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EXPANSIONS OF THE COVERAGE PROBABILITIES OF PREDICTION REGION BASED ON A SHRINKAGE ESTIMATOR

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An expansion formula for the coverage probability of prediction region based on a shrinkage estimator proposed by Joshi [Joshi, V. M. (1967). Inadmissibility of the usual confidence sets for the mean of a multivariate normal population. *Ann. Math. Statist.*, **38**, 1868–1875.] is obtained. Its error bound is evaluated in terms of a function of an unknown parameter. Applying this result, three types of asymptotic expansions are derived. These expansions show inadmissibility of the usual prediction region.

Keywords: Asymptotic expansion; Malliavin calculus; Prediction region; Inadmissibility

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1 INTRODUCTION

Suppose that X and Y are independently and identically distributed p-variate normal random vectors with unknown mean vector θ and identity covariance matrix I. For any $a > 0$ and $b > 0$, let $\delta_{a,b}(X)$ be the shrinkage estimator defined by

$$\delta_{a,b}(X) = 1\left(1 - \frac{b}{a + b + \|X\|^2}\right)X, \tag{1}$$

and $S(X)$ be the prediction region for Y centred about $\delta_{a,b}(X)$, *i.e.*,

$$S(X) = \{y;\ \|y - \delta_{a,b}(X)\| < c\}. \tag{2}$$

In this article, we will present an expansion formula for the coverage probability $\Pr_\theta\{Y \in S(X)\}$ with the help of Malliavin calculus. This expansion takes another form than the Edgeworth expansion. More precisely, this expansion consists of rational functions multiplying the normal density, while the Edgeworth expansion relies on the Hermite polynomials. The formula yields

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three types of expansions for large a, for small b, and for large θ. As error bounds of these expansions are given by functions of a, b, and θ, it can be shown that the prediction region $S(X)$ improves the usual prediction region defined by

$$S_0(X) = \{y;\ \|y - X\| < c\}.$$

The admissibility of confidence region was discussed by Joshi (1967, 1969), and Hwang and Casella (1982, 1984) among others. Joshi (1969) showed that if $p = 1$ or $p = 2$, then the confidence region $C_0(X)$ defined by

$$C_0(X) = \{\theta;\ \|\theta - X\| < c\}$$

is admissible, and Joshi (1967) considered the confidence set

$$C(X) = \{\theta;\ \|\theta - \delta_{a,b}(X)\| < c\},$$

and proved that if $p \geq 3$, $C(X)$ improves the coverage probability of $C_0(X)$ for sufficiently large a and sufficiently small b. Hence, it turns out that $C_0(X)$ is inadmissible if $p \geq 3$.[1] Moreover, Takada (1998) discussed inadmissibility of usual confidence region in the case where the population variance is unknown, and showed that a shrinkage type of confidence region with an estimator of variance improves a usual one.

In general, a prediction region $S(X)$ is evaluated by the coverage probability $P_\theta\{Y \in S(X)\}$, and the volume $\mu\{S(X)\}$ with respect to the Lebesgue measure μ. A prediction region $S(X)$ is said to be admissible if there exists no other prediction region $S'(X)$ such that for all θ

$$P_\theta\{Y \in S'(X)\} \geq P_\theta\{Y \in S(X)\} \tag{3}$$

and

$$E_\theta\{\mu\{S'(X)\}\} \leq E_\theta\{\mu\{S(X)\}\}, \tag{4}$$

and the strict inequality holds for at least one θ in Eq. (3) or in Eq. (4). In this sense, Takada (1995a, b) proved that $S_0(X)$ is admissible if $p = 1$ or $p = 2$. The corollaries of our expansion formulas show that if $p \geq 3$, then the prediction region $S(X)$ based on the shrinkage estimator improves the usual one S_0, *i.e.*, S_0 is inadmissible for $p \geq 3$.

In Section 2, we present several expansion formula concerned with the coverage probability of the prediction region $S(X)$ by means of the Malliavin calculus (Sakamoto and Yoshida, 1994, 1996) and prove the inadmissibility of S_0. Their proofs are deferred to Section 3.

The aim of this article is to derive asymptotic expansion formulas and to show the possibility of the approach by expansions to the decision theory, rather than the results themselves stated in the corollaries.

2 EXPANSION FORMULA AND INADMISSIBILITY

First, we give a fundamental result obtained from Theorem 2 in Sakamoto and Yoshida (1996) (or Theorem 5.1.3 of Sakamoto and Yoshida (1994)), which allows us to obtain other results in this article.

[1] The method used in this article can be applied to the confidence region, and supplies another approach to the result of Joshi (1967); see Sakamoto and Yoshida (1996).

THEOREM 2.1 *Let* $g_i(x) = x_i/(a+b+\|x\|^2)$, $i = 1, \ldots, p$, *and* $g(x) = (g_1(x), \ldots, g_p(x))$. *For any multi-index* $\mathbf{n} = (n_1, \ldots, n_p) \in \mathbf{Z}^{+p}$ *and any* $x = (x_1, \ldots, x_p) \in \mathbf{R}^p$, *let* $\mathbf{n}! = n_1! \cdots n_p!$, $|\mathbf{n}| = n_1 + \cdots + n_p$, $x^{\mathbf{n}} = x_1^{n_1} \cdots x_p^{n_p}$ *and* $\partial_x^{\mathbf{n}} = (\partial/\partial x_1)^{n_1} \cdots (\partial/\partial x_p)^{n_p}$. *The remainder term* $r(a, b, c, p, K, \theta)$ *of the asymptotic expansion of* $\Pr_\theta\{Y \in S(X)\}$ *is defined by*

$$r(a, b, c, p, K, \theta) = \Pr_\theta\{Y \in S(X)\} - \sum_{|\mathbf{n}| \le K-1} \frac{b^{|\mathbf{n}|}}{\mathbf{n}!} \int_{\|w-z\|<c} (\partial_z)^{\mathbf{n}}\{(g(z+\theta))^{\mathbf{n}}\phi(z)\} \phi(w)\, \mathrm{d}z\, \mathrm{d}w,$$

where $\phi(z)$ *is the* p*-dimensional standard normal density. Then, there exists a polynomial* $P_{p,K}^{(1)}(x, y)$ *with positive coefficients depending only on* p *and* K *such that*

$$|r(a, b, c, p, K, \theta)| \le P_{p,K}^{(1)}(a^{-1/2}, b) \left\{ \left(\frac{b}{\sqrt{a+b+\|\theta\|^2}} \right)^K + \sqrt{\Pr\left\{ \frac{2bp^2}{a+b+\|Z+\theta\|^2} \ge \frac{1}{2} \right\}} \right\}.$$

Proof See Section 3.1. ∎

From this theorem, we can obtain three asymptotic formulas for small b, for large a, and for large $\|\theta\|$, respectively.

THEOREM 2.2

(1) *Define* $\nu_1(a, b, c, \theta)$ *and* $h(c)$ *by*

$$\begin{aligned}\nu_1(a, b, c, \theta) = {} & \int_{\|w-z\| < c} \int_0^1 \sum_{i,j}^{p} z_j \frac{\partial^2 g_i}{\partial z_i \partial z_j}(\theta + uz)\, \mathrm{d}u \phi(z)\phi(w)\, \mathrm{d}z\, \mathrm{d}w \\ & - \int_{\|w-z\| < c} \int_0^1 (1-u) \sum_{i,j,k}^{p} z_i z_j z_k \frac{\partial^2 g_i}{\partial z_j \partial z_k}(\theta + uz)\, \mathrm{d}u \phi(z)\phi(w)\, \mathrm{d}z\, \mathrm{d}w,\end{aligned}$$

$$h(c) = 1 - \alpha - \frac{c^p \mathrm{e}^{-c^2/4}}{p^{2^p}\Gamma(p/2)}.$$

If $4bp^2 < a + b$, *then it holds that*

$$\Pr_\theta\{Y \in S(X)\} = 1 - \alpha + \frac{b}{a+b+\|\theta\|^2}\left[\{1 - \alpha - h(c)\}\left\{ p - \frac{2\|\theta\|^2}{a+b+\|\theta\|^2} \right\} + (a+b+\|\theta\|^2)\nu_1(a, b, c, \theta) + r^{(1)}(a, b, c, p, \theta) \right],$$

where $|r^{(1)}(a, b, c, p, \theta)| \le bP_{p,2}^{(1)}(a^{-1/2}, b)$. *Moreover, define* β_a *by*

$$\beta_a = \int_{\|w-z\| < c} \left(1 + \frac{\|z\|}{\sqrt{a}}\right)^2 \left(1 + \frac{\|z\|^2}{2}\right) \|z\|\phi(z)\phi(w)\, \mathrm{d}z\, \mathrm{d}w.$$

Then it holds that $|(a+b+\|\theta\|^2)\nu_1(a,b,c,\theta)| \leq 7p\sqrt{p}\beta_a/\sqrt{a}$.

(2) *If* $4bp^2 < a+b$, *it holds that*

$$\Pr_\theta\{Y \in S(X)\} = 1-\alpha+\frac{b}{a+b+\|\theta\|^2}\Bigg[\{1-\alpha-h(c)\} \times\left\{p-\frac{2\|\theta\|^2}{a+b+\|\theta\|^2}-\frac{b\|\theta\|^2}{2(a+b+\|\theta\|^2)}\right\}+r^{(2)}(a,b,c,p,\theta)\Bigg],$$

where $|r^{(2)}(a,b,c,p,\theta)| \leq a^{-1/2}P_p^{(2)}(a^{-1/2},b)$ *for some polynomial* $P_p^{(2)}(\xi_1,\xi_2)$ *with positive coefficient.*

(3) *As* $\|\theta\| \to \infty$,

$$\Pr_\theta\{Y \in S(X)\} = 1-\alpha+\frac{b}{\|\theta\|^2}\{1-\alpha-h(c)\}\left\{p-2-\frac{b}{2}\right\}+\mathrm{O}(\|\theta\|^{-3}).$$

Proof See Section 3.2. ■

The part (1) of Theorem 2.2 implies that

$$\Pr_\theta\{Y \in S(X)\} > \Pr_\theta\{Y \in S_0(X)\} \quad \text{for all } \theta, \tag{5}$$

if the constants a and b satisfy the inequality

$$\{1-\alpha-h(c)\}(p-2)-7p\sqrt{p}\frac{\beta_a}{\sqrt{a}}-bP_{p,2}^{(1)}(a^{-1/2},b) \geq 0.$$

In addition, from (2) of Theorem 2.2, we see that Eq. (5) holds if the constants a and b satisfy the inequality

$$\{1-\alpha-h(c)\}\left(p-2-\frac{b}{2}\right)-a^{-1/2}P_p^{(2)}(a^{-1/2},b) \geq 0.$$

Hence we get the following corollaries.

COROLLARY 2.1

(1) *If* $p \geq 3, 7p\sqrt{p}\beta_a/\sqrt{a} < (1-\alpha-h(c))(p-2)$, *then there exists* $b > 0$ *satisfying Eq.* (5).

(2) *If* $p \geq 3$, $0 < b < 2(p-2)$, *then there exists* $a > 0$ *satisfying Eq.* (5).

Now from the definition of admissibility, we have the following corollary.

COROLLARY 2.2 *If* $p \geq 3$, *then the usual prediction region* $S_0(X)$ *is inadmissible.*

It is also seen easily that the condition $b \leq 2(p-2)$ is necessary for $S(X)$ to improve $S_0(X)$.

THEOREM 2.3 *If* $p \geq 3$, $b > 2(p-2)$, *then for sufficiently large* $\|\theta\|$

$$\Pr_\theta\{Y \in S(X)\} < \Pr_\theta\{Y \in S_0(X)\}.$$

Remark 2.1 The same results concerning the confidence region can also be obtained in an analogous way; see Section 3.4 of Sakamoto and Yoshida (1996) for the details.

3 PROOFS

Throughout the proofs of theorems we will apply the following conventions: $C^{(i)}(\alpha_1, \dots, \alpha_m)$, $i \in \mathbf{N}$, denotes positive constants depending only on $\alpha_1, \dots, \alpha_m$, and $P^{(i)}_{\alpha_1,\dots,\alpha_m}(\xi_1, \dots, \xi_m)$, $i \in \mathbf{N}$, denotes polynomials of $\xi_1, \dots, \xi_m$ with positive coefficients depending only on $\alpha_1, \dots, \alpha_m$. We shall often use the following relations:

$$(\partial_x)^{\mathbf{n}} g_i(x) = \sum_{|\mathbf{m}| \le (|\mathbf{n}|+1)/2} \frac{Q^{(1)}_{\mathbf{n},\mathbf{m}}(x)}{(a+b+\|x\|^2)^{|\mathbf{n}|+1-|\mathbf{m}|}} \tag{6}$$

and

$$(\partial_x)^{\mathbf{n}} \left(\frac{1}{a+b+\|x\|^2} \right) = \sum_{|\mathbf{m}| \le |\mathbf{n}|/2} \frac{Q^{(2)}_{\mathbf{n},\mathbf{m}}(x)}{(a+b+\|x\|^2)^{|\mathbf{n}|+1-|\mathbf{m}|}}, \tag{7}$$

where $Q^{(1)}_{\mathbf{n},\mathbf{m}}(x)$ and $Q^{(2)}_{\mathbf{n},\mathbf{m}}(x)$ are homogeneous polynomials of degree $|\mathbf{n}| + 1 - 2|\mathbf{m}|$ and $|\mathbf{n}| - 2|\mathbf{m}|$, respectively. As $|x/(a+b+\|x\|^2)| \le (a+b+\|x\|^2)^{-1/2}$, we also see that for some constants $C^{(1)}(\mathbf{n})$ and $C^{(2)}(\mathbf{n})$,

$$|(\partial_x)^{\mathbf{n}} g_i(x)| \le C^{(1)}(\mathbf{n})(a+b+\|x\|^2)^{-(|\mathbf{n}|+1)/2} \tag{8}$$

and

$$\left| (\partial_x)^{\mathbf{n}} \left(\frac{1}{a+b+\|x\|^2} \right) \right| \le C^{(2)}(\mathbf{n})(a+b+\|x\|^2)^{-(|\mathbf{n}|+2)/2}. \tag{9}$$

Moreover, it follows from the Leibniz's rule that

$$\partial_x^{\mathbf{n}}(g_i(x))^m = \sum_{\mathbf{j}_1+\cdots+\mathbf{j}_m=\mathbf{n}} \frac{\mathbf{n}!}{\mathbf{j}_1! \cdots \mathbf{j}_m!} \partial_x^{\mathbf{j}_1} g_i(x) \cdots \partial_x^{\mathbf{j}_m} g_i(x)$$

and

$$\partial_x^{\mathbf{n}}(g(x))^{\mathbf{m}} = \sum_{\mathbf{j}_1+\cdots+\mathbf{j}_p=\mathbf{n}} \frac{\mathbf{n}!}{\mathbf{j}_1! \cdots \mathbf{j}_p!} \partial_x^{\mathbf{j}_1}(g_1(x))^{m_1} \cdots \partial_x^{\mathbf{j}_p}(g_p(x))^{m_p}.$$

Therefore, we see from Eq. (8) that

$$|\partial_x^{\mathbf{n}}(g_i(x))^m| \le C^{(3)}(\mathbf{n}, m)(a+b+\|x\|^2)^{-(|\mathbf{n}|+m)/2}$$

and

$$|\partial_x^{\mathbf{n}}(g(x))^{\mathbf{m}}| \le C^{(4)}(\mathbf{n}, \mathbf{m})(a+b+\|x\|^2)^{-(|\mathbf{n}|+|\mathbf{m}|)/2}. \tag{10}$$

3.1 Proof of Theorem 2.1

Let $Z = X - \theta$ and $W = Y - \theta$. Define the indicator functions $1_B(x)$ by

$$1_B(x) = \begin{cases} 1 & (x \in B) \\ 0 & (x \notin B). \end{cases}$$

Then we have

$$\Pr_\theta\{Y \in S(X)\} = \Pr_\theta\{\|W - (Z - bg(Z+\theta))\| < c\} = \int_{\mathbf{R}^p} I(w)\phi(w)\,\mathrm{d}w,$$

where

$$I(w) = \int_{\mathbf{R}^p} 1_{B_w}(z - bg(z+\theta))\phi(z)\,\mathrm{d}z$$

and $B_w = \{z;\ \|z - w\| < c\}$.

Now, we shall give the expansion formula for $I(w)$ applying Theorem 2 of Sakamoto and Yoshida (1996) based on the Malliavin calculus. Let a random variable $\sigma_{Z-bg(Z+\theta)}$ be the Malliavin covariance matrix of $Z - bg(Z+\theta)$, then we have

$$\begin{aligned}\sigma_{Z-bg(Z+\theta)} &= \left(\sum_{i=1}^{p} \frac{\partial}{\partial z_i}(z_k - b g_k(z+\theta))\bigg|_{z=Z} \cdot \frac{\partial}{\partial z_i}(z_l - b g_l(z+\theta))\bigg|_{z=Z}\right)_{k,l=1}^{p} \\ &= \left(\delta_{kl} - b\frac{\partial g_l}{\partial z_k}(Z+\theta) - b\frac{\partial g_k}{\partial z_l}(Z+\theta) + b^2 \sum_{i=1}^{p} \frac{\partial g_k}{\partial z_i}(Z+\theta)\cdot\frac{\partial g_l}{\partial z_i}(Z+\theta)\right)_{k,l=1}^{p} \\ &= I - b(G + {}^tG) + b^{2t}GG = {}^t(I - bG)(I - bG),\end{aligned}$$

where δ_{kl} is the Kronecker delta and G is the $p \times p$ random matrix of the (i, j) elements of which equal to $\partial g_i(x)/\partial x_j|_{x=Z+\theta}$. As

$$\det(I - bG) = 1 + \sum_{k=1}^{p} \frac{(-b)^k}{k!} \sum_{j_1,\ldots,j_k} \det \begin{vmatrix} G_{j_1,j_1} & \cdots & G_{j_1,j_k} \\ \vdots & & \vdots \\ G_{j_k,j_1} & \cdots & G_{jk,jk} \end{vmatrix}$$

and

$$|G_{j_s,j_t}| \le \frac{2}{a + b + \|Z+\theta\|^2} \quad \text{for all } j_s, j_t,$$

we see that if $2bp^2/(a + b + \|Z+\theta\|^2) < 3/4$, then $\det\sigma_{Z-bg(Z+\theta)} > 1/16$. Define the function $\psi_{a,b}(z)$ by $\psi_{a,b}(z) = \psi(2bp^2/(a + b + \|z+\theta\|^2))$, where ψ is a C^∞-function: $\mathbf{R} \to [0, 1]$ such that $\psi(x) = 1$ for $|x| \le 1/2$ and $\psi(x) = 0$ for $|x| \ge 3/4$, then it follows from Eq. (9) that if $2bp^2/(a + b + \|z+\theta\|^2) < 3/4$,

$$|\partial_z^{\mathbf{n}}\psi_{a,b}(z)| \le P_{\mathbf{n}}^{(3)}(a^{-1/2}, b)$$

for some polynomial $P_{\mathbf{n}}^{(3)}(\xi_1, \xi_2)$. Hence it holds that for any $\mathbf{n} \in \mathbf{Z}^{+p}$, $j > 0$, $q > 1$, and $u \in [0, 1]$,

$$\|(\det\sigma_{Z-ubg(Z+\theta)})^{-j}\partial_z^{\mathbf{n}}\psi_{a,b}(Z)\|_q < 16^j P_{\mathbf{n}}^{(3)}(a^{-1/2}, b),$$

where $\|\cdot\|_q$ is the L_q-norm with respect to the measure $\phi(z)\,\mathrm{d}z$. Therefore, Theorem 2 of Sakamoto and Yoshida (1996) can be applied to $I(w)$ and its asymptotic formula can be given by

$$I(w) = \sum_{|\mathbf{n}| \le K-1} \frac{1}{\mathbf{n}!} \int_{B_w} (\partial_z)^{\mathbf{n}}\{(bg(z+\theta))^{\mathbf{n}}\phi(z)\}\,\mathrm{d}z + \tilde{r}(a, b, c, p, w, \theta),$$

where $\tilde{r}(a, b, c, p, w, \theta)$ is the remainder term defined by (2.4) of Sakamoto and Yoshida (1996), and it can be estimated as follows. Note that for any p-dimensional vectors μ and λ,

$$\|\mu\|^2 \le \|\mu + \lambda\|^2 + 2\|\mu + \lambda\| \cdot \|\lambda\| + \|\lambda\|^2 \tag{11}$$

and that for any $A > 0$ and $X \ge 0$,

$$\frac{x}{A + x^2} \le \frac{1}{2\sqrt{A}}. \tag{12}$$

These inequalities yield that

$$\begin{aligned}\frac{a + b + \|\theta\|^2}{a + b + \|\theta + z\|^2} &\le 1 + \frac{2\|\theta + z\| \cdot \|z\| + \|z\|^2}{a + b + \|\theta + z\|^2} \\ &\le 1 + 4\|z\| \frac{\|\theta + z\|}{a + b + \|\theta + z\|^2} + \frac{\|z\|^2}{a + b} \le \left(\frac{\|z\|}{\sqrt{a + b}} + 1\right)^2. \end{aligned} \tag{13}$$

Combining Eqs. (10) with (13), we see that

$$|\partial_z^{\mathbf{n}}(g(z + \theta))^{\mathbf{m}}| \le C^{(4)}(\mathbf{n}, \mathbf{m}) \left(\frac{1}{\sqrt{a + b + \|\theta\|^2}}\right)^{|\mathbf{n}|+|\mathbf{m}|} \left(\frac{\|z\|}{\sqrt{a + b}} + 1\right)^{|\mathbf{n}|+|\mathbf{m}|}. \tag{14}$$

From Eq. (8), it follows that for some polynomial $P_{\mathbf{n}}^{(4)}(\xi_1, \xi_2, \xi_3)$,

$$|(\partial_z^{\mathbf{n}})(z_i - bg_i(z + \theta))| \le P_{\mathbf{n}}^{(4)}(a^{-1/2}, b, |z|).$$

By using these inequalities and formula (5.7) in Sakamoto and Yoshida (1996) [see also Sakamoto and Yoshida (1994), Lemma 3.2.1 for the details], we can evaluate the functional Ψ in the remainder term $\tilde{r}(a, b, c, p, w, \theta)$, and therefore we can estimate the remainder term: there exist two polynomials $P_{\mathbf{n},p,K}^{(5)}(\xi_1, \xi_2, \xi_3)$ and $P_{p,K}^{(6)}(\xi_1, \xi_2, \xi_3)$ satisfying that

$$\begin{aligned}|\tilde{r}(a, b, c, p, w, \theta)| \le E|\psi_{a,b}(Z) - 1| &+ \sum_{|\mathbf{n}| \le p+K+1} E\left[P_{\mathbf{n},p,K}^{(5)}(a^{-1/2}, b, Z)|\partial_z^{\mathbf{n}}(\psi_{a,b} - 1) \circ Z|\right] \\ &+ \left(\frac{b}{\sqrt{a + b + \|\theta\|^2}}\right)^K E|P_{p,K}^{(6)}(a^{-1/2}, b, Z)|.\end{aligned}$$

As

$$E|\psi_{a,b}(Z) - 1| \le \Pr\left\{\frac{2bp^2}{a + b + \|Z + \theta\|^2} \ge \frac{1}{2}\right\}$$

and

$$\begin{aligned}E[P_{\mathbf{n},p,K}^{(5)}(a^{-1/2}, b, Z)|\partial_z^{\mathbf{n}}(\psi_{a,b} - 1) \circ Z|] &\le P_{\mathbf{n},p,K}^{(7)}(a^{-1/2}, b) \\ &\times \left(\Pr\left\{\frac{2bp^2}{a + b + \|Z + \theta\|^2} \ge \frac{1}{2}\right\}\right)^{1/2}\end{aligned}$$

for some polynomial $P_{\mathbf{n},p,K}^{(7)}(\xi_1, \xi_2)$, the proof of Theorem 2.1 is complete. ■

3.2 Proof of Theorem 2.2

First, we consider the result of Theorem 2.1 for $K = 2$. Expanding $\partial/\partial z_i\{g_i(z+\theta)\phi(z)\}$ around $z = 0$, we have

$$\begin{aligned}\Pr_\theta\{Y \in S(X)\} &= \int_{\|w-z\|<c} \phi(z)\phi(w)\,\mathrm{d}z\,\mathrm{d}w \\ &\quad + b\sum_{i=1}^{p}\int_{\|w-z\|<c} \frac{\partial}{\partial z_i}\{g_i(z+\theta)\phi(z)\}\phi(w)\,\mathrm{d}z\,\mathrm{d}w + r(a,b,c,p,2,\theta) \\ &= 1-\alpha + b(\eta_1(a,b,c,\theta) + \nu_1(a,b,c,\theta)) + r(a,b,c,p,2,\theta),\end{aligned}$$

where

$$\eta_1(a,b,c,\theta) = \sum_{i=1}^{p}\int_{\|w-z\|<c}\left[\frac{\partial g_i}{\partial z_i}(\theta) - z_i\left\{g_i(\theta) + \sum_{j=1}^{p}\frac{\partial gi}{\partial z_j}(\theta)z_j\right\}\right]\phi(z)\phi(w)\,\mathrm{d}z\,\mathrm{d}w.$$

As

$$\int_{\|w-z\|<c}\langle z,\theta\rangle\phi(z)\phi(w)\,\mathrm{d}z\,\mathrm{d}w = 0$$

and

$$\int_{\|w-z\|<c}\langle z,\theta\rangle^2\phi(z)\phi(w)\,\mathrm{d}z\,\mathrm{d}w = p^{-1}\|\theta\|^2\int_{\|w-z\|<c}\|z\|^2\phi(z)\phi(w)\,\mathrm{d}z\,\mathrm{d}w = \|\theta\|^2 h(c), \tag{15}$$

it follows that

$$\begin{aligned}\eta_1(a,b,c,\theta) &= \int_{\|w-z\|<c}\frac{1}{a+b+\|\theta\|^2}\left[p - 2\frac{\|\theta\|^2}{a+b+\|\theta\|^2}\right. \\ &\qquad \left. - \langle z,\theta\rangle - \|z\|^2 + 2\frac{\langle z,\theta\rangle^2}{a+b+\|\theta\|^2}\right]\phi(z)\phi(w)\,\mathrm{d}z\,\mathrm{d}w \\ &= \{1-\alpha-h(c)\}\frac{1}{a+b+\|\theta\|^2}\left(p - \frac{2\|\theta\|^2}{a+b+\|\theta\|^2}\right).\end{aligned}$$

If $4bp^2 < a+b$, then for $K \in \mathbf{N}$,

$$|r(a,b,c,p,K,\theta)| < P_{p,K}^{(1)}(a^{-1/2},b)\left(\frac{b}{\sqrt{a+b+\|\theta\|^2}}\right)^K. \tag{16}$$

Therefore, it follows that when $4bp^2 < a+b$,

$$|r^{(1)}(a,b,c,p,\theta)| = \left|\frac{a+b+\|\theta\|^2}{b}r(a,b,c,p,2,\theta)\right| \le bP_{p,2}^{(1)}(a^{-1/2},b).$$

On the other hand, the inequalities (12) and (13) imply that

$$\begin{aligned}\left|\frac{\partial^2 g_i}{\partial z_j \partial z_k}(\theta + uz)\right| &\le \frac{6\|\theta + uz\|}{(a+b+\|\theta+uz\|^2)^2} + \frac{8\|\theta+uz\|^3}{(a+b+\|\theta+uz\|^2)^3} \\ &\le \frac{14\|\theta+uz\|}{(a+b+\|\theta+uz\|^2)^2} \le \frac{7}{(a+b+\|\theta+uz\|^2)\sqrt{a+b}} \\ &\le \frac{7}{(a+b+\|\theta\|^2)\sqrt{a}}\left(\frac{\|z\|}{\sqrt{a}}+1\right)^2\end{aligned}$$

for $0 \le u \le 1$. Hence

$$\begin{aligned}|(a+b+\|\theta\|^2)\nu_1(a,b,c,\theta)| &\le \frac{7}{\sqrt{a}}\int_{\|w-z\|<c}\left(\frac{\|z\|}{\sqrt{a}}+1\right)^2 \\ &\quad \times \sum_{i=1}^{p}|z_i|\left\{p+\frac{1}{2}\left(\sum_{i=1}^{p}|z_i|\right)^2\right\}\phi(z)\phi(w)\,\mathrm{d}z\,\mathrm{d}w \\ &\le \frac{7p\sqrt{p}}{\sqrt{a}}\beta_a. \end{aligned} \tag{17}$$

Thus, we have proved (1) of Theorem 2.2.

Next, we shall consider the result of Theorem 2.1 for $K = 3$. Let the Kronecker delta be denoted by $\delta_{i,j}$. Expanding $\Pr_\theta\{Y \in S(X)\}$ in the same fashion as above, we have

$$\begin{aligned}\Pr_\theta\{Y \in S(X)\} &= 1-\alpha+b(\eta_1(a,b,c,\theta)+\nu_1(a,b,c,\theta)) \\ &\quad + \frac{b^2}{2}\sum_{i,j}^{p}\int_{\|w-z\|<c}\frac{\partial^2}{\partial z_i \partial z_j}\{g_i(z+\theta)g_j(z+\theta)\phi(z)\}\phi(w)\,\mathrm{d}z\,\mathrm{d}w \\ &\quad + r(a,b,c,p,3,\theta) \\ &= 1-\alpha+b(\eta_1(a,b,c,\theta)+\nu_1(a,b,c,\theta)) \\ &\quad + \frac{b^2}{2}(\eta_2(a,b,c,\theta)+\nu_2(a,b,c,\theta)) + r(a,b,c,p,3,\theta),\end{aligned} \tag{18}$$

where

$$\begin{aligned}\eta_2(a,b,c,\theta) &= \sum_{i,j}^{p}\int_{\|w-z\|<c} g_i(\theta)g_j(\theta)(z_iz_j-\delta_{i,j})\phi(z)\phi(w)\,\mathrm{d}z\,\mathrm{d}w,\ \nu_2(a,b,c,\theta) \\ &= \sum_{i,j}^{p}\int_{\|w-z\|<c}\left[\frac{\partial^2}{\partial z_i\partial z_j}\{g_i(z+\theta)g_j(z+\theta)\} - 2z_j\frac{\partial}{\partial z_i}\{g_i(z+\theta)g_j(z+\theta)\}\right. \\ &\quad \left. + \{2g_i(\theta)\zeta_j+\zeta_i\zeta_j\}(z_iz_j-\delta_{i,j})\right]\phi(z)\phi(w)\,\mathrm{d}z\,\mathrm{d}w,\end{aligned}$$

and

$$\zeta_i = \sum_{k=1}^{p} z_k \int_0^1 \frac{\partial g_i}{\partial z_k}(\theta+uz)\,\mathrm{d}u.$$

It follows immediately from Eq. (15) that

$$\eta_2(a, b, c, \theta) = \frac{1}{(a + b + \|\theta\|^2)^2} \int_{\|w-z\|<c} (\langle z, \theta\rangle^2 - \|\theta\|^2)\phi(z)\phi(w)\, \mathrm{d}z\, \mathrm{d}w$$

$$= -\{1 - \alpha - h(c)\} \frac{\|\theta\|^2}{(a + b + \|\theta\|^2)^2}.$$

From Eqs. (13) and (14), it holds that

$$|(a + b + \|\theta\|^2)\nu_2(a, b, c, \theta)| \leq \frac{1}{\sqrt{a}} P_p^{(8)}(a^{-1/2})$$

for some polynomial $P_p^{(8)}(\xi)$. Combining this and the inequalities (15) and (17), we have that when $4bp^2 < a + b$,

$$|r^{(2)}(a, b, c, p, \theta)| = (a + b + \|\theta\|^2) \left| \nu_1(a, b, c, \theta) + \frac{b}{2}\nu_2(a, b, c, \theta) + \frac{r(a, b, c, p, 3, \theta)}{b} \right|$$

$$\leq \frac{7p\sqrt{p}\beta_a}{\sqrt{a}} + \frac{b}{2\sqrt{a}} P_p^{(8)}(a^{-1/2}) + \frac{b^2}{\sqrt{a}} P_{p,3}^{(1)}(a^{-1/2}, b),$$

which shows that (2) of Theorem 2.2 holds.

Finally, we shall derive (3) of Theorem 2.2 from Eq. (18). From Eq. (14), we see that as $\|\theta\| \to \infty$,

$$\nu_1(a, b, c, \theta) = \mathrm{O}(\|\theta\|^{-3}), \quad \nu_2(a, b, c, \theta) = \mathrm{O}(\|\theta\|^{-3}).$$

It is easy to show that as $\|\theta\| \to \infty$,

$$b\eta_1(a, b, c, \theta) + \frac{b^2}{2}\eta_2(a, b, c, \theta) = \frac{b}{\|\theta\|^2}\{1 - \alpha - h(c)\} \left\{p - 2 - \frac{b}{2}\right\} + \mathrm{O}(\|\theta\|^{-4})$$

and

$$\Pr\left\{\frac{2bp^2}{a + b + \|Z + \theta\|^2} \geq \frac{1}{2}\right\} \leq \Pr\left\{\|z\| \geq \frac{\|\theta\|}{2}\right\} = \mathrm{O}(\|\theta\|^{-6}).$$

Thus, the proof of Theorem 2.2 is complete. ■

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