_{|x| \to \infty} |A^{-m} T(x)| = 0$. In this way, it is shown that $T \in \tilde{C}^{-2m}(\mathbf{R}^d)$. ∎

From Lemma 1, we see easily that if $m > |\mathbf{n}|/2$, $m \in \mathbf{Z}^+$, then for any Borel set $B \subset \mathbf{R}^d$,

$$\partial_x^{\mathbf{n}} 1_B(x) \in \tilde{C}^{-2m}(\mathbf{R}^d),$$

and

$$\sup_{B \in \mathbf{B}^d} \|A^{-m} \partial_x^{\mathbf{n}} 1_B(x)\|_\infty < \infty.$$

Note that

$$\partial_x^{\mathbf{n}} \delta_a(x) = (-1)^d \, \partial_x^{\mathbf{n}+\mathbf{1}} \, 1_{B_a}(x),$$

where $\mathbf{1} = (1, ..., 1)$ and $B_a = (-\infty, a_1) \times \cdots \times (-\infty, a_d)$. Then it also follows from Lemma 1 that for Dirac delta function $\delta_a(x)$ on $\mathbf{R}^d$,

$$m > (|\mathbf{n}| + d)/2, \qquad m \in \mathbf{Z}^+ \Rightarrow \partial_x^{\mathbf{n}} \delta_a(x) \in \tilde{C}^{-2m}(\mathbf{R}^d). \tag{4.6}$$

In this case, it holds that

$$\sup_{a \in \mathbf{R}^d} \|A^{-m} \partial_x^{\mathbf{n}} \delta_a(x)\|_\infty < \infty. \tag{4.7}$$

Lemma 2. *Let $q > 1$. Let d and $m \in \mathbf{Z}^+$ satisfy $m > d/2$. Assume that $F \in \bigcap_{p>1} \mathbf{D}_p^{2m+1}(\mathbf{R}^d)$ and $\psi \in \bigcup_{p>1} \mathbf{D}_p^{2m}$ satisfy (5.10). If $1 < p < q$ and $1/p + 1/p' = 1$, then $(\psi \cdot \delta_x) \circ F \in \mathbf{D}_p^{-2m}$, and it holds that for any $T \in \mathscr{S}(\mathbf{R}^d)$ and any $G \in \mathbf{D}_{p'}^{2m}$,*

$$\int_{\mathbf{R}^d} T(x) \; {}_{D_p^{-2m}}\langle (\psi \cdot \delta_x) \circ F, G \rangle_{D_{p'}^{2m}} \, dx = \int_{\mathbf{R}^d} T(x) \, E[\psi G \mid F = x] \, \mu^F(dx), \tag{4.8}$$

where μ^F is the induced measure of F; hence

$${}_{D_p^{-2m}}\langle (\psi \cdot \delta_x) \circ F, G \rangle_{D_{p'}^{2m}} \, dx = E[\psi G \mid F = x] \, \mu^F(dx)$$

as finite signed measures. In particular, if $F \in \bigcap_{p>1} \mathbf{D}_p^{2m+1}(\mathbf{R}^d)$ satisfies

$$j \leqslant 4m, \qquad j \in \mathbf{Z}^+ \Rightarrow (\det \sigma_F)^{-j} \in \mathbf{L}_q, \tag{4.9}$$

then there exists a density $p_F(x)$ of F defined by $p_F(x) = {}_{D^{-\infty}}\langle \delta_x \circ F, 1 \rangle_{D^\infty}$ and

$$E[\psi \mid F = x] \, p_F(x) = {}_{D_p^{-2m}}\langle (\psi \cdot \delta_x) \circ F, 1 \rangle_{D_{p'}^{2m}}.$$

Proof. For any $T \in \mathscr{S}(\mathbf{R}^d)$ and any $G \in \mathbf{D}_{p'}^{2m}$, it follows from Proposition 4 in the Appendix that

$$\int_{\mathbf{R}^d} T(x)\ {}_{D_p^{-2m}}\langle (\psi\delta_x) \circ F, G \rangle_{D_{p'}^{2m}}\, dx$$

$$= \int_{\mathbf{R}^d} T(x) \int_W (A^{-m}\delta_x)(F)\ \Psi_{2m}^F(w; \psi G)\ P(dw)\, dx.$$

By (4.7), $\sup_{x,y} |A^{-m}\delta_x(y)| < \infty$. Therefore it follows from Fubini's theorem and Proposition 2 in the Appendix that

$$\int_{\mathbf{R}^d} T(x) \int_W (A^{-m}\delta_x)(F)\ \Psi_{2m}^F(w; \psi G)\ P(dw)\, dx$$

$$= \int_W \Psi_{2m}^F(w; \psi G) \int_0^\infty \frac{t^{m-1}e^{-t}}{\Gamma(m)} \int_{\mathbf{R}^d} T(x)\, p(t, F, x)\, dx\, dt\, P(dw)$$

$$= \int_W \Psi_{2m}^F(w; \psi G)(A^{-m}T)(F)\ P(dw)$$

$$= \int_{\mathbf{R}^d} T(x)\ E[\psi G \mid F = x]\ \mu^F(dx).$$

It is not difficult to show from this equation that

$${}_{D_p^{-2m}}\langle (\psi \cdot \delta_x) \circ F, G \rangle_{D_{p'}^{2m}}\, dx = E[\psi G \mid F = x]\ \mu^F(dx)$$

as finite signed measures. Note that if we set $\psi = 1$ in (5.10), then (5.10) becomes equivalent to (4.9). Setting $G = 1$, we have

$$\int_{\mathbf{R}^d} T(x)\ {}_{D^{-\infty}}\langle \delta_x \circ F, 1 \rangle_{D^\infty}\, dx = \int_{\mathbf{R}^d} T(x)\, \mu^F(dx).$$

Therefore F has a density p_F defined by $p_F(x) = {}_{D^{-\infty}}\langle \delta_x \circ F, 1 \rangle_{D^\infty}$. Obviously,

$$E[\psi \mid F = x]\ p_F(x) = {}_{D_p^{-2m}}\langle (\psi \cdot \delta_x) \circ F, 1 \rangle_{D_{p'}^{2m}}.$$

The proof is complete. ∎

LEMMA 3. *Let $q > 1$. Let d and $m \in \mathbf{Z}^+$ satisfy $m > d/2$. Assume that $F \in \bigcap_{p>1} \mathbf{D}_p^{2m+1}(\mathbf{R}^d)$ and $\psi \in \bigcup_{p>1} \mathbf{D}_p^{2m}$ satisfy (5.10). If $1 < p < q$ and $1/p + 1/p' = 1$, then it holds that for any bounded measurable function f and any $G \in \mathbf{D}_{p'}^{2m}$,*

$${}_{D_p^{-2m}}\langle (\psi \cdot f) \circ F, G \rangle_{D_{p'}^{2m}} = E[\psi f(F) G].$$

Proof. Let $f_N = f \cdot 1_{\{|x| \leqslant N\}}$. Then for any $\varepsilon > 0$, there exists $N \in \mathbf{N}$,

$$n \geqslant N \Rightarrow |E[\psi f(F)\, G] - E[\psi f_n(F)\, G]| < \varepsilon.$$

On the other hand, for some fixed N, there exist functions $T_{\rho, N}(x) \in \mathscr{S}(\mathbf{R}^d)$, $\rho > 0$, such that for any $s > 1$,

$$\lim_{\rho \to 0} \left(\int_{\mathbf{R}^d} |f_N(x) - T_{\rho, N}(x)|^s \, dx \right)^{1/s} = 0$$

and $\operatorname{supp} T_{\rho, N} \subset \{|x| \leqslant N + \rho\}$. Set $s > \max(d/2, 1)$ and $s' > 1$ so that $1/s + 1/s' = 1$. If f_N is fixed, then there exists $\rho = \rho(\varepsilon, N) > 0$ satisfying that

$$\left(\int_{|x| \leqslant N + \rho} dx \right)^{1/s'} \left(\int_{\mathbf{R}^d} |f_N(x) - T_{\rho, N}(x)|^s \, dx \right)^{1/s} < \varepsilon.$$

Let $r(x)$ be defined by

$$r(x) = {}_{D_p^{-2m}}\langle (\psi \cdot \delta_x) \circ F, G \rangle_{D_{p'}^{2m}},$$

then it follows from Lemma 2 that for any bounded measurable function g,

$$E[\psi g(F)\, G] = \int_{\mathbf{R}^d} g(x)\, r(x)\, dx.$$

Since $\sup_{x, y} |A^{-m} \delta_x(y)| < \infty$ by (4.7), it follows from Proposition 4 in the Appendix that $\sup_x |r(x)| < \infty$. Thus we have

$$\begin{aligned} &|E[\psi f(F)\, G] - E[\psi T_{\rho, N}(F)\, G]| \\ &\quad \leqslant \varepsilon + \sup |r(x)| \left(\int_{\mathbf{R}^d} |f_N(x) - T_{\rho, N}(x)|^s \, dx \right)^{1/s} \left(\int_{|x| \leqslant N + \rho} dx \right)^{1/s'} \\ &\quad \leqslant \varepsilon (1 + \sup |r(x)|). \end{aligned} \tag{4.10}$$

On the other hand, we have

$$\begin{aligned} &|A^{-1}(f_N(x) - T_{\rho, N}(x))| \\ &\quad \leqslant \int_0^\infty \frac{e^{-t} g(x, t)}{\sigma(t)^d} \int_{\mathbf{R}^d} |f_N(y) - T_{\rho, N}(y)|\, \phi\left(\frac{y - x \operatorname{sech} \sqrt{2}\, t}{\sigma(t)} \right) dy\, dt \\ &\quad \leqslant \int_0^\infty \frac{e^{-t}}{\sigma(t)^{d/s}} \, dt \left(\int_{\mathbf{R}^d} |\phi(z)|^{s'} \, dz \right)^{1/s'} \left(\int_{\mathbf{R}^d} |f_N(y) - T_{\rho, N}(y)|^s \, dy \right)^{1/s}. \end{aligned}$$

Since $s > d/2$, we see that $e^{-t}/\sigma(t)^{d/s}$ is integrable. Thus we have for some constant c_s,

$$\|A^{-1}(f_N - T_{\rho, N})\|_\infty \leqslant c_s \varepsilon. \tag{4.11}$$

Since $\lim_{N\to\infty} \|A^{-1}(f-f_N)\|_\infty = 0$, inequalities (4.10) and (4.11) imply that there exist $T_n \in \mathscr{S}(\mathbf{R}^d)$ satisfying that

$$\lim_{n\to\infty} E[\psi T_n(F)G] = E[\psi f(F)G]$$

and

$$\lim_{n\to\infty} \|A^{-1}(f-T_n)\|_\infty = 0.$$

Consequently, we see

$$E[\psi f(F)G] = \lim_{n\to\infty} E[\psi T_n(F)G] = {}_{D_p^{-2m}}\langle (\psi\cdot f)\circ F, G\rangle_{D_{p'}^{2m}}.$$

In this way, the proof is complete. ∎

Lemma 4. *For any Borel set $B\subset \mathbf{R}^d$, there exists a sequence $\{T_n\}\subset \mathscr{S}(\mathbf{R}^d)$ such that if $m\in \mathbf{Z}^+$, $\mathbf{n}\in \mathbf{Z}^{+d}$, and $m>|\mathbf{n}|/2$, then*

$$\lim_{n\to\infty} \|A^{-m}\partial^{\mathbf{n}}(1_B - T_n)\|_\infty = 0.$$

Furthermore,

$$m>k/2,\ m,k\in\mathbf{Z}^+ \Rightarrow 1_B \in \tilde{C}^{k,-2m}(\mathbf{R}^d) \qquad \textit{for any Borel set } B\subset \mathbf{R}^d. \tag{4.12}$$

Proof. Let $0<\delta<1/2$ and let $p>d/\delta$. Clearly, $p>2d>1$. Then there exist $T_n\in\mathscr{S}(\mathbf{R}^d)$ such that

$$\lim_{n\to\infty} \int_{\mathbf{R}^d} |(1+|x|^2)^{-\delta/2}(1_B(x)-T_n(x))|^p\,dx = 0. \tag{4.13}$$

Furthermore, we see that T_n satisfy $\|A^{-m}\partial^{\mathbf{n}}(1_B - T_n)\|_\infty \to 0$. Indeed, taking $q>1$ such that $1/p+1/q=1$, we have

$$\begin{aligned}
&|A^{-m}\partial_x^{\mathbf{n}}(1_B - T_n)(x)| \\
&\quad\leqslant \int_0^\infty \frac{t^{m-1}e^{-t}}{\Gamma(m)} |\langle (1_B - T_n)(y), (-\partial_y)^{\mathbf{n}}\, p(t,x,y)\rangle|\,dt \\
&\quad\leqslant \int_0^\infty \frac{t^{m-1}e^{-t}g(x,t)}{\Gamma(m)(\sigma(t))^{|\mathbf{n}|+d/p}} \\
&\qquad\quad \times\left(\int |(1+|z\sigma(t)+x\operatorname{sech}\sqrt{2}\,t|^2)^{\delta/2} H_{\mathbf{n}}(z)\,\phi(z)|^q\,dz\right)^{1/q} dt \\
&\qquad\quad \times\left(\int |(1+|y|^2)^{-\delta/2}(1_B(y)-T_n(y))|^p\,dy\right)^{1/p}.
\end{aligned}$$

Since there exists a positive constant c_0 such that

$$\left(\int |(1+|z\sigma(t)+x \operatorname{sech} \sqrt{2}\, t|^2)^{\delta/2} H_{\mathbf{n}}(z)\, \phi(z)|^q\, dz\right)^{1/q} \leqslant c_0(1+|x|^{\delta}),$$

we have

$$\begin{aligned} c_1 &:= \sup_{x,\, t} \left| g(x, t)\, \sigma(t)^{\delta} \left(\int |(1+|z\sigma(t)+x \operatorname{sech} \sqrt{2}\, t|^2)^{\delta/2} H_{\mathbf{n}}(z)\, \phi(z)|^q\, dz\right)^{1/q} \right| \\ &\leqslant c_0 \sup_{x,\, t} |g(x, t)\, \sigma(t)^{\delta}\, (1+|x|^{\delta})| < \infty. \end{aligned}$$

Since $|\mathbf{n}|+d/p+\delta<|\mathbf{n}|+2\delta<|\mathbf{n}|+1 \leqslant 2m$, we see

$$\begin{aligned} \sup_x & \left| \int_0^{\infty} \frac{t^{m-1} e^{-t} g(x, t)}{\Gamma(m)(\sigma(t))^{|\mathbf{n}|+d/p}} \right. \\ & \qquad \left. \times \left(\int |(1+|z\sigma(t)+x \operatorname{sech} \sqrt{2}\, t|^2)^{\delta/2} H_{\mathbf{n}}(z)\, \phi(z)|^q\, dz\right)^{1/q} dt \right| \\ & \leqslant c_1 \int_0^{\infty} \frac{t^{m-1} e^{-t}}{\Gamma(m)(\sigma(t))^{|\mathbf{n}|+d/p+\delta}}\, dt < \infty. \end{aligned}$$

Consequently we see $\lim_{n \to \infty} \|A^{-m} \partial^{\mathbf{n}}(1_B - T_n)\|_{\infty} = 0$.

On the other hand, from the notice just after Lemma 1, we see $A^{-m} \partial_x^{\mathbf{n}} 1_B(x) \in \hat{C}(\mathbf{R}^d)$ if $|\mathbf{n}| \leqslant k$. Hence, by the definition of $\tilde{C}^{k,-2m}(\mathbf{R}^d)$, it is clear that $1_B(x) \in \tilde{C}^{k,-2m}(\mathbf{R}^d)$. ∎

4.2. *Proof*

We are now on the point of proving Theorem 2. From Lemma 2 we see that F_0 has the density $p_{F_0}(x)$ defined by $p_{F_0}(x) = {}_{D^{-\infty}}\langle \delta_x \circ F_0, 1 \rangle_{D^{\infty}}$. On the other hand, as $1_B \in \tilde{C}^{K,-2m}(\mathbf{R}^d)$ from Lemma 4, Theorem 1 can be applied to the case $T = 1_B$. Therefore setting $G = 1$ and using Lemma 3, we have

$$E[\psi 1_B(F)] = \sum_{|\mathbf{n}| \leqslant K-1} \frac{1}{\mathbf{n}!}\, {}_{D^{-\infty}}\langle (\psi R^{\mathbf{n}} \cdot \partial^{\mathbf{n}} 1_B) \circ F_0, 1 \rangle_{D^{\infty}} + r_K^1(1_B). \tag{4.14}$$

Since

$$\int_{\mathbf{R}^d} |1_B(y)(-\partial_y)^{\mathbf{n}}\, p(t, F_0, y)|\, dy \leqslant \frac{1}{\sigma(t)^{|\mathbf{n}|}} \int_{\mathbf{R}^d} |H_{\mathbf{n}}(x)\, \phi(x)|\, dx$$

and

$$|(-\partial_y)^{\mathbf{n}}\, p(t, F_0, y)| \leqslant \frac{1}{\sigma(t)^{|\mathbf{n}|+d}} \sup_x |H_{\mathbf{n}}(x)\, \phi(x)|,$$

it follows from Proposition 4 in the Appendix and Lemma 2 that if $J=R^{\mathbf{n}}$ or $R^{\mathbf{n}}\psi$, then for some $p, p'>1$ $(1/p+1/p'=1)$,

$$\begin{aligned}
&{}_{D_p^{-2m}}\langle (J\cdot\partial^{\mathbf{n}}1_B)\circ F_0, 1\rangle_{D_{p'}^{2m}}\\
&\quad=\int_W (A^{-m}\partial^{\mathbf{n}}1_B)(F_0)\,\Psi_{2m}^{F_0}(w; J)\,P(dw)\\
&\quad=\int_W\int_0^\infty \frac{t^{m-1}e^{-t}}{\Gamma(m)}\int_{\mathbf{R}^d} 1_B(y)(-\partial_y)^{\mathbf{n}}\,p(t, F_0, y)\,dy\,dt\,\Psi_{2m}^{F_0}(w; J)\,P(dw)\\
&\quad=\int_{\mathbf{R}^d} 1_B(y)(-\partial_y)^{\mathbf{n}}\left\{\int_W \Psi_{2m}^{F_0}(w; J)\int_0^\infty \frac{t^{m-1}e^{-t}}{\Gamma(m)}\,p(t, F_0, y)\,dt\,P(dw)\right\}dy\\
&\quad=\int_B (-\partial_y)^{\mathbf{n}}\left\{\int_W (A^{-m}\delta_y)(F_0)\,\Psi_{2m}^{F_0}(w; J)\,P(dw)\right\}dy\\
&\quad=\int_B (-\partial_y)^{\mathbf{n}}\left\{{}_{D^{-\infty}}\langle (J\cdot\delta_y)\circ F_0, 1\rangle_{D^\infty}\right\}dy\\
&\quad=\int_B (-\partial_y)^{\mathbf{n}}\left\{E[J\mid F_0=y]\,p_{F_0}(y)\right\}dy.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left|{}_{D_p^{-2m}}\langle (R^{\mathbf{n}}\psi\cdot\partial^{\mathbf{n}}1_B)\circ F_0, 1\rangle_{D_{p'}^{2m}}-\int_B (-\partial_x)^{\mathbf{n}}\left\{E[R^{\mathbf{n}}\mid F_0=x]\,p_{F_0}(x)\right\}dx\right|\\
&\qquad\leqslant \|A^{-m}\partial^{\mathbf{n}}1_B\|_\infty\,\|\Psi_{2m}^{F_0}(\cdot;(\psi-1)\,R^{\mathbf{n}})\|_1.
\end{aligned}$$

Thus it follows from (4.14) that $\tilde r_K(1_B)$ can be estimated by (2.4). In this way, the assertion follows. ▌

APPENDIX

In this Appendix, the integration-by-parts formulas under truncation will be presented. Those are more-or-less well-known.

LEMMA 5. *Assume that for some $p>1$ and $q>1$ satisfying that $1/p+1/q=1$, (F, G) is in $\mathbf{D}_p^1(E)\times\mathbf{D}_q^1(H\otimes E)$. Then*

$$\int_W \langle DF(w), G\rangle_{H\otimes E}\,P(dw)=\int_W \langle F(w), D^*G(w)\rangle_E\,P(dw), \tag{5.1}$$

where D and D^* are H-derivative and its dual operator.

Proof. If F and G are polynomial functionals, (5.1) holds; see the equation (8.23) in Ikeda and Watanabe [6]. Choosing $F_n \in \mathbf{P}(E)$ and $G_n \in \mathbf{P}(H \otimes E)$ such that $\|F_n - F\|_{p,1} \to 0$, $\|G_n - G\|_{q,1} \to 0$, we can show (5.1) under the assumption. ▮

LEMMA 6. *If $F \in \bigcap_{p>1} \mathbf{D}_p^2(\mathbf{R}^d)$ and $\psi \in \bigcup_{p>1} \mathbf{D}_p^1$ satisfy $\psi(\det \sigma_F)^{-1} \in \bigcup_{q>1} \mathbf{D}_q^1$, then it holds that*

$$\int_W (\partial_i T)(F)\, \psi P(dw) = \int_W T(F)\, \Phi_i^F(w; \psi)\, P(dw) \tag{5.2}$$

for any $T \in \mathscr{S}(\mathbf{R}^d)$, where $\Phi_i^F(\cdot; \psi) = \sum_{j=1}^d D^*[(\gamma_F^{ij}\psi)\, DF^j]$ and γ_F^{ik} is the (i, k)-element of $\gamma_F = \sigma_F^{-1}$. Here $\partial_i T(x) = \partial T/\partial x^i$, $x = (x^i)$.

Proof. Applying the chain rule to $DT(F)$, we obtain that

$$\int_W \left\langle DT(F), \sum_{j=1}^d (\psi\gamma_F^{ij})\, DF^j \right\rangle_H P(dw) = \int_W (\partial_i T)(F) \cdot \psi P(dw). \tag{5.3}$$

For some $q > 1$ satisfying $\|\psi(\det \sigma_F)^{-1}\|_{q,1} < \infty$, there exist $q' > 1$ and $r > 1$ such that $1/q + 1/q' = 1/r$. Since the cofactors $\tilde{\sigma}_F^{ij}$ of σ_F are in $\bigcap_{p>1} \mathbf{D}_p^1$, we have

$$\|(\psi\gamma_F^{ij})\, DF^j\|_{r,1} \leqslant \|\psi(\det \sigma_F)^{-1}\|_{q,1}\, \|\tilde{\sigma}_F^{ij} DF^j\|_{q',1} < \infty.$$

From this, we see that couple $(T(F), (\psi\gamma_F^{ij})\, DF^j)$ satisfy the assumption of Lemma 5. If we apply Lemma 5 to the left-hand side of (5.3), we can prove (5.2). ▮

Let $\Phi_{i_1, ..., i_k}^F(\cdot; \psi) = \Phi_{i_k}^F(\cdot; \Phi_{i_1, ..., i_{k-1}}^F(\cdot; \psi))$, then it can be shown by induction that

$$\Phi_{i_1, ..., i_k}^F(\cdot; \psi) = \sum_{j=0}^{k} \sum_{i=k}^{2k-j} \langle (\det \sigma_F)^{-i}\, D^j\psi, P_{(i,j;\, i_1, ..., i_k)}^F \rangle_{H^{\otimes j}}, \tag{5.4}$$

where $P_{(i,j;\, i_1, ..., i_k)}^F$ is a polynomial in F, DF, D^2F, ..., LF, ... Moreover it is seen that if F is in $\bigcap_{p>1} \mathbf{D}_p^{l+1}(\mathbf{R}^d)$ for some $l \geqslant k$, then $P_{(i,j;\, i_1, ..., i_k)}^F \in \bigcap_{p>1} \mathbf{D}_p^s(H^{\otimes j})$ for any $s \leqslant l - k$.

PROPOSITION 1 (Integration-by-Parts Formula 1). *For some $k \in \mathbf{N}$, assume that $F \in \bigcap_{p>1} \mathbf{D}_p^{k+1}(\mathbf{R}^d)$ and $\psi \in \bigcup_{p>1} \mathbf{D}_p^k$ satisfy the following condition:*

$$i \leqslant k,\ i + j \leqslant 2k,\ i, j \in \mathbf{Z}^+ \Rightarrow (\det \sigma_F)^{-j}\, D^i\psi \in \bigcup_{q>1} \mathbf{L}_q(H^{\otimes i}). \tag{5.5}$$

Then it holds that for any $T \in \mathscr{S}(\mathbf{R}^d)$ *and* $0 \leqslant i_1, ..., i_k \leqslant d$,

$$\int_W (\partial_{i_1} \cdots \partial_{i_k} T)(F)\, \psi P(dw) = \int_W T(F)\, \Phi^F_{i_1, ..., i_k}(w; \psi)\, P(dw). \qquad (5.6)$$

Proof. By (5.5), there exists $q > 1$ such that $\|(\det \sigma_F)^{-j} D^i \psi\|_q < \infty$. If $q > r > 1$, we can choose $q' > 1$ such that $1/q + 1/q' = 1/r$. Moreover, we choose $p > 1$ and $r' > 1$ satisfying $1/r + 1/r' = 1/p$. Then, for any positive integer $l \leqslant k - 1$, with (5.4), we obtain

$$\begin{aligned}
&\|\Phi^F_{i_1, ..., i_l}(\cdot; \psi)(\det \sigma_F)^{-1}\|_{p,1} \\
&\quad \leqslant \sum_{i=0}^{l} \sum_{j=l}^{2l-i} C_{r,r'}\, \|(\det \sigma_F)^{-(j+1)} D^i \psi\|_{r,1}\, \|P^F_{(j,i;i_1, ..., i_l)}\|_{r',1} \\
&\quad \leqslant \sum_{i=0}^{l} \sum_{j=l}^{2l-i} \bar{C}_{r,r'}\, \|P^F_{(j,i;i_1, ..., i_l)}\|_{r',1} \{\|(\det \sigma_F)^{-(j+1)} D^i \psi\|_r \\
&\qquad + \|(\det \sigma_F)^{-(j+1)} D^{i+1} \psi\|_r \\
&\qquad + (j+1)\, \|(\det \sigma_F)^{-(j+2)} D^i \psi \otimes D(\det \sigma_F)\|_r\} \\
&\quad \leqslant \sum_{i=0}^{l} \sum_{j=l}^{2l-i} \bar{C}_{r,r'}\, \|P^F_{(j,i;i_1, ..., i_l)}\|_{r',1} \{\|(\det \sigma_F)^{-(j+1)} D^i \psi\|_r \\
&\qquad + \|(\det \sigma_F)^{-(j+1)} D^{i+1} \psi\|_r \\
&\qquad + (j+1)\, \|(\det \sigma_F)^{-(j+2)} D^i \psi\|_q\, \|D(\det \sigma_F)\|_{q'}\}.
\end{aligned}$$

Since $i + 1 \leqslant l + 1 \leqslant k$, $i + j + 2 \leqslant 2l + 2 \leqslant 2k$ and $r < q$, it follows from (5.5) that

$$\|\Phi^F_{i_1, ..., i_l}(\cdot; \psi)(\det \sigma_F)^{-1}\|_{p,1} < \infty.$$

[Note that approximating sequence argument and the same estimate as above lead to $\Phi^F_{i_1, ..., i_l}(\cdot; \psi)(\det \sigma_F)^{-1} \in \mathbf{D}^1_p$.]

Therefore Lemma 6 can be applied and so

$$\begin{aligned}
&\int_W (\partial_{i_{l+1}} \cdots \partial_{i_k} T)(F)\, \Phi^F_{i_1, ..., i_l}(w; \psi)\, P(dw) \\
&\qquad = \int_W (\partial_{i_{l+2}} \cdots \partial_{i_k} T)(F)\, \Phi^F_{i_1, ..., i_{l+1}}(w; \psi)\, P(dw).
\end{aligned}$$

Consequently we have the result by induction. ∎

Next, we give the integration-by-parts formula for the differential operator A defined by $A = 1 + |x|^2 - \Delta/2$ where $\Delta = \sum_{i=1}^{d} (\partial/\partial x^i)^2$. Let $\Psi_2^F(\cdot\,;\psi) = (1+|F|^2)\,\psi - \sum_{i=1}^{d} \Phi_{ii}^F(\cdot\,;\psi)/2$ and $\Psi_{2(m+1)}^F(\cdot\,;\psi) = \Psi_2^F(\cdot\,;\Psi_{2m}^F(\cdot\,;\psi))$, $m \in \mathbf{N}$. Then we can show by induction that

$$\Psi_{2m}^F(\cdot\,;\psi) = \sum_{i \leqslant 2m,\, i+j \leqslant 4m} \langle (\det \sigma_F)^{-j} D^i \psi,\, Q_{(i,j;\,2m)}^F \rangle_{H^{\otimes i}}, \tag{5.7}$$

where $Q_{(i,j;\,2m)}^F$ is a polynomial in F, DF, D^2F, ..., LF, ... It is also shown that if F is in $\bigcap_{p>1} \mathbf{D}_p^{l+1}(\mathbf{R}^d)$ for some $l \geqslant 2m$, then $Q_{(i,j;\,2m)}^F \in \bigcap_{p>1} \mathbf{D}_p^s(H^{\otimes i})$ for any $s \leqslant l - 2m$.

Proposition 2 (Integration-by-Parts Formula 2). *For some $m \in \mathbf{N}$, suppose that $F \in \bigcap_{p>1} \mathbf{D}_p^{2m+1}(\mathbf{R}^d)$ and $\psi \in \bigcup_{p>1} \mathbf{D}_p^{2m}$ satisfy the condition*

$$i \leqslant 2m,\ i+j \leqslant 4m,\ i, j \in \mathbf{Z}^+ \Rightarrow (\det \sigma_F)^{-j} D^i \psi \in \bigcup_{q>1} \mathbf{L}_q(H^{\otimes i}). \tag{5.8}$$

Then it holds that for any $T \in \mathscr{S}(\mathbf{R}^d)$,

$$\int_W (A^m T)(F)\, \psi P(dw) = \int_W T(F)\, \Psi_{2m}^F(w;\psi)\, P(dw). \tag{5.9}$$

Proof. From (5.8), we can choose $q > 1$ such that $\|(\det \sigma_F)^{-j} D^i \psi\|_q < \infty$. If $q > r > 1$, there exists $q' > 1$ such that $1/q + 1/q' = 1/r$. Moreover choose $r' > 1$ and $p > 1$ such that $1/r + 1/r' = 1/p$. Then, by (5.7) and Hölder's inequality, it follows that for any non-negative integer l satisfying $0 \leqslant l \leqslant m-1$,

$$\begin{aligned}
\|\Psi_{2l}^F(\cdot\,;\psi)\|_{p,2} &\leqslant \sum_{s \leqslant 2l,\, s+t \leqslant 4l} C_{r,r'} \|(\det \sigma_F)^{-t} D^s \psi\|_{r,2} \|Q_{(s,t;\,2l)}^F\|_{r',2} \\
&\leqslant \sum_{\substack{s \leqslant 2l \\ s+t \leqslant 4l}} \bar{C}_{r,r'} \|Q_{(s,t;\,2l)}^F\|_{r',2} \sum_{k=0}^{2} \|D^k \{(\det \sigma_F)^{-t} D^s \psi\}\|_r \\
&\leqslant \sum_{\substack{s \leqslant 2l \\ s+t \leqslant 4l}} \bar{C}_{r,r'} \|Q_{(s,t;\,2l)}^F\|_{r',2} \sum_{k=0}^{2} \sum_{v=0}^{k} \binom{k}{v} \\
&\qquad \times \sum_{u=0}^{v} \|(\det \sigma_F)^{-t-u} D^{k-v+s} \psi\|_q \|R_{v,t,u}(\sigma_F)\|_{q'},
\end{aligned}$$

where $R_{v,t,u}(\sigma_F)$ is a polynomial in the H-derivatives of σ_F. Since $k - v + s \leqslant 2m$, $t + u + k - v + s \leqslant 4m - 2$, $r < q$ and $F \in \bigcap_{p>1} \mathbf{D}_p^{2m+1}(\mathbf{R}^d)$, it follows that $\|\Psi_{2l}^F(\cdot\,;\psi)\|_{p,2} < \infty$.

Similarly we see that if $i \leqslant 2$, $i+j \leqslant 4$, then

$$\|(\det \sigma_F)^{-j} D^i \Psi^F_{2l}(\cdot\,; \psi)\|_p < \infty.$$

Since $A^{m-(l+1)} T \in \mathscr{S}(\mathbf{R}^d)$, we obtain, from Proposition 1,

$$\int_W (\partial_i \partial_i A^{m-(l+1)} T)(F)\, \Psi^F_{2l}(w; \psi)\, P(dw)$$

$$= \int_W (A^{m-(l+1)} T)(F)\, \Phi^F_{ii}(w; \Psi^F_{2l}(w; \psi))\, P(dw).$$

Consequently,

$$\int_W (A^{m-l} T)(F)\, \Psi^F_{2l}(w; \psi)\, P(dw) = \int_W (A^{m-(l+1)} T)(F)\, \Psi^F_{2(l+1)}(w; \psi)\, P(dw).$$

As a result, we obtain (5.9) by induction. ∎

Proposition 3. *Let $q > 1$ and $m \in \mathbf{Z}^+$. Assume that $F \in \bigcap_{p>1} \mathbf{D}^{2m+1}_p(\mathbf{R}^d)$ and $\psi \in \bigcup_{p>1} \mathbf{D}^{2m}_p$ satisfy the condition*

$$i \leqslant 2m,\ i+j \leqslant 4m,\ i, j \in \mathbf{Z}^+ \Rightarrow (\det \sigma_F)^{-j} D^i \psi \in \mathbf{L}_q(H^{\otimes^i}). \tag{5.10}$$

Then, for $p (q > p > 1)$, there exists a constant $C^{F,\psi}_{p,2m}$ such that

$$\|\psi T(F)\|_{p,-2m} \leqslant C^{F,\psi}_{p,2m} \|A^{-m} T\|_\infty \tag{5.11}$$

for any $T \in \mathscr{S}(\mathbf{R}^d)$.

Proof. Take $p' > 1$ so that $1/p + 1/p' = 1$. As $1/p' + 1/q < 1$, we can choose $r' > 1$ such that $1/p' + 1/q = 1/r'$. Moreover choose r so that $1/r + 1/r' = 1$. Then we see that for any $G \in \mathbf{D}^\infty$,

$$\begin{aligned}\|(\det \sigma_F)^{-j} D^i(\psi G)\|_{r'} &\leqslant \sum_{k=0}^{i} \binom{i}{k} \|(\det \sigma_F)^{-j} D^{i-k}\psi \otimes D^k G\|_{r'} \\ &\leqslant c_{p',m} \sum_{k=0}^{i} \binom{i}{k} \|(\det \sigma_F)^{-j} D^{i-k}\psi\|_q \|G\|_{p',2m} < \infty,\end{aligned} \tag{5.12}$$

where $c_{p',m}$ is a positive constant. Therefore if we replace ψ by ψG in Proposition 2, we see that F and ψG satisfy (5.8); we can apply (5.9). Hence, from (5.7) and (5.12), it follows that

$$\left|\int_W \psi T(F)\, GP(dw)\right| = \left|\int_W (A^{-m}T)(F)\, \Psi^F_{2m}(w; \psi G)\, P(dw)\right|$$

$$\leqslant \|A^{-m}T\|_\infty \int_W |\Psi^F_{2m}(w; \psi G)|\, P(dw)$$

$$\leqslant \|A^{-m}T\|_\infty \sum_{\substack{i \leqslant 2m \\ i+j \leqslant 4m}} \|Q^F_{(i,j;\,2m)}\|_r$$

$$\times \|(\det \sigma_F)^{-j}\, D^i(\psi G)\|_{r'}$$

$$\leqslant c_{p',\,m}\, \|A^{-m}T\|_\infty \sum_{\substack{i \leqslant 2m \\ i+j \leqslant 4m}} \|Q^F_{(i,j;\,2m)}\|_r$$

$$\times \sum_{k=0}^{i} \binom{i}{k} \|(\det \sigma_F)^{-j}\, D^{i-k}\psi\|_q\, \|G\|_{p',\,2m}.$$

In this way, (5.11) follows from the duality. ▮

PROPOSITION 4 (Integration-by-Parts Formula 3). *Let* $m \in \mathbf{Z}^+$ *and* $q > 1$. *Assume that* $F \in \bigcap_{p>1} \mathbf{D}^{2m+1}_p(\mathbf{R}^d)$ *and* $\psi \in \bigcup_{p>1} \mathbf{D}^{2m}_p$ *satisfy* (5.10). *If* $1 < p < q$ *and* $1/p + 1/p' = 1$, *then for any* $T \in \tilde{C}^{-2m}(\mathbf{R}^d)$ *and* $G \in \mathbf{D}^{2m}_{p'}$,

$$_{D^{-2m}_p}\langle (\psi \cdot T) \circ F, G \rangle_{D^{2m}_{p'}} = \int_W (A^{-m}T)(F)\, \Psi^F_{2m}(w; \psi G)\, P(dw). \quad (5.13)$$

Proof. Let $T_n \in \mathcal{S}(\mathbf{R}^d)$ satisfy $\|A^{-m}T - A^{-m}T_n\|_\infty \to 0$ $(n \to \infty)$. From (2.1) and Proposition 2, it follows that

$$_{D^{-2m}_p}\langle (\psi \cdot T) \circ F, G \rangle_{D^{2m}_{p'}} = \lim_{n \to \infty} \int_W (A^m A^{-m} T_n)(F)\, \psi GP(dw)$$

$$= \lim_{n \to \infty} \int_W (A^{-m} T_n)(F)\, \Psi^F_{2m}(w; \psi G)\, P(dw). \quad (5.14)$$

Since $\lim_{n \to \infty} \|A^{-m}T_n - A^{-m}T\|_\infty = 0$, there exists a positive constant C independent of n such that $|(A^{-m}T_n)(F)\, \Psi^F_{2m}(\cdot; \psi G)| \leqslant C\, |\Psi^F_{2m}(\cdot; \psi G)|$. As $\Psi^F_{2m}(\cdot; \psi G)$ is integrable, we obtain the desired result. ▮

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