Probab. Theory Relat. Fields 129, 559–624 (2004)
Digital Object Identifier (DOI) 10.1007/s00440-003-0325-8

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Partial mixing and Edgeworth expansion

Received: 24 April 2001 / Revised version: 6 November 2003 /
Published online: 25 May 2004 –

Abstract. Introducing a conditional mixing property, Götze and Hipp's theory is generalized to a continuous-time conditional ϵ-Markov process satisfying this property. The Malliavin calculus for jump processes applies to random-coefficient stochastic differential equations with jumps with the aid of the support theorem to verify the non-degeneracy condition, i.e., a conditional type Cramér condition.

1. Introduction

In this article, we will first introduce a *partial mixing process*, and derive asymptotic expansions for a functional of that process. A partial mixing process is a process which in part possesses a nice mixing property. In order to explain a motivation from statistics, let us consider a linear stochastic regression model having a long-memory explanatory variable. This is a simple example rather than our main purpose. Suppose that the stochastic process $Y = (Y_t)_{t\in\mathbf{Z}_+}$ is defined by the linear model:

$$Y_t = \theta X_t + e_t,$$

where $e = (e_t)$ is a zero-mean i.i.d. sequence and $X = (X_t)$ is an explanatory process which is independent of e and may have a long-range dependence. The parater θ is unknown in a statistical context, and it is very common to examine the asymptotics of the least-square estimator $\hat{\theta}_T$:

$$\mathcal{X}_T = \sqrt{T}\left(\hat{\theta}_T - \theta\right) = \frac{\sum_{t=1}^{T} X_t e_t/\sqrt{T}}{\sum_{t=1}^{T} X_t^2/T}$$

as $T \to \infty$.

Because of the possible long-range dependence of X, the strong mixing coefficient of the full process (X_t, e_t) does not decay so rapidly as to induce asymptotic expansion. For the usual condition under which asymptotic expansion is derived involves a fairly-high-order of polynomial decay. This condition however is broken

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This work was in part supported by the Research Fund for Scientists of the Ministry of Science, Education and Culture, and by Cooperative Research Program of the Institute of Statistical Mathematics.

Mathematics Subject Classification (2000): 60H07, 60F05, 60J25, 60J75, 62E20

Key words or phrases: Asymptotic expansion – Malliavin calculus – Partial mixing – Stochastic differential equation – Support theorem

when X has an autocovariance which is not integrable and the full mixing coefficient inevitably decreases slowly, as this fact follows from the covariance inequality for mixing processes.

On the other hand, clearly, e satisfies an ideal mixing condition under the conditional probability given the whole process X. Under this conditioning, $\mathcal{X}_T$ is just a sum of independent random variables and it is easy to derive the conditional asymptotic expansion: what we need to do is to integrate the conditional expansion with respect to the variable X in order to obtain a usual expansion for $\mathcal{X}_T$ under the original probability measure. In this example, the resulting expansion is a fractional expansion which consists of the familiar power $1/\sqrt{T}$ and another fractional power $1/T^{a/2}$, a being an index related with the Hurst number; see [71].

As already suggested, the first aim of this article is to derive asymptotic expansion for the partial mixing process. We will treat this problem within a more general framework. It applies to conditional ϵ-Markov processes, and as a special case, to a random-coefficient stochastic differential equation with jumps.

Secondly, we will present a *precise estimate of the conditional characteristic function under slower-than-exponential decays* of the conditional mixing coefficients. Giving careful consideration to such slow decays is of importance even in unconditional situations: see Veretennikov [66, 67]. While the possibility of weakening the exponential decay was suggested or proved, e.g., in Jensen [24], Lahiri [33], we will show it in the conditional situation. In order to obtain a conditional estimate of the conditional characteristic function over a moderately large region, we carried out a conditional version of estimate similar to that of the prominent work by Götze and Hipp [18]. In contrast to the unconditional case, it is necessary to construct a suitable truncation functional (localization), as seen in the proof. Roughly speaking, the expansion finally obtained is to be a mixture of unconditional expansions, but the real position we are in is not as simple as expected. If one would apply directly the estimates from the unconditional cases, the resulting conditional error bounds would be infinity in vain: consider the example at the beginning of this introduction. We need, in our conditional situation, precise estimates of those that were deterministic in the unconditional case and that could successfully be estimated by Götze-Hipp's subtle induction technique in the deterministic case.

As for the estimates of the conditional characteristic function over a large region, a conditional type Cramér condition under suitable truncation is crucial. The use of truncation functionals seems to be inevitable for our later applications, and it turns out to be essential to look into to what extent those truncations retain positive probability. In order to measure the nondegeneracy, certain *counting processes measurable with respect to the conditioning variables* will play an important role.

Since we are aiming not only at discrete-time processes but also at continuous-time processes like semimartingales, the estimate of the conditional characteristic function over the large region requires an *infinite-dimensional stochastic analysis*. Various formulations have been presented by many authors like Bismut, Stroock, Bichteler-Gravereaux-Jacod, Norris, Nualart, Privault, Picard. See the references of Bichteler et al. [9]. Also see Norris [40], Nualart and Vives[41], Privault [46–48], Carlen and Pardoux [12], and Elliott and Tsoi [16]. In the present article, we adopted the Malliavin calculus formulated by Bichteler, Gravereaux and Jacod [9] among other possibilities such as the essentially important work recently developed by Picard [44, 45]. One reason is that it is easy to handle in computations showing non-degeneracy under path-dependent truncations. In an example in the author's previous paper [29], a result on regularity, which was rather standard in the Malliavin calculus, was directly applied to the local estimate of the characteristic functions. In that case, the used truncation functional essentially selected a good set of the *initial* values, not a good path set. We will present another sufficient condition suitable for *support theorems*. An application of the support theorem to such a distributional expansion is seen in martingale expansion of [70]. We will show that the path-dependent truncation method with the aid of the support theorem enables us to weaken considerably the non-degeneracy conditions. As the third aim, at the final stage, *the conditional expansion, the Malliavin calculus for jump processes, and the support theorem with the help of the stability of stochastic integrations will fuse to give a valid asymptotic expansion to a functional from a stochastic differential equation with jumps*. This is a new result even in the unconditional case.

We conclude this section by mentioning related literature. Bhattacharya and Rao [8] is a systematic exposition of Edgeworth expansions for independent random variables, which treats also the lattice case. A mixture of lattice and non-lattice cases was examined in Babu and Singh [4]. For Markov chains, see Nagaev [39], Statulevicius [58], and for an abstract framework for an envelope process of a Markov chain, Götze and Hipp [18]. See also Jensen [25] for a particular Harris recurrent Markov chain. Götze and Hipp [19] studied conditional type Cramér conditions for more concrete Markov chains. Jensen [24] is, with an extension to random fields, a nice instruction for Götze-Hipp theory. Lahiri [32] gave a refinement. Datta and McCormick [14] obtained the first-order expansion under a conditional non-latticeness condition. Hipp [21] studied compound processes. Kusuoka and Yoshida [29] considered an abstract ϵ-Markov mixing process and proved the validity of the asymptotic expansion based on the Malliavin calculus. All those works treated unconditional cases. For near-independent structures, the idea of conditioning was used; e.g., Albers et al. [3], Hipp [21], Bickel et al. [10].

As for a random variable admitting a stochastic expansion, Bhattacharya and Ghosh [7] proved the validity of the formal Edgeworth expansion for independent variables, introducing the Bhattacharya-Ghosh map. Götze and Hipp [19] transferred the idea to their setting. Sakamoto and Yoshida [52, 53] presented an explicit expansion formula for ϵ-Markov processes with continuous time parameter, and applied it to diffusion functionals. In the same setting, Sakamoto [51] presented expansions for various test statistics with applications to diffusion processes. A study of information criteria for statistical model selection of stochastic processes was done by Uchida and Yoshida [65] in the light of the asymptotic expansions.

The mixing property is explicitly or implicitly a key condition in the above works. Doukhan [15] is an excellent text book including extensive references. Diffusion processes over non-compact area are of practical importance. Then the Doeblin type condition does not hold and advanced methods are needed. See Bhattacharya [6] for the derivation of exponential decay using spectral theory; also see Stroock [59]. Veretennikov [66, 67] studied various mixing rates for nondegenerate diffusion processes. Recently, Kusuoka and Yoshida [29] proved an exponential rate using operational calculus and perturbation theory.

In standard arguments based on the mixing property (*mixing approach* or *local approach*), the component functionals of the stochastic expansion are asymptotically normal. Consequently, this approach excludes expansions involving non-central limit theorems. For example, the least square estimator with a strongly dependent explanatory variable has a non-central component in the second-order, so that the usual mixing approach does not work. It is possible to obtain a second-order expansion ([69]), however, we need another method which is called the *global approach* (or *martingale approach*). The estimation of volatility over finite time interval is another example. There, the martingale problem method is necessary to obtain the limit distribution of the second-order term. A mixture of Gaussian distributions, therefore non-Gaussian distribution, appears and it cannot be a consequence of a mixing approach. Even if the central limit theorem cannot apply, the global method still provides an approach to the asymptotic expansion. See [70] for details, and also a series of Mykland's works [36–38], which inspired me.

Historically, the theory of asymptotic expansions has been oriented toward statistical applications. We refer the interested reader to Akahira and Takeuchi [2], Pfanzagl [43], Taniguchi [62], Hall [20], Ghosh [17], Barndorff-Nielsen and Cox [5], Pace and Salvan [42], Kutoyants [30], and Taniguchi and Kakizawa [63].

2. Conditional α-mixing and asymptotic expansion

Given a standard probability space $(\Omega, \mathcal{F}, P)$, suppose that there are increasing sub σ-fields $\mathcal{B}_I$ $(I \subset \mathbf{R}_+)$ [1] with respect to the partial order of sets, i.e., that $I \subset J$ implies that $\mathcal{B}_I \subset \mathcal{B}_J$, and there is a sub σ-field $\mathcal{C}$ of $\mathcal{F}$. For $s, t \in \mathbf{R}_+$, $s \le t$, the random number $\alpha(s, t|\mathcal{C})$ which satisfies

$$1 \ge \alpha(s, t|\mathcal{C}) \ge \sup_{B_1 \in \mathcal{B}_{[0,s]} \vee \mathcal{C},\, B_2 \in \mathcal{B}_{[t,\infty)} \vee \mathcal{C}} |P_{\mathcal{C}}[B_1 \cap B_2] - P_{\mathcal{C}}[B_1]\, P_{\mathcal{C}}[B_2]|$$

is called the conditional α-mixing coefficient given $\mathcal{C}$. Suppose that $1 \ge \alpha(h|\mathcal{C}) \ge \sup_{h' \ge h, s \in \mathbf{R}_+} \alpha(s, s + h'|\mathcal{C})$. Henceforth, we assume $\mathcal{C}$-measurability of $\alpha(s, t|\mathcal{C})$ and $\alpha(h|\mathcal{C})$. This is without essential loss of generality as we can replace them by their measurable envelopes. We put $\alpha(h|\mathcal{C}) = 1$ for negative h.

We will consider a d-dimensional process $(Z_t)_{t \in \mathbf{R}_+}$ which satisfies the measurability condition:

$$Z_0 \in \mathcal{F}(\mathcal{B}_{[0]} \vee \mathcal{C}) \ \text{ and } \ Z_t^s \equiv Z_t - Z_s \in \mathcal{F}(\mathcal{B}_{[s,t]} \vee \mathcal{C})$$

for every $s, t \in \mathbf{R}_+$ $(s \le t)$. Here $\mathcal{F}(\mathcal{B})$ denotes the set of $\mathcal{B}$-measurable functions. The aim of the present article is to derive asymptotic expansion of the expectation $P[f(Z_T/\sqrt{T})]$ for measurable functions f.

[1] $\mathbf{R}_+$ denotes the set of nonnegative real numbers. Similarly, $\mathbf{N}$ denotes the set of positive integers, and $\mathbf{Z}_+$ the set of nonnegative integers.

In order to obtain the error bounds for the asymptotic expansion, we need to specify the rate of convergence to zero for the conditional mixing coefficient $\alpha(h|\mathcal{C})$. We will consider three situations. The exponential decay is assumed in the first case, and the polynomial decay in the third case together with an additional condition. The second one is intermediate.

[A1′] **(exponential)** There exists a constant $a \in (0, \infty)$ such that $\|\alpha(h|\mathcal{C})\|_1 \le a^{-1}e^{-ah}$ for all $h \in (0, \infty)$.

Next, we will consider a case where the expected decay of α is faster than T^{-L} but it may be slower than the exponential order. In this case, we need a stronger integrability condition. Fix two positive numbers δ and $\bar{\delta}$ as $\bar{\delta} \ge 2\delta$. Denote by $\mathcal{I}$ the set of partitions $\mathbf{I} = (t_j)_{j=0}^{\infty}$ of $\mathbf{R}_+$ such that $0 = t_0 < t_1 < \cdots$ and $\delta \le t_{j+1} - t_j \le \bar{\delta}$. For $\mathbf{I} \in \mathcal{I}$, let $\mathcal{H}(\mathbf{I}) = \{H : \mathbf{R}_+ \to \{0, 1\}$ be the indicator function of a (partial) union of the intervals among [0] and $[t_j, t_{j+1}]$'s for $\mathbf{I}$ }. Let $H \cdot Z_T$ denote the primitive stochastic integration of $H \in \mathcal{H}$ over $[0, T]$ with respect to Z. Finally, set

$$\Phi_T(\mathbf{I}) = \max_{H \in \mathcal{H}(\mathbf{I})} |\mathrm{Var}_{\mathcal{C}}[T^{-1/2} H \cdot Z_T]|.$$

[A1″] **(hyper-polynomial)** There exist constants $a, b, c, C > 0$ such that $ab > 1$, $\|\alpha(h|\mathcal{C})\|_1 \le C \exp\left(-c(\log h)^{1+a}\right)$ $(h \in \mathbf{R}_+)$, and $\limsup_{T\to\infty} \sup_{\mathbf{I}\in\mathcal{I}} P\left[\exp\left(\Phi_T(\mathbf{I})^b\right)\right] < \infty$.

The final condition is an extreme case. It requires the boundedness of the conditional variance but it works well for bounded energy additive functionals as well as in unconditional (Markovian) cases.

[A1‴] **(polynomial)** For any $L > 0$, $\|\alpha(h|\mathcal{C})\|_1 = O(h^{-L})$, and $\limsup_{T\to\infty} \sup_{\mathbf{I}\in\mathcal{I}} \|\Phi_T(\mathbf{I})\|_\infty < \infty$.[2]

For some $p \in \mathbf{N}$ $(p \ge 3)$, we also assume:

[A2] There exists a positive number h_0 such that for every $L \in \mathbf{N}$, $\sup_{t,h:t\in\mathbf{R}_+, 0\le h\le h_0} P\left[\left|P_{\mathcal{C}}\left[\left|Z^t_{t+h}\right|^{p+1}\right]\right|^L\right] < \infty$, and the same inequality with Z^t_{t+h} replaced by Z_0 holds. Furthermore, $P_{\mathcal{C}}[Z_t] = 0$ for all $t \in \mathbf{R}_+$.[3]

[2] Obviously from the proof presented below, we only need to assume this condition for a certain finite number L to prove our results. Lahiri [32] gave a proof under polynomial rate in unconditional case, assuming an additional condition. In unconditional case, as seen in the proof, it is not a so simple matter to write out an exact rate because it depends on the conditioning field $\mathcal{C}$.

[3] It is possible to reduce "every L" to some sufficiently large L in the condition. The interested reader would find a necessary number. When $\mathcal{C}$ is trivial, this condition will be a usual condition of finite moments. Our interest is mainly in practically applied models such as SDE's, and for such models, the condition for the existence of solutions often ensures the existence of moments of arbitrary order. For this reason, we did not pursue a minimal moment condition here.

In this paper, we shall devote our attention to a certain Markovian structure, mainly, like Götze and Hipp [19] did, rather than the more general nearly Markovian process treated in [18]. Thus we do not need conditions for approximation to the original process by another process that yields the reduction.

Definition 1. *A double sequence of intervals* $I(j) = [u(j), v(j)] \subset [0, T]$ *(*$j = 1, ..., n'(T)$*;* $T > 0$*) is called* $h(T)$*-***sparse reduction intervals** *with* $(\hat{\mathcal{C}}(j))_{j=1}^{n'(T)}$ *if:*

(i) $\hat{\mathcal{C}}(j)$ *are sub* σ*-fields of* $\mathcal{F}$ *with* $\hat{\mathcal{C}}(j) \supset \mathcal{C}$*.*
(ii) $v(j) + h(T) \le u(j+1)$ *for* $j = 1, ..., n'(T) - 1$*,* $\delta \le v(j) - u(j) \le \bar{\delta} < \infty$ *for any* j*,* T*, and* $\liminf_{T\to\infty} n'(T)(h(T)+1)/T > 0$*.*
(iii) *For any subsequence* $j_1, ..., j_{n''(T)}$ *of* $1, ..., n'(T)$*, any* $A \in b\left(\mathcal{B}_{[0,T]\setminus\cup_{l=1}^{n''(T)} I(j_l)} \vee \mathcal{C}\right)$[4] *and* $A_{j_l} \in b\left(\mathcal{B}_{I(j_l)} \vee \mathcal{C}\right)$ *(*$l = 1, ..., n''(T)$*),*

$$P_{\mathcal{C}}\left[AA_{j_1}\cdots A_{j_{n''(T)}}\right] = P_{\mathcal{C}}\left[AP_{\hat{\mathcal{C}}(j_1)}\left[A_{j_1}\right]\cdots P_{\hat{\mathcal{C}}(j_{n''(T)})}\left[A_{j_{n''(T)}}\right]\right].$$

In the above definition, $I(j)$ and $\hat{\mathcal{C}}(j)$ may depend on T. In general, $h(T)$ may be bounded. 0-sparse reduction intervals are called **dense reduction intervals**.

Denote by Z_J the increment of Z over the interval J. The following is a conditional type Cramér condition.

[A3] For every $L > 0$, there exist truncation functionals $\psi_j : (\Omega, \mathcal{F}) \to ([0,1], \mathbf{B}([0,1]))$ and constants $a', a \in (0,1)$ and $B > 0$ such that $4a' < (a-1)^2$ and
(i) $P\left[\frac{1}{n'(T)}\sum_j P_{\mathcal{C}}\left[\sup_{u:|u|\ge B}\left|P_{\hat{\mathcal{C}}(j)}\left[e^{iu\cdot Z_{I(j)}}\psi_j\right]\right|\right] \ge a'\right] = o(\frac{1}{T^L})$.
(ii) $P\left[\frac{1}{n'(T)}\sum_j P_{\mathcal{C}}\left[1-\psi_j\right] \ge a\right] = o(\frac{1}{T^L})$.

Condition $[A3]$ may seem to be slightly stronger than the usual condition in independent cases. It is possible to weaken this condition though it will become more complicated; see Remark 15. In the above condition, ψ_j may depend on T.

The $\mathcal{C}$-conditional cumulant functions $\chi_{T,r,\mathcal{C}}(u)$ of $Z_T/\sqrt{T}$ are defined by

$$\chi_{T,r,\mathcal{C}}(u) = \left(\frac{d}{d\epsilon}\right)^r \Big|_{\epsilon=0} \log P_{\mathcal{C}}\left[\exp\left(i\epsilon u \cdot \frac{1}{\sqrt{T}} Z_T\right)\right].$$

With the formal expansion:

$$\exp\left(\sum_{r=2}^{\infty}\frac{1}{r!}\epsilon^{r-2}\chi_{T,r,\mathcal{C}}(u)\right) = \exp\left(\frac{1}{2}\chi_{T,2,\mathcal{C}}(u)\right) + \sum_{r=1}^{\infty}\epsilon^r T^{-\frac{r}{2}}\tilde{P}_{T,r,\mathcal{C}}(u),$$

we define the function $\hat{\Psi}_{T,p,\mathcal{C}}(u)$ by the partial sum:

$$\hat{\Psi}_{T,p,\mathcal{C}}(u) = \exp\left(\frac{1}{2}\chi_{T,2,\mathcal{C}}(u)\right) + \sum_{r=1}^{p-2} T^{-\frac{r}{2}}\tilde{P}_{T,r,\mathcal{C}}(u).$$

Let $\Psi_{T,s,\mathcal{C}} = \mathcal{F}^{-1}[\hat{\Psi}_{T,s,\mathcal{C}}]$, the Fourier inversion of $\hat{\Psi}_{T,s,\mathcal{C}}$.

[4] $b(\mathcal{B})$ is the set of bounded $\mathcal{B}$-measurable functions.

Let $p_0 = 2[p/2]$ and set $\mathcal{E}(M,\gamma) = \{f : \mathbf{R}^d \to \mathbf{R}$, measurable, $|f(x)| \le M(1+|x|^\gamma)$ $(x \in \mathbf{R}^d)\}$. We will in the sequel assume that

$$\limsup_{T\to\infty} \left\| \left|\Psi_{T,p,\mathcal{C}}\right| [1+|\cdot|^{p_0}] \right\|_{L^q} < \infty \tag{1}$$

for some constant $q \in (1,\infty)$. One will see that under $[A1]$ in Section 6.1 and $[A2]$, Condition (1) is satisfied, e.g., if $\{\det(Cov_{\mathcal{C}}(Z_T/\sqrt{T}))^{-1}\}$ is L^R-bounded uniformly in large T for some large constant R. This is clear if one recalls the form of $\Psi_{T,p,\mathcal{C}}$ and the boundedness of moments of $Z_T/\sqrt{T}$. We should note that such a global nondegeneracy of $Z_T/\sqrt{T}$ is a problem completely different from the local nondegeneracy of the increments of Z.

It is sometimes convenient to consider the following weaker condition rather than (1):
$\limsup_{T\to\infty}\sup_{f\in\mathcal{E}'} \left\| |\Psi_{T,p,\mathcal{C}}|[|f|] \right\|_{L^q} < \infty$, where $\mathcal{E}'$ is a subclass of $\mathcal{E}(M,p_0)$. Under this condition, the coming results will be valid for $\mathcal{E}'$ in place of $\mathcal{E}(M,p_0)$.

For measurable function $f : \mathbf{R}^d \to \mathbf{R}$, $\epsilon > 0$ and a Borel measure ν on $\mathbf{R}^d$, define $\omega(f;\epsilon,\nu)$ by $\omega(f;\epsilon,\nu) = \int \omega_f(x;\epsilon)d\nu$, where $\omega_f(x;r) = \sup\{|f(x+h)-f(x)|; |h| \le r\}$. For a signed measure ν, ν^+ denotes the positive part of ν. Put $\omega_2(f;\epsilon,\nu) = \sqrt{\int_{\mathbf{R}^d}\omega_f(x;\epsilon)^2\nu(dx)}$. Denote $\Delta_T(f) := \left\| P_{\mathcal{C}}\left[f\left(\frac{1}{\sqrt{T}}Z_T\right)\right] - \Psi_{T,p,\mathcal{C}}[f] \right\|_1$. Here $r_T(f) = \bar{o}(T^{-a})$ means that $\sup_f r_T(f) = o(T^{-a})$.

We will show the following inequalities:

$$\begin{aligned}\Delta_T(f) \le M^*\Big[& P\left[\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right] < s_T I_d\right]^\theta + P\left[\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right] > u_T I_d\right]^\theta \\ & + u_T^{\gamma(1)} s_T^{-\gamma(2)} \omega_2\left(f; T^{-K}, \phi(x;0,u_T I_d)dx\right) + \bar{o}(T^{-(p-2+\delta^*)/2})\Big]\end{aligned} \tag{2}$$

and

$$\begin{aligned}\Delta_T(f) \le M^*\Big[& P\left[\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right] < s_T I_d\right]^\theta \\ & + u_T^{\gamma(1)} s_T^{-\gamma(2)} \omega_2\left(f; T^{-K}, \phi(x;0,u_T I_d)dx\right) + \bar{o}(T^{-(p-2+\delta^*)/2})\Big],\end{aligned} \tag{3}$$

where $\gamma(1) = [3(p-2)+d]/2$ and $\gamma(2) = 3(p-2)d + 2^{-1}d$. Let $I^\varrho = \{t \in \mathbf{R}_+;\ \mathrm{dist}(t,I) \le \varrho\}$ for $I \subset \mathbf{R}_+$ and $\varrho > 0$. In the following theorem, c' denotes a sufficiently small positive constant which will be implicitly specified in the proof.

Theorem 1. *Let $I(j)$ be dense reduction intervals with $(\hat{\mathcal{C}}(j))_{j=1}^{n'(T)}$. Suppose that there exists a positive constant ϱ such that $\hat{\mathcal{C}}(j) \subset \mathcal{B}_{I(j)^\varrho} \vee \mathcal{C}$. Suppose that* $[A3]$ *and* $[A2]$ *are satisfied. Let $K, M > 0$, $\theta \in (0,1)$, and let s_T, u_T be any sequences of positive numbers with* $\liminf_{T\to\infty} u_T > 0$. *Assume either of the following conditions:*

(a) $[A1']$ *is satisfied and $s_T \ge T^{-c'}$ for large T;*
(b) $[A1'']$ *is satisfied and $s_T \ge (\log T)^{-c'}$ for large T;*
(c) $[A1''']$ *is satisfied and* $\liminf_{T\to\infty} s_T > 0$.

Then there exist positive constants M^* and δ^* for which the inequality (2) holds uniformly in $f \in \mathcal{E}(M, p_0)$. Moreover, if $\liminf_{T\to\infty} u_T/T^{c''} > 0$ *for some positive constant c'', then (3) holds uniformly.*

The proof will be presented in Section 6. The quantity ω_2 can be replaced by the usual ω (cf. [29]) in the error term in the unconditional setting.

It is also possible to obtain the same results for $\Delta'_T(f) := \| P_{\mathcal{C}}[f(T^{-1/2}Z_T)] - \Psi_{T,p,\mathcal{C}}[f] 1_{\{\mathrm{Var}_{\mathcal{C}}[T^{-1/2}Z_T] \geq s_T I_d\}} \|_1$ in place of $\Delta_T(f)$ if Condition (1) is replaced by

$$\limsup_{T\to\infty} \left\| |\Psi_{T,p,\mathcal{C}}|[1 + |\cdot|^{p_0}] 1_{\{\mathrm{Var}_{\mathcal{C}}[T^{-1/2}Z_T] \geq s_T I_d\}} \right\|_q < \infty. \tag{4}$$

See Remark 13. This is also the case for $\mathcal{E}'$. Large deviation like techniques can be used to verify Condition (4): it is easier than verifying the integrability of $\Psi_{T,p,\mathcal{C}}$ without truncation. It is sometimes helpful to take advantage of $T^{-k/2}$-factors in the representation of $\Psi_{T,p,\mathcal{C}}$ to show the uniform boundedness of the norm. In case (4), we can use $P[\Psi_{T,p,\mathcal{C}}[f] 1_{\{\mathrm{Var}_{\mathcal{C}}[T^{-1/2}Z_T] \geq s_T I_d\}}]$ to make approximation to $P[f(T^{-1/2}Z_T)]$.

3. General conditional ϵ-Markov process

3.1. ϵ-Markov property and mixing coefficients

In this section, we consider a conditional ϵ-Markov process and its functional. Let X and Y be separable stochastic processes of d_1 and d_2 dimension, respectively, defined on a given probability space $(\Omega, \mathcal{F}, P)$. Here $d_1, d_2 \in \mathbf{N} \cup \{\infty\}$. Assume that $X_0 = 0$. Define sub σ-fields $\mathcal{B}_I^{dX}$ and $\mathcal{B}_I^Y$ of $\mathcal{F}$ for $I \subset \mathbf{R}_+$ by $\mathcal{B}_I^{dX} = \sigma[X_t - X_s; s, t \in I] \vee \mathcal{N}$ and $\mathcal{B}_I^Y = \sigma[Y_t; t \in I] \vee \mathcal{N}$, respectively, where $\mathcal{N}$ is the σ-field generated by null sets in $\mathcal{F}$. Moreover, let $\mathcal{B}_I = \mathcal{B}_I^{dX} \vee \mathcal{B}_I^Y$. As before, $\mathcal{C}$ denotes a sub σ-field of $\mathcal{F}$. We assume that X is a process with independent increments, more precisely, $\mathcal{B}_{[0,r]} \vee \mathcal{C}$ is independent of $\mathcal{B}_{[r,\infty)}^{dX}$ for all $r \in \mathbf{R}_+$. In particular, $\mathcal{C}$ is independent of $\mathcal{B}_{\mathbf{R}_+}^{dX}$. Furthermore, we assume that the process Y is a **$\mathcal{C}$-conditional ϵ-Markov process driven by** X in the sense that $Y_t \in \mathcal{F}\Big(\mathcal{B}_{[s-\epsilon,s]}^Y \vee \mathcal{B}_{[s,t]}^{dX} \vee \mathcal{C}\Big)$ for any $s, t \in \mathbf{R}_+$ with $\epsilon \leq s \leq t$.

Example 1. (Generalization of random coefficient models) Let ξ_j $(j \in \mathbf{Z}_+)$ be a sequence of independent random variables independent of $\mathcal{C} = \sigma[c]$, c being a random element possibly infinite-dimensional. Let $X_n = \sum_{j=1}^n \xi_j$. We consider a process Y_n defined by

$$Y_{n+1} = F_{n+1}(c, Y_{[n-M(c),n]}, \xi_{n+1})$$

for some measurable function F_{n+1}, where $M(c)$ is a $\mathcal{C}$-measurable $\mathbf{N}$-valued random variable, and $Y_{[n-m,n]} = (Y_{n-m}, Y_{n-m+1}, ..., Y_n)$. In this case, the ϵ-Markov property holds if $M(c)$ is uniformly bounded: $M(c) \leq M$. Processes X and Y are naturally extended over $\mathbf{R}_+$ as $X_{\nu(t)}$ and $Y_{\nu(t)}$, $\nu(t)$ being the maximum integer which is not greater than t.

Example 2. Assume that (X, Y) is independent of $\mathcal{C} = \sigma[c]$, and that Y is a Markov chain (i.e., $\epsilon = 0$) driven by X in Example 1: $Y_{n+1} = F_{n+1}(Y_n, \xi_{n+1})$. For $f(Y_{[n,\infty)}, c) \in b(\mathcal{B}^Y_{[n,\infty)} \vee \mathcal{C})$, there exists a measurable function H_n such that $H_n(Y_n, c) = P_{\mathcal{B}^Y_{[n]}\vee\mathcal{C}}[f]$, $[n] = [n, n]$. With Lemma 1, we obtain $P_{\mathcal{B}^Y_{[m]}\vee\mathcal{C}}[f] = P_{\mathcal{B}_{[0,m]}\vee\mathcal{C}}[f] = P_{\mathcal{B}_{[0,m]}\vee\mathcal{C}}[P_{\mathcal{B}^Y_{[n]}\vee\mathcal{C}}[f]] = P_{\mathcal{B}^Y_{[m]}\vee\mathcal{C}}[P_{\mathcal{B}^Y_{[n]}\vee\mathcal{C}}[f]] = P_{\mathcal{B}^Y_{[m]}\vee\mathcal{C}}[H_n(Y_n, c)] = (P_{n-m}[H_n(\cdot, c)])(Y_m)$ for $m < n$, where P_n is the semigroup of process Y (independent of $\mathcal{C}$). If Y is a stationary Markov chain with invariant distribution ν (independent of $\mathcal{C}$), $P_{\mathcal{C}}[f] = \nu[H_n(\cdot, c)]$ and

$$P_{\mathcal{C}}\left[\left|P_{\mathcal{B}^Y_{[m]}\vee\mathcal{C}}[f] - P_{\mathcal{C}}[f]\right|\right] = \|P_{n-m}[H_n(\cdot, c)] - \nu[H_n(\cdot, c)]\|_{L^1(\nu)}.$$

The right-hand side can often be estimated from above by $Be^{-a(m-n)}\|H_n\|_\infty$ with constants a and B independent of $\mathcal{C}$.

A conditional α-mixing condition for Y will be expressed by the inequality

$$P_{\mathcal{C}}\left[\left|P_{\mathcal{B}^Y_{[s-\epsilon,s]}\vee\mathcal{C}}[f] - P_{\mathcal{C}}[f]\right|\right] \le \tilde{\alpha}_Y(s, t|\mathcal{C})\|f\|_\infty \tag{5}$$

for $s \le t$ and $f \in b(\mathcal{B}^Y_{[t,\infty)} \vee \mathcal{C})$. We have:

Lemma 1. *Let Y be a $\mathcal{C}$-conditional ϵ-Markov process driven by a process X with independent increments. Then, for $f \in b(\mathcal{B}_{[t,\infty)} \vee \mathcal{C})$, $s \le t$,*

$$P_{\mathcal{B}_{[0,s]}\vee\mathcal{C}}[f] = P_{\mathcal{B}_{[s-\epsilon,s]}\vee\mathcal{C}}[f] = P_{\mathcal{B}^Y_{[s-\epsilon,s]}\vee\mathcal{C}}[f].$$

In particular,

$$|P_{\mathcal{C}}[ef] - P_{\mathcal{C}}[e]P_{\mathcal{C}}[f]| \le \tilde{\alpha}_Y(s, t-\epsilon|\mathcal{C})\|e\|_\infty\|f\|_\infty$$

for any $e \in b(\mathcal{B}_{[0,s]} \vee \mathcal{C})$ *and* $f \in b(\mathcal{B}_{[t,\infty)} \vee \mathcal{C})$*,* $s \le t - \epsilon$*.*

Proof. Denote $Y_I = (Y_t)_{t\in I}$. Suppose that $\mathcal{C}$ is generated by some random element c. Since $f \in b(\mathcal{B}_{[t,\infty)} \vee \mathcal{C})$, there exists a measurable function F such that $f = F(c, Y_{[s-\epsilon,s]}, dX_{[s,\infty)})$, where F depends on the at most countable data in Y_I, dX_I by separability, and we used the ϵ-Markov property of Y. The fact that X has independent increments implies

$$P_{\mathcal{B}_{[0,s]}\vee\mathcal{C}}[f] = \int F(c, Y_{[s-\epsilon,s]}, x)P^{dX_{[s,\infty)}}(dx) \in \mathcal{B}^Y_{[s-\epsilon,s]} \vee \mathcal{C}. \tag{6}$$

Note that (6) is easily obtained, by monotone class argument, without using random element c and separability. Taking conditional expectations $P_{\mathcal{B}^Y_{[s-\epsilon,s]}\vee\mathcal{C}}$ and $P_{\mathcal{B}_{[s-\epsilon,s]}\vee\mathcal{C}}$ for (6), we obtain the first assertion. For the second assertion, we see

$$\begin{aligned}|P_{\mathcal{C}}[ef] - P_{\mathcal{C}}[e]P_{\mathcal{C}}[f]| &= \left|P_{\mathcal{C}}\left[eP_{\mathcal{B}_{[0,s]}\vee\mathcal{C}}[f - P_{\mathcal{C}}[f]]\right]\right| \\ &\le \|e\|_\infty P_{\mathcal{C}}\left[\left|P_{\mathcal{B}_{[0,s]}\vee\mathcal{C}}[f - P_{\mathcal{C}}[f]]\right|\right],\end{aligned}$$

and by using the first part twice,

$$P_{\mathcal{B}_{[0,s]}\vee\mathcal{C}}\left[f - P_{\mathcal{C}}[f]\right] = P_{\mathcal{B}_{[0,s]}\vee\mathcal{C}}\left[P_{\mathcal{B}_{[0,t]}\vee\mathcal{C}}[f] - P_{\mathcal{C}}[f]\right]$$
$$= P_{\mathcal{B}^Y_{[s-\epsilon,s]}\vee\mathcal{C}}\left[P_{\mathcal{B}^Y_{[t-\epsilon,t]}\vee\mathcal{C}}[f] - P_{\mathcal{C}}[f]\right].$$

Take $P_{\mathcal{B}^Y_{[t-\epsilon,t]}\vee\mathcal{C}}[f]$ as f in (5), and we obtain the result. □

In view of Lemma 1 for indicator functions e and f, we will hereafter take $\alpha(s,t|\mathcal{C})$ as

$$\alpha(s,t|\mathcal{C}) = \begin{cases} \tilde{\alpha}_Y(s,t-\epsilon|\mathcal{C}) \wedge 1 & \text{if } s \le t-\epsilon, \\ \\ 1 & \text{if } s > t-\epsilon. \end{cases}$$

3.2. Reduction formula

Let $u, v \in \mathbf{R}_+, u \le v-\epsilon$. Let $F = [0,u], G = [u,v], H = [v,\infty), U = [u-\epsilon, u]$ and $V = [v-\epsilon, v]$. Assume that $f \in b(\mathcal{B}_F \vee \mathcal{C}), g \in b(\mathcal{B}_G \vee \mathcal{C})$ and $h \in b(\mathcal{B}_H \vee \mathcal{C})$. Moreover assume that for some sub σ-field $\mathcal{B}'_V$ of $\mathcal{B}_V$, $P_{\mathcal{B}_V\vee\mathcal{C}}[h] = P_{\mathcal{B}'_V\vee\mathcal{C}}[h]$ for all $h \in b(\mathcal{B}_H \vee \mathcal{C})$.

In particular, for any $c \in b\mathcal{C}, s \in b\mathcal{B}_U$ and $t \in b\mathcal{B}'_V$, $P[csth] = P[cst P_{\mathcal{B}_{[0,v]}\vee\mathcal{C}}[h]] = P[cst P_{\mathcal{B}'_V\vee\mathcal{C}}[h]]$, and hence, $P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[h] = P_{\mathcal{B}'_V\vee\mathcal{C}}[h]$ since finite sums of the terms taking the form of cst can approximate any bounded $\mathcal{B}_U \vee \mathcal{B}'_V \vee \mathcal{C}$-measurable functions in L^1-sense (if $\mathcal{B}'_V$ and $\mathcal{C}$ are good). In the same way, due to the ϵ-Markov property, $P[cstf] = P[csf P_{\mathcal{B}_U\vee\mathcal{C}}[t]] = P[f P_{\mathcal{B}_U\vee\mathcal{C}}[cst]] = P[cst P_{\mathcal{B}_U\vee\mathcal{C}}[f]]$; thus, $P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[f] = P_{\mathcal{B}_U\vee\mathcal{C}}[f]$.

The ϵ-Markov property implies that

$$P_{\mathcal{B}_U\vee\mathcal{C}}[gh] = P_{\mathcal{B}_U\vee\mathcal{C}}[g P_{\mathcal{B}_{[0,v]}\vee\mathcal{C}}[h]] = P_{\mathcal{B}_U\vee\mathcal{C}}[g P_{\mathcal{B}_V\vee\mathcal{C}}[h]]$$
$$= P_{\mathcal{B}_U\vee\mathcal{C}}[g P_{\mathcal{B}'_V\vee\mathcal{C}}[h]] = P_{\mathcal{B}_U\vee\mathcal{C}}[P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[g] P_{\mathcal{B}'_V\vee\mathcal{C}}[h]]. \quad (7)$$

For any $c \in b\mathcal{C}, s \in b\mathcal{B}_U$ and $t \in b\mathcal{B}'_V$,

$$\begin{aligned} P[cst P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[f] P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[h]] &= P[P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[sf]cth] \\ &= P[P_{\mathcal{B}_U\vee\mathcal{C}}[sf]cth] = P[sf P_{\mathcal{B}_U\vee\mathcal{C}}[cth]] \\ &= P[sf P_{\mathcal{B}_F\vee\mathcal{C}}[cth]] \ (\epsilon\text{-Markov property}) \\ &= P[cstfh]. \end{aligned}$$

Therefore, $P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[fh] = P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[f] P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[h] = P_{\mathcal{B}_U\vee\mathcal{C}}[f] P_{\mathcal{B}'_V\vee\mathcal{C}}[h]$.

Since $P_{\mathcal{B}_U\vee\mathcal{C}}[fgh] = P_{\mathcal{B}_U\vee\mathcal{C}}[f P_{\mathcal{B}_{[0,u]}\vee\mathcal{C}}[gh]] = P_{\mathcal{B}_U\vee\mathcal{C}}[f P_{\mathcal{B}_U\vee\mathcal{C}}[gh]]$ (ϵ-Markov property) $= P_{\mathcal{B}_U\vee\mathcal{C}}[f] P_{\mathcal{B}_U\vee\mathcal{C}}[gh]$,

$$\begin{aligned} P[cfgh] &= P[cP_{\mathcal{B}_U\vee\mathcal{C}}[fgh]] = P[cP_{\mathcal{B}_U\vee\mathcal{C}}[f] P_{\mathcal{B}_U\vee\mathcal{C}}[gh]] \\ &= P[cP_{\mathcal{B}_U\vee\mathcal{C}}[f] P_{\mathcal{B}_U\vee\mathcal{C}}[P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[g] P_{\mathcal{B}'_V\vee\mathcal{C}}[h]]] \ (\text{by } (7)) \\ &= P[cP_{\mathcal{B}_U\vee\mathcal{C}}[f] P_{\mathcal{B}'_V\vee\mathcal{C}}[h] P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[g]] \\ &= P[cP_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[fh] P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[g]] \\ &= P[cfh P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[g]] = P[cP_{\mathcal{C}}[fh P_{\mathcal{B}_U\vee\mathcal{B}'_V\vee\mathcal{C}}[g]]]. \end{aligned}$$

Thus, if $P_{\mathcal{B}_{[v-\epsilon,v]}\vee\mathcal{C}} = P_{\mathcal{B}'_{[v-\epsilon,v]}\vee\mathcal{C}}$ on $b(\mathcal{B}_{[v,\infty)} \vee \mathcal{C})$, then

$$P_{\mathcal{C}}[fgh] = P_{\mathcal{C}}[f P_{\mathcal{B}_{[u-\epsilon,u]}\vee\mathcal{B}'_{[v-\epsilon,v]}\vee\mathcal{C}}[g]h] \tag{8}$$

for any $f \in b(\mathcal{B}_{[0,u]} \vee \mathcal{C})$, $g \in b(\mathcal{B}_{[u,v]} \vee \mathcal{C})$, $h \in b(\mathcal{B}_{[v,\infty)} \vee \mathcal{C})$.

Let $(u(j), v(j))_{j\in\mathbf{N}}$ be a sequence of numbers satisfying that $\epsilon < u(j) \leq u(j) + \epsilon < v(j) \leq v(j) + \epsilon < u(j+1)$. We assume that there exist good sub σ-fields $\mathcal{B}'_{[v(j)-\epsilon,v(j)]}$ of $\mathcal{B}_{[v(j)-\epsilon,v(j)]}$ such that $P_{\mathcal{B}_{[v(j)-\epsilon,v(j)]}\vee\mathcal{C}} = P_{\mathcal{B}'_{[v(j)-\epsilon,v(j)]}\vee\mathcal{C}}$ on $b(\mathcal{B}_{[v(j),\infty)} \vee \mathcal{C})$. Suppose that $K_j \in b(\mathcal{B}_{[v(j-1),u(j)]} \vee \mathcal{C})$, $v(0) := 0$, and $L_j \in b(\mathcal{B}_{[u(j),v(j)]} \vee \mathcal{C})$. Then, by repeated use of (8), one obtains

$$\begin{aligned}
P_{\mathcal{C}}[K_1L_1\cdots K_nL_nK_{n+1}] &= P_{\mathcal{C}}[K_1L_1\cdots K_{n-1}L_{n-1}K_n \\
&\quad \times P_{\mathcal{B}_{[u(n)-\epsilon,u(n)]}\vee\mathcal{B}'_{[v(n)-\epsilon,v(n)]}\vee\mathcal{C}}[L_n]K_{n+1}] \\
&= P_{\mathcal{C}}[K_1L_1\cdots K_{n-1}P_{\mathcal{B}_{[u(n-1)-\epsilon,u(n-1)]}\vee\mathcal{B}'_{[v(n-1)-\epsilon,v(n-1)]}\vee\mathcal{C}} \\
&\quad [L_{n-1}]K_n P_{\mathcal{B}_{[u(n)-\epsilon,u(n)]}\vee\mathcal{B}'_{[v(n)-\epsilon,v(n)]}\vee\mathcal{C}}[L_n]K_{n+1}] \\
&= P_{\mathcal{C}}[(\Pi_{j=1}^{n+1}K_j)(\Pi_{j=1}^{n}P_{\mathcal{B}_{[u(j)-\epsilon,u(j)]}\vee\mathcal{B}'_{[v(j)-\epsilon,v(j)]}\vee\mathcal{C}} \\
&\quad [L_j])].
\end{aligned} \tag{9}$$

3.3. *Asymptotic expansions*

As before, Y denotes a $\mathcal{C}$-conditional ϵ-Markov process driven by process X with independent increments. We consider a process Z which is adapted to $\mathcal{B}_I$ (generated by Y and dX), i.e., $Z_t^s = Z_t - Z_s \in \mathcal{F}\left(\mathcal{B}_{[s,t]} \vee \mathcal{C}\right)$ for any $s, t \in \mathbf{R}_+$ $(s \leq t)$. The $\mathcal{C}$-conditional mixing coefficient $\alpha(s, t|\mathcal{C})$ and hence $\alpha(h|\mathcal{C})$ are now defined in terms of $\tilde{\alpha}(s, t|\mathcal{C})$ for $\mathcal{B}_I^Y$. If there exists an interval sequence $I(j) = [u(j), v(j)]$ $(j = 1, .., n(T))$ for which sub σ-fields $\mathcal{B}'_{[v(j)-\epsilon,v(j)]}$ of $\mathcal{B}_{[v(j)-\epsilon,v(j)]}$ have the property mentioned in Section 3.2, and if $\liminf_{T\to\infty} n(T)/T > 0$ and $0 < \delta \leq v(j) - u(j) \leq \bar{\delta} < \infty$, then $(I(j))$ forms a set of dense reduction intervals with $\hat{\mathcal{C}}(j) = \mathcal{B}_{[u(j)-\epsilon,u(j)]} \vee \mathcal{B}'_{[v(j)-\epsilon,v(j)]} \vee \mathcal{C}$.

If α (for $\tilde{\alpha}_Y(s, t|\mathcal{C})$) satisfies Condition $[A1']$, $[A1'']$ or $[A1''']$, and if those reduction intervals have the properties in $[A3]$, then under moment condition $[A2]$, we obtain asymptotic expansions (2) and (3) for $Z_T/\sqrt{T}$ as Theorem 1.

The truncation functional ψ_j may depend on the path. All what we have to do is to show the existence of a subset in path space on which the variables $Z_{I(j)}$ and the variables that determine $\hat{\mathcal{C}}(j)$ have a locally nondegenerate, regular distribution. For this purpose, the Malliavin calculus will be applied later. However, we should notice that there are many other possibilities of obtaining the conditional type Cramér condition for specific simple models. For instance, if the model in question has a discrete time parameter, we may not need such an infinite-dimensional method unless an infinite-dimensional structure is hiding behind the randomness of the model. In order to show the positivity of the remaining set after truncation, the support theorem can apply as in [70].

We will return to conditional ϵ-Markov processes in Section 4.2 in the light of the Malliavin calculus.

4. Malliavin calculus and asymptotic expansion

4.1. A general model equipped with the Malliavin operator that does not shift the conditioning variable

In order to verify the conditional type Cramér condition such as $[A3]$, we will apply the Malliavin calculus. We refer the reader to [69] or Kusuoka and Yoshida [29] for necessary notation in the jump type Malliavin calculus.

We first consider the general model described in Section 2 in terms of $\mathcal{B}_I$. Suppose that $I(j) = [u(j), v(j)]$ $(j = 1, ..., n(T))$ are dense reduction intervals with $(\hat{\mathcal{C}}(j))_{j=1}^{n(T)}$. Assume that $\hat{\mathcal{C}}(j)$ is generated by a random variable C_j taking values in a measurable space $S(j)$ with σ-field $\mathcal{S}(j)$, and that $\mathcal{C}$ is generated by a random variable C taking values in a measurable space S' with σ-field $\mathcal{S}'$. Moreover, we assume that for every j, there exists a **distributional equivalent** $(\hat{\psi}_j, \hat{Z}_j, \hat{C}_j, \hat{C})$ of $(\psi_j, Z_{I(j)}, C_j, C)$, where ψ_j is a truncation functional given in $[A3^M]$ below. In other words, there exists a probability space $(\hat{\Omega}(j), \hat{\mathcal{B}}(j), \hat{P}(j))$ which has a random variable $(\hat{\psi}_j, \hat{Z}_j, \hat{C}_j, \hat{C})$ taking values in $([0, 1] \times \mathbf{R}^d \times S(j) \times S', \mathbf{B}([0, 1]) \otimes \mathbf{B}_d \otimes \mathcal{S}(j) \otimes \mathcal{S}')$, and $\mathcal{L}\{(\hat{\psi}_j, \hat{Z}_j, \hat{C}_j, \hat{C})|\hat{P}(j)\} = \mathcal{L}\{(\psi_j, Z_{I(j)}, C_j, C)|P\}$.

Suppose that for every j, a Malliavin operator $(L_j, \mathcal{D}(L_j))$ in the sense of Bichteler et al. [9] p.102 exists on $\hat{\Omega}(j)$. The operator $(L_j, \mathcal{D}(L_j))$ has an extension to the Banach space $D_{2,p}^{L_j}$. We use the same letter L_j for this extension. In this subsection, we assume that for any bounded measurable function $F : S(j) \to \mathbf{R}$, $F(\hat{C}_j) \in D_{2,p}^{L_j}$ for any $p \geq 2$, and $L_j F(\hat{C}_j) = 0$.

Let $\sigma_{\hat{Z}_j}^{kl} = \Gamma_{L_j}(\hat{Z}_{j,k}, \hat{Z}_{j,l})$, where Γ_{L_j} is the bilinear form corresponding to L_j, and also let $\Delta_{\hat{Z}_j} = \det \sigma_{\hat{Z}_j}$ for $\sigma_{\hat{Z}_j} = (\sigma_{\hat{Z}_j}^{kl})$. Set $S_{1,j} = \{\Delta_{\hat{Z}_j}^{-1}\hat{\psi}_j, \sigma_{\hat{Z}_j}^{kl}, L_j\hat{Z}_{j,k}, \Gamma_{L_j}(\sigma_{\hat{Z}_j}^{kl}, \hat{Z}_{j,m}), \Gamma_{L_j}(\Delta_{\hat{Z}_j}^{-1}\hat{\psi}_j, \hat{Z}_{j,l})\}$ for operator L_j. We have an integration-by-parts formula:

$$\hat{P}(j)\left[\partial_i f(\hat{Z})\hat{\psi}|\sigma(\hat{C}_j)\right] = \hat{P}(j)\left[f(\hat{Z})\Psi_i^{\hat{Z}}(\hat{\psi})|\sigma(\hat{C}_j)\right]$$

for $f \in C_{\uparrow}^{\infty}(\mathbf{R}^d)$, and $\hat{Z}, \hat{\psi} \in D_{2,\infty-}^{L_j}$ [5], under the nondegeneracy for $\hat{Z}$, i.e., $S_1[\hat{\psi}; \hat{Z}] \subset D_{2,\infty-}^{L_j}$. We know that

$$\Psi_i^{\hat{Z}}(\hat{\psi}) = -\sum_{i'=1}^{d}\{2\gamma_{\hat{Z}}^{ii'}\hat{\psi}L_j\hat{Z}_{i'} + \Gamma_{L_j}(\gamma_{\hat{Z}}^{ii'}\hat{\psi}, \hat{Z}_{i'})\},$$

$\gamma_{\hat{Z}}$ being the inverse matrix of $\sigma_{\hat{Z}}$. It is easy to verify the above integration-by-parts formula with the fact that the shift of L_j has no effect on $\hat{C}_j$. [6]

The following condition is usually verified with boundedness of L^p-norms of $\{S_{1,j}\}$ and a kind of ergodicity of the conditioning stochastic process generating $\mathcal{C}$.

[5] We omit $(\mathbf{R}^d)$ even for multi-dimensional functionals.

[6] It is not the case with the Malliavin operator in Section 4.2.

[A3^M] There exist truncation functionals $\psi_j : (\Omega, \mathcal{F}) \to ([0,1], \mathbf{B}([0,1]))$ satisfying the following conditions:

(i) For any $L > 0$, there exists $A > 0$ such that

$$P\left[\frac{1}{n(T)}\sum_{j=1}^{n(T)} \hat{P}(j)\left[\left|\Psi^{\hat{Z}_j}(\hat{\psi}_j)\right| |\hat{C} = C\right] \geq A\right] = o\left(\frac{1}{T^L}\right).$$

(ii) For any $L > 0$, there exists $a \in (0,1)$ such that

$$P\left[\frac{1}{n(T)}\sum_{j=1}^{n(T)} P_{\mathcal{C}}\left[1 - \psi_j\right] \geq a\right] = o\left(\frac{1}{T^L}\right).$$

In the above conditions, "any L" can be replaced by "sufficiently large L", as shown in the proof.

We then easily obtain the following theorem.

Theorem 2. *The same assertions as in Theorem 1 are valid under the same assumptions even if* [A3] *is replaced by* [A3^M].

Remark 1. Here Condition [$A3^M$] tacitly implies that $\hat{Z}_j, \hat{\psi}_j \in D^{L_j}_{2,\infty-}$ and $S_1[\hat{\psi}_j; \hat{Z}_j] \subset D^{L_j}_{2,\infty-}$. The Malliavin operator L_j does not shift $\hat{\mathcal{C}}(j)$-measurable functionals, intuitively. This theorem applies to m-dependent sequences. Moreover it could apply to ϵ-Markov processes (Section 4.2) though the raw Malliavin operator given there may shift $\hat{\mathcal{C}}(j)$-measurable functionals in general. As a matter of fact, it is possible to construct another Malliavin operator L_j having the above property from the raw Malliavin operator. This is what we will implicitly carry out in Section 4.2.

4.2. Conditional ϵ-Markov process

Let us return to the càdlàg conditional ϵ-Markov process Y driven by a càdlàg process X with independent increments (see Section 3). As before, we shall consider a sequence of intervals $I(j) = [u(j), v(j)]$ $(j = 1, ..., n(T))$ such that $u(j)+\epsilon < v(j) \leq u(j+1)-\epsilon$. Put $I_j = [u(j)-\epsilon, u(j)]$ and $J_j = [v(j)-\epsilon, v(j)]$. Let $M \in \mathbf{N}$. Suppose that for each j, for some $M_j \leq M$, there is an M_j-dimensional random variable $\mathcal{Y}_j$ which satisfies:

(1) $\mathcal{B}'_{J_j} := \sigma[\mathcal{Y}_j] \subset \mathcal{B}_{J_j} \vee \mathcal{C}$;
(2) $P_{\mathcal{B}_{[0,v(j)]}\vee\mathcal{C}} = P_{\mathcal{B}'_{J_j}\vee\mathcal{C}}$ on $b\left(\mathcal{B}_{[v(j),\infty)} \vee \mathcal{C}\right)$.

Put $\mathcal{Z}_j = (Z_{I(j)}, \mathcal{Y}_j)$. Let $\hat{\mathcal{C}}(j) := \mathcal{B}_{I_j} \vee \mathcal{B}'_{J_j} \vee \mathcal{C}$. We assume that $\liminf_{T\to\infty} n(T)/T > 0$, and then find that $I(j) = [u(j), v(j)]$ $(j = 1, ..., n(T))$ form dense reduction intervals with $\hat{\mathcal{C}}(j)$. Let $C_j = ((Y_t, X_t - X_{u(j)-\epsilon} : t \in I_j), \mathcal{Y}_j, C)$, where C is a measurable space $(S', \mathcal{S}')$-valued random variable such that $\mathcal{C} = \sigma[C]$.

Now, we further assume that a *distributional equivalent* $(\hat{\psi}_j, \hat{Z}_j, \hat{C}_j)$ of $(\psi_j, Z_{I(j)}, C_j)$ exists: there exists a probability space $(\hat{\Omega}(j), \hat{\mathcal{B}}(j), \hat{P}(j))$ on which there is a random variable $(\hat{\psi}_j, \hat{Z}_j, \hat{C}_j)$ taking values in $([0,1] \times \mathbf{R}^d \times S(j), \mathbf{B}([0,1]) \otimes \mathbf{B}_d \otimes \mathcal{S}(j))$, where $S(j)$ is the value space of C_j and $\mathcal{S}(j)$ is its σ-field, and

$$\mathcal{L}\{(\hat{\psi}_j, \hat{Z}_j, \hat{C}_j)|\hat{P}(j)\} = \mathcal{L}\{(\psi_j, Z_{I(j)}, C_j)|P\}. \tag{10}$$

C_j and $\hat{C}_j$ are random variables taking values in the product space $S(j) = \mathbf{D}(I_j; \mathbf{R}^{d_1+d_2}) \times \mathbf{R}^{M_j} \times S'$ equipped with the product σ-field $\mathcal{S}(j) = \mathcal{D}(I_j; \mathbf{R}^{d_1+d_2}) \otimes \mathbf{B}_{M_j} \otimes \mathcal{S}'$.

Assume that $(L_j, \mathcal{D}(L_j))$ is a Malliavin operator over $(\hat{\Omega}(j), \hat{\mathcal{B}}(j), \hat{P}(j))$, and that for any bounded measurable function F, $F(\pi\hat{C}_j) \in D^{L_j}_{2,\infty-}$ and $L_j F(\pi\hat{C}_j) = 0$, where $\pi\hat{C}_j$ stands for any finite projection of $((\hat{Y}_t, \hat{X}_t - \hat{X}_{u(j)-\epsilon} : t \in I_j), \hat{C})$ (a part of $\hat{C}_j$ without $\hat{\mathcal{Y}}_j$, and not whole $\hat{C}_j$, where $\hat{\mathcal{Y}}_j$ is a distributional equivalent to $\mathcal{Y}_j$). Here $\hat{C}$ denotes the distributional equivalent part in $\hat{C}_j$ corresponding to C.

Let $\hat{\mathcal{Z}}_j = (\hat{Z}_j, \hat{\mathcal{Y}}_j)$. Suppose that $\hat{\mathcal{Z}}_j \subset D^{L_j}_{2,\infty-}$. Put $S^*_1[\hat{\psi}_j; \hat{\mathcal{Z}}_j] = \{\sigma^{pq}_{\hat{\mathcal{Z}}_j}, \Delta^{-1}_{\hat{\mathcal{Z}}_j} \Delta^{-(d-1)}_{\hat{\mathcal{Y}}_j} \hat{\psi}_j\}$, where $\Delta_{\hat{\mathcal{Z}}_j} = \det \sigma_{\hat{\mathcal{Z}}_j}$ and $\Delta_{\hat{\mathcal{Y}}_j} = \det \sigma_{\hat{\mathcal{Y}}_j}$. We assume that

$$S^*_1[\hat{\psi}_j; \hat{\mathcal{Z}}_j] \subset D^{L_j}_{2,\infty-}. \tag{11}$$

The matrix $(\gamma^{mn}_{\hat{\mathcal{Y}}_j})$ denotes the inverse matrix of $\sigma_{\hat{\mathcal{Y}}_j}$. Assume that $\phi \in D^{L_j}_{2,\infty-}$ and $\phi \det \sigma^{-1}_{\hat{\mathcal{Y}}_j} \in D^{L_j}_{2,\infty-}$; then $\phi\gamma^{mn}_{\hat{\mathcal{Y}}_j}$ makes sense. It follows from the chain rule for the Γ-bilinear form that

$$\begin{aligned}
&\sum_{m,n=1}^{M_j} \phi\Gamma_{L_j}(\hat{Z}_{j,l}, \hat{\mathcal{Y}}_{j,m})\gamma^{mn}_{\hat{\mathcal{Y}}_j}\Gamma_{L_j}(\hat{\mathcal{Y}}_{j,n}, F)\\
&\quad= \sum_{m,n=1}^{M_j}\sum_{m'} \phi\Gamma_{L_j}(\hat{Z}_{j,l}, \hat{\mathcal{Y}}_{j,m})\gamma^{mn}_{\hat{\mathcal{Y}}_j}\Gamma_{L_j}(\hat{\mathcal{Y}}_{j,n}, \hat{\mathcal{Y}}_{j,m'})\partial_{m'} g(\hat{\mathcal{Y}}_j)H\\
&\quad= \sum_{m=1}^{M_j} \phi\Gamma_{L_j}(\hat{Z}_{j,l}, \hat{\mathcal{Y}}_{j,m})\partial_m g(\hat{\mathcal{Y}}_j)H\\
&\quad= \phi\Gamma_{L_j}(\hat{Z}_{j,l}, g(\hat{\mathcal{Y}}_j))H = \phi\Gamma_{L_j}(\hat{Z}_{j,l}, F)
\end{aligned} \tag{12}$$

for functionals F taking the form of $F = g(\hat{\mathcal{Y}}_j)H$ for $g \in C^\infty_B(\mathbf{R}^{M_j})$[7] and $H = h_1(\hat{X}_{u_k} - \hat{X}_{u_{k-1}}, \hat{Y}_{u_k} : 1 \le k \le m_1)h_2(\hat{C})$, with $h_1 \in b\mathbf{B}_{(d_1+d_2)m_1}$, $h_2 \in b\mathcal{S}'(j)$ and $u(j) - \epsilon \le u_0 \le u_1 \le \cdots \le u_{m_1} \le u(j)$. In order to distinguish the

[7] C^∞_B denotes the space of all bounded smooth functions with bounded derivatives.

probability measure for the distributional equivalents from the original probability measure P, we shall here use the bold letter $\mathbf{P}$ for the equivalent probability measure $\hat{P}(j)$ on the space on which the distributional equivalents are defined. The integration-by-parts formula yields

$$\sum_p iu_p \mathbf{P}[e^{iu\cdot\hat{Z}_j}\sigma^{pq}_{\hat{Z}_j}\phi F] = \mathbf{P}[e^{iu\cdot\hat{Z}_j}\{-2\phi F L_j \hat{Z}_{j,q} - \Gamma_{L_j}(\phi F, \hat{Z}_{j,q})\}]. \quad (13)$$

By using the formulas: $\Gamma_{L_j}(\mathbf{A},\mathbf{BC}) = \Gamma_{L_j}(\mathbf{A},\mathbf{B})\mathbf{C} + \Gamma_{L_j}(\mathbf{A},\mathbf{C})\mathbf{B}$, $\mathbf{P}[\mathbf{A}L_j\mathbf{B}] = \mathbf{P}[\mathbf{B}L_j\mathbf{A}](= -\mathbf{P}[\Gamma_{L_j}(\mathbf{A},\mathbf{B})]/2)$ for $\mathbf{A},\mathbf{B},\mathbf{C} \in D^{L_j}_{2,\infty-}$ and $L_j 1 = 0$, we see that

$$\begin{aligned}\mathbf{P}[\mathbf{A}\Gamma_{L_j}(\mathbf{B},\mathbf{C})] &= \mathbf{P}[\Gamma_{L_j}(\mathbf{AC},\mathbf{B}) - \Gamma_{L_j}(\mathbf{A},\mathbf{B})\mathbf{C}]\\ &= \mathbf{P}[L_j(\mathbf{ACB}) - \mathbf{AC}L_j\mathbf{B} - \mathbf{B}L_j(\mathbf{AC}) - \Gamma_{L_j}(\mathbf{A},\mathbf{B})\mathbf{C}]\\ &= \mathbf{P}[-\mathbf{AC}L_j\mathbf{B} - \mathbf{AC}L_j\mathbf{B} - \Gamma_{L_j}(\mathbf{A},\mathbf{B})\mathbf{C}]\\ &= \mathbf{P}[\{-\Gamma_{L_j}(\mathbf{A},\mathbf{B}) - 2\mathbf{A}L_j\mathbf{B}\}\mathbf{C}].\end{aligned}$$

Applying this formula to $\mathbf{A} = \gamma^{kl}_{\hat{\mathfrak{Y}}_j}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,l}, \hat{Z}_{j,q})\phi F$, $\mathbf{B} = \hat{\mathfrak{Y}}_{j,k}$ and $\mathbf{C} = e^{iu\cdot\hat{Z}_j}$, we obtain

$$\begin{aligned}&\sum_p iu_p \mathbf{P}[e^{iu\cdot\hat{Z}_j}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,k}, \hat{Z}_{j,p})\gamma^{kl}_{\hat{\mathfrak{Y}}_j}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,l}, \hat{Z}_{j,q})\phi F]\\ &\quad = \mathbf{P}[\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,k}, e^{iu\cdot\hat{Z}_j})\gamma^{kl}_{\hat{\mathfrak{Y}}_j}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,l}, \hat{Z}_{j,q})\phi F]\\ &\quad = \mathbf{P}[e^{iu\cdot\hat{Z}_j}\{-\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,k}, \gamma^{kl}_{\hat{\mathfrak{Y}}_j}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,l}, \hat{Z}_{j,q})\phi F)\\ &\qquad -2\gamma^{kl}_{\hat{\mathfrak{Y}}_j}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,l}, \hat{Z}_{j,q})\phi F L_j\hat{\mathfrak{Y}}_{j,k}\}]. \end{aligned} \quad (14)$$

On the event $\{\phi \neq 0\}$, define $\bar{\sigma}_{\hat{Z}_j}$ by

$$\bar{\sigma}^{pq}_{\hat{Z}_j} = \sigma^{pq}_{\hat{Z}_j} - \sum_{k,l}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,k}, \hat{Z}_{j,p})\gamma^{kl}_{\hat{\mathfrak{Y}}_j}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,l}, \hat{Z}_{j,q}).$$

The partial covariance matrix $\bar{\sigma}_{\hat{Z}_j} = (\bar{\sigma}^{pq}_{\hat{Z}_j})$ is nonnegative definite, and $\phi\,\bar{\sigma}^{pq}_{\hat{Z}_j}$ is well defined, indeed, in $D^{L_j}_{2,\infty-}$. It follows from (13), (14) and (12) that

$$\sum_p iu_p \mathbf{P}[e^{iu\cdot\hat{Z}_j}\bar{\sigma}^{pq}_{\hat{Z}_j}\phi F] = \mathbf{P}[e^{iu\cdot\hat{Z}_j}\Psi^q_j(\phi)F], \quad (15)$$

where

$$\begin{aligned}\Psi^q_j(\phi) &= \sum_{k,l}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,k}, \gamma^{kl}_{\hat{\mathfrak{Y}}_j}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,l}, \hat{Z}_{j,q})\phi) - \Gamma_{L_j}(\phi, \hat{Z}_{j,q})\\ &\quad -2\phi L_j\hat{Z}_{j,q} + 2\sum_{k,l}\gamma^{kl}_{\hat{\mathfrak{Y}}_j}\Gamma_{L_j}(\hat{\mathfrak{Y}}_{j,l}, \hat{Z}_{j,q})\phi L_j\hat{\mathfrak{Y}}_{j,k}.\end{aligned}$$

Put $\phi' = (\det \bar{\sigma}_{\hat{Z}_j})^{-1}\bar{\sigma}_{j,[q,s]}\hat{\psi}_j$, where $\bar{\sigma}_{j,[q,s]}$ is the (q,s)-cofactor of $\bar{\sigma}_{\hat{Z}_j}$. Then

$$\phi' = (\det \bar{\sigma}_{\hat{Z}_j})^{-1}(\det \sigma_{\hat{\mathcal{Y}}_j})^{-(d-1)}\hat{\psi}_j\,\Xi$$

for some functional $\Xi \in D^{L_j}_{2,\infty-}$. By Assumption (11), $(\det \sigma_{\hat{\mathcal{Y}}_j})^{-1}\phi' \in D^{L_j}_{2,\infty-}$ since $\det \sigma^{-1}_{\hat{\mathcal{Y}}_j} \cdot \det \bar{\sigma}^{-1}_{\hat{Z}_j} = \det \sigma^{-1}_{\hat{\mathcal{Z}}_j}$. In particular, $\phi' \in D^{L_j}_{2,\infty-}$.

Substituting ϕ' into ϕ of (15), and summing up, we obtain

$$iu_p\mathbf{P}[e^{iu\cdot\hat{Z}_j}\hat{\psi}_jF] = \mathbf{P}[e^{iu\cdot\hat{Z}_j}\Psi_p^{\hat{Z}_j}(\hat{\psi}_j)F], \tag{16}$$

where

$$\Psi_p^{\hat{Z}_j}(\hat{\psi}_j) = \sum_q \Psi_j^q((\det \bar{\sigma}_{\hat{Z}_j})^{-1}\bar{\sigma}_{j,[q,p]}\hat{\psi}_j).$$

We see that (16) holds for any $F = 1_B(\hat{C}_j)$ $(B \in \mathcal{S}(j))$ by the monotone class theorem. Consequently,

$$iu_p\mathbf{P}[e^{iu\cdot\hat{Z}_j}\hat{\psi}_j|\hat{C}_j = t] = \mathbf{P}[e^{iu\cdot\hat{Z}_j}\Psi_p^{\hat{Z}_j}(\hat{\psi}_j)|\hat{C}_j = t]\ \mathbf{P}^{\hat{C}_j} - \text{a.s. } t. \tag{17}$$

The distributional equivalence yields that for any bounded measurable functions ζ and η,

$$\begin{aligned} P_{\hat{C}(j)}[\zeta(Z_{I(j)},\psi_j)] &= \mathbf{P}[\zeta(\hat{Z}_j,\hat{\psi}_j)|\hat{C}_j = C_j] \\ \text{and } P_{\mathcal{C}}\left[\eta(C_j)\right] &= \mathbf{P}\left[\eta(\hat{C}_j)|\hat{C} = C\right]. \end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$\begin{aligned} P_{\mathcal{C}}\left[\sup_{u:|u|\geq B}\left|P_{\hat{C}(j)}[e^{iu\cdot Z_{I(j)}}\psi_j]\right|\right] &= \mathbf{P}_{\sigma[\hat{C}]}\left[\sup_{u:|u|\geq B}\left|\mathbf{P}_{\sigma[\hat{C}_j]}\left[e^{iu\cdot\hat{Z}_j}\hat{\psi}_j\right]\right|\right]|_{\hat{C}=C} \\ &\leq \mathbf{P}_{\sigma[\hat{C}]}\Bigg[\sum_{p=1}^d \sup_{u:|u_p|\geq B/d}\Bigg|(iu_p)^{-1} \\ &\qquad \times\mathbf{P}_{\sigma[\hat{C}_j]}\left[e^{iu\cdot\hat{Z}_j}\Psi_p^{\hat{Z}_j}(\hat{\psi}_j)\right]\Bigg|\Bigg]|_{\hat{C}=C} \\ &\leq \frac{d}{B}\sum_{p=1}^d \mathbf{P}_{\sigma[\hat{C}]}\left[\left|\Psi_p^{\hat{Z}_j}(\hat{\psi}_j)\right|\right]|_{\hat{C}=C}. \end{aligned}$$

Thus, from Theorem 1, we have obtained:

Theorem 3. *Let Y be a $\mathcal{C}$-conditional ϵ-Markov process driven by a process X with independent increments. Assume that* $\liminf_{T\to\infty} n(T)/T > 0$. *Then Inequalities (2) and (3) hold under* $[A2]$ *and* $[A3^M]$ *in place of* $[A3]$.

Here "Inequalities (2) and (3) hold" means that "Let $K, M > 0, \theta \in (0,1), ...,$ then (3) holds uniformly" (in Theorem 1).

Remark 2. The above Malliavin operator L_j may shift $\hat{\mathcal{C}}(j)$-measurable random variable, while the Malliavin operator for Theorem 2 does not. Condition $[A3^M]$ here is implicitly assuming that $\hat{\mathcal{Z}}_j \in D^{L_j}_{2,\infty-}$ and $S_1^*[\hat{\psi}_j; \hat{\mathcal{Z}}_j] \subset D^{L_j}_{2,\infty-}$. The condition $[A3^M]$ in Theorem 3 is formally the same as that in Theorem 2. However, note that the same notation $\Psi^{\hat{Z}_j}(\hat{\psi}_j)$ is used in different senses in Theorem 2 and in Theorem 3. In Theorem 3, we just assumed that $(I(j))$ is a sequence of intervals mentioned above, and we did not assume that it is accompanied with certain reduction σ-fields $(\hat{\mathcal{C}}(j))$. That is, the reduction property is automatically satisfied by the ϵ-Markov property.

Let

$$\begin{aligned} S_{1,j} = \{&(\Delta_{\hat{\mathcal{Z}}_j})^{-1}(\Delta_{\hat{\mathcal{Y}}_j})^{-(d-1)}\hat{\psi}_j, \sigma^{kl}_{\hat{\mathcal{Z}}_j}, L_j\hat{\mathcal{Z}}_{j,k}, \Gamma_{L_j}(\sigma^{kl}_{\hat{\mathcal{Z}}_j}, \hat{\mathcal{Z}}_{j,m}),\\ &\Gamma_{L_j}((\Delta_{\hat{\mathcal{Z}}_j})^{-1}(\Delta_{\hat{\mathcal{Y}}_j})^{-(d-1)}\hat{\psi}_j, \hat{\mathcal{Z}}_{j,l})\}. \end{aligned}$$

Similarly as in [29], Condition $[A3^M]$(i) can be reduced to the L^p-boundedness of $\{S_{1,j}\}$ and a property such as ergodicity of C if it is a stochastic process.

Remark 3. For the distributional equivalent $(\hat{\psi}_j, \hat{Z}_j, \hat{C}_j)$, we may assume the equivalence $\mathcal{L}\{(\hat{\psi}_j, 1_{\{\hat{\psi}_j>0\}}\hat{Z}_j, \hat{C}_j)|\hat{P}(j)\} = \mathcal{L}\{(\psi_j, 1_{\{\psi_j>0\}}Z_{I(j)}, C_j)|P\}$ instead of (10). Sometimes we meet a stochastic differential equation whose coefficient has a singularity. In such a case, it is not possible to make a differentiable $\hat{Z}_j$ the distribution of which equals to that of Z_j. However, we can make $\hat{Z}_j$ so that its distribution on the event $\{\hat{\psi}_j > 0\}$ coincides with that of Z_j on $\{\psi_j > 0\}$. This truncation enables us to avoid the singularity. Masuda and Yoshida [34] took this route to derive an expansion for a stochastic volatility model having an Ornstein-Uhlenbeck process driven by a Lévy process as a volatility process.

5. Stochastic differential equations

We will apply our results to stochastic differential equations. For simplicity, we shall start with unconditional expansions.

5.1. Stochastic differential equation with jumps

For each γ ($\gamma = 1, ..., m$), let E_γ be an open set in $\mathbf{R}^{b_\gamma}$ equipped with the Borel σ-field, and let $(E_0, \mathcal{E}_0)$ be another measurable space. Let us consider a stochastic process $(Y_t, Z_t)_{t\in\mathbf{R}_+}$ which satisfies the following stochastic integral equations:

$$\begin{aligned} Y_t = Y_0 + \int_0^t A(Y_{s-})ds + \sum_{\beta=1}^r \int_0^t B_\beta(Y_{s-})dw_s^\beta\\ + \sum_{\gamma=0}^m \int_{[0,t]\times E_\gamma} C_\gamma(Y_{s-}, v^\gamma)\tilde{\mu}^\gamma(ds, dv^\gamma) \end{aligned} \tag{19}$$

$$Z_t = Z_0 + \int_0^t A'(Y_{s-})ds + \sum_{\beta=1}^r \int_0^t B'_\beta(Y_{s-})dw_s^\beta$$
$$+ \sum_{\gamma=0}^m \int_{[0,t]\times E_\gamma} C'_\gamma(Y_{s-}, v^\gamma)\tilde{\mu}^\gamma(ds, dv^\gamma), \tag{20}$$

where Z_0 is $\sigma[Y_0]$-measurable, $A \in \mathcal{F}(\mathbf{R}^{d_2}; \mathbf{R}^{d_2})$ (i.e., $\mathbf{R}^{d_2}$-valued measurable functions), $B_\beta \in \mathcal{F}(\mathbf{R}^{d_2}; \mathbf{R}^{d_2})$, $C_\gamma \in \mathcal{F}(\mathbf{R}^{d_2} \times E_\gamma; \mathbf{R}^{d_2})$, and similarly, $A' \in \mathcal{F}(\mathbf{R}^{d_2}; \mathbf{R}^d)$, $B'_\beta \in \mathcal{F}(\mathbf{R}^{d_2}; \mathbf{R}^d)$, $C'_\gamma \in \mathcal{F}(\mathbf{R}^{d_2} \times E_\gamma; \mathbf{R}^d)$. Each w^β is a one-dimensional Wiener process, and each $\tilde{\mu}^\gamma$ is a compensated Poisson random measure on $\mathbf{R}_+ \times E_\gamma$: $\tilde{\mu}^\gamma = \mu^\gamma - \nu^\gamma$, where μ^γ is an integer-valued random measure on $\mathbf{R}_+ \times E_\gamma$ which satisfies $\mu^\gamma(\{t\} \times E_\gamma) \leq 1$ for all $t \in \mathbf{R}_+$, with compensator $\nu^\gamma(dt, dv^\gamma) = dt \otimes dv^\gamma$, dv^γ being the Lebesgue measure on E_γ for $\gamma \in \{1, ..., m\}$, and $\nu^0(dt, dv^0) = dt \otimes \lambda^0(dv^0)$ for a σ-finite measure λ^0 on $(E_0, \mathcal{E}_0)$. We assume that $(w^\beta, \tilde{\mu}^\gamma)$ are independent. Without loss of generality, we may assume that the Lebesgue measure $|E_\gamma| = \infty$ for every $\gamma \in \{1, .., m\}$.

Denote by $\bar{X}(t, \bar{x}) = (Y(t, y), Z(t, z))$ the flow corresponding to the set of stochastic integral equations (19)–(20) starting at $\bar{x} = (y, z)$ instead of (Y_0, Z_0), namely, $\bar{d}$-dimensional stochastic integral equation, $\bar{d} = d_2 + d$:

$$\bar{X}(t, \bar{x}) = \bar{x} + \int_0^t \bar{A}(\bar{X}(s-, \bar{x}))ds + \sum_{\beta=1}^r \int_0^t \bar{B}_\beta(\bar{X}(s-, \bar{x}))dw_s^\beta$$
$$+ \sum_{\gamma=0}^m \int_{[0,t]\times E_\gamma} \bar{C}_\gamma(\bar{X}(s-, \bar{x}), v^\gamma)\tilde{\mu}^\gamma(ds, dv^\gamma), \tag{21}$$

where $\bar{A} = (A, A')$, $\bar{B}_\beta = (B_\beta, B'_\beta)$ and $\bar{C}_\gamma = (C_\gamma, C'_\gamma)$.

We shall apply the result in Subsection 4.2 in the present situation. To this end, let us prepare a sequence of Malliavin operators L_j over particular distributional equivalents and consider several conditions from which the uniform nondegeneracy of the Malliavin covariances of $\hat{\mathcal{Z}}_j$ follows. In this case, the key variables are given by $\mathcal{Y}_j = Y_{v(j)}$ and $\mathcal{Z}_j = (Z_{v(j)}^{u(j)}, Y_{v(j)})$.

Let $I(j) = [u(j), v(j)]$ $(j \in \mathbf{N})$ be a sequence of intervals in $\mathbf{R}_+$ such that $\delta \leq v(j) - u(j) \leq \bar{\delta}$ for some fixed positive numbers δ and $\bar{\delta}$. Define the canonical space $(\hat{\Omega}(j), \hat{\mathcal{B}}(j), \hat{P}(j))$ as follows. Take $(\hat{\Omega}(j), \hat{\mathcal{B}}(j))$ as the product measurable space of the spaces $(\mathbf{R}^{d_2}, \mathbf{B}_{d_2})$ and $(\tilde{\Omega}(j), \tilde{\mathcal{B}}(j))$. Here $(\mathbf{R}^{d_2}, \mathbf{B}_{d_2})$ is the Borel space and $(\tilde{\Omega}(j), \tilde{\mathcal{B}}(j))$ is the canonical product Wiener-Poisson space over time-interval $[0, v(j) - u(j)]$. Define a probability measure $\hat{P}(j)$ so that under $\hat{P}(j)$, the projection to the first space yields the same law as $Y_{u(j)}$, and the canonical projections $(w^\beta; \beta = 1, ..., r)$ form an r-dimensional Wiener process on $[0, v(j) - u(j)]$, the canonical projections $(\mu^\gamma; \gamma = 0, 1, ..., m)$ are independent Poisson random measures on $[0, v(j) - u(j)] \times E_\gamma$ for each, and $(w^\beta, \mu^\gamma | \beta = 1, ..., r; \gamma = 0, 1, ..., m)$ are independent. In the sequel, the distributional equivalent $(\hat{\psi}_j, \hat{Z}_j, \hat{C}_j)$ is assumed to be constructed on the canonical space $(\hat{\Omega}(j), \hat{\mathcal{B}}(j), \hat{P}(j))$, and we will often neglect the "hat" convention of distributional equivalents for simplicity.

Let $E = \sum_{\gamma=0}^{m} E_\gamma$ (direct sum) and $\mu = \sum_{\gamma=0}^{m} \mu_\gamma$. The j-th Malliavin operator is defined as follows. The domain $\mathcal{R}_j = \mathcal{D}(L_j)$ is the set of functionals Φ of the form

$$\Phi = F(y, w_{t_1} - w_{t_0}, \ldots, w_{t_N} - w_{t_{N-1}}, \mu(f_1), \ldots, \mu(f_n)), \tag{22}$$

where $0 = t_0 \le t_1 \le \cdots \le t_N \le v(j) - u(j)$, $f_i \in C^2_{K,v}([0, v(j) - u(j)] \times E)$ (bounded Borel measurable with compact support, of class C^2 in the $v^\gamma \in E_\gamma$-direction, $\gamma = 0, 1, ..., m$, admitting uniformly bounded derivatives) , and F is bounded measurable and $F(y, \cdot) \in C^2_B(\mathbf{R}^{Nr+n})$ with derivatives uniformly bounded in $y \in \mathbf{R}^{d_2}$. Clearly, $\mathcal{R}_j$ generates $\hat{\mathcal{B}}(j)$. Define a function α as $\alpha = \alpha^\gamma$ on E_γ ($\gamma = 1, ..., m$) and $\alpha = 0$ on E_0, where α^γ ($\gamma = 1, ..., m$) are auxiliary functions which satisfy 10–1 of Bichteler et al. [9] (also see p.147 (11–2)). With the auxiliary function $\alpha : E \to \mathbf{R}_+$, we define L_j by $L_j\Phi = L_j^{(1)}\Phi + L_j^{(2)}\Phi$, where

$$L_j^{(1)}\Phi = \frac{1}{2}\sum_{i=1}^{N} \mathrm{trace}\frac{\partial^2 F}{\partial \mathbf{x}_i^2}(t_i - t_{i-1}) - \frac{1}{2}\sum_{i=1}^{N} \frac{\partial F}{\partial \mathbf{x}_i} \cdot (w_{t_i} - w_{t_{i-1}})$$

and

$$\begin{aligned} L_j^{(2)}\Phi = &\frac{1}{2}\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\mu\Big(\alpha\Delta_v f_i + (\partial_v\alpha)\cdot\partial_v f_i\Big) \\ &+\frac{1}{2}\sum_{i,k=1}^{n} \frac{\partial^2 F}{\partial x_i \partial x_k}\mu\Big(\alpha(\partial_v f_i)\cdot(\partial_v f_k)\Big) \end{aligned}$$

for $\Phi \in \mathcal{R}_j$ having the form of (22). Here $\Delta_v = \Delta_{v^\gamma}$ on E_γ. Put $Q_t(\bar{x}) = \det U_t(\bar{x})$ with $U_t^{kl}(\bar{x}) = \Gamma_{L_j}(\bar{X}^k(t, \bar{x}), \bar{X}^l(t, \bar{x}))$. Under the condition that for a certain truncation functional $\hat{\psi}_j \in D^{L_j}_{2,\infty-}$, $S_1^*[\hat{\psi}_j; \hat{\mathcal{Z}}_j] \subset D^{L_j}_{2,\infty-}$, the integration-by-parts formula holds (Section 4.2, also [69, 29]). Put $\Delta t_j = v(j) - u(j)$. In the following condition, we adopt a condition $(\tilde{A}' - r)$ which Bichteler et al. [9] were based on. Roughly speaking, Condition $(\tilde{A}' - r)$ is the condition for differentiability and integrability of the coefficients in the stochastic differential equation. For a detailed description, see p.60 and p. 147 of [9].

Let $\hat{\xi}_j \in D^{L_j}_{2,\infty-}$. Fix a function $\varphi_1 \in C^\infty(\mathbf{R})$ which satisfies that $\varphi_1(x) = 1$ if $|x| \le 1/2$ and $\varphi_1(x) = 0$ if $|x| \ge 1$. Then by using Schwarz inequality for Γ-bilinear form, it is easy to see that

$$|\Psi^{\hat{\mathcal{Z}}_j}(\varphi_1(\hat{\xi}_j))| \le \mathbf{p}(1_{\{|\hat{\xi}_j|\le 1\}}Q_{\Delta t_j}((y,0))^{-1}, |\sigma_{\hat{\mathcal{Z}}_j}|, |L_j\hat{\mathcal{Z}}_j|, |\sigma_{\sigma_{\hat{\mathcal{Z}}_j}}|, |\sigma_{\hat{\xi}_j}|) =: \mathbf{p}_j,$$

with $\hat{\mathcal{Z}}_j = \bar{X}(\Delta t_j, (y, 0))$, where $\mathbf{p}$ is a polynomial independent of j, and y denotes the canonical projection on the canonical space, corresponding to $Y_{u(j)}$.

[A3^Q] (i) For each j, $1_{\{|\hat{\xi}_j|\le 1\}}Y_{u(j)} \in \cap_{p>1} L^p(P)$, and $(\tilde{A}' - 4)$ is satisfied for the coefficients (as functions of (y, v^γ)) of the stochastic differential equation.

(ii) $\limsup_{T\to\infty} \sum_{j=1}^{n(T)} \hat{P}(j)\left[\mathbf{p}_j\right]/n(T) < \infty$.
(iii) $\liminf_{T\to\infty} \sum_{j=1}^{n(T)} \hat{P}(j)\left[|\hat{\xi}_j| \le \frac{1}{2}\right]/n(T) > 0$.

We notice that $\Psi^{\hat{Z}_j}(\varphi_1(\hat{\xi}_j))$ makes sense thanks to the truncation functional $\varphi_1(\hat{\xi}_j)$ although we did not assume L^p-integrability of Y_0. Applying Theorem 3 to $\hat{\psi}_j = \varphi_1(\hat{\xi}_j)$, we obtain:

Theorem 4. *Let (Y, Z) be a solution of the stochastic differential equation (19) and (20). Assume that* $\liminf_{T\to\infty} n(T)/T > 0$. *Then Inequalities (2) and (3) hold under* $[A2]$ *and* $[A3^Q]$ *in place of* $[A3]$.

Note that in this unconditional case, $[A1''']$ is sufficient to validate expansions. Since $\mathcal{C}$ is trivial, $[A2]$ becomes a familiar condition. In case $\mathrm{Var}[Z_T/\sqrt{T}]$ tends to a positive definite matrix, we can take appropriate constants as s_T and u_T so that the probability-terms in (2) and (3) vanish. Let us briefly discuss Condition $[A3^Q]$. The functionals $\Psi^{\hat{Z}_j}$ are well defined by Condition $[A3^Q]$(i). It follows from Lemma 10–17 of Bichteler et al. [9] that $(\bar{X}, \nabla\bar{X}, U, V)$ $(V = L_j\bar{X})$ satisfies a graded stochastic differential equation of $(\tilde{A}' - 2)$. Similarly, from the same lemma, $(\bar{X}, \nabla\bar{X}, U)$ satisfies a graded stochastic differential equation of $(\tilde{A}' - 3)$. By applying the same lemma once again, we see that the process $U^\dagger$ given by

$$U^\dagger = ((\bar{X}, \nabla\bar{X}, U), \nabla(\bar{X}, \nabla\bar{X}, U), \Gamma((\bar{X}, \nabla\bar{X}, U), (\bar{X}, \nabla\bar{X}, U)))$$

satisfies an equation of $(\tilde{A}' - 2)$, and hence we get L^p-estimates for $U^\dagger$ under truncation by $\hat{\psi}_j$ ($V_0 = 0$ etc. and apply 5–10 of the same book). In particular, we obtain L^p-estimates for $\sigma_{\sigma_{\hat{\mathcal{Z}}_j}}$. Those estimates help to check $[A3^Q]$(ii).

While it is rather general, the condition (ii) of $[A3^Q]$ seems slightly abstract. It is possible to rephrase it more simply if we restrict the objects of study. For simplicity, we will focus our attention to a stationary case in Section 5.2.

5.2. Stationary case

We still consider the process (Y, Z) satisfying (19) and (20). Also, assume that Y is stationary. The conditions $(SB - (\zeta, \theta))$ and (SC) in Bichteler et al. [9] will be used in order to ensure the nondegeneracy.

Theorem 5. *Suppose that $(\tilde{A}' - 4)$, $(SB - (\zeta, \theta))$ and (SC) are satisfied. Moreover, assume that $Y_0, Z_0 \in \cap_{p>1} L^p(P)$ and that $P[Z_t] = 0$ for any $t \in \mathbf{R}_+$. Then Inequalities (2) and (3) hold.*

Proof. Take $I(j) = [j\bar{\delta}, (j+1)\bar{\delta}]$ with some large $\bar{\delta}$. Fix a bounded open set $B \in \mathbf{B}_{d_2}$ satisfying that $P^{Y_0}[B] > 0$. Let $\tilde{\varphi} \in C_K^\infty(\mathbf{R}^{d_2}; [0, 1])$ be a smooth function such that $\tilde{\varphi} = 1$ on B, and let $\hat{\xi}_j = 1 - \tilde{\varphi}(Y_{u(j)})$. Then it is possible to verify Condition $[A3^Q]$ through a similar path as Bichteler et al. did. Note that the choice of $\bar{\delta}$ depends on ζ, θ. $\square$

Here we do not assume $[A2]$ explicitly. This result was first given in [29] under the stronger condition $[A1']$; ω_2 can be replaced by ω in the error term in the present unconditional setting. In the above proof, good initial values $Y_{u(j)}$ were selected. On the other hand, it is possible to select a good path set, and it is what we will later consider with the help of the support theorem.

Differently than Bichteler et al. did, we do not need higher-order differentiability because what is necessary in our situation is not integrability but convergence of the conditional characteristic function to zero.

In diffusion case, though it is too strong, the Hörmander condition is a practical, convenient one in our nondegeneracy problem ([29]).

5.3. *Support theorem and asymptotic expansion*

The conditions in the previous result are relatively strong and too sufficient for our purpose. As suggested above, the existence of a good skeleton is sufficient to find a good subset of path space with the aid of a support theorem.

We introduce the following notation:

$$
\begin{aligned}
B^*(\bar{x}) &= \bar{B}(\bar{x})\bar{B}(\bar{x})' \\
C^*_\gamma(\bar{x}, v^\gamma) &= \begin{cases} \left\{I + \partial_{\bar{x}}\bar{C}_\gamma(\bar{x}, v^\gamma)\right\}^{-1}(\partial_{v^\gamma}\bar{C}_\gamma)(\partial_{v^\gamma}\bar{C}_\gamma)'(\bar{x}, v^\gamma) \\ \quad \cdot\left\{I + \partial_{\bar{x}}\bar{C}_\gamma(\bar{x}, v^\gamma)'\right\}^{-1} \quad (\text{if } I + \partial_{\bar{x}}\bar{C}_\gamma(\bar{x}, v^\gamma) \text{ is invertible}) \\ 0 \quad (\text{otherwise}) \end{cases} \\
S_t(\bar{x}) &= \int_0^t (\nabla\bar{X}(s-, \bar{x}))^+ B^*(\bar{X}(s-, \bar{x}))(\nabla\bar{X}(s-, \bar{x})')^+ ds \\
&\quad + \sum_{\gamma=1}^m \int_0^t \int_{E_\gamma} (\nabla\bar{X}(s-, \bar{x}))^+ C^*_\gamma(\bar{X}(s-, \bar{x}), v^\gamma)(\nabla\bar{X}(s-, \bar{x})')^+ \\
&\quad \cdot\alpha^\gamma(v^\gamma)\mu^\gamma(ds, dv^\gamma), \qquad (+ \text{ means the Moore-Penrose g-inverse}) \\
U_t(\bar{x}) &= \nabla\bar{X}(t, \bar{x})\ S_t(\bar{x})\nabla\bar{X}(t, \bar{x})', \\
Q_t(\bar{x}) &= \det U_t(\bar{x})
\end{aligned}
$$

Here M' denotes the transposed matrix of the matrix M. $U_t(\bar{x})$ coincides with the former one.

For $\mathbf{D}(\mathbf{R}_+; \mathbf{R}^d)$-valued stochastic process Y, denote by $\text{Supp}(Y)$ the support of the probability distribution of Y and define it by

$$
\text{Supp}(Y) = \left\{\varphi \in \mathbf{D}(\mathbf{R}_+; \mathbf{R}^d);\ P(\mathbf{d}(Y, \varphi) < \epsilon) > 0 \quad \text{for all } \epsilon > 0\right\}.
$$

Here $\mathbf{d}$ denotes a metric on $\mathbf{D}(\mathbf{R}_+; \mathbf{R}^d)$ compatible with the Skorohod topology.

On the Wiener-Poisson space $\tilde{\Omega} = \{(w^\beta, \mu^\gamma)\}$ $(\beta = 1, ..., r; \gamma = 0, 1, ..., m)$ on $[0, t_0]$, t_0 being a positive constant, we consider the $\check{d}(:= \bar{d} + \sum_{\gamma=1}^m b_\gamma + \bar{d}^2)$-dimensional flow $\Phi(\xi) = (\Phi_t(\xi))_{t\in[0,t_0]}$ defined by the stochastic integral equation enlarging (21):

$$\Phi_t(\xi) = \xi + \int_0^t \check{A}(\Phi_{s-}(\xi))ds + \sum_{\beta=1}^{r} \int_0^t \check{B}_\beta(\Phi_{s-}(\xi))dw_s^\beta$$
$$+ \sum_{\gamma=0}^{m} \int_{[0,t]\times E_\gamma} \check{C}_\gamma(\Phi_{s-}(\xi), v^\gamma)\tilde{\mu}^\gamma(ds, dv^\gamma). \tag{23}$$

Here $\xi \in \mathbf{R}^{\check{d}}$ and the coefficients $\check{A}$, $\check{B}_\beta$ and $\check{C}_\gamma$ are the liftings of $\bar{A}$, $\bar{B}_\beta$ and $\bar{C}_\gamma$ given by

$$\check{A}(\xi) = \begin{bmatrix} \bar{A}(\pi_1\xi) \\ (G_\gamma[1_{R_\gamma}(v^\gamma)v^\gamma])_{\gamma=1,\ldots,m} \\ \nabla\bar{A}(\pi_1\xi)\pi_3\xi \end{bmatrix},$$

G_γ being the Lebesgue measure on E_γ and π_i being the projection to the i-th block,

$$\check{B}_\beta(\xi) = \begin{bmatrix} \bar{B}_\beta(\pi_1\xi) \\ 0 \\ \nabla\bar{B}_\beta(\pi_1\xi)\pi_3\xi \end{bmatrix} \text{ and } \quad \check{C}_\gamma(\xi, v^\gamma) = \begin{bmatrix} \bar{C}_\gamma(\pi_1\xi, v^\gamma) \\ \mathbf{e}_\gamma \otimes 1_{R_\gamma}(v^\gamma)v^\gamma \\ \nabla\bar{C}_\gamma(\pi_1\xi, v^\gamma)\pi_3\xi \end{bmatrix},$$

where $\mathbf{e}_\gamma$ is the γ-th unit vector of the standard basis of $\mathbf{R}^m$ for $\gamma = 1, ..., m$, $\mathbf{e}_0 = 0 \in \mathbf{R}^m$, and each R_γ is a bounded open set in $\mathbf{R}^{b_\gamma}$. The third argument of (23) is the variational equation for $\bar{X}$. We will in the sequel let $\pi_3\xi = I_{\bar{d}}$ for the initial value ξ of (23).

Let $\mathcal{S}_+^d$ be the set of $d \times d$-nonnegative matrices, and $\bar{\mathcal{S}}_+^d = \mathcal{S}_+^d \cup \{\infty\}$ a one-point conpactification of $\mathcal{S}_+^d$. Define $\bar{\mathcal{S}}_+^{\bar{d}}$- valued mapping $\mathcal{Q}$ on $\mathbf{D}([0, t_0]; \mathbf{R}^{\check{d}})$ by

$$\mathcal{Q}(\phi) = \int_0^{t_0} (\pi_3\phi_{s-})^+ B^*(\pi_1\phi_{s-})(\pi_3\phi'_{s-})^+ ds$$
$$+ \sum_{s:s\le t_0} \sum_{\gamma=1}^{m} (\pi_3\phi_{s-})^+ C_\gamma^*(\pi_1\phi_{s-}, \Delta\phi_s^\gamma)(\pi_3\phi'_{s-})^+(\alpha^\gamma 1_{R_\gamma})(\Delta\phi_s^\gamma)$$

for $\phi = (\pi_1\phi, \pi_2\phi, \pi_3\phi) \in \mathbf{D}([0, t_0]; \mathbf{R}^{\check{d}})$, $\pi_2\phi = (\phi^\gamma)_{\gamma=1,\ldots,m}$. We put $\mathcal{Q}(\phi) = \infty$ if some element of the second terms on the right-hand side does not converge absolutely. Assume that $|\det(I + \partial_{\bar{x}}\bar{C}_0)| > \delta_2 > 0$.

Let R_γ^* be a bounded open set in $\mathbf{R}^{b_\gamma}$ such that $\overline{R}_\gamma \subset R_\gamma^*$. Moreover, set $\check{\xi}(y) = ((y, 0), 0, I_{\bar{d}})$.

[A3^S] There exist positive constants δ_i $(i = 1, 2)$ and a measurable set $B \in \mathbf{B}_{d_2}$ for which the following conditions are fulfilled:

(i) $P^{Y_0}[B] > 0$.

(ii) For every $y \in B$, there exists a skeleton $\phi(\check{\xi}(y)) \in \mathrm{Supp}(\Phi(\check{\xi}(y)))$ with $\pi_2\phi$ admitting at most finite jumps such that $\Delta\phi_t^\gamma\left(\check{\xi}(y)\right) \in R_\gamma$ for all $t \in [0, t_0]$,

$$\mathcal{Q}\left(\phi\left(\check{\xi}(y)\right)\right) \geq \delta_1 I$$

and

$$\inf_{\substack{v^\gamma \in R^*_\gamma \\ t\in[0,t_0]}} \left|\det\left(I + \partial_{\bar{x}}\bar{C}_\gamma\left(\pi_1\phi_t\left(\check{\xi}(y)\right), v^\gamma\right)\right)\right| > \delta_2.$$

Remark 4. Condition $[A3^S]$ may still look abstract. However, it is in general easy to verify this condition with the aid of a support theorem. Really, it is sufficient to find a deterministic nice control variable satisfying the nondegeneracy. We will return to this point after presenting our result and its proof.

Remark 5. It may seem in appearance that the skeleton $\phi(\check{\xi}(y))$ can be chosen very freely. However, no jump times of elements of $\pi_2\phi(\check{\xi}(y))$ should coincide with each other. The condition that $\phi(\check{\xi}(y))$ is in the support inevitably imposes such restrictions on the choice of $\phi(\check{\xi}(y))$. For later use, it is sufficient to find a skeleton $\phi(\check{\xi}(y))$ defined on some adequately larger interval than $[0, t_0]$.

Remark 6. We do not exclude the case where $\bar{X}$ has infinitely many jumps over finite time intervals. For example, if the Lévy measure for small jumps diverges, we can split that part of Lévy measure as E_0, and it is usually possible in practical situations that the resudual part of finite number of jumps assures the nondegeneracy.

Remark 7. It is possible that the nondegeneracy condition is satisfied even when the factors B^* and C^*_γ degenerate in part. If uniform ellipticity is assumed to a sum of $B^*(\bar{x})$ and $C^*_\gamma(\bar{x}, v^\gamma)$ like 2–24 of Bichteler et al. [9], Condition (ii) of $[A3^S]$ is obviously satisfied. It is different from their nondegeneracy because we need only *local* nondegeneracy in the present situation.

Theorem 6. *Suppose that* (Y, Z) *satisfy (19) and (20) and that* Y *is strongly stationary. Assume* $(\tilde{A}' - 4)$ *and* $[A3^S]$. *Moreover, assume that* $Y_0, Z_0 \in \cap_{p>1} L^p(P)$ *and that* $P[Z_t] = 0$ *for any* $t \in \mathbf{R}_+$. *Then Inequalities (2) and (3) hold.*

Proof. Let $\varphi_1 \in C^\infty(\mathbf{R}_+; [0, 1])$ be a truncation function satisfying $\varphi_1(x) = 1$ if $x \leq 1/2$ and $\varphi_1(x) = 0$ if $x \geq 1$. Let $R^*_0 = E_0$. In the notation introduced above, we define truncation functionals ψ_j by

$$\psi_j = \varphi_1\left(5\left[1 + \left(\frac{6Q_{t_0}((Y_{(j-1)\Delta}, 0))}{\delta}\right)^2\right]^{-1}\right)$$

for an adequately fixed positive constant Δ such that $\Delta \geq t_0$. In particular, if $Q_{t_0}((Y_{(j-1)\Delta}, 0)) \leq \delta/3$, then $\psi_j = 0$, and if $Q_{t_0}((Y_{(j-1)\Delta}, 0)) \geq \delta/2$, then

$\psi_j = 1$. Clearly, ψ_j (or more rigorously, their distributinal equivalents) are differentiable in Malliavin's sense. Let us show that the truncation ψ_j retains positive probability uniformly in j. Because of the stationarity, we may only consider ψ_1. We will show the existence of a positive event $\Omega_0 \subset Y_0^{-1}(B)$ on which

(i) $\mathcal{Q}\left(\Phi\left(\check{\xi}(Y_0)\right)\right) \geq \delta_1 I/2$,

(ii) $\inf_{\substack{v^\gamma \in R_\gamma^* \\ t\in[0,t_0] \\ \gamma=0,1,\dots,m}} \left|\det\left(I + \partial_{\bar{x}}\bar{C}_\gamma\left(\pi_1\Phi_t\left(\check{\xi}(Y_0)\right), v^\gamma\right)\right)\right| \geq \delta_2/2.$

For $y \in B$, let

$$\begin{aligned}\epsilon(y) = \sup\Big\{\epsilon;\ &\mathcal{Q}(\phi) \geq \delta_1 I/2 \text{ and}\\ &\inf_{\substack{v^\gamma \in R_\gamma^* \\ t\in[0,t_0] \\ \gamma=0,1,\dots,m}} \left|\det\left(I + \partial_{\bar{x}}\bar{C}_\gamma(\pi_1\phi_t, v^\gamma)\right)\right| \geq \delta_2/2\\ &\text{for all}\quad \phi \in \mathbf{D}([0,t_0]; \mathbf{R}^{\check{d}}) \text{ satisfying } \mathbf{d}(\phi, \phi(\check{\xi}(y))) < \epsilon\Big\}\end{aligned}$$

It follows from Lemma 2.1 of Kurz and Protter [27] and similar reasoning as the note after it that the mappings $\phi \mapsto \mathcal{Q}(\phi)$ and $\phi \mapsto \inf_{\substack{v^\gamma \in R_\gamma^* \\ t\in[0,t_0];\gamma=1,\cdots,m}} |\det(I + \partial_{\bar{x}}\bar{C}_\gamma(\pi_1\phi_t, v^\gamma))|$ are continuous at $\phi(\check{\xi}(y))$ with respect to the Skorohod topology; if t_0 is a jump point of the skeleton $\pi_2\phi(\check{\xi}(y))$, then we can change t_0 to a bigger number at the beginning without changing our nondegeneracy conditions. For the continuity of the ds-integral term, the right-continuous simple function approximation to càdlàg functions would help us. Therefore, the two inequalities of $[A3^S]$(ii) imply that $\epsilon(y) > 0$ for each y. By the assumption that $\phi(\check{\xi}(y)) \in \mathrm{Supp}(\Phi(\check{\xi}(y)))$ in $[A3^S]$(ii), we see that

$$\begin{aligned}\mathbf{P}^{WP}\Big[&\mathcal{Q}\left(\Phi\left(\check{\xi}(y)\right)\right) \geq \delta_1 I/2 \text{ and}\\ &\inf_{\substack{v^\gamma \in R_\gamma^* \\ t\in[0,t_0] \\ \gamma=0,1,\dots,m}} \left|\det\left(I + \partial_{\bar{x}}\bar{C}_\gamma\left(\pi_1\Phi_t\left(\check{\xi}(y)\right), v^\gamma\right)\right)\right| \geq \delta_2/2\Big] > 0\end{aligned}$$

for every $y \in B$. Here $\mathbf{P}^{WP}$ stands for the Wiener-Poisson measure. By using the Markovian property, integrating the LHS of the above inequality, for $\bar{X}_t = \bar{X}(t, (Y_0, 0))$, we obtain

$$\begin{aligned}P\Big[&\mathcal{Q}\left(\Phi(\check{\xi}(Y_0))\right) \geq \delta_1 I/2 \text{ and}\\ &\inf_{\substack{v^\gamma \in R_\gamma^* \\ t\in[0,t_0] \\ \gamma=0,1,\dots,m}} \left|\det\left(I + \partial_{\bar{x}}\bar{C}_\gamma(\bar{X}_t, v^\gamma)\right)\right| \geq \delta_2/2\Big] > 0.\end{aligned} \tag{24}$$

On the event

$$\{\inf_{\substack{v^\gamma \in R_\gamma^* \\ t\in[0,t_0] \\ \gamma=0,1,\dots,m}} \left|\det\left(I + \partial_{\bar{x}}\bar{C}_\gamma(\bar{X}_t, v^\gamma)\right)\right| \geq \delta_2/2\},$$

the L^p-norm of $\sup_{t\in[0,t_0]}|(\nabla\bar{X}_t)^{-1}|$ is finite, and there exists a constant M such that the probability of the event

$$\{\sup_{t\in[0,t_0]}|(\nabla\bar{X}_t)^{-1}|+\sup_{t\in[0,t_0]}|\nabla\bar{X}_t|>M\}$$

becomes arbitrarily small, especially, smaller than the probability of (24). Consequently, taking the intersection of those events, we see that on this event, the minimum eigenvalue of $U_{t_0}(\bar{X}_0)$ admits the following estimate:

$$\begin{aligned}\lambda_1\left(U_{t_0}(\bar{X}_0)\right) &= \inf_{z\in\mathbf{R}^{\bar{d}}:|z|=1} z'\nabla\bar{X}(t_0,\bar{X}_0)S_{t_0}(\bar{X}_0)\nabla\bar{X}(t_0,\bar{X}_0)'z\\ &\geq \inf_{z\in\mathbf{R}^{\bar{d}}:|z|=1} z'S_{t_0}(\bar{X}_0)z\cdot\inf_{z_1\in\mathbf{R}^{\bar{d}}:|z_1|=1}\left|\nabla\bar{X}(t_0,\bar{X}_0)'z_1\right|^2.\end{aligned}$$

For the second factor on the RHS, we have the estimate:

$$\left|\nabla\bar{X}(t_0,\bar{X}_0)'z_1\right|\geq\left\|\left(\nabla\bar{X}(t_0,\bar{X}_0)\right)^{-1}\right\|_{op}^{-1}\geq\mathbf{c_1}\left|\left(\nabla\bar{X}(t_0,\bar{X}_0)\right)^{-1}\right|^{-1}\geq\frac{\mathbf{c_1}}{M},$$

where $\mathbf{c_1}$ is a universal positive constant. For the first factor,

$$\inf_{z\in\mathbf{R}^{\bar{d}}:|z|=1} z'S_{t_0}(\bar{X}_0)z\geq\inf_{z\in\mathbf{R}^{\bar{d}}:|z|=1} z'\mathcal{Q}\left(\Phi(\check{\xi}(Y_0))\right)z\geq\delta_1/2.$$

Therefore, $\lambda_1\left(U_{t_0}(\bar{X}_0)\right)$ is bounded from below on the good event, and we see that for a sufficiently small $\delta>0$, $P[\psi_1]>0$. This implies Condition $[A3^M]$ and hence the desired result follows from Theorem 4. We note that on the nice event we took, the Malliavin covariance matrix $U_{t_0}(\bar{x})$ admits the representation with $S_t(\bar{x})$ given above. □

Let us discuss briefly the use of support theorems. Let $\mathcal{U}$ be a space of control processes for the stochastic integral equation (23). Denote by ϕ^u the deterministic solution corresponding to the control variable $u\in\mathcal{U}$ and the initial state $\check{\xi}(y)$. Then the support theorem asserts that

$$\mathrm{Supp}\left(\Phi(\check{\xi}(y))\right)=\overline{\{\phi^u;u\in\mathcal{U}\}},\tag{25}$$

where the bar means the closure with respect to the Skorohod topology. In this way, once such a support theorem is established, in order to check Condition $[A3^S]$, it suffices to find a control variable $u\in\mathcal{U}$ for which $\phi(\check{\xi}(y))=\phi^u$ satisfies (ii) of $[A3^S]$.

For illustration, we shall consider the case where $B_\beta\equiv0$, $m=1$ and $C_0\equiv0$. In this situation, the set $\mathcal{U}$ is the set of sequences $\{(t_n,v_n)\}$, where $\{t_n\}$ is a strictly increasing sequence of positive numbers tending to the infinity and $\{v_n\}$ is a sequence in the support of the spatial intensity measure G_1. For this $\mathcal{U}$, Simon [55] obtained a support theorem (25). Our nondegeneracy problem is after all turned to the problem of selecting a finite sequence $\{(t_n,v_n)\}$ with $t_n\in\mathbf{R}_+$ (but essentially over $[0, t_0]$) such that the resulting path ϕ^u satisfies the nondegeneracy (ii) of $[A3^S]$.[8]

For other support theorems, we refer the reader to Stroock and Varadhan [60], Aida, Kusuoka and Stroock [1], Kunita [26], Millet and Nualart [35], and Ishikawa [22, 23].

5.4. Stochastic differential equation with random coefficients

We shall consider a stochastic integral equation with random coefficients:

$$\begin{aligned} Y_t &= Y_0 + \int_0^t A(\mathbf{c}_s, Y_{s-})ds + \sum_{\beta=1}^{r} \int_0^t B_\beta(\mathbf{c}_s, Y_{s-})dw_s^\beta \\ &\quad + \sum_{\gamma=0}^{m} \int_0^t \int_{E_\gamma} C_\gamma(\mathbf{c}_s, Y_{s-}, v^\gamma)\tilde{\mu}(ds, dv^\gamma) \end{aligned} \tag{26}$$

$$\begin{aligned} Z_t &= Z_0 + \int_0^t A'(\mathbf{c}_s, Y_{s-})ds + \sum_{\beta=1}^{r} \int_0^t B'_\beta(\mathbf{c}_s, Y_{s-})dw_s^\beta \\ &\quad + \sum_{\gamma=0}^{m} \int_0^t \int_{E_\gamma} C'_\gamma(\mathbf{c}_s, Y_{s-}, v^\gamma)\tilde{\mu}(ds, dv^\gamma) \end{aligned} \tag{27}$$

and will validate the asymptotic expansion for $Z_T/\sqrt{T}$. Here $\mathbf{c} = (\mathbf{c}_t)$ denotes a conditioning stochastic process independent of (w^β, μ^γ).

If we want to check the nondegeneracy in the same way as before using the Malliavin calculus, we will need the Malliavin calculus for time-dependent equations. It is a routine job by approximation methods to compute the Malliavin covariances if the coefficients are smooth in c-component and $\mathbf{c}_t$ has nice continuity, or not bad discontinuity over $I(j)$. For example, it is the case in each of the following situations: (i) A measurable space valued process $\mathbf{c}_t$ is random but constant on $I(j)$. (ii) An $\mathbf{R}^{d_3}$-valued process $\mathbf{c}_t$ is also defined by another stochastic differential equation driven by independent noises, with jumps, in general. (iii) A process $\mathbf{c}_t$ is Lipschitz (in t on every compact a.s.) and it appears in coefficients as a product form as $f(\mathbf{c}_t)g(Y_t)$ through differentiable function f (e.g., Taniguchi [64]). On the other hand, without such a regularity condition, we could not expect anything general. Because Equations (26) and (27) include a too wide diversity of models, and because here we do not want to go into those models distinctively, it would be a feasible way for us to assume the possible final form of the Malliavin covariance.

In order to include periodic stationary models, which include continuous-time representation of time series models, we keep a sequence of intervals $I(j) = [u(j), v(j)]$. For simplicity, let us assume that $v(j) = u(j) + \tau$ and $((\mathbf{c}_t)_{t \in I(j)}, Y_{u(j)})$ have the same distribution for all j and T. Consequently, we may only consider

[8] Null-control is also a possible control. It often works. Deterministic simulation by computer is another practical solution.

the interval $I(1)$ and its distributional equivalent. As a convention, we keep the same notation even for the equivalent. Let $u(1) = 0$ for notational simplicity. We will assume that the process $\mathbf{c} = (\mathbf{c}_t)_{t \in I(1)}$ takes its values in a topological space $\mathbf{S}$ and that paths are càdlàg . In the present case, the coefficients A, B_β, C_γ have $\mathbf{S}$-component and they are assumed to be continuous. According to this change, we define $\bar{A}$, $\bar{B}_\beta$, $\bar{C}_\gamma$, B^*, C^*_γ, S_t, U_t, Q_t, $\check{A}$, $\check{B}_\beta$, $\check{C}_\gamma$, $\mathcal{Q}$ in a natural way.

In stead of $[A3^S]$, we now consider

[A3^{SC}] There exist positive constants δ_i $(i = 1, 2)$ and subsets $A \in \mathcal{D}(I(1); \mathbf{S})$, $B \in \mathbf{B}_{d_2}$ for which the following conditions are satisfied:

(i) $P^{(\mathbf{c}, Y_0)}[A \times B] > 0$.
(ii) For every $(c, y) \in A \times B$, there exists a skeleton $\phi(c, \check{\xi}(y)) \in \mathrm{Supp}(\Phi(c, \check{\xi}(y)))$ with $\pi_2\phi$ admitting at most finite jumps such that $\Delta\phi_t^\gamma\left(c, \check{\xi}(y)\right) \in R_\gamma$ for all $t \in I(1)$,

$$\mathcal{Q}\left(c, \phi\left(c, \check{\xi}(y)\right)\right) \geq \delta_1 I$$

and

$$\inf_{\substack{v^\gamma \in R^*_\gamma \\ t \in I(1)}} \left|\det\left(I + \partial_{\bar{x}}\bar{C}_\gamma\left(c_t, \pi_1\phi_t\left(c, \check{\xi}(y)\right), v^\gamma\right)\right)\right| \geq \delta_2.$$

As mentioned above, a basic assumption here is that the Malliavin covariance matrix $U_t(\mathbf{c}, (Y_0, 0))$ has the same expression as Section 5.3 with $\pi_1\Phi(\mathbf{c}, \check{\xi}(Y_0))$ for $\bar{X}$ and $\pi_3\Phi(\mathbf{c}, \check{\xi}(Y_0))$ for $\nabla\bar{X}$. Moreover, if $(\mathbf{c}_t)_{t \in I(j)}$-dependent processes $U^{\dagger}$[9] for the initial value $Y_{u(j)}$ admit $L^p(P)$-estimates, $P[\left|\Psi^{\hat{Z}_j}(\hat{\psi}_j)\right| \mid \mathbf{c}_t; t \in I(j)]$ can be represented as $g(\mathbf{c}_t; t \in I(j))$, where $g : \mathbf{D}([0, \tau]; \mathbf{S}) \to \mathbf{R}$ is measurable, and $g(\mathbf{c}_t; t \in I(j))$ have the same distribution and are in $L^\tau(P)$ for some $\tau \geq 1$. Thus, Theorem 3 leads to the asymptotic expansion if $(\mathbf{c}_t)$ satisfies the ergodic property:

$$P\left[\left|\frac{1}{n(T)}\sum_{j=1}^{n(T)} \tilde{g}(\mathbf{c}_t; t \in I(j))\right| > \epsilon\right] = O\left(\frac{1}{T^L}\right)$$

for any L^τ-function $\tilde{g}(\mathbf{c}_t; t \in I(j))$ with mean zero, and any positive ϵ and L.

6. Proof of Theorem 1

6.1. Unifying three mixing conditions

In order to carry out a similar procedure as Götze and Hipp [18] even for our conditional mixing case, we reform the original Z_t^s as follows. Let $\varphi : \mathbf{R}^d \to [0, 1]$ be a measurable function satisfying that $\varphi(x) = 1$ if $|x| \leq 1/2$ and $\varphi(x) = 0$ if $|x| \geq 1$. Put $\varphi_T(x) = x\varphi(x/(2T^\beta))$. Then $\varphi_T(x) = x$ if $|x| \leq T^\beta$ and $\varphi_T(x) = 0$

[9] See Section 5.1.

if $|x| \geq 2T^{\beta}$. Here $\beta \in (0, 1/2)$ is a constant and it will later be taken sufficiently close to 1/2. Next, fix some constants $\delta, \bar{\delta}$ such that $0 < \delta < \bar{\delta}$. For each $T > 0$, consider a sequence $(t_{T,j})_{j=1}^{N_T}$ such that $0 = t_{T,0} < t_{T,1} < \ldots < t_{T,N_T} = T$ and $\delta < t_{T,j} - t_{T,j-1} < \bar{\delta}$ ($j = 1, ..., N_T - 1$) and $t_{T,N_T} - t_{T,N_T-1} < \bar{\delta}$. Write $\tilde{I}_j = [t_{T,j-1}, t_{T,j}]$, and set $Z_{\tilde{I}_j} = Z_{t_{T,j}} - Z_{t_{T,j-1}}$. Let $\tilde{Z}_{T,j} = \varphi_T(Z_{\tilde{I}_j}) - P_{\mathcal{C}}[\varphi_T(Z_{\tilde{I}_j})]$ for $j \geq 1$, and similarly $\tilde{Z}_{T,0} = \varphi_T(Z_0) - P_{\mathcal{C}}[\varphi_T(Z_0)]$. Furthermore, let $\tilde{Z}_T = \sum_{j=0}^{N_T} \tilde{Z}_{T,j}$, and $S_T^* = \tilde{Z}_T/\sqrt{T}$. The $\mathcal{C}$-conditional characteristic function of S_T^* is given by $H_T(u, \mathcal{C}) = P_{\mathcal{C}}[e^{iu \cdot S_T^*}]$ for $u \in \mathbf{R}^d$. In order to derive asymptotic expansion for $Z_T/\sqrt{T}$, we will deal an expansion of $H_T(u, \mathcal{C})$.

For $I = (i_1, ..., i_r)$, $i_1, ..., i_r \in \{0, 1, ..., N_T\}$, denote $\tilde{Z}_I = \tilde{Z}_{T,i_1} \otimes \cdots \otimes \tilde{Z}_{T,i_r}$. Let $k(T), m(T), l(T)$ be increasing sequences of positive integers. For x_1 ($0 < x_1 < 1$), let

$$\Omega_1(T) = \{\alpha(\delta m(T)|\mathcal{C})^{1/k(T)} \leq \frac{x_1}{4}\}.$$

Take ϵ_1 ($0 < \epsilon_1 < 1/2$). For $I \in \{0, 1, ..., N_T\}^r$, let

$$\Omega_2(T, I) = \left\{ \max_{k: 1 \leq k \leq k(T)} P_{\mathcal{C}} \left[\left| \sum_{j: m(T)(k-1) \leq d(j,I) < m(T)k} \frac{\tilde{Z}_{T,j}}{T^{\frac{1}{2}-\epsilon_1}} \right| \right] \leq \frac{x_1^2}{2} \right\},$$

where $d(j, I) = \min\{|j - i_\alpha|; \alpha = 1, ..., r\}$. Further, let

$$\Omega_2(T) = \left\{ \max_{\substack{\hat{I}: \#\hat{I} \leq C_0 m(T), \\ \hat{I} \subset \{0,1,...,N_T\}}} P_{\mathcal{C}} \left[\left| \sum_{j \in \hat{I}} \frac{\tilde{Z}_{T,j}}{T^{\frac{1}{2}-\epsilon_1}} \right| \right] \leq \frac{x_1^2}{2} \right\},$$

where C_0 is a constant greater than $2(\bar{l} + p + 1)$ for some constant $\bar{l}$ we specify later on. Fix an positive integer $p \geq 3$. For $\epsilon_2 > 0$, let $\Omega_3(T) = \left\{T^{-\epsilon_2} M_T(\omega) \leq 1\right\}$, where $M_T(\omega) = \max_{j: 0 \leq j \leq N_T} P_{\mathcal{C}} \left[\left| Z_{\tilde{I}_j} \right|^{(p+1)} \right] + 1$. Moreover, for x_2 ($0 < x_2 < 1$), let

$$\Omega_4(T) = \left\{ \max_{\hat{I} \subset \{0,1,...,N_T\}} \left| \mathrm{Var}_{\mathcal{C}} \left[T^{-\frac{1}{2}} \sum_{j \in \hat{I}} \tilde{Z}_{T,j} \right] \right| \leq -2(\log x_2) \frac{k(T)}{l(T)^2} \right\}$$

and put $\Omega_0(T) = \cap_{i=1}^4 \Omega_i(T)$.

Under each one of the three conditions $[A1']$–$[A1''']$, we will obtain a conditional expansion of the characteristic function by estimating the events $\Omega_i(T)$. In order to treat those cases in a unified way, we will consider the following Condition $[A1]$, which looks slightly involved. Let $\bar{l}$ be an integer not less than $d + 2[p/2] + 1$ such that $\bar{l} + p + 1$ is even, and let $p' = \bar{l} + (p + 1)^2$. (More precisely, $\bar{l}$ is a large number which is specified by $[A1]$ (iv) below.) We adopt a particular partition $(\tilde{I}_j)$, which will be specified. Let $K \geq 1/2$, $q > 1$ and $q' = q/(q-1)$. Set [10]

$$\hat{L}_1 = \max\{\frac{1}{2}(\bar{l}+p+1), (\bar{l}+p+1)(1+\beta)\left(\log\frac{x_2}{x_1}\right)\left(\log\frac{4x_2^2}{x_1}\right)^{-1},$$
$$(p+1)(p-1-(p-1)\epsilon+\beta)\left(\log\frac{x_2}{x_1}\right)\left(\log\frac{4}{x_1}\right)^{-1},$$
$$\left[\frac{p-2}{2}+\left(\frac{3}{2}-\beta\right)\bar{l}+Kd\right]\left(\log\frac{x_2}{x_1}\right)\left(\log\frac{4}{x_1}\right)^{-1}\}.$$

[A1] There exist positive constants $L_1, L_2 > 0$, x_1, x_2 $(0 < x_1 < x_2^2 < 1/4)$, $\epsilon_1, \epsilon_2, \epsilon, \epsilon^*, \beta$ $(0 < \epsilon_1, \epsilon_2, \epsilon, \epsilon^*, \beta < 1/2)$, $\bar{l} \in \mathbf{N}$, and increasing sequences $m(T)$, $k(T)$ and $l(T)$ satisfying the following conditions:
(i) $k(T)/\log T \geq L_1/\log(x_2/x_1)$ for large T, and $L_1 > \hat{L}_1$.
(ii) $k(T) \leq m(T) = o(T^{\epsilon})$ as $T \to \infty$.
(iii) $l(T) = o(T^{\epsilon^*})$.
(iv) $\epsilon_2 + \frac{5}{2}\epsilon + \epsilon^* < \frac{1}{2} - \beta$, $\epsilon^* \leq \epsilon_1$, $(p'+d)\epsilon^* + 2(\bar{l}+p)\epsilon + \epsilon_2 < \frac{1}{4p}$, $(\bar{l}+1+p)\epsilon_2 < 1$ and

$$\beta > \max\left\{\frac{p+2\epsilon_2}{2(p+1)}[1+p_0\left(\bar{l}+1+p-p_0\right)^{-1}], \frac{1}{2}\left[1-\frac{2p-1}{2p(p+1)}\right]\right\}.$$

(v) $T^{L_2}P\left[\Omega_0(T)^c\right] = O(1)$ $(T \to \infty)$, and $L_2 > q'(p-2)/2$.

The following three lemmas show that we may start with $[A1]$ instead of $[A1']$, $[A1'']$ or $[A1''']$.

Lemma 2. *Assume* $[A1']$ *and* $[A2]$. *Let* $p \in \mathbf{N}$ $(p \geq 3)$, $K \geq 1/2$, $q > 1$, *and* $x_1, x_2 > 0$ $(x_1 < x_2^2 < 1/4)$. *Take sufficiently small positive* $\epsilon_1, \epsilon_2, \epsilon, \epsilon^*$; $\epsilon^* < \epsilon_1$; *sufficiently large* $\beta < 1/2$; $k(T) = [T^{\epsilon_3}]$, $l(T) = [T^{\epsilon_4}]$ *and* $m(T) = [T^{\epsilon_5}]$ *with* $0 < 2\epsilon_4 < \epsilon_3 < \epsilon_5 < \epsilon$ *and* $\epsilon_4 < \epsilon^*$. *Then for any* $L_1, L_2 > 0$ *satisfying the last inequalities in* $[A1]$*(i) and (v), all the conditions in* $[A1]$ *are satisfied.*

Lemma 3. *Assume* $[A1'']$ *and* $[A2]$. *Fix* $a' > 0$ *satisfying* $a'b > 1$ *and* $a > a'$. *Let* $K \geq 1/2$, $q > 1$, $x_1, x_2 > 0$ $(x_1 < x_2^2 < 1/4)$. *Take sufficiently small positive* $\epsilon_1, \epsilon_2, \epsilon, \epsilon^*$; $\epsilon^* < \epsilon_1$; *sufficiently large* $\beta < 1/2$. *Take* $L_1, L_2, L_3 > 0$ *so that* L_1, L_2 *satisfy the last inequalities in* $[A1]$*(i) and (v). Take sufficiently large* M_1 *(depending on* x_2 *and* L_3*);* $k(T) = [M_1(\log T)^{1+a'}]$, $l(T) = [(2L_3\log T)^{(1+a'-1/b)/2}]$, *and* $m(T) = [T^{\epsilon_5}]$ *with* $0 < \epsilon_5 < \epsilon$. *Then all the conditions in* $[A1]$ *are satisfied.*

Lemma 4. *Assume* $[A1''']$ *and* $[A2]$. *Let* $K \geq 1/2$, $q > 1$, $L_1, L_2, L_3 > 0$, L_1, L_2 *satisfying the last inequalities in* $[A1]$*(i) and (v) independently of* β *and* ϵ. *Let* $x_1 < x_2^2 < 1/4$ *and suppose that* x_2 *is small so that*

$$\frac{(-\log x_2)L_1}{2(\log 2)L_3} > \limsup_{T\to\infty}\sup_{\mathbf{I}\in\mathcal{I}}\|\Phi_T(\mathbf{I})\|_\infty + 2.$$

[10] "ϵ" is distinct from the one of ϵ-Markov process.

Moreover, let

$$k(T)=\left[\frac{L_1\log T}{\log 2}\right],\ l(T)=\left[\sqrt{2L_3\log T}\right],\ m(T)=[T^{\epsilon_5}],$$

where we take a sufficiently small $\epsilon_5\in(0,1/2-\beta)$. *Then there exist positive* $\epsilon_1,\epsilon_2,\epsilon,\epsilon^*$ *such that* $\epsilon_5<\epsilon$ *and all the conditions in* $[A1]$ *are fulfilled.*

Proof of Lemma 2. It suffices to show $[A1]$(v). Let $\alpha(h|\mathcal{C})=1$ if $h<0$ for convenience. Define a random variable G by

$$G=\sum_{g=-1}^{N_T}\alpha(\delta g|\mathcal{C})^{1/3}\cdot\max_{j:0\le j\le N_T}\left(P_{\mathcal{C}}\left[\left|\tilde{Z}_{T,j}\right|^3\right]\right)^{2/3}.$$

Then

$$\begin{aligned}\max_{\substack{I:I\subset\{0,1,\dots,N_T\}\\ \#I\le n}}P_{\mathcal{C}}\left[\left(\frac{1}{\sqrt{T}}\sum_{j\in I}\tilde{Z}^{(a)}_{T,j}\right)^2\right]&=\max_{\substack{I:I\subset\{0,1,\dots,N_T\}\\ \#I\le n}}T^{-1}\sum_{i,j\in I}Cov_{\mathcal{C}}\left[\tilde{Z}^{(a)}_{T,i},\tilde{Z}^{(a)}_{T,j}\right]\\&\lesssim\max_{\substack{I:I\subset\{0,1,\dots,N_T\}\\ \#I\le n}}T^{-1}\sum_{g=-1}^{N_T}|I|\alpha(\delta g|\mathcal{C})^{1/3}\\&\quad\cdot\max_{j:0\le j\le N_T}\left(P_{\mathcal{C}}\left[|\tilde{Z}_{T,j}|^3\right]\right)^{2/3}\le\frac{n}{T}G.\end{aligned}\tag{28}$$

Let us estimate $P[\Omega_0(T)^c]$. First,

$$P[\Omega_1(T)^c]\le a^{-1}(\frac{4}{x_1})^{k(T)}e^{-a\delta m(T)}\lesssim e^{-\frac{a}{2}\delta T^{\epsilon_5}}.$$

With (28), we have

$$\begin{aligned}P\left[\max_{\substack{I:I\subset\{0,1,\dots,N_T\}\\ \#I\le n}}P_{\mathcal{C}}\left[\left(\frac{1}{\sqrt{T}}\sum_{j\in I}\tilde{Z}^{(a)}_{T,j}\right)^2\right]\ge A\right]&\le P\left[\frac{n}{T}G\ge A\right]\\&\le\left(\frac{n}{TA}\right)^L P\left[G^L\right].\end{aligned}$$

For any $L\in\mathbf{N}$ $(L\ge 2)$ and any $\epsilon'''>0$,

$$\begin{aligned}P\left[G^L\right]&\le\left\{P\left[\left(\sum_{g=-1}^{N_T}\alpha(\delta g|\mathcal{C})^{\frac{1}{3}}\right)^{2L}\right]\right\}^{\frac{1}{2}}\\&\quad\cdot\left\{P\left[\max_{j:0\le j\le N_T}\left(P_{\mathcal{C}}\left[\left|\tilde{Z}_{T,j}\right|^3\right]\right)^{\frac{4L}{3}}\right]\right\}^{\frac{1}{2}}\\&\lesssim\left\{P\left[\left(\sum_{g=-1}^{N_T}\alpha(\delta g|\mathcal{C})^{\frac{1}{3}}\right)^{2L}\right]\right\}^{\frac{1}{2}}\cdot T^{\epsilon'''}\end{aligned}$$

from [A2]. Furthermore, for ϵ' $(0 < \epsilon' < 1)$,

$$
\begin{aligned}
P\left[\left(\sum_{g=-1}^{N_T} \alpha(\delta g|\mathcal{C})^{\frac{1}{3}}\right)^{2L}\right] &\leq 2^{2L-1}\left\{P\left[\left(\sum_{g=-1}^{[T^{\epsilon'}]} \alpha(\delta g|\mathcal{C})^{\frac{1}{3}}\right)^{2L}\right]\right. \\
&\qquad \left. + P\left[\left(\sum_{g=[T^{\epsilon'}]+1}^{N_T} \alpha(\delta g|\mathcal{C})^{\frac{1}{3}}\right)^{2L}\right]\right\} \\
&\lesssim T^{2L\epsilon'} + T^{2L} P\left[T^{-1} \sum_{g=[T^{\epsilon'}]+1}^{N_T} \alpha(\delta g|\mathcal{C})^{\frac{2L}{3}}\right] \\
&\leq T^{2L\epsilon'} + T^{2L-1}\left\|\sum_{g=[T^{\epsilon'}]+1}^{N_T} \alpha(\delta g|\mathcal{C})\right\|_1 .
\end{aligned}
$$

It then follows from $[A1']$ that

$$
\left\|\sum_{g=[T^{\epsilon'}]+1}^{N_T} \alpha(\delta g|\mathcal{C})\right\|_1 \lesssim \sum_{g=[T^{\epsilon'}]+1}^{N_T} e^{-a\delta g} \lesssim e^{-a\delta T^{\epsilon'}/2}.
$$

After all, for any L and any $\epsilon'' > 0$, if ϵ' and ϵ''' are taken sufficiently small, we obtain

$$
P\left[\max_{\substack{I:I\subset\{0,1,\dots,N_T\} \\ \#I\leq n}} P_{\mathcal{C}}\left[\left(\frac{1}{\sqrt{T}}\sum_{j\in I}\tilde{Z}^{(a)}_{T,j}\right)^2\right] \geq A\right] \lesssim \left(\frac{n}{TA}\right)^L T^{\epsilon''}.
$$

Therefore, we have $P[\Omega_2(T)^c] = O(T^{-L_2})$ when $n = CT^{\epsilon_5}$ and $A = x_1^4 T^{-2\epsilon_1}/4$, and $P[\Omega_4(T)^c] = O(T^{-L_2})$ when $n = N_T$ and $A = -(\log x_2)T^{\epsilon_3 - 2\epsilon_4}$. Showing $P[\Omega_3(T)^c] = O(T^{-L_2})$ from [A2] is a simple matter. □

Proof of Lemma 3. We will show $T^{L_2}P[\Omega_i(T)^c] = O(1)$ $(i = 1, 2, 3, 4)$. For $\Omega_2(T)$ and $\Omega_3(T)$, the proof is the same as that of Lemma 2. We see that

$$
P\left[\Omega_1(T)^c\right] \leq \left(\frac{4}{x_1}\right)^{k(T)} \|\alpha(\delta m(T)|\mathcal{C})\|_1 \lesssim \exp\left(-c'(\log T)^{1+a}\right)
$$

for large T, with a certain positive constant c'. By using $[A2]$, one has

$$
\max_{I:I\subset\{0,1,\dots,N_T\}} P_{\mathcal{C}}\left[\left(\frac{1}{\sqrt{T}}\sum_{j\in I}\tilde{Z}^{(a)}_{\tilde{I}_j} - \frac{1}{\sqrt{T}}\sum_{j\in I} Z^{(a)}_{\tilde{I}_j}\right)^2\right] \lesssim 1
$$

uniformly on $\Omega_3(T)\cap\Omega_1(T)$ for large T. [This estimate can be obtained if one splits this quadratic form into two parts according to the maximal gap and uses the covariance inequalities.] Therefore, with $[A1'']$,

$$P\left[\Omega_4(T)^c \cap \Omega_3(T)\cap\Omega_1(T)\right] \lesssim \exp\left(-(-2\log x_2)^b \frac{M_1^b}{(4L_3)^{b(1+a'-1/b)}} \log T\right) \times \sup_{T>T_0} P\left[\exp\left(\Phi_T^b((\tilde{I}_j))+1\right)\right].$$

It completes the proof. □

Proof of Lemma 4. By the choice of x_2, $\Omega_4(T)^c \cap \Omega_3(T) = \phi$ for large T, and it suffices to estimate $P[\Omega_i(T)^c]$, $i = 1, 3$.

$$P\left[\Omega_1(T)^c\right] \lesssim \left(\frac{4}{x_1}\right)^{k(T)} \quad \frac{1}{T^L} \leq T^{L_1(\log 2)^{-1}\log(4/x_1)-L} \lesssim T^{-r}$$

as $T \to \infty$, if one chooses large L for every $r > 0$. The same estimate as before applies to $P[\Omega_3(T)^c]$. □

6.2. Expansion of the conditional characteristic function

For real random variables X and V, define **complex-valued conjugate conditional expectation** (CVCCE) $P_{\mathcal{C}}[X](V)$ by

$$P_{\mathcal{C}}[X](V) = P_{\mathcal{C}}^0[X](V)/P_{\mathcal{C}}^0[1](V)$$

if $\quad P_{\mathcal{C}}[1](V) \neq 0$, where

$$P_{\mathcal{C}}^0[X](V) = P_{\mathcal{C}}[Xe^{iV}].$$

Suppose that μ is a C-valued probability measure on $(\Omega, \mathcal{F})$, i.e., $\mu = \mu_1 + i\mu_2$, μ_1, μ_2 being real-valued finite signed measures, and $\mu[\Omega] = 1$. For real random variables $X_1, ..., X_r$, the complex cumulants $\kappa^\mu[X_1, ..., X_r]$ are defined by

$$\kappa^\mu[X_1, ..., X_r] = (-i)^r(\partial_{\epsilon_1})_0 \cdots (\partial_{\epsilon_r})_0 \log \mu\left[\exp\left(i\epsilon_1 X_1 + \cdots + i\epsilon_r X_r\right)\right].$$

Complex cumulant $\kappa^\mu[X_1, ..., X_r]$ is symmetric in $X_1, ..., X_r$ and multi-linear, and it satisfies the usual cumulant-moment relations and the cumulant-covariance relations. For real random variables $X_1, .., X_r$ and V, $\kappa_{\mathcal{C}}[X_1, ..., X_r](V)$ is defined as the complex cumulant for the complex $\mathcal{C}$-conditional probability measure $\mu = P_{\mathcal{C}}[\cdot](V)$.

Let $I \in \{0, 1, ..., N_T\}^r$. For $u \in \mathbf{R}^d$, define $S_I^{[\alpha]}$ by

$$S_I^{[\alpha]} = iT^{-\frac{1}{2}} \sum_{j:d(j,I)\geq\alpha} u \cdot \tilde{Z}_{T,j}.$$

Let ψ_T be a truncation functional ([0, 1]-valued random variable) which satisfies $\psi_T \leq 1_{\Omega_0(T)}$. The following lemmas will be proved in Section 6.5.

Lemma 5. *Suppose that* [$A1$] *and* [$A2$] *hold. Then for any* $r \in \mathbf{N}$ $(r \leq \bar{l} + p + 1)$ *and* $a_1, ..., a_r \in \{1, ..., d\}$*, there exist* $\delta_1 \in (0, 1/(4p))$ *and* $c_1 > 0$ *such that*

$$1_{\{\psi_T>0\}} 1_{\tilde{U}_T}(u)|u|^j \left|\kappa_{\mathcal{C}}\left[S_T^{*(a_1)}, ..., S_T^{*(a_r)}\right](\eta u \cdot S_T^*)\right| \lesssim \begin{cases} T^{-(r-2)/2+\delta_1} & \text{if } r \leq p \\ T^{-(p-2)/2-(\frac{1}{2}-\delta_1)} & \text{if } p+1 \leq r \leq \bar{l}+p+1 \end{cases}$$

uniformly for $\eta \in [0, 1]$ *and* j $(0 \leq j \leq p')$*, where* $\tilde{U}_T = \{u \in \mathbf{R}^d; |u| \leq \min\{c_1 T^{\frac{1}{2}-\beta-2\epsilon}, l(T)\}\}$.

Set $S_T = Z_T/\sqrt{T}$.

Lemma 6. *Let* $r \leq p$*. Then*

$$\begin{aligned} &\left|\kappa_{\mathcal{C}}\left[S_T^{*(a_1)}, ..., S_T^{*(a_r)}\right](0) - \kappa_{\mathcal{C}}\left[S_T^{(a_1)}, ..., S_T^{(a_r)}\right](0)\right| \\ &\lesssim M_T(\omega)\left\{T^{-\frac{r}{2}+1-(p+1-r)\beta} m(T)^{r-1} + T^{\frac{r}{2}}\{\alpha(\delta m(T)|\mathcal{C})\}^{1/(p+1)}\right\} \\ &\lesssim M_T(\omega)\left\{T^{-\frac{r}{2}+1+(r-1)\epsilon-(p+1-r)\beta} + T^{\frac{r}{2}}\{\alpha(\delta m(T)|\mathcal{C})\}^{1/(p+1)}\right\} \quad \textit{under } [A1]. \end{aligned}$$

Lemma 7. *There exists* $\delta_2 > 0$ *such that*

$$1_{\{\psi_T>0\}} 1_{\tilde{U}_T}(u)\left|(\partial_u)^l\left\{H_T(u, \mathcal{C}) - \hat{\Psi}_{T,p,\mathcal{C}}(u)\right\}\right| \lesssim T^{-\frac{p-2}{2}-\delta_2-d\epsilon^*}$$

for $l \leq \bar{l}$.

Remark 8. The estimate in Lemma 7 holds uniformly in all partitions $(\tilde{I}_j)$ each subinterval of which is greater than δ and less than $\bar{\delta}$ in length.

6.3. Conditional type Cramér conditions and the proof of Theorem 1

We will prove Theorem 1 with Lemma 8, Proposition 1 and Lemma 9 below. We will give proof of Lemmas 8 and 9 in Section 6.4.

[A3$^\flat$] There exist positive constants $\eta_1, \eta_2, \eta_3, B$ $(\eta_1 + \eta_2 < 1, \eta_3 < 1)$, and truncation functionals $\psi_j : (\Omega, \mathcal{F}) \to ([0, 1], \mathbf{B}([0, 1]))$ [11] such that,

(i) $\sup_{u:|u|\geq B}\left|P_{\hat{\mathcal{C}}(j)}\left[\psi_j e^{iu\cdot Z_{I(j)}}\right]\right| \leq \eta_1$ a.s. for every j.

(ii) For functionals

$$p_j(\hat{\mathcal{C}}(j)) := P_{\hat{\mathcal{C}}(j)}\left[\left(1-\psi_j\right) + 2\left(1-\varphi(T^{-\beta}Z_{I(j)})\right)\right],$$

$$P\left[\#\left\{j; p_j(\hat{\mathcal{C}}(j)) \geq \eta_2\right\} \geq \eta_3 n'(T)\right] = o(T^{-M_1}).$$

[11] ψ_j are distinct from 'ψ_T'.

Remark 9. It is possible to state $[A3^\flat]$ with a truncation ψ_T. However, we can make $\mathcal{C}$-conditional ψ_j have that truncation factor from the beginning. Consequently, our set of conditions includes such truncated version of conditions.

$[\mathrm{A1}^\#]$ For any $L_3 > 0$, there exist increasing sequences $m(T), k(T), l(T)$, and positive constants $L_1, L_2, x_1, x_2, \epsilon_1, \epsilon_2, \epsilon, \epsilon^*$ for which $[A1]$ is satisfied and
(#) $l(T)^2(\log T)^{-1} \geq L_3$ for large T.

Lemma 8. *Let $p \in \mathbf{N}$ $(p \geq 3)$, $K \geq \frac{1}{2}$, $M > 0$, $q > 1$. Let $q' = q/(q-1)$ and $k_0 = (p-2)(3d-2)$. Let $I(j) = [u(j), v(j)]$ $(j = 1, ..., n'(T); T > 0)$ be dense reduction intervals with $(\hat{\mathcal{C}}(j))_{j=1}^{n'(T)}$. Suppose that $[A1]$, and $[A3^\flat]$ are satisfied for some $M_1 > \bar{l} + \frac{p-2}{2} + Kd$ and some β. Moreover, assume the compatibility condition that $(I(j)) \subset (\tilde{I}_j)$.*

(i) Suppose that $l(T) \sim T^{\epsilon_4}$ for some $\epsilon_4 > 0$. If there exists a positive constant a such that $\|\alpha(h|\mathcal{C})\|_1 \leq a^{-1}e^{-ah}$ for all $h > 0$, and if $[A2]$ with $L > k_0qp(p+1)^{-1}$ holds, then for a sequence of positive numbers s_T such that $s_T > T^{-c'}$ for sufficiently small $c' > 0$, and for any sequence of positive numbers u_T with $\liminf_{T\to\infty} u_T > 0$, Inequality (2) with $\theta = 1/q'$ holds for any $f \in \mathcal{E}(M, p_0)$, for some positive constants M^* and δ^*.
(ii) Suppose that $l(T)^2 \sim (\log T)^{1+c_1}$ for some $c_1 > 0$. If there exist positive constants a, c, C such that $\|\alpha(h|\mathcal{C})\|_1 \leq C\exp\left(-c(\log h)^{1+a}\right)$ for all $h > 0$, and if $[A2]$ with $L > k_0qp(p+1)^{-1}$ is satisfied, then Inequality (2) with $\theta = 1/q'$ holds for a sequence s_T satisfying $s_T \geq (\log T)^{-c'}$ for sufficiently small positive c', and any sequence u_T of positive numbers with $\liminf_{T\to\infty} u_T > 0$.
(iii) Assume $[A1^\#]$. If for some $\xi \geq k_0q(p+1)$, $\|\alpha(h|\mathcal{C})\|_1 = O(h^{-L'})$ for some constant $L' \geq \xi p(p-1)/(p-2)$, and $[A2]$ is satisfied for some $L \geq p[(p+1)((k_0q)^{-1} - \xi^{-1})]^{-1}$, then Inequality (2) with $\theta = 1/q'$ holds for a sequence s_T satisfying $\liminf_T s_T > 0$ and any sequence u_T of positive numbers with $\liminf_{T\to\infty} u_T > 0$.

Remark 10. In the above lemma, besides $[A1]$, we specified the rate of convergence of $\|\alpha(h|\mathcal{C})\|_1$. It is not redundant since $[A1]$ does not regulate the contribution of the truncated events to the moments.

We provided a precise relation between the speed at which the random mixing coefficient tends to zero, the degree of nondegeneracy of the conditional variance of $Z_T/\sqrt{T}$, and the order of necessary moments, however Condition $[A1^\#]$ is still cumbersome. We will present more simple statements in cases $[A1']$, $[A1'']$ and $[A1''']$.

Corresponding to $[A1']$, $[A1'']$ and $[A1''']$, under $[A3^\flat]$, we obtain the following results as corollaries to Lemma 8. As it is essentially much more general than Theorem 1 (it is the case when the maximum length of the reduction intervals tends to infinity as $T \to \infty$, for example), we will state the results as a proposition. Proof is easy and omitted, however we only note that a partition $(\tilde{I}_j)$ compatible with $(I(j))$ can be constructed and we fix one in the following proposition.

Proposition 1. *Let $I(j)$ be dense reduction intervals with $(\hat{\mathcal{C}}(j))_{j=1}^{n'(T)}$. Suppose that Conditions* $[A1']$ *[resp.* $[A1'']$, $[A1''']$*],* $[A2]$ *(for any $L > 0$) and* $[A3^\flat]$ *(for every $M_1 > 0$ and for $\beta \in (0, 1/2)$ chosen in Lemma 2 [resp. Lemma 3, Lemma 4]) are satisfied. Then for any $K, M > 0$, $\theta \in (0, 1)$ and any sequences s_T satisfying that $s_T \geq T^{-c'}$ for sufficiently small positive c' [resp. $s_T \geq (\log T)^{-c'}$ for sufficiently small positive c', $\liminf_{T\to\infty} s_T > 0$], and any sequence u_T of positive numbers with $\liminf_{T\to\infty} u_T > 0$, there exist positive constants M^* and δ^* for which (2) holds. Moreover, if $\liminf_{T\to\infty} u_T / T^{c''} > 0$ for some constant $c'' > 0$, then (3) holds.*

Condition $[A3^\flat]$ is stated in terms of a $\vee_j \hat{\mathcal{C}}(j)$-measurable counting process. On the other hand, it is possible to replace it with a condition in terms of a $\mathcal{C}$-measurable process under a weak additional condition.

Lemma 9. *Assume* $[A2]$ *(for any $L > 0$) and assume* $[A1']$ *[resp.* $[A1'']$, $[A1''']$ *]. Let $I(j)$ be dense reduction intervals with $(\hat{\mathcal{C}}(j))_{j=1}^{n'(T)}$. Suppose that there exists a positive constant ϱ such that $\hat{\mathcal{C}}(j) \subset \mathcal{B}_{I(j)^\varrho} \vee \mathcal{C}$, where $I^\varrho = \{t; dist(t, I) \leq \varrho\}$. Then Condition* $[A3]$ *implies that there exists a compatible partition $(\tilde{I}_j)$ with $(I(j))$ for which Condition* $[A3^\flat]$ *is satisfied for any $M_1 > 0$ and $\beta \in (0, 1/2)$ chosen in Lemma 2 [resp. Lemma 3, Lemma 4].*

Proof of Theorem 1. Theorem 1 follows from Proposition 1 and Lemma 9. □

Remark 11. With an additional moment condition, $[A3^\flat]$ can be replaced by: $[A3^+]$ There exist positive constants $\eta_1, \eta_2, \eta_3, B$ $(\eta_1 + \eta_2 < 1,\ \eta_3 < 1)$, and truncation functionals $\psi_j : (\Omega, \mathcal{F}) \to ([0, 1], \mathbf{B}([0, 1]))$ such that, (i) $\sup_{u:|u|\geq B} \left| P_{\hat{\mathcal{C}}(j)}\left[\psi_j e^{iu\cdot Z_{I(j)}}\right]\right| \leq \eta_1$ a.s. for every j. (ii) For functionals $p_j(\hat{\mathcal{C}}(j)) := P_{\hat{\mathcal{C}}(j)}\left[1 - \psi_j\right]$, $P\left[\#\left\{j; p_j(\hat{\mathcal{C}}(j)) \geq \eta_2\right\} \geq \eta_3 n'(T)\right] = o(T^{-M_1})$ for every $M_1 > 0$. We then see that it is possible to prove the same assertion as in Theorem 1 under $[A1']$ [resp. $[A1'']$, $[A1''']$], $[A3^+]$ and the moment condition $[A2^*]$: For every $L > 0$, $\sup_{\substack{t,h,T:0\leq t\leq t+h\leq T\\ 0\leq h\leq\bar{\delta}, T\in\mathbf{R}_+}} P\left[\left|Z_{t+h}^t\right|^L\right] < \infty$, and the same inequality with Z_{t+h}^t replaced by Z_0 holds. Furthermore, $P_{\mathcal{C}}[Z_{t+h}^t] = 0$ and $P_{\mathcal{C}}[Z_0] = 0$.

Remark 12. Proof of theorems will be presented in the next Section 6.4. We treated conditional ϵ-Markov processes in Section 3, showing the reduction property, and as an example, discussed a random-coefficient stochastic differential equation with jumps in Section 5. Hidden Markov models, and more generally, latent-variable models with strongly dependent components are also examples to those later sections. The results in this section are more general: to say nothing of the conditional ϵ-Markov process, it also applies to non-Markovian processes like *m-dependent sequences*. We do not give the details but let us briefly discuss a cluster model for rainfalls. The centers T_j^c $(j \in \mathbf{Z})$ of storms are dispersed on the time-axis $\mathbf{R}$ according to a Poisson random measure with rate λ. The subsidiary points T_{jk}^s $(k \in \mathbf{N})$ and the rainfall intensity processes $X_{jk}(t)$ are attached to each T_j^c. Assume that those processes are independent, and that the occurrence of the subsidiaries and the influence of the intensity are limited within a bounded length. The rainfall $Y(t)$ at time t is modeled by $Y(t) = \sum_{j,k} X_{jk}(t - T_j^c - T_{jk}^s)$. The real data Y_n^h are given by aggregating $Y(t)$: $Y_n^h = \int_{nh}^{(n+1)h} Y(t)dt$. Then it is not difficult to find a reduction sequence. Moreover, the conditioning enables us to deal with a doubly stochastic Poisson process (T_j^c) with strongly dependent intensity $\lambda(c,t)$, which would be natural to explain real data. See Cox and Isham [13] for the cluster models (Neyman-Scott model and Bartlett-Lewis model), and see Sakamoto and Yoshida [54] for asymptotic expansion of the Yule-Walker estimator.

6.4. Proof of Lemmas 8 and 9

Let $p_1 \in (1,\infty)$.

Lemma 10. *Suppose that* $[A1^{\#}]$ [12] *and* $[A3^{\flat}]$ *are satisfied for some* $M_1 > p_1 \left[\bar{l} + \frac{p-2}{2} + Kd\right]$ *and some* β. *Assume the compatibility condition* $(I(j)) \subset (\tilde{I}_j)$. *Let* $a_1, ..., a_l \in \{a \in \mathbf{R}^d; |a| \le 1\}$. *Then there exists a positive constant* δ''' *such that*

$$\left\| (\partial_u)^l H_T(u, \mathcal{C})[a_1, ..., a_l] \right\|_{p_1} \lesssim T^{-(\frac{p-2}{2} + \delta''' + Kd)}$$

for u $(|u| \ge l(T))$ *and* $l \le \bar{l}$.

Proof. (a) Put $v(0) = 0$ and $u(n'(T)+1) = T$. Let $C = \{j; \tilde{I}_j = I(k_j)$ for some k_j $(1 \le k_j \le n'(T))\ \}$. For $i_1, ..., i_l \in \{0, 1, ..., N_T\}$, let $C(i_1, ..., i_l) = C \setminus \{i_1, ..., i_l\}$. Moreover, let (c_k) $(c_1 < c_2 < \cdots < c_{\#C(i_1,...,i_l)})$ denote all the elements of $C(i_1, ..., i_l)$, and let $c_0 := -1$ and $c_{\#C(i_1,...,i_l)+1} := N_T + 1$. Define $j_n \in \{1, ..., n'(T)\}$ so that $I(j_n) = \tilde{I}_{c_n}$ for $c_n \in C(i_1, ..., i_l)$, and let $C^*(i_1, ..., i_l) = \{j_n; n = 1, ..., \#C(i_1, ..., i_l)\} \subset \{1, ..., n'(T)\}$. Take K_n and L_n as

$$K_n = \prod_{m: c_{n-1} < m < c_n} \left\{ \left(\tilde{Z}_{T,m}^{(a_1)}\right)^{1_{\{i_1\}}(m)} \cdots \left(\tilde{Z}_{T,m}^{(a_l)}\right)^{1_{\{i_l\}}(m)} \exp\left(iT^{-\frac{1}{2}} u \cdot \tilde{Z}_{T,m}\right) \right\},$$

$a_1, ..., a_l \in \{1, 2, ..., d\}$, and

$$L_n = \exp\left(iT^{-\frac{1}{2}} u \cdot \tilde{Z}_{T,c_n}\right) = \tilde{e}_{j_n}(T^{-\frac{1}{2}} u),$$

where

$$\tilde{e}_j(u) = \exp\left\{ iu \cdot \left(\varphi_T\left(Z_{I(j)}\right) - P_{\mathcal{C}}[\varphi_T\left(Z_{I(j)}\right)]\right) \right\}$$

with $Z_{I(j)} = Z_{v(j)} - Z_{u(j)}$.

[12] We only need (#) for some sufficiently large L_3.

(b) Put $\tilde{Z}^{\star}_{I(j)} = \varphi_T(Z_{I(j)}) = \tilde{Z}_{I(j)} + P_{\mathcal{C}}[\varphi_T(Z_{I(j)})]$. Since $\tilde{Z}^{\star}_{I(j)} = Z_{I(j)}$ when $|Z_{I(j)}| \le T^{\beta}$,

$$
\begin{aligned}
\left|P_{\hat{\mathcal{C}}(j)}\left[\tilde{e}_j(u)\right]\right| &= \left|P_{\hat{\mathcal{C}}(j)}\left[e^{iu\cdot\tilde{Z}^{\star}_{I(j)}}\right]\right| \\
&\le \left|P_{\hat{\mathcal{C}}(j)}\left[\psi_j e^{iu\cdot Z_{I(j)}}\right]\right| + \left|P_{\hat{\mathcal{C}}(j)}\left[(1-\psi_j)e^{iu\cdot Z_{I(j)}}\right]\right| \\
&\quad + \left|P_{\hat{\mathcal{C}}(j)}\left[\left\{1-\varphi\left(T^{-\beta}Z_{I(j)}\right)\right\}\left(e^{iu\cdot\tilde{Z}^{\star}_{I(j)}} - e^{iu\cdot Z_{I(j)}}\right)\right]\right| \\
&\le \eta_1 + P_{\hat{\mathcal{C}}(j)}\left[1-\psi_j\right] + 2P_{\hat{\mathcal{C}}(j)}\left[1-\varphi\left(T^{-\beta}Z_{I(j)}\right)\right]
\end{aligned}
$$

for $|u| \ge B$. Then, if $p_j(\hat{\mathcal{C}}(j)) < \eta_2$, then $\left|P_{\hat{\mathcal{C}}(j)}\left[\tilde{e}_j(u)\right]\right| \le \eta_4$ for $|u| \ge B$, where $\eta_4 = \eta_1 + \eta_2 < 1$. Petrov's lemma yields $|P_{\hat{\mathcal{C}}(j)}\left[\tilde{e}_j(u)\right]| \le e^{-\eta_5|u|^2}$ for $|u| \le B$ on the same events. Note that η_5 is a universal constant independent of ω. [We are in a slightly different situation than Lemma (3.2) of Götze and Hipp [18] while it can be proved in a similar way: let $p(\omega, dx)$ be a regular conditional distribution of $\tilde{Z}_{I(j)}$ given $\hat{\mathcal{C}}_j$. Then $g(u) := P_{\mathcal{C}}\left[\left|P_{\hat{\mathcal{C}}_j}\left[e^{iu\cdot\tilde{Z}_{I(j)}}\right]\right|^2\right]$ admits a representation

$$
g(u) = P_{\mathcal{C}}\left[\int_{\mathbf{R}^d}\int_{\mathbf{R}^d} p(\omega, dx)p(\omega, dy)\cos\left(u\cdot(x-y)\right)\right],
$$

and hence an iterating inequality $1 - g(2^m u) \le 4^m(1 - g(u))$. The rest is the same as in Götze and Hipp [18].]

(c) On the other hand, we obtain the following estimate by using a property of the dense reduction sequence : for $a_1, ..., a_l \in \{1, ..., d\}$,

$$
\begin{aligned}
\left|\partial_{u_{a_1}}\cdots\partial_{u_{a_l}} H_T(u, \mathcal{C})\right| &\le T^{-\frac{l}{2}} \sum_{i_1,\dots,i_l=0}^{N_T} \left|P_{\mathcal{C}}\left[e^{iu\cdot S_T^*}\tilde{Z}^{(a_1)}_{T,i_1}\cdots\tilde{Z}^{(a_l)}_{T,i_l}\right]\right| \\
&= T^{-\frac{l}{2}} \sum_{i_1,\dots,i_l=0}^{N_T} \left|P_{\mathcal{C}}\left[(\Pi_{j=1}^{\#C(i_1,\dots,i_l)+1} K_j)\right.\right. \\
&\qquad \left.\left.\times\ (\Pi_{j\in C^*(i_1,\dots,i_l)} P_{\hat{\mathcal{C}}(j)}[\tilde{e}_j(T^{-\frac{1}{2}}u)])\right]\right| \\
&\lesssim T^{-\frac{l}{2}+l\beta} \sum_{i_1,\dots,i_l=0}^{N_T} P_{\mathcal{C}}\left[\Pi_{j\in C^*(i_1,\dots,i_l)}\right. \\
&\qquad \left.\times \left|P_{\hat{\mathcal{C}}(j)}[\tilde{e}_j(T^{-\frac{1}{2}}u)]\right|\right].
\end{aligned}
$$

(d) Let $\Omega^*(T)=\left\{\#\{j;\,p_j(\hat{\mathcal{C}}(j))\geq\eta_2\}<\eta_3 n'(T)\right\}$. Then

$$\begin{aligned}
&\left\|\sum_{i_1,\dots,i_l=0}^{N_T}P_{\mathcal{C}}\left[\Pi_{j\in C^*(i_1,\dots,i_l)}\left|P_{\hat{\mathcal{C}}(j)}\left[\tilde{e}_j(T^{-\frac{1}{2}}u)\right]\right|\right]\right\|_{p_1}\\
&\quad\leq\sum_{i_1,\dots,i_l=0}^{N_T}\left\{\left\|P_{\mathcal{C}}\left[1_{\Omega^*(T)}\Pi_{j\in C^*(i_1,\dots,i_l)}\left|P_{\hat{\mathcal{C}}(j)}\left[\tilde{e}_j(T^{-\frac{1}{2}}u)\right]\right|\right]\right\|_{p_1}\right.\\
&\qquad\left.+\left\|P_{\mathcal{C}}\left[1_{\Omega^*(T)^c}\right]\right\|_{p_1}\right\}\\
&\quad\lesssim T^{\bar{l}}\left(e^{-\eta_5 l(T)^2/T}\vee\eta_4\right)^{\eta_6 n'(T)}+o(T^{\bar{l}-M_1/p_1})\\
&\quad\lesssim T^{-(\frac{p-2}{2}+\delta'''+Kd)}
\end{aligned}$$

for u ($|u|\geq l(T)$), where $[A1^{\#}]$ was used. We may use the inequality in Step (c) to complete the proof. □

Proof of Lemma 8. The desired result will be obtained by showing a series of inequalities. Set

$$e_T=\frac{1}{\sqrt{T}}\left\{P_{\mathcal{C}}\left[\varphi_T\left(Z_0\right)\right]+\sum_{j=1}^{N_T}P_{\mathcal{C}}\left[\varphi_T\left(Z_{\tilde{I}_j}\right)\right]\right\}.$$

(a) *Estimate of* e_T.

$$\begin{aligned}
\left|P_{\mathcal{C}}\left[\varphi_T\left(Z_{\tilde{I}_j}\right)\right]\right|&=\left|P_{\mathcal{C}}\left[\varphi_T\left(Z_{\tilde{I}_j}\right)-Z_{\tilde{I}_j}\right]\right|\leq P_{\mathcal{C}}\left[\left|\varphi_T\left(Z_{\tilde{I}_j}\right)-Z_{\tilde{I}_j}\right|\right]\\
&\leq P_{\mathcal{C}}\left[\left|Z_{\tilde{I}_j}\right|1_{\{\left|Z_{\tilde{I}_j}\right|\geq T^{\beta}\}}\right]\\
&\leq T^{-p\beta}M_T(\omega)\lesssim T^{-p\beta+\epsilon_2}\leq T^{-\frac{p-2}{2}-\frac{1}{2}-\epsilon_8}
\end{aligned}$$

on $\{\psi_T>0\}$, for some $\epsilon_8>0$. Here we used $[A1]$(iv) and the inequality: $-p\beta+\epsilon_2<-p\beta+1/2-\beta=-(p+1)\beta+1/2<-p/2+1/2-\epsilon_8$. Thus, though it is too strong an estimate for our use, we obtain

$$1_{\{\psi_T>0\}}\left|e_T\right|\lesssim T^{-\frac{p-2}{2}-\epsilon_8}=o(1). \tag{29}$$

(b) *Estimate for* $P_{\mathcal{C}}\left[f\left(T^{-\frac{1}{2}}Z_T\right)\right]$. We will first estimate the cumulants of $Z_T/\sqrt{T}$ without truncation.

Suppose that either of the following conditions holds true:

(i) (exponential)+($[A2]$ with $L>qp(p+1)^{-1}$ for some $q>1$),
(ii) (hyper-polynomial)+($[A2]$ with $L>qp(p+1)^{-1}$ for some $q>1$),
(iii) ((polynomial) with $L'\geq\xi p(p-1)/(p-2)$) +($[A2]$ with $L\geq p[(p+1)(q^{-1}-\xi^{-1})]^{-1}$ for some $q>1$ and $\xi\geq q(p+1)$).

Let $Z_{T,j} = Z_{\tilde{I}_j}$. Let $r = (p+1)/(p-1)$ and $r' = (p+1)/2$. By using the covariance inequality, for $\xi > q, \xi \geq (p+1)(p-1)^{-1}$, and $\zeta \geq 2[(p+1)(q^{-1} - \xi^{-1})]^{-1}$, we have

$$\begin{aligned}\left\| P_{\mathcal{C}}\left[\left|T^{-\frac{1}{2}} Z_T^{(a)}\right|^2\right]\right\|_q &\leq T^{-1} \sum_{j,k=0}^{N_T} \left\| P_{\mathcal{C}}\left[Z_{T,j}^{(a)} Z_{T,k}^{(a)}\right]\right\|_q \\ &\lesssim T^{-1} \sum_{j,k=0}^{N_T} \left\| \alpha(\delta(|k-j|-1)|\mathcal{C})^{\frac{1}{r}} \left(P_{\mathcal{C}}\left[\left|Z_{T,j}^{(a)}\right|^{2r'}\right]\right)^{\frac{1}{2r'}} \right. \\ &\qquad \left. \times \left(P_{\mathcal{C}}\left[\left|Z_{T,k}^{(a)}\right|^{2r'}\right]\right)^{\frac{1}{2r'}} \right\|_q \\ &\lesssim T^{-1} N_T \left(\sum_{k=0}^{\infty} \|\alpha(\delta k|\mathcal{C})\|_1^{\frac{1}{\xi}} + 1\right) \\ &\qquad \times \sup_j \left\| P_{\mathcal{C}}\left[\left|Z_{T,j}^{(a)}\right|^{p+1}\right]\right\|_\zeta^{2(p+1)^{-1}}.\end{aligned}$$

We then use the conditions to obtain

$$\left\| P_{\mathcal{C}}\left[\left|T^{-\frac{1}{2}} Z_T^{(a)}\right|^2\right]\right\|_q = O(1);$$

Make ξ sufficiently large and ζ sufficiently small in (i) and (ii).

Next, we will estimate higher-order cumulants of $Z_T/\sqrt{T}$ without truncation. A modified proof of Lemma 6 can apply. Let $e_r = \frac{r-2}{2(r-1)} \in [\frac{1}{4}, \frac{1}{2})$ for $r = 3, ..., p$. Put

$$\Phi_1 = T^{-\frac{r}{2}} \sum_{\substack{j_1,...,j_r:0\leq j_1,...,j_r\leq N_T \\ maxgap(j_1,...,j_r)\leq T^{e_r}}} \left|\kappa_{\mathcal{C}}\left[Z_{T,j_1}^{(a_1)}, ..., Z_{T,j_r}^{(a_r)}\right](0)\right|$$

and

$$\Phi_2 = T^{-\frac{r}{2}} \sum_{\substack{j_1,...,j_r:0\leq j_1,...,j_r\leq N_T \\ maxgap(j_1,...,j_r)> T^{e_r}}} \left|\kappa_{\mathcal{C}}\left[Z_{T,j_1}^{(a_1)}, ..., Z_{T,j_r}^{(a_r)}\right](0)\right|.$$

For $j_1, ..., j_r \in \{0, 1, ..., N_T\}$ and a decomposition $I_1 + \cdots + I_l = \{1, ..., r\}$, $3 \leq r \leq p$,

$$\left|\Pi_{m=1}^l P_{\mathcal{C}}[Z_{(j_\alpha;\alpha\in I_m)}]\right| \lesssim \Pi_{i=1}^r P_{\mathcal{C}}\left[\left|Z_{T,j_i}^{(a_i)}\right|^{p+1}\right]^{1/(p+1)}.$$

In view of (40), we see that

$$\|\Phi_1\|_q \lesssim T^{-\frac{r}{2}+1} T^{e_r(r-1)} \sup_{j_i} \left\| \Pi_{i=1}^r P_{\mathcal{C}}\left[\left|Z_{T,j_i}^{(a_i)}\right|^{p+1}\right]^{1/(p+1)}\right\|_q,$$

hence it follows from $[A1]$ and $[A2]$ for $L \geq qr/(p+1)$ that

$$\|\Phi_1\|_q \lesssim \sup_{j_i,i,T} \left\| P_{\mathcal{C}} \left[\left| Z_{T,j_i}^{(a_i)} \right|^{p+1} \right] \right\|_{qr/(p+1)}^{r/(p+1)} = O(1).$$

Let $\xi \geq q$, $\xi \geq (p+1)(p+1-r)^{-1}$ and $\zeta \geq r[(p+1)(q^{-1}-\xi^{-1})]^{-1}$. As for Φ_2, from a similar formula to (41) and the covariance inequality, it holds that

$$\left|\kappa_{\mathcal{C}}\left[Z_{T,j_1}^{(a_1)}, ..., Z_{T,j_r}^{(a_r)}\right](0)\right| \lesssim \left(\Pi_{i=1}^r P_{\mathcal{C}}\left[\left|Z_{T,j_i}^{(a_i)}\right|^{p+1}\right]^{1/(p+1)}\right) \times \left\{\alpha(\delta T^{e_r}|\mathcal{C})\right\}^{(p+1-r)/(p+1)},$$

and hence,

$$\begin{aligned} \|\Phi_2\|_q &\lesssim T^{\frac{r}{2}} \left\|\alpha(\delta T^{e_r}|\mathcal{C})\right\|_1^{1/\xi} \left(\sup_{i,j} \left\| P_{\mathcal{C}}\left[\left|Z_{T,j_i}^{(a_i)}\right|^{p+1}\right]\right\|_{\zeta}\right)^{r/(p+1)} \\ &\leq T^{\frac{r}{2}-\frac{L'e_r}{\xi}} = O(1) \end{aligned}$$

if $L' \geq \frac{r(r-1)}{r-2}\xi$ in (polynomial) and $L \geq r[(p+1)(q^{-1}-\xi^{-1})]^{-1}$ in $[A2]$. In case (i) or (ii), we may take $\xi \to \infty$.

After all, we have proved the boundedness of the L^q-norm of $f(Z_T/\sqrt{T})$ for $f \in \mathcal{E}(M, p_0)$ since the possible maximum number of the cumulant factors in the expression of the moments up to p_0-th order is $p_0/2 \leq k_0$. We here notice that if $3 \leq r \leq p$, then $r(r-1)/(r-2) \leq p(p-1)/(p-2)$.

(c) Each term in the conditional density of $\Psi_{T,p,\mathcal{C}}(dz)/dz$ is written out in terms of the cumulants of $Z_T/\sqrt{T}$ of degree at most p, $(\det\mathrm{Var}_{\mathcal{C}}[Z_T/\sqrt{T}])^{-k}$ $(k = 0, 1, ..., 3(p-2))$ and the Gaussian kernel $\phi(z; 0, \mathrm{Var}_{\mathcal{C}}[Z_T/\sqrt{T}])$. In order to estimate the L^q-norm of each term except for the determinant-factor, considering the integration of a polynomial with respect to a Gaussian kernel, we deduce that it is sufficient to estimate

$$\left\| |\kappa_{\mathcal{C}}[(Z_T/\sqrt{T})^{\otimes p_1}]| \cdots |\kappa_{\mathcal{C}}[(Z_T/\sqrt{T})^{\otimes p_k}]| |\mathrm{Var}_{\mathcal{C}}[Z_T/\sqrt{T}]|^j \right\|_q$$

for $p_1 + \cdots + p_k \leq \bar{p} := 3(p-2)$ and $j \leq \bar{j} := 3(p-2)(d-1)$. Let $k_0 = (p-2) + \bar{j} = (p-2)(3d-2)$. From Step (b), we see that each term in $\Psi_{T,p,\mathcal{C}}[f]$ is $O(1)$ in L^q-norm under any one of the following conditions:

(i) (exponential)+([$A2$] with $L > k_0qp(p+1)^{-1}$ for some $q > 1$),
(ii) (hyper-polynomial)+([$A2$] with $L > k_0qp(p+1)^{-1}$ for some $q > 1$),
(iii) ((polynomial) with $L' \geq \xi p(p-1)/(p-2)$) +([A2] with $L \geq p[(p+1)((k_0q)^{-1}-\xi^{-1})]^{-1}$ for some $q > 1$ and $\xi \geq k_0q(p+1)$).

(d) Let us complete the proof. It follows from Lemmas 11 (below, and Remark 14), 7, 10 and Steps (a) and (b) that $\Delta_T(f) \equiv \left\| P_{\mathcal{C}}\left[f\left(\frac{1}{\sqrt{T}}Z_T\right)\right] - \Psi_{T,p,\mathcal{C}}[f]\right\|_1$ satisfies

$$\Delta_T(f) \leq 2C\,\|1-\psi_T\|_{q'} + C_d M \sum_{\alpha:|\alpha|\leq d+1+p_0} \int_{|u|\geq l(T)} \left\| \psi_T \partial_u^\alpha \left[\hat{\Psi}_{T,p,\mathcal{C}}(u)\hat{\mathcal{K}}(T^{-K_1}u)e^{iu\cdot e_T}\right]\right\|_1 du$$
$$+C'\left\|\psi_T\omega\left(f;T^{-K},\Psi^+_{T,p,\mathcal{C}}\right)\right\|_1 + \bar{o}(T^{-(p-2+\delta'')/2}),$$

where $K_1(>K)$ is a constant sufficiently close to K, $q'=q/(q-1)$ and δ'' is some positive constant.

Since $\hat{\Psi}_{T,p,\mathcal{C}}(u)$ takes the form:

$$\hat{\Psi}_{T,p,\mathcal{C}}(u) = \exp\left(-\frac{1}{2}\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right][u,u]\right)\sum_{|\mathbf{n}|\leq 3(p-2)} c_{\mathbf{n}}(\mathcal{C})u^{\mathbf{n}}$$

($c_{\mathbf{n}}(\mathcal{C})$ depend on T), the derivatives $\partial_u^l\hat{\Psi}_{T,p,\mathcal{C}}(u)$ $(l\leq\bar{l})$ are dominated by

$$C\exp\left(-\frac{1}{2}\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right][u,u]\right)\sum_{\mathbf{n}}|c_{\mathbf{n}}(\mathcal{C})|\left(\left|\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]\right|+1\right)^{\bar{l}}(|u|^{p''}+1),$$

$p''=3(p-2)+\bar{l}$. Therefore, for a sequence $s_T>0$, on the event $\{\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]\geq s_T I_d\}$, the integration of $\partial_u^l\hat{\Psi}_{T,p,\mathcal{C}}(u)$ over the region $\{|u|\geq l(T)\}$ is dominated by

$$C\sum_{\mathbf{n}}|c_{\mathbf{n}}(\mathcal{C})|\left(\left|\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]\right|+1\right)^{\bar{l}}\int_{|u|\geq l(T)} e^{-\frac{1}{2}s_T|u|^2}(|u|^{p''}+1)du$$
$$\leq C'\sum_{\mathbf{n}}|c_{\mathbf{n}}(\mathcal{C})|\left(\left|\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]\right|+1\right)^{\bar{l}}\left(\frac{1}{\sqrt{s_T}}+1\right)^{p''+d}$$
$$\cdot\exp\left(-\frac{1}{4}s_T l(T)^2\right).$$

For Lemma 8 (i), $s_T=T^{-c'}$ for some $c'>0$. For Lemma 8 (ii), we take $l(T)^2\sim(\log T)^{1+c_1}$ and $s_T=(\log T)^{-c'}$ for some $c_1>c'>0$. For Lemma 8 (iii), $l(T)^2\geq L_3\log T$ and $\liminf_T s_T>0$ by assumption: see $[A1^{\#}]$. In any case, the right-hand side of the above inequality turns out to be asymptotically smaller than T^{-N} for every $N>0$: the Covariance factors are dominated by T^{ϵ} on $\{\psi_T>0\}$, and we also know the L^q-boundedness of $c_{\mathbf{n}}(\mathcal{C})$. If the truncation functional ψ_T is chosen to be the indicator function of the set

$$\Omega_0(T)\ \cap\ \{s_T I_d\leq\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]\leq u_T I_d\},$$

then we have an estimate:

$$\sum_{\alpha:|\alpha|\leq d+1+p_0}\int_{|u|\geq l(T)}\left\|\psi_T\partial_u^\alpha\left[\hat{\Psi}_{T,p,\mathcal{C}}(u)\hat{\mathcal{K}}(T^{-K_1}u)e^{iu\cdot e_T}\right]\right\|_1 du \lesssim T^{-N}$$

as $T\to\infty$, where N is a constant satisfying $N>(p-2)/2$.

It follows from $[A1]$ that

$$\|1-\psi_T\|_{q'}\leq P\left[\Omega_0(T)^c\right]^{1/q'}+P\left[\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]<s_TI_d\right]^{1/q'}$$
$$+P\left[\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]>u_TI_d\right]^{1/q'}.$$

In a similar manner as $\hat{\Psi}_{T,p,\mathcal{C}}$, the measure $\Psi^+_{T,p,\mathcal{C}}$ is expressed by the density

$$\phi(x;0,\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right])\left(\sum_{|\mathbf{n}|\leq 3(p-2)}d_{\mathbf{n}}(\mathcal{C})x^{\mathbf{n}}\right)^+.$$

With this,

$$\begin{aligned}
&\left\|\psi_T\omega\left(f;T^{-K},\Psi^+_{T,p,\mathcal{C}}\right)\right\|_1\\
&\leq\sum_{|\mathbf{n}|\leq 3(p-2)}\left\|\psi_T|d_{\mathbf{n}}(\mathcal{C})|\int_{\mathbf{R}^d}\omega_f(x;T^{-K})|x|^{|\mathbf{n}|}\phi(x;0,\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right])dx\right\|_1\\
&\leq u_T^{d/2}s_T^{-d/2}\sum_{|\mathbf{n}|\leq 3(p-2)}\|\psi_Td_{\mathbf{n}}(\mathcal{C})\|_1\int_{\mathbf{R}^d}\omega_f(x;T^{-K})|x|^{|\mathbf{n}|}\phi(x;0,u_TI_d)dx.\\
&\leq C'u_T^{\gamma(1)}s_T^{-\gamma(2)}\omega_2\left(f;T^{-K},\phi(x;0,u_TI_d)dx\right).
\end{aligned}$$

Notice that the L^1-norms of $d_{\mathbf{n}}(\mathcal{C})$ were evaluated in Step (c).

By using estimates thus far, we obtain:

$$\Delta_T(f)\lesssim P\left[\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]<s_TI_d\right]^{1/q'}+P\left[\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]>u_TI_d\right]^{1/q'}$$
$$+u_T^{\gamma(1)}s_T^{-\gamma(2)}\omega_2\left(f;T^{-K},\phi(x;0,u_TI_d)dx\right)+\bar{o}(T^{-(p-2+\delta'')/2})$$

for some positive constant δ''. □

Remark 13. Under Condition (4), the above proof is modified as follows. Denote $v_{T,\mathcal{C}}=\mathrm{Var}_{\mathcal{C}}[T^{-1/2}Z_T]$. Then

$$\begin{aligned}
\Delta'_T(f)&\leq\left\|P_{\mathcal{C}}[f(T^{-1/2}Z_T)]1_{\{v_{T,\mathcal{C}}<s_TI_d\}}\right\|_1\\
&\quad+\left\|\left(P_{\mathcal{C}}[f(T^{-1/2}Z_T)]-\Psi_{T,p,\mathcal{C}}[f]\right)\psi_T\right\|_1\\
&\quad+\left\|P_{\mathcal{C}}[|f(T^{-1/2}Z_T)|](1_{\Omega_0(T)^c}+1_{\{v_{T,\mathcal{C}}>u_TI_d\}})\right\|_1\\
&\quad+\left\||\Psi_{T,p,\mathcal{C}}|[|f|]1_{\{v_{T,\mathcal{C}}\geq s_TI_d\}}(1_{\Omega_0(T)^c}+1_{\{v_{T,\mathcal{C}}>u_TI_d\}})\right\|_1\\
&\lesssim\|1-\psi_T\|_{q'}+\left\|\left(P_{\mathcal{C}}[f(T^{-1/2}Z_T)]-\Psi_{T,p,\mathcal{C}}[f]\right)\psi_T\right\|_1.
\end{aligned}$$

Thus, for $\Delta'_T(f)$, we obtain the same inequality (up to a constant factor) as the first inequality in Step (d) of the above proof.

With a smooth probability kernel $\mathcal{K}$ on $\mathbf{R}^d$ whose Fourier transform $\hat{\mathcal{K}}$ has a compact support, we obtain a conditional smoothing lemma as follows.

Lemma 11. *For $K_1 > K > 0$, there exist positive constants C_d and δ'' such that*

$$
\begin{aligned}
&\left\| \psi_T \left(P_{\mathcal{C}} \left[f\left(\frac{1}{\sqrt{T}} Z_T \right) \right] - \Psi_{T,p,\mathcal{C}}[f] \right) \right\|_1 \\
&\le C_d M \sum_{\alpha:|\alpha|\le d+1+p_0} \int \left\| \psi_T \partial_u^\alpha \left[\left(H_T(u,\mathcal{C}) e^{iu\cdot e_T} - \hat{\Psi}_{T,p,\mathcal{C}}(u) \right) \hat{\mathcal{K}}(T^{-K_1}u) \right] \right\|_1 du \\
&\quad + C_d \left\| \psi_T \omega\left(f; T^{-K}, \Psi^+_{T,p,\mathcal{C}} \right) \right\|_1 + \bar{o}(T^{-(p-2+\delta'')/2})
\end{aligned}
$$

for any $f \in \mathcal{E}(M, p_0)$, where ψ_T is a truncation functional and $\Psi_{T,p,\mathcal{C}}$ is the $\mathcal{C}$-conditional Edgeworth expansion of the conditional law $\mathcal{L}_{\mathcal{C}}\{Z_T/\sqrt{T}\}$. Here $\mathcal{K}$ is chosen as suggested in the proof.

Proof. The proof of the above lemma is a conditional version of that of Lemma (3.3) of Götze and Hipp [18], under truncation. For a positive number η, let $A = \{|Z_T/\sqrt{T}| \le T^\eta\}$ and $B = \{|Z'_T/\sqrt{T}| \le T^\eta\}$, where $Z'_T = \sum_{j=0}^{N_T} \varphi_T(Z_{\tilde{I}_j})$. Let

$$
D(f,T) = \left| f(Z_T/\sqrt{T}) - f(Z'_T/\sqrt{T}) \right| = \left| f(Z_T/\sqrt{T}) - f(Z'_T/\sqrt{T}) \right| 1_{\{Z_T \ne Z'_T\}}.
$$

We observe

$$
\begin{aligned}
P_{\mathcal{C}}[D(f,T)1_{A\cap B}] &\le 2M(1+T^{\eta p_0}) P_{\mathcal{C}}\left[Z_T \ne Z'_T \right], \\
P_{\mathcal{C}}[D(f,T)1_{A\cap B^c}] &\le M(1+T^{\eta p_0}) P_{\mathcal{C}}\left[Z_T \ne Z'_T \right] \\
&\quad + P_{\mathcal{C}}\left[M(1+|Z'_T/\sqrt{T}|^{p_0}) 1_{B^c} 1_{\{Z_T \ne Z'_T\}} \right], \\
P_{\mathcal{C}}[D(f,T)1_{A^c\cap B}] &\le P_{\mathcal{C}}\left[M(1+|Z_T/\sqrt{T}|^{p_0}) 1_{A^c} 1_{\{Z_T \ne Z'_T\}} \right] \\
&\quad + M(1+T^{\eta p_0}) P_{\mathcal{C}}\left[Z_T \ne Z'_T \right], \\
P_{\mathcal{C}}[D(f,T)1_{A^c\cap B^c}] &\le P_{\mathcal{C}}\left[M(1+|Z_T/\sqrt{T}|^{p_0}) 1_{\{Z_T \ne Z'_T\}} 1_{A^c} \right] \\
&\quad + P_{\mathcal{C}}\left[M(1+|Z'_T/\sqrt{T}|^{p_0}) 1_{\{Z_T \ne Z'_T\}} 1_{B^c} \right].
\end{aligned}
$$

Thus one obtains an estimate uniform in $\mathcal{E}(M, p_0)$:

$$
\begin{aligned}
1_{\{\psi_T>0\}} P_{\mathcal{C}}[D(f,T)] &\lesssim 1_{\{\psi_T>0\}} \Bigg\{ T^{\eta p_0} P_{\mathcal{C}}\left[Z_T \ne Z'_T \right] + P_{\mathcal{C}}\left[\left| \frac{Z_T}{\sqrt{T}} \right|^{p_0} 1_{A^c} \right] \\
&\quad + P_{\mathcal{C}}\left[\left| \frac{Z'_T}{\sqrt{T}} \right|^{p_0} 1_{B^c} \right] \Bigg\}. \qquad (30)
\end{aligned}
$$

Noticing that p_0 is even and applying Lemma 6, we see that the second term on the right-hand side of (30) is less than or equal to:

$$
\begin{aligned}
&1_{\{\psi_T>0\}}\left\{\left|P_{\mathcal{C}}\left[\left|\frac{Z'_T}{\sqrt{T}}\right|^{p_0}\right]-P_{\mathcal{C}}\left[\left|\frac{Z_T}{\sqrt{T}}\right|^{p_0}\right]\right|+P_{\mathcal{C}}\left[\left|\frac{Z'_T}{\sqrt{T}}\right|^{p_0}1_{B^c}\right]\right.\\
&\quad\left.+\left|P_{\mathcal{C}}\left[\left|\frac{Z'_T}{\sqrt{T}}\right|^{p_0}1_B\right]-P_{\mathcal{C}}\left[\left|\frac{Z_T}{\sqrt{T}}\right|^{p_0}1_A\right]\right|\right\}\\
&\lesssim 1_{\{\psi_T>0\}}M_T\left\{T^{-R}+T^{(1+\epsilon)p_0/2}\alpha(\delta m(T)|\mathcal{C})^{1/(p+1)}\right\}+1_{\{\psi_T>0\}}|e_T|T^{p_0\epsilon/2}\\
&\quad+1_{\{\psi_T>0\}}P_{\mathcal{C}}\left[\left|\frac{Z'_T}{\sqrt{T}}\right|^{p_0}1_{B^c}\right]+1_{\{\psi_T>0\}}\left\{P_{\mathcal{C}}\left[\left|\frac{Z'_T}{\sqrt{T}}\right|^{p_0}1_{B\cap A^c}\right]\right.\\
&\quad\left.+P_{\mathcal{C}}\left[\left|\frac{Z_T}{\sqrt{T}}\right|^{p_0}1_{A\cap B^c}\right]+2T^{\eta p_0}P_{\mathcal{C}}\left[Z_T\neq Z'_T\right]\right\},
\end{aligned}
\tag{31}
$$

where $R=-\{1+2^{-1}p_0\epsilon-(p+1)\beta\}$. [We obtain the first term on the right-hand side as follows. Lemma 6 together with $[A1]$ implies that $|\mathrm{Var}_{\mathcal{C}}[S^*_T]-\mathrm{Var}_{\mathcal{C}}[S_T]|$ is bounded on $\{\psi_T>0\}$ uniformly in (ω,T). Since $|\mathrm{Var}_{\mathcal{C}}[S^*_T]|\lesssim k(T)\lesssim T^{\epsilon}$ by $[A1]$ and the definition of $\Omega_4(T)$, $|\mathrm{Var}_{\mathcal{C}}[S_T]|\lesssim T^{\epsilon}$ on $\{\psi_T>0\}$. On the other hand, it follows from Lemmas 5 and 6 that for r $(3\le r\le p_0)$, the r-th conditional cumulants of S^*_T and S_T are bounded on $\{\psi_T>0\}$ uniformly in (ω,T): one can show this fact by using estimates appearing below in this proof. Since the possible maximum order of $\mathrm{Var}_{\mathcal{C}}$ in the expression of p_0-th moments is $p/2$, we obtain the bound in question.]

Since on the event $B\cap A^c$, $|Z'_T/\sqrt{T}|\le T^{\eta}$ and $Z_T\neq Z'_T$, and similarly, on the event $A\cap B^c$, $|Z_T/\sqrt{T}|\le T^{\eta}$ and $Z_T\neq Z'_T$, the first two terms in the second braces $\{\cdots\}$ at the end of (31) are evaluated by $T^{\eta p_0}P_{\mathcal{C}}[Z_T\neq Z'_T]$. Thus we obtained

$$
\begin{aligned}
1_{\{\psi_T>0\}}P_{\mathcal{C}}[D(f,T)]\lesssim{}&1_{\{\psi_T>0\}}\left\{T^{\eta p_0}P_{\mathcal{C}}\left[Z_T\neq Z'_T\right]+P_{\mathcal{C}}\left[\left|\frac{Z'_T}{\sqrt{T}}\right|^{p_0}1_{B^c}\right]\right\}\\
&+|e_T|T^{p_0\epsilon/2}+1_{\{\psi_T>0\}}M_T\\
&\times\left\{T^{-R}+T^{(1+\epsilon)p_0/2}\alpha(\delta m(T)|\mathcal{C})^{1/(p+1)}\right\}.
\end{aligned}
\tag{32}
$$

First, $|e_T|T^{p_0\epsilon/2}\lesssim T^{-p\beta+\frac{1}{2}+\epsilon_2+\frac{1}{2}p_0\epsilon}$, and $-p\beta+\frac{1}{2}+\epsilon_2+\frac{1}{2}p_0\epsilon<-(p-2)/2+(\frac{1}{2}-\beta)(p+1)-\frac{1}{2}+(4p)^{-1}<-(p-2)/2$. On the other hand, since by $[A1]$(iv), we can choose η so as

$$
\frac{p+2\epsilon_2}{2(\bar{l}+1+p-p_0)}<\eta<[\beta(p+1)-1-\epsilon_2-\frac{1}{2}(p-2)]/p_0.
$$

Therefore,

$$
\begin{aligned}
1_{\{\psi_T>0\}} P_{\mathcal{C}}\left[Z_T \neq Z_T'\right] &\leq 1_{\{\psi_T>0\}} \sum_{j=0}^{N_T} P_{\mathcal{C}}\left[|Z_{\tilde{I}_j}| \geq T^{\beta}\right] \\
&\lesssim 1_{\{\psi_T>0\}} T^{-\beta(p+1)+1} M_T(\omega) \\
&\leq 1_{\{\psi_T>0\}} T^{-\beta(p+1)+1+\epsilon_2} \leq T^{-\frac{1}{2}(p-2)-\epsilon_9-\eta p_0}
\end{aligned}
$$

for some $\epsilon_9 > 0$. Lemma 14 for $u = 0$ (in this case, $|\theta_T(0,\omega)| \leq 2$) together with (29) yields

$$
\begin{aligned}
&1_{\{\psi_T>0\}} P_{\mathcal{C}}\left[\left|\frac{Z_T'}{\sqrt{T}}\right|^{p_0} 1_{B^c}\right] \\
&\lesssim 1_{\{\psi_T>0\}} \left\{ P_{\mathcal{C}}\left[\left|S_T^*\right|^{p_0} 1_{\{|S_T^*| \geq T^{\eta}/2\}}\right] + |e_T|^{p_0} \right\} \\
&\lesssim 1_{\{\psi_T>0\}} T^{-\eta(\bar{l}+1+p-p_0)} \\
&\quad \times \sum_{r:2\leq r\leq \bar{l}+1+p} \left(1 + T^{-\frac{r}{2}+\beta(r-p-1)^{+}+1+2(r-1)\epsilon+\epsilon_2}\right)^{(\bar{l}+1+p)/2} \\
&\lesssim 1_{\{\psi_T>0\}} T^{-\eta(\bar{l}+1+p-p_0)} T^{(2\epsilon+\epsilon_2)(\bar{l}+1+p)/2} \\
&\lesssim T^{-\frac{p-2}{2}-\epsilon_{10}}
\end{aligned}
$$

for some positive constant ϵ_{10}. [Here we used the inequalities $-r/2 + 1 + 2(r-1)\epsilon+\epsilon_2 \leq 2\epsilon+\epsilon_2$ for $r \leq p+1$ and $-r/2+\beta(r-p-1)+1+2(r-1)\epsilon+\epsilon_2 \leq \epsilon_2$ for $r > p+1$. $[A1]$ (iv) was used at the last line.]

Using the inequalities in $[A1]$, we have the inequality:

$$
\begin{aligned}
&1 + 2^{-1} p_0\epsilon - (p+1)\beta + \epsilon_2 \\
&\quad < 1 + 2^{-1} p_0\epsilon + \epsilon_2 - (p+1)[1 - (2p-1)/(2p(p+1))]/2 \\
&\quad = -(p-2)/2 + 2^{-1} p_0\epsilon + \epsilon_2 - 1/(4p) \\
&\quad < -\frac{p-2}{2} - \epsilon_{11}
\end{aligned}
$$

for some positive constant ϵ_{11}, which is easily shown for even and odd p's, and hence,

$$
1_{\{\psi_T>0\}} M_T(\omega) T^{-R} \lesssim T^{-\frac{p-2}{2}-\epsilon_{11}}.
$$

Furthermore, it is easily seen that on the event $\{\psi_T > 0\}$,

$$
\begin{aligned}
T^{\epsilon_2+\frac{1}{2}p_0(1+\epsilon)} \alpha(\delta m(T)|\mathcal{C})^{1/(p+1)} &\lesssim T^{(\epsilon_2+\frac{1}{2}p_0(1+\epsilon))-(p-1-(p-1)\epsilon+\beta)} \\
&\lesssim T^{\frac{1}{4p}+\frac{1}{2}p-(p-1)-\frac{p}{2(p+1)}} \\
&\lesssim T^{-\frac{1}{2}(p-2)-\epsilon_{12}}
\end{aligned}
$$

for some $\epsilon_{12} > 0$.

After all, in particular we saw from those estimates and (32) that it suffices to show the same bound for

$$\left\| \psi_T \left(P_{\mathcal{C}} \left[f \left(\frac{Z'_T}{\sqrt{T}} \right) \right] - \Psi_{T,p,\mathcal{C}}[f] \right) \right\|_1$$

to prove the lemma.

Let $\mathcal{K}$ be a probability measure on $\mathbf{R}^d$ whose Fourier transform has a compact support, and choose a constant a such that

$$\alpha := \mathcal{K}(\{x : |x| < a\}) > \frac{1}{2}.$$

The scaled measure $\mathcal{K}_\epsilon$ is defined by $\mathcal{K}_\epsilon(A) = \mathcal{K}(\epsilon^{-1}A)$ for $A \in \mathbf{B}_d$ and $\epsilon > 0$. For a finite measure P, a finite singed measure Q on $\mathbf{R}^d$, and $f \in \mathcal{F}\mathbf{B}_d$, define $\gamma_f(\epsilon)$, $\zeta_f(r)$ and $\tau(t)$ by

$$\begin{aligned}
\gamma_f(\epsilon) &= \|f^\star\|_\infty \int_{\mathbf{R}^d} h(|x|)|\mathcal{K}_\epsilon * (P - Q)|(dx), \\
\zeta_f(r) &= \|f^\star\|_\infty \int_{x:|x|\geq ar} h(|x|)\mathcal{K}(dx), \\
\tau(t) &= \sup_{x:|x|\leq ta\epsilon'} \int \omega_f(x + y, 2a\epsilon)Q^+(dy),
\end{aligned}$$

where $f^\star(x) = f(x)/h(|x|)$, $h(x) = 1 + x^{p_0}$ $(x \in \mathbf{R})$, and Q^+ is the positive part of the Jordan decomposition of Q. The following is well known Sweeting's smoothing inequality (Sweeting [61], Bhattacharya and Rao [8], Yoshida [72]):

$$\begin{aligned}
|(P - Q)[f]| \leq \frac{1}{2\alpha - 1}&[A_0\gamma_f(\epsilon) + A_1\zeta_f(\epsilon'/\epsilon) \\
&+\tau(t)] + \left(\frac{1-\alpha}{\alpha}\right)^t A_2\|f^\star\|_\infty,
\end{aligned} \tag{33}$$

for ϵ, ϵ', t satisfying $0 < \epsilon < \epsilon' < a^{-1}$ and $t \in \mathbf{N}$ $(a\epsilon' t \leq 1)$, where the constant $A_0 = a_0(p, d)$ depends only on p and d, and A_i $(i = 1, 2)$ take the form of:

$$A_i = a_i(p, d)(P + |Q|)[h(|\cdot|)], \quad (i = 1, 2)$$

with some constants $a_i(p, d)$ depending only on p and d.

For given $K_1 > K > 0$, let $\zeta = (K_1 - K)/2$, $\epsilon = T^{-K_1}$, $\epsilon' = T^{-(K+\zeta)}$, $t = [(2a)^{-1}T^{\zeta}]$. Applying (33) to $P = P_{\mathcal{C}}^{Z'_T/\sqrt{T}}$ and $Q = \Psi_{T,p,\mathcal{C}}$, we obtain:

$$\begin{aligned}
&\left\|\psi_T\left(P_{\mathcal{C}}\left[f\left(\frac{1}{\sqrt{T}}Z'_T\right)\right] - \Psi_{T,p,\mathcal{C}}[f]\right)\right\|_1 \\
&\leq CM\Big[\left\|\psi_T\int_{\mathbf{R}^d} h(|x|)|\mathcal{K}_{T^{-K_1}} * (P_{\mathcal{C}}^{Z'_T/\sqrt{T}} - \Psi_{T,p,\mathcal{C}})|(dx)\right\|_1 \\
&\quad + \left\|(P_{\mathcal{C}}^{Z'_T/\sqrt{T}} + |\Psi_{T,p,\mathcal{C}}|)[h(|\cdot|)]\right\|_1 \int_{x:|x|\geq aT^{\zeta}} h(|x|)\mathcal{K}(dx) \\
&\quad + \left\|(P_{\mathcal{C}}^{Z'_T/\sqrt{T}} + |\Psi_{T,p,\mathcal{C}}|)[h(|\cdot|)]\right\|_1 \delta(\alpha)^{[a^{-1/2}T^{\zeta}]}\Big] \\
&\quad + C'\left\|\psi_T\int \sup_{x:|x|\leq aT^{-(K+\zeta)}[(2a)^{-1}T^{\zeta}]} \omega_f(x+y, 2aT^{-K_1})\Psi^+_{T,p,\mathcal{C}}(dy)\right\|_1, \qquad (34)
\end{aligned}$$

where $\delta(\alpha) = \alpha/(1-\alpha) \in (0,1)$, C is a constant depending on p, d, α, and $C' = (2\alpha - 1)^{-1}$.

We will estimate four terms appearing on the right-hand side of (34). It is easy to show that if we choose $\mathcal{K}$ appropriately, the second and the third terms are of $o(T^{-(p-2)/2-\epsilon_{13}})$ for some positive constant ϵ_{13} under our assumptions for $\Psi_{T,p,\mathcal{C}}$. If we took the kernel $\mathcal{K}$ appropriately, a can be small, and the last term is evaluated by

$$\left\|\psi_T \int \omega_f(z, T^{-K})\Psi^+_{T,p,\mathcal{C}}(dz)\right\|_1.$$

Moreover, the first term can be estimated by integration of the derivatives of Fourier transform $H_T(u,\mathcal{C})e^{iu\cdot e_T} - \hat{\Psi}_{T,p,\mathcal{C}}$ if we use Lemma 11.6 of Bhattacharya and Rao [8]. □

Remark 14. With the estimate of $\hat{\Psi}_{T,p,\mathcal{C}}(\cdot)$ and e_T, it is possible to replace the first term on the right-hand side of the inequality in the above lemma by

$$CM\textstyle\sum_{\alpha:|\alpha|\leq d+1+p_0}\int\left\|\psi_T\partial_u^{\alpha}\left[\left(H_T(u,\mathcal{C}) - \hat{\Psi}_{T,p,\mathcal{C}}(u)\right)\hat{\mathcal{K}}(T^{-K_1}u)e^{iu\cdot e_T}\right]\right\|_1 du$$

for ψ_T given in Lemma 8. In fact, it can be proved in a similar way as in Step (d) of the proof of Lemma 8, with the help of an estimate of $|e_T|$.

Proof of Lemma 9. (a) Let $\eta_1, \eta_2 > 0$. For ψ_j given in $[A3]$, define Ψ_j by

$$\Psi_j = \psi_j 1_{\left\{\sup_{|u|\geq B}\left|P_{\hat{\mathcal{C}}(j)}\left[\psi_j e^{iu\cdot Z_{I(j)}}\right]\right|<\eta_1\right\}}.$$

Since the indicator in the above equation is $\hat{\mathcal{C}}(j)$-measurable, $[A3^{\flat}]$(i) clearly holds for Ψ_j replacing ψ_j.

We will verify $[A3^{\flat}]$(ii) for Ψ_j. In our case, $p_j(\hat{\mathcal{C}}(j))$ in $[A3^{\flat}]$ is defined by

$$p_j(\hat{\mathcal{C}}(j)) = P_{\hat{\mathcal{C}}(j)}\left[(1-\Psi_j) + 2\left(1 - \varphi(T^{-\beta}Z_{I(j)})\right)\right]$$

for some fixed $\beta \in (0, 1/2)$. Put $\zeta_j = 1_{\{p_j(\hat{\mathcal{C}}(j))\geq\eta_2\}}$. Let $\eta_3 \in (0, 1)$; we will later make η_3 sufficiently large. Then

$$
\begin{aligned}
P\left[\#\left\{j;\, p_j(\hat{\mathcal{C}}(j)) \geq \eta_2\right\} \geq \eta_3 n'(T)\right] &= P\left[\sum_j \zeta_j \geq \eta_3 n'(T)\right] \\
&= P\left[\frac{1}{\sqrt{n'(T)}}\sum_j \left\{\zeta_j - P_{\mathcal{C}}\left[\zeta_j\right]\right\}\right. \\
&\qquad\left.\geq \frac{1}{\sqrt{n'(T)}}\sum_j \left\{\eta_3 - P_{\mathcal{C}}\left[\zeta_j\right]\right\}\right].
\end{aligned}
$$

Since $P\left[\Omega_0(T)^c\right] = o(T^{-M_1})$ from assumption, in order to show

$$
P\left[\#\left\{j;\, p_j(\hat{\mathcal{C}}(j)) \geq \eta_2\right\} \geq \eta_3 n'(T)\right] = o(T^{-M_1}),
$$

it suffices to show that

$$
\begin{aligned}
P_1 := P\Bigg[\Omega_0(T) \cap \Bigg\{&\frac{1}{\sqrt{n'(T)}}\sum_j \left\{\zeta_j - P_{\mathcal{C}}\left[\zeta_j\right]\right\} \\
&\geq \frac{1}{\sqrt{n'(T)}}\sum_j \left\{\eta_3 - P_{\mathcal{C}}\left[\zeta_j\right]\right\}\Bigg\}\Bigg] = o(T^{-M_1}).
\end{aligned}
$$

If we apply the same argument as Step (b) in the proof of Lemma 8 to bounded $\zeta_j - P_{\mathcal{C}}\left[\zeta_j\right]$, which is measurable to $\hat{\mathcal{C}}(j) \subset \mathcal{B}_{I(j)^\varrho} \vee \mathcal{C}$ (therefore, we can use the mixing property to ζ_j), we obtain the boundedness of the moments:

$$
\sup_T \left\| \kappa_{\mathcal{C}}^r \left[\sum_{j=1}^{n'(T)} \frac{\zeta_j - P_{\mathcal{C}}\left[\zeta_j\right]}{\sqrt{n'(T)}}\right]\right\|_q < \infty \quad (q > 1,\ r \geq 1).
$$

Let $W_1 = n'(T)^{-\frac{1}{2}}\sum_j \left\{\zeta_j - P_{\mathcal{C}}[\zeta_j]\right\}$ and $W_2 = n'(T)^{-\frac{1}{2}}\sum_j \left\{\eta_3 - P_{\mathcal{C}}[\zeta_j]\right\}$. Using the fact that W_2 and $\Omega_0(T)$ are $\mathcal{C}$-measurable, we see that for $\epsilon > 0$,

$$
\begin{aligned}
P_1 &\leq P\left[\Omega_0(T) \cap \{W_2 \leq \epsilon\}\right] + P\left[1_{\Omega_0(T)} 1_{\{W_2>\epsilon\}} \cdot P_{\mathcal{C}}\left[W_1 \geq W_2\right]\right] \\
&\leq P\left[\Omega_0(T) \cap \{W_2 \leq \epsilon\}\right] + P\left[1_{\Omega_0(T)} 1_{\{W_2>\epsilon\}} \cdot W_2{}^{-L} P_{\mathcal{C}}\left[|W_1|^L\right]\right] \\
&\lesssim P\left[\Omega_0(T) \cap \{W_2 \leq \epsilon\}\right] + \left(P\left[1_{\Omega_0(T)} \min\left\{\left(\frac{1}{\epsilon}\right)^{2L},\ (W_2)_+{}^{-2L}\right\}\right]\right)^{\frac{1}{2}}.
\end{aligned}
$$

(We can use Lemma 5 on $\Omega_0(T)$ for r-th cumulants if $r \geq 3$, however it is not sufficient for $r = 2$ in the present case.)

(b) Let us show that $P_{\mathcal{C}}[\zeta_j]$ are small. For any $\eta_2' \in (0, \eta_2)$,

$$\begin{aligned} P_{\mathcal{C}}[\zeta_j] &\le P_{\mathcal{C}}\left[P_{\hat{\mathcal{C}}(j)}\left[1-\Psi_j\right] \ge \eta_2'\right] \\ &\quad + P_{\mathcal{C}}\left[2P_{\hat{\mathcal{C}}(j)}\left[1-\varphi(T^{-\beta}Z_{I(j)})\right] \ge \eta_2 - \eta_2'\right] \\ &=: \Phi_1(j) + \Phi_2(j) \text{ (say).} \end{aligned}$$

By using the inequality $1 - ab \le (1-a) + (1-b)$ for $a, b \in [0, 1]$, we have

$$\begin{aligned} \Phi_1(j) &\le P_{\mathcal{C}}\left[P_{\hat{\mathcal{C}}(j)}\left[1-\psi_j\right] + 1_{\left\{\sup_{|u|\ge B}\left|P_{\hat{\mathcal{C}}(j)}\left[\psi_j e^{iu\cdot Z_{I(j)}}\right]\right| \ge \eta_1\right\}} \ge \eta_2'\right] \\ &\le \frac{1}{\eta_2'}\left\{P_{\mathcal{C}}\left[1-\psi_j\right] + P_{\mathcal{C}}\left[\sup_{|u|\ge B}\left|P_{\hat{\mathcal{C}}(j)}\left[\psi_j e^{iu\cdot Z_{I(j)}}\right]\right| \ge \eta_1\right]\right\} \\ &\le \frac{1}{\eta_2'}\left\{P_{\mathcal{C}}\left[1-\psi_j\right] + \frac{1}{\eta_1}P_{\mathcal{C}}\left[\sup_{|u|\ge B}\left|P_{\hat{\mathcal{C}}(j)}\left[\psi_j e^{iu\cdot Z_{I(j)}}\right]\right|\right]\right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Phi_2(j) &\le P_{\mathcal{C}}\left[P_{\hat{\mathcal{C}}(j)}\left[Z_{I(j)} \ge \frac{1}{2}T^\beta\right] \ge \frac{\eta_2-\eta_2'}{2}\right] \\ &\le \left(\frac{\eta_2-\eta_2'}{2}\right)^{-1} P_{\mathcal{C}}\left[Z_{I(j)} \ge \frac{1}{2}T^\beta\right] \\ &\le \left(\frac{\eta_2-\eta_2'}{2}\right)^{-1}\left(\frac{2}{T^\beta}\right)^{p+1} P_{\mathcal{C}}\left[\left|Z_{I(j)}\right|^{p+1}\right] \\ &\le \left(\frac{\eta_2-\eta_2'}{2}\right)^{-1}\frac{2^{p+1}}{T^{\beta(p+1)-\epsilon_2}}. \end{aligned}$$

The last inequality holds on $\Omega_0(T)$. Choose $c = c(\eta_1, \eta_2)$ sufficiently large, and $c' = \beta(p+1) - \epsilon_2 - 1$. Fix $a'' \in (0, 1)$ such that

$$a'' > \frac{1}{\eta_2'}\left(a + \frac{a'}{\eta_1}\right).$$

In fact, by assumption for a and a', it is possible to choose such η_1, η_2' and η_2 such that

$$\begin{aligned} \eta_1 &> 0, \qquad \eta_1 + \eta_2 < 1, \\ \frac{1}{\eta_2'}\left(a + \frac{a'}{\eta_1}\right) &< 1, \qquad 0 < \eta_2' < \eta_2 < 1. \end{aligned}$$

Roughly speaking, one may choose

$$\begin{aligned} \eta_2' \approx \eta_2 &\approx \frac{1}{2}\left(a + 1 - \sqrt{(a+1)^2 - 4(a+a')}\right) \in (0, 1) \\ \eta_1 \approx 1 - \eta_2 &\approx \frac{1}{2}\left(2 - (a+1) + \sqrt{(a+1)^2 - 4(a+a')}\right) \in (0, 1). \end{aligned}$$

We then have

$$P\left[\Omega_0(T), \sum_j P_{\mathcal{C}}[\zeta_j] > a''n'(T) + cT^{-c'}\right]$$
$$\leq P\left[\Omega_0(T), \sum_j \frac{1}{\eta_2'}\left\{P_{\mathcal{C}}\left[1-\psi_j\right] + \frac{1}{\eta_1}P_{\mathcal{C}}\left[\sup_{|u|\geq B}\left|P_{\hat{\mathcal{C}}(j)}\left[\psi_j e^{iu\cdot Z_{I(j)}}\right]\right|\right]\right\}\right.$$
$$\left. > \frac{1}{\eta_2'}\left(a + \frac{a'}{\eta_1}\right)n'(T)\right]$$
$$\leq P\left[\sum_j P_{\mathcal{C}}\left[1-\psi_j\right] > an'(T)\right] + P\left[\sum_j P_{\mathcal{C}}\left[\sup_{|u|\geq B}\left|P_{\hat{\mathcal{C}}(j)}\left[\psi_j e^{iu\cdot Z_{I(j)}}\right]\right|\right]\right.$$
$$\left. > a'n'(T)\right] = o(\frac{1}{T^L})$$

by assumption.
(c) Take η_3' and η_3 so that $a'' < \eta_3' < \eta_3 < 1$. Obviously, from Step (b),

$$P\left[\Omega_0(T),\ \sum_j P_{\mathcal{C}}[\zeta_j] > \eta_3'n'(T)\right] = o(\frac{1}{T^L}).$$

Hence,

$$P\left[1_{\Omega_0(T)}\min\left\{\left(\frac{1}{\epsilon}\right)^{2L},\ (W_2)_+{}^{-2L}\right\}\right]$$
$$= P\left[1_{\Omega_0(T)}\min\left\{\left(\frac{1}{\epsilon}\right)^{2L},\ (W_2)_+{}^{-2L}\right\}1_{\left\{\sum_j P_{\mathcal{C}}[\zeta_j]\leq\eta_3'n'(T)\right\}}\right]$$
$$+\epsilon^{-2L}o\left(\frac{1}{T^L}\right)$$
$$\leq \left(\sqrt{n'(T)}(\eta_3-\eta_3')\right)^{-2L} + \epsilon^{-2L}o\left(\frac{1}{T^L}\right)$$
$$= o\left(\frac{1}{T^L}\right).$$

Finally,

$$P\left[\Omega_0(T)\cap\{W_2\leq\epsilon\}\right] = P\left[\Omega_0(T),\ W_2\leq\epsilon,\ \sum_j P_{\mathcal{C}}[\zeta_j]\leq\eta_3'n'(T)\right] + o\left(\frac{1}{T^L}\right)$$
$$= o\left(\frac{1}{T^L}\right)$$

since $\eta_3' < \eta_3$. After all, we arrived at the desired result through Step (a). □

Remark 15. The assertions in Theorem 1 hold true even if $[A3]$ is replaced by: $[A3^\natural]$ For every $L > 0$, there exist truncation functionals $\psi_j : (\Omega, \mathcal{F}) \to ([0,1], \mathbf{B}([0,1]))$, and there exist positive constants $\eta_1^\natural, \eta_2^\natural, \eta_3^\natural, B$ $(\eta_1^\natural + \eta_2^\natural < 1, \eta_3^\natural < 1)$ such that $P[\sum_j P_\mathcal{C}\left[\bar{\Phi}(j) \le 1 - \eta_2^\natural\right] > \eta_3^\natural n'(T)] = o(T^{-L})$, where $\bar{\Phi}(j) = P_{\hat{\mathcal{C}}(j)}[\psi_j] \cdot 1_{\{\sup_{u:|u|\ge B} |P_{\hat{\mathcal{C}}(j)}[\psi_j e^{iu\cdot Z_{I(j)}}]| < \eta_1^\natural\}}$.

6.5. Proof of Lemmas 5, 6 and 7

We will give a proof which is a conditional version of the proof in Götze and Hipp [18] and, differently, ($\mathcal{C}$-measurable) truncations play an essential role to make conditional estimates.

First, we give a conditional moment estimate for CVCCE $P_\mathcal{C}[\tilde{Z}_I](u \cdot S_T^*)$:

Lemma 12. *Assume $[A1](i)$. For any ϵ_1 $(0 < \epsilon_1 < 1/2)$ and any $r \in \mathbf{N}$ $(r \le \bar{l} + p + 1)$, there exist constants $C_2, C_2', T_1 > 0$ such that*

$$
\begin{aligned}
&1_{\{|u|\le T^{\epsilon_1}\}} 1_{\Omega_1(T)\cap\Omega_2(T,I)} \left| H_T(u,\mathcal{C}) P_\mathcal{C}\left[\tilde{Z}_I\right](u\cdot S_T^*)\right| \\
&\le C_2 P_\mathcal{C}\left[\left|\tilde{Z}_I\right|\right] \left\{ \max_{k:1\le k\le k(T)} \left| P_\mathcal{C}\left[\exp\left(S_I^{[m(T)k]}\right)\right]\right| + x_1^{k(T)} \right\} + T^{|I|/2} x_1^{k(T)} \\
&\le C_2'\left(P_\mathcal{C}\left[\left|\tilde{Z}_I\right|\right] + 1\right) \left\{ \max_{k:1\le k\le k(T)} \left| P_\mathcal{C}\left[\exp\left(S_I^{[m(T)k]}\right)\right]\right| + x_2^{k(T)} \right\}
\end{aligned}
$$

for any $T \ge T_1$ *and any* $I \in \{0, 1, ..., N_T\}^n$, $n \le r$.

Proof. Suppose that bounded random variables $A_i (i = 1, ..., r)$ are measurable with respect to $\mathcal{B}_{[a_i', a_i'']}$, bounded random variables $B_i (i = 1, ..., r)$ are measurable with respect to $\mathcal{B}_{[b_i', b_i'']}$, respectively, and $a_i' < a_i'' < a_i'' + h \le b_i' < b_i'' < b_i'' + h \le a_{i+1}'$. Then

$$
|P_\mathcal{C}[A_1 B_1 ... A_r B_r] - P_\mathcal{C}[A_1 ... A_r] P_\mathcal{C}[B_1 ... B_r]| \le C_r \alpha(h|\mathcal{C}),
$$

where the coefficient C_r is proportional to $\Pi_{i=1}^r \|A_i\|_\infty \cdot \Pi_{i=1}^r \|B_i\|_\infty$. In fact, the difference between $|P_\mathcal{C}[A_1 B_1 ... A_r B_r]$ and $\Pi_{i=1}^r P_\mathcal{C}[A_i] \cdot \Pi_{i=1}^r P_\mathcal{C}[B_i]$ can be estimated with α, and the difference between $P_\mathcal{C}[A_1 ... A_r] P_\mathcal{C}[B_1 ... B_r]$ and $\Pi_{i=1}^r P_\mathcal{C}[A_i] \cdot \Pi_{i=1}^r P_\mathcal{C}[B_i]$ can also be estimated with α.

Set $\Delta(n_1, n_2; I) = \exp\left(S_I^{[n_1]} - S_I^{[n_2]}\right) - 1$ for $n_1, n_2 \in \mathbf{Z}_+$, $n_1 \le n_2$. Then

$$
\begin{aligned}
\exp(S_I^{[0]}) &= \sum_{k=1}^K \left(\Pi_{j=1}^{k-1} \Delta((j-1)m, jm; I)\right) \exp(S_I^{[km]}) \\
&\quad + \left(\Pi_{j=1}^K \Delta((j-1)m, jm; I)\right) \exp(S_I^{[Km]}),
\end{aligned} \tag{35}
$$

where $\Pi_{j=1}^0 \cdots = 1$.

Using the covariance inequality successively, we have

$$\left|P_{\mathcal{C}}\left[\tilde{Z}_I\left(\Pi_{j=1}^{k-1}\Delta((j-1)m,\, jm;\, I)\right)\right]\right|$$
$$\leq 2^{1+k/2}P_{\mathcal{C}}\left[\left|\tilde{Z}_I\right|\right]\cdot\Pi_{\substack{j:even,\\ 4\leq j\leq k-1}}P_{\mathcal{C}}\left[\left|\Delta\left((j-1)m,\, jm;\, I\right)\right|\right]+Ck2^kT^{|I|\beta}\alpha(\delta m|\mathcal{C}).$$

On the $\mathcal{C}$-measurable event $\Omega_2(T,\, I)$,

$$P_{\mathcal{C}}\left[\left|\Delta((j-1)m(T),\, jm(T);\, I)\right|\right]\leq P_{\mathcal{C}}\left[\left|S_I^{[(j-1)m(T)]}-S_I^{[jm(T)]}\right|\right]\leq\frac{x_1^2}{2}\frac{|u|}{T^{\epsilon_1}}$$

for $j(1\leq j\leq k(T))$. Thus we have

$$\begin{aligned}&1_{\{|u|\leq T^{\epsilon_1}\}}1_{\Omega_2(T,I)}\left|P_{\mathcal{C}}\left[\tilde{Z}_I\Pi_{j=1}^{k-1}\Delta((j-1)m(T),\, jm(T);\, I)\right]\right|\\&\quad\leq CP_{\mathcal{C}}\left[\left|\tilde{Z}_I\right|\right]x_1^k+Ck2^kT^{|I|\beta}\alpha(\delta m(T)|\mathcal{C}).\end{aligned}\tag{36}$$

Moreover, in the same fashion, we see

$$\begin{aligned}&1_{\{|u|\leq T^{\epsilon_1}\}}1_{\Omega_2(T,I)}\left|P_{\mathcal{C}}\left[\tilde{Z}_I\left(\Pi_{j=1}^{k-1}\Delta((j-1)m(T),\, jm(T);\, I)\right)\exp\left(S_I^{[m(T)k]}\right)\right]\right|\\&\quad\leq C\left\{P_{\mathcal{C}}\left[\left|\tilde{Z}_I\right|\right]x_1^k+k2^kT^{|I|\beta}\alpha(\delta m(T)|\mathcal{C})\right\}\left|P_{\mathcal{C}}\left[\exp\left(S_I^{[m(T)k]}\right)\right]\right|\\&\qquad+CT^{|I|\beta}2^k\alpha(\delta m(T)|\mathcal{C}).\end{aligned}\tag{37}$$

On the other hand, in a similar way as (36), we obtain

$$\begin{aligned}&1_{\{|u|\leq T^{\epsilon_1}\}}1_{\Omega_2(T,I)}\left|P_{\mathcal{C}}\left[\tilde{Z}_I\left(\Pi_{j=1}^{k(T)}\Delta((j-1)m(T),\, jm(T);\, I)\right)\exp\left(S_I^{[m(T)k(T)]}\right)\right]\right|\\&\quad\leq CP_{\mathcal{C}}\left[\left|\tilde{Z}_I\right|\right]x_1^{k(T)}+Ck(T)2^{k(T)}T^{|I|\beta}\alpha(\delta m(T)|\mathcal{C}).\end{aligned}\tag{38}$$

It follows from (35), (37) and (38) that

$$\begin{aligned}&1_{\{|u|\leq T^{\epsilon_1}\}}1_{\Omega_2(T,I)}\left|H_T(u,\mathcal{C})P_{\mathcal{C}}\left[\tilde{Z}_I\right](u\cdot S_T^*)\right|\\&\quad\leq C'P_{\mathcal{C}}\left[\left|\tilde{Z}_I\right|\right]\left\{\left(\max_{k:1\leq k\leq k(T)}\left|P_{\mathcal{C}}\left[\exp\left(S_I^{[m(T)k]}\right)\right]\right|\right)+x_1^{k(T)}\right\}\\&\qquad+C(2k(T)+1)2^{k(T)+1}T^{|I|\beta}\alpha(\delta m(T)|\mathcal{C}).\end{aligned}$$

For every $C>0$, there exists $T_1>0$ such that for $T\geq T_1$, on $\Omega_1(T)$,

$$\rho_1\leq\frac{x_1}{2}\{C(k(T)+1)\}^{1/k(T)}\leq x_1<1,\tag{39}$$

where $\rho_1=\rho_1(T|\mathcal{C}):=\left\{C(k(T)+1)2^{k(T)}\alpha(\delta m(T)|\mathcal{C})\right\}^{1/k(T)}$.

Finally, from (39), we see that on $\Omega_1(T)$, the last term on the right-hand side of the above inequality is less than $c''T^{|I|\beta}x_1^{k(T)}$ if $T\geq T_1$. Thus we obtained the desired result. □

The following lemma provides a covariance inequality for CVCCE.

Lemma 13. *Let $n_0 \in \mathbf{N}$. There exist constants C_1, c_1 and x_3 $(0 < x_3 < 1)$ such that*

$$1_{\{|u| \le c_1 T^{\frac{1}{2}-\beta} m^{-1}\}} |H_T(u, \mathcal{C})|^2 \left| Cov_{\mathcal{C}} \left[\tilde{Z}_{I_1}, \tilde{Z}_{I_2} \right] (u \cdot S_T^*) \right|$$
$$\le C_1 T^{\beta(|I_1|+|I_2|)} \left\{ \alpha(\delta\sqrt{m}|\mathcal{C}) + x_3^{\sqrt{m}}/[\sqrt{m}]! \right\}$$

for any $m \in \mathbf{N}$, $I_1, I_2 \in \cup_{n \le n_0} \{0, 1, ..., N_T\}^n$ satisfying $\min \bar{I}_2 - \max \bar{I}_1 \ge m$, bar making sets, where

$$Cov_{\mathcal{C}} \left[\tilde{Z}_{I_1}, \tilde{Z}_{I_2} \right] (u \cdot S_T^*) = P_{\mathcal{C}} \left[\tilde{Z}_{I_1} \tilde{Z}'_{I_2} \right] (u \cdot S_T^*)$$
$$- P_{\mathcal{C}} \left[\tilde{Z}_{I_1} \right] (u \cdot S_T^*) P_{\mathcal{C}} \left[\tilde{Z}'_{I_2} \right] (u \cdot S_T^*).$$

For sufficiently small $c_1 > 0$, it is possible to take x_3 as $x_3 < x_2$.

Proof. For each $T > 0$, let $\mathcal{Z}^T = \mathbf{R}^{d(N_T+1)}$ regarded as a measurable space with Borel σ-field. Denote by $p(\omega, \cdot)$ the regular conditional distribution of $(\tilde{Z}_{T,j})_{j=0}^{N_T}$ given $\mathcal{C}$. Over $\bar{\Omega} = \Omega \times \mathcal{Z}^T \times \mathcal{Z}^T$, we define a probability measure $\bar{P}$ on $\mathcal{C} \otimes \mathbf{B}[\mathcal{Z}^T] \otimes \mathbf{B}[\mathcal{Z}^T]$ by $\bar{P}(d\omega, d\omega', d\omega'') = P(d\omega)p(\omega, d\omega')p(\omega, d\omega'')$. Then the canonical projections $Z' = (Z'_j)_{j=0}^{N_T} : (\omega, \omega', \omega'') \to \omega'$ and $Z'' = (Z''_j)_{j=0}^{N_T} : (\omega, \omega', \omega'') \to \omega''$ are (extended) $\mathcal{C}$-conditionally independent, and have the same $\mathcal{C}$-conditional distribution as $\tilde{Z} = (\tilde{Z}_{T,j})_{j=0}^{N_T}$.

Since the rest of the proof is quite the same as Götze and Hipp [18], we will give a sketch. Put $\zeta_j = iT^{-1/2}u \cdot (Z'_j + Z''_j)$, $T_1 = \sum_{j=0}^{\max I_1} \zeta_j$, $T_2 = \sum_{j=\max I_1+m}^{N_T} \zeta_j$, $T_3 = \sum_{j=\max I_1+1}^{\max I_1+m-1} \zeta_j$, and $U_i = (Z'_{I_i} - Z''_{I_i})e^{T_i}$ $(i = 1, 2)$. Then the complex-valued conjugate conditional covariance has the representation: $2H_T(u, \mathcal{C})^2 Cov_{\mathcal{C}}[Z_{I_1}, Z_{I_2}](u \cdot S_T^*) = P_{\mathcal{C}}[U_1 e^{T_3} U'_2]$. Suppose that $\max I_1 + 1 \le j_1, ..., j_r \le \max I_1 + m - 1$. With the covariance inequality for a biggest gap between $\max I_1, j_1, ..., j_r, \max I_1 + m$, and the fact that $P_{\mathcal{C}}[U_1 \zeta_{j_1} \cdots \zeta_{j_q}] = 0$ $(0 \le q \le r)$ due to symmetry, we see that for $|u| \le c_1 T^{1/2-\beta}/m$, $|P_{\mathcal{C}}[U_1 \zeta_{j_1} \cdots \zeta_{j_r} U_2]| \le Cm^{-r} T^{\beta(|I_1|+|I_2|)} \alpha(\delta m/(r+1) - 3\delta)$. Taylor's formula yields that for some C'_1, $\exp(T_3) = \sum_{r=0}^{m'-1} T_3^r/r! + \theta(C'_1|u|T^{\beta-1/2}m)^{m'}/m'!$, where $|\theta| \le 1$. Choose $c_1 > 0$ sufficiently small so that $x_3 := c_1 C'_1 < 1$. We may consider $m \ge 2$. Thus, $|P_{\mathcal{C}}[U_1 e^{T_3} U_2]| \le CT^{\beta(|I_1|+|I_2|)} \alpha(\delta m/m' - 3\delta) + T^{\beta(|I_1|+|I_2|)} x_3^{m'}/m'! \le CT^{\beta(|I_1|+|I_2|)} \{\alpha(\delta\sqrt{m}|\mathcal{C}) + x_3^{\sqrt{m}}/[\sqrt{m}]!\}$ if m' is chosen as $m' = [\sqrt{m}]/2$. □

Define a random set U_T by $U_T = \{u; H_T(u, \mathcal{C}) \neq 0\}$ and set

$$B_T = \left\{ u; |u| \le c_1 T^{\frac{1}{2}-\beta} m(T)^{-2} \right\} \cap U_T \cap \left\{ u; |u| \le T^{\epsilon_1} \right\}.$$

Lemma 14. *Assume $[A1](i), (ii)$. Let $r \in \mathbf{N}$ $(r \le \bar{l} + p + 1)$ and $a_1, ..., a_r \in \{1, ..., d\}$. Then*

$$1_{\{u \in B_T\}} \left| \kappa_{\mathcal{C}} \left[S_T^{*(a_1)}, ..., S_T^{*(a_r)} \right] \left(u \cdot S_T^* \right) \right| \lesssim T^{-\frac{r}{2}+\beta(r-p-1)^+ +1} m(T)^{2(r-1)} M_T(\omega)$$
$$\cdot (\theta_T(u, \omega) + 1)^r \, 1_{\Omega_1(T)}$$

on $\Omega_1(T) \cap \Pi_{I:|I|\le r, I\in\mathcal{I}_T} \Omega_2(T, I)$*, where* $\mathcal{I}_T = \cup_{n\in\mathbf{N}}\{0, 1, ..., N_T\}^n$ *and*

$$\theta_T(u,\omega) = \left\{ \left(\max_{\substack{k,I:1\le k\le k(T),\\ |I|\le r, I\in\mathcal{I}_T}} \left| P_{\mathcal{C}}\left[\exp\left(S_I^{[m(T)k]}\right)\right]\right| \right) + x_2^{k(T)} \right\} |H_T\,(u,\mathcal{C})|^- .$$

Here – denotes the g-inverse.

Proof. For $a_1, ..., a_r \in \{1, .., d\}$ and a subsequence I' of $I = (j_1, .., j_r) \in \mathcal{I}_T$, denote $\tilde{Z}_{I'} = \Pi_{k=1}^r (\tilde{Z}_{T,j_k}^{(a_k)})^{\delta_k}$, where $\delta_k = 1$ if the k-th number j_k appears in the subsequence I' of I and zero otherwise with convention that $x^0 = 1$ for $x \in \mathbf{R}$. The conditional cumulant $\kappa_{\mathcal{C}}\left[\tilde{Z}_{T,j_1}^{(a_1)}, ..., \tilde{Z}_{T,j_r}^{(a_r)}\right]\left(u \cdot S_T^*\right)$ has two representations:

$$\kappa_{\mathcal{C}}\left[\tilde{Z}_{T,j_1}^{(a_1)}, ..., \tilde{Z}_{T,j_r}^{(a_r)}\right](u\cdot S_T^*) = \sum_{l=1}^r \sum_{\substack{I_1,...,I_l:\\ I_1+\cdots+I_l=\{1,...,r\}}} \frac{(-1)^{l-1}}{l} \Pi_{m=1}^l P_{\mathcal{C}}\left[\tilde{Z}_{(j_\alpha;\alpha\in I_l)}\right](u\cdot S_T^*), \tag{40}$$

[where $I_1+\cdots+I_l = \{1, ..., r\}$ means that $I_1, ..., I_m$ form a partition of $\{1, ..., r\}$; we make a distinction between two partitions $\{1\}+\{2, 3\} = \{1, 2, 3\}$ and $\{2, 3\}+\{1\} = \{1, 2, 3\}$, for example], and another one is

$$\begin{aligned}\kappa_{\mathcal{C}}\left[\tilde{Z}_{T,j_1}^{(a_1)}, ..., \tilde{Z}_{T,j_r}^{(a_r)}\right](u\cdot S_T^*) = \sum_{l=1}^r \sum_{\substack{I_1,...,I_l:\\ I_1+\cdots+I_l=\{1,...,r\}}} \frac{(-1)^{l-1}}{l} \sum_{m=1}^l \Bigg\{ \Bigg(& \Pi_{m'=1}^{m-1} \\ & P_{\mathcal{C}}\left[\tilde{Z}_{(j_\alpha;\alpha\in I_{m'})}\right](u\cdot S_T^*)\Bigg) \\ & \times \Bigg(P_{\mathcal{C}}\left[\tilde{Z}_{(j_\alpha;\alpha\in I_m)}\right](u\cdot S_T^*) \\ & - P_{\mathcal{C}}\left[\tilde{Z}_{(j_\alpha;\alpha\in I_m\cap J_1)}\right](u\cdot S_T^*) \\ & \times \quad P_{\mathcal{C}}\left[\tilde{Z}_{(j_\alpha;\alpha\in I_m\cap J_2)}\right](u\cdot S_T^*)\Bigg) \\ & \times \Bigg(\Pi_{m'=m+1}^l \Bigg\{ P_{\mathcal{C}}\left[\tilde{Z}_{(j_\alpha;\alpha\in I_{m'}\cap J_1)}\right](u\cdot S_T^*) \\ & \times \; P_{\mathcal{C}}\left[\tilde{Z}_{(j_\alpha;\alpha\in I_{m'}\cap J_2)}\right](u\cdot S_T^*)\Bigg\}\Bigg)\Bigg\},\end{aligned} \tag{41}$$

where J_1, J_2 form a partition of $\{1, ..., r\}$: $J_1 + J_2 = \{1, ..., r\}$, $\#J_1, \#J_2 \ge 1$.

Denote by $maxgapI$ ($I = (j_1, ..., j_r)$) the maximal gap between consecutive pairs in $j_{(1)} \le \cdots \le j_{(r)}$. Put

$$\Phi_1 = 1_{\{u\in B_T\}} T^{-\frac{r}{2}} \sum_{\substack{j_1,...,j_r:0\le j_1,...,j_r\le N_T\\ maxgap(j_1,...,j_r)\le m(T)^2}} \left|\kappa_{\mathcal{C}}\left[\tilde{Z}_{T,j_1}^{(a_1)}, ..., \tilde{Z}_{T,j_r}^{(a_r)}\right](u\cdot S_T^*)\right|$$

and

$$\Phi_2 = 1_{\{u\in B_T\}} T^{-\frac{r}{2}} \sum_{\substack{j_1,\dots,j_r:0\le j_1,\dots,j_r\le N_T\\ maxgap(j_1,\dots,j_r)>m(T)^2}} \left|\kappa_{\mathcal{C}}\left[\tilde{Z}^{(a_1)}_{T,j_1},\dots,\tilde{Z}^{(a_r)}_{T,j_r}\right](u\cdot S^*_T)\right|.$$

We write $I \in part(j_1, \dots, j_r)$ if $I = (i_1, \dots, i_k)$ is a subsequence of $(j_1, \dots, j_r)$. From (40) and Lemma 12 for the summation on the event $\Omega_1(T) \cap \Pi_{I\in part(j_1,\dots,j_r)} \Omega_2(T, I)$, we obtain an estimate:

$$\begin{aligned}\Phi_1 \le{}& CT^{-\frac{r}{2}} \sum_{\substack{j_1,\dots,j_r:0\le j_1,\dots,j_r\le N_T\\ maxgap(j_1,\dots,j_r)\le m(T)^2}} \left(\max_{j:0\le j\le N_T} P_{\mathcal{C}}\left[\left|\tilde{Z}_{T,j}\right|^{(p+1)\wedge r}\right]+1\right) T^{\beta(r-p-1)_+}\\ &\cdot\left(\left\{\left(\max_{\substack{k,I:\\ 1\le k\le k(T),\\ I\in part(j_1,\dots,j_r)}} \left|P_{\mathcal{C}}\left[\exp\left(S_I^{[m(T)k]}\right)\right]\right|\right)+x_2^{k(T)}\right\}|H_T(u,\mathcal{C})|^- +1\right)^r\\ \lesssim{}& T^{-\frac{r}{2}+\beta(r-p-1)_++1} m(T)^{2(r-1)} M_T(\omega)\left(\theta_T(u,\omega)+1\right)^r 1_{\Omega_1(T)}. \end{aligned}\tag{42}$$

On the other hand, an estimate for Φ_2 follows from (41) and Lemma 13 (the case $r = 1$ is already included in the above part for Φ_1): by [A1](i),(ii), $m(T) \ge k(T) \ge (L_1/\log(x_2/x_1)) \log T$, and hence, $T^{r(1+\beta)}/[m(T)]! \le 1$ for large T; moreover, by using $k(T)/\log T \ge L_1/\log(x_2/x_1)$, we obtain

$$\begin{aligned}\Phi_2 \lesssim{}& T^{-\frac{r}{2}} \sum_{\substack{j_1,\dots,j_r:\\ maxgap(j_1,\dots,j_r)>m(T)^2}} 1_{\Omega_1(T)\cap\Pi_{I\in part(j_1,\dots,j_r)}\Omega_2(T,I)}\\ &\cdot\left|\kappa_{\mathcal{C}}\left[\tilde{Z}^{(a_1)}_{T,j_1},\dots,\tilde{Z}^{(a_r)}_{T,j_r}\right](u\cdot S^*_T)\right| 1_{\{u\in B_T\}}\\ \lesssim{}& T^{-\frac{r}{2}} \sum_{\substack{j_1,\dots,j_r:\\ maxgap(j_1,\dots,j_r)>m(T)^2}} 1_{\Omega_1(T)} \sum_{r'\le r-2} (\text{product of } P[|\tilde{Z}_{T,j_\cdot}|\cdots]+1)\\ &\times(\theta_T(u,\omega)+1)^{r'} T^{(r-r')\beta}\left\{\alpha\left(\delta m(T)|\mathcal{C}\right)+\frac{x_3^{m(T)}}{[m(T)]!}\right\}\left(|H_T(u,\mathcal{C})|^-\right)^2\\ \lesssim{}& 1_{\Omega_1(T)} M_T(\omega) T^{-r/2}\left(\theta_T(u,\omega)+1\right)^r, \end{aligned}\tag{43}$$

since $|H_T(u, \mathcal{C})| \le 1$. Here we took $x_3 < x_2^2$ and also used the fact that on $\Omega_1(T)$,

$$T^{r(1+\beta)}\alpha(\delta m(T)|\mathcal{C}) \le T^{r(1+\beta)}\left(\frac{x_1}{4}\right)^{k(T)} \le x_2^{2k(T)}$$

if $L_1 \ge (\bar{l}+p+1)(1+\beta)\left(\log\frac{x_2}{x_1}\right)\left(\log\frac{4x_2}{x_1}\right)^{-1}$. We then obtain the desired result from (42) and (43). □

Lemma 15. *Assume* $[A1](i), (ii), (iii), (iv)$. *Then there exists* $T_2 > 0$ *such that* $\sup_{\eta\in(0,1)} \theta_T(\eta u, \omega) < 3$ *for* u *satisfying* $|u| \le \min\{c_1 T^{\frac{1}{2}-\beta-2\epsilon}, l(T)\}$ *on* $\{\psi_T > 0\}$ *for* $T > T_2$.

Proof. For n ($n = 0, ..., N_T$), $f = (f_0, f_1, ..., f_n)$ denotes an arbitrary sequence of integers satisfying $0 \le f_0 < f_1 < \cdots < f_n \le N_T$. Denote $F(n, T)$ the set of such sequences f; in particular, $F(N_T, T) = \{(0, 1, ..., N_T)\}$. Let $S_T^*(f) = T^{-1/2}\sum_{j=0}^n \tilde{Z}_{T,f_j}$ for $f \in F(n, T)$, and let $\hat{S}_{I_n}^{[\alpha]}(f) = T^{-1/2}\sum_{j:d(j,\bar{I}_n)\ge\alpha} \tilde{Z}_{T,f_j}$ for $I_n \in \{0, 1, ..., n\}^r$. Put $H_T(u, \mathcal{C}, f) = P_{\mathcal{C}}\left[\exp\left(iu \cdot S_T^*(f)\right)\right]$ for $f \in F(n, T)$. Moreover, let

$$\theta_T(u, \omega, f) = \left\{\left(\max_{\substack{k, I_n: 1\le k\le k(T),\\ |I_n|\le r, I_n\in\{0,1,...,n\}^r}} \left|P_{\mathcal{C}}\left[\exp\left(iu \cdot \hat{S}_{I_n}^{[m(T)k]}(f)\right)\right]\right|\right) + x_2^{k(T)}\right\} \times |H_T(u, \mathcal{C}, f)|^- .$$

Since the sequence $\{\tilde{Z}_{T,f_j}\}$ satisfies the same mixing condition as the original sequence $\{\tilde{Z}_{T,j}\}$ (in fact, the former has more rapid decay than the latter) and the truncation functional ψ_T is sufficient for $\{\tilde{Z}_{T,f_j}^{(a_j)}\}$, we can apply the estimate obtained thus far. Namely, for $\delta' := 4\epsilon + \epsilon_2$, which is less than $1/2$ from [A1](iv), there exist some constants $C > 0$ such that

$$1_{\{u\in B_T(f)\}} H_T(u, \mathcal{C}, f) = 1_{\{u\in B_T(f)\}} \exp\left\{ -\frac{1}{2}\kappa_{\mathcal{C}}\left[u \cdot S_T^*(f), u \cdot S_T^*(f)\right] + R_3(u, \mathcal{C}, f)\right\} \tag{44}$$

for any $f \in F(n, T)$ on $\{\psi_T > 0\}$, where

$$|R_3(u, \mathcal{C}, f)| \le C(1 + |u|^3)T^{-\frac{1}{2}+\delta'} \sup_{\eta\in[0,1]} (\theta(\eta u, \omega, f) + 1)^3 \tag{45}$$

and $B_T(f) = \left\{u; |u| \le c_1 T^{\frac{1}{2}-\beta} m(T)^{-2}\right\} \cap \{u; H_T(\eta u, \mathcal{C}, f) \ne 0\ (\forall \eta \in [0, 1])\} \cap \{u; |u| \le T^{\epsilon_1}\}$. In fact, under [A1](ii), it follows that $m(T)^4 M_T(\omega) \le T^{4\epsilon+\epsilon_2}$ on $\Omega_3(T)$, and Lemma 14 yields the estimate (45). It should be noted that θ-term may be infinity at this stage, while we will show the boundedness under truncation in the next step.

As Götze and Hipp [18], we will show the lemma by induction. Suppose that

$$\bar{\theta}_T(\omega, n') := 1_{\{\psi_T>0\}} \sup\left\{\theta_T(\eta u, \omega, f) : \eta \in [0, 1], f \in F(n', T), |u| \le \min\left\{c_1 T^{\frac{1}{2}-\beta-2\epsilon}, l(T)\right\}\right\} < 3 \tag{46}$$

for $n' \le n - 1$ on $\{\psi_T > 0\}$ when $T > T_2$. Under [A1](iii),(ii), it follows that

$$\left\{u; |u| \le \min\{c_1 T^{\frac{1}{2}-\beta-2\epsilon}, l(T)\}\right\} \subset \left\{u; |u| \le c_1 T^{\frac{1}{2}-\beta} m(T)^{-2}\right\} \cap \left\{u; |u| \le T^{\epsilon_1}\right\}$$

for large T since $\epsilon^* \leq \epsilon_1$ by [A1](iv). If $\bar{\theta}_T(\omega, n) \geq 3$ for some $\omega \in \{\psi_T > 0\}$, then by continuity, there exist $u \in B_T(f)$[13] and $f \in F(n, T)$ such that $1_{\{\psi_T>0\}}\theta_T(u, \omega, f) = 3$ and that $1_{\{\psi_T>0\}}\theta_T(u', \omega, f) < 3$ if $0 \leq |u'| < |u|$.

By definition, for some k $(1 \leq k \leq k(T))$ and some $I_n \in \{0, 1, ..., n\}^r$,

$$
\begin{aligned}
3 = \theta_T(u, \omega, f) &= \left\{\left|P_{\mathcal{C}}\left[\exp\left(iu \cdot \hat{S}_{I_n}^{[m(T)k]}(f)\right)\right]\right| + x_2^{k(T)}\right\} |H_T(u, \mathcal{C}, f)|^- \\
&\leq \exp\left\{\frac{1}{2}\kappa_{\mathcal{C}}\left[u \cdot S_T^*(f), u \cdot S_T^*(f)\right] + C\left(1 + l(T)^3\right)T^{-\frac{1}{2}+\delta'}\right. \\
&\quad \left. \cdot \sup_{\eta\in[0,1]} \left(\theta_T\left(\eta u, \omega, f\right) + 1\right)^3\right\} \\
&\quad \cdot\left[\exp\left\{-\frac{1}{2}\kappa_{\mathcal{C}}\left[u \cdot \hat{S}_{I_n}^{[m(T)k]}(f), u \cdot \hat{S}_{I_n}^{[m(T)k]}(f)\right]\right.\right. \\
&\quad \left.\left. +64C\left(1 + l(T)^3\right)T^{-\frac{1}{2}+\delta'}\right\} + x_2^{k(T)}\right] \\
&\leq \exp\left\{\frac{l(T)^2}{2}\left|Cov_{\mathcal{C}}\left[S_T^*(f), S_T^*(f)\right]\right.\right. \\
&\quad \left. -Cov_{\mathcal{C}}\left[\hat{S}_{I_n}^{[m(T)k]}(f), \hat{S}_{I_n}^{[m(T)k]}(f)\right]\right| \\
&\quad \left. +128C\left(1 + l(T)^3\right)T^{-\frac{1}{2}+\delta'}\right\} \\
&\quad +x_2^{k(T)}\exp\left\{\frac{l(T)^2}{2}\left|Cov_{\mathcal{C}}\left[S_T^*(f), S_T^*(f)\right]\right|\right. \\
&\quad \left. +64C\left(1 + l(T)^3\right)T^{-\frac{1}{2}+\delta'}\right\}.
\end{aligned}
\tag{47}
$$

Here we used the assumption of induction for $S_{I_n}^{[m(T)k]}$, and the definition of u.

Since clearly $\left|S_T^*(f) - \hat{S}_{I_n}^{[m(T)k]}(f)\right| \lesssim T^{-\frac{1}{2}+\beta}m(T)k(T)$, it follows from [A1] (iv) that

$$
\begin{aligned}
&1_{\Omega_4(T)}l(T)^2\left|Cov_{\mathcal{C}}\left[S_T^*(f), S_T^*(f)\right] - Cov_{\mathcal{C}}\left[\hat{S}_{I_n}^{[m(T)k]}(f), \hat{S}_{I_n}^{[m(T)k]}(f)\right]\right| \\
&\leq 1_{\Omega_4(T)}l(T)^2\left\{P_{\mathcal{C}}\left[\left|S_T^*(f)\right|^2 + \left|\hat{S}_{I_n}^{[m(T)k]}(f)\right|^2\right]\right. \\
&\quad \left. \cdot P_{\mathcal{C}}\left[\left|S_T^*(f) - \hat{S}_{I_n}^{[m(T)k]}(f)\right|^2\right]\right\}^{\frac{1}{2}}
\end{aligned}
$$

[13] In fact, one may assume that at that u, $H_T(\eta u, \mathcal{C}, f) \neq 0$ $(\eta \in [0, 1])$. For, if u_0 is a point at which $|u|$ is minimum in $\left\{u;\ |u| \leq c_1 T^{\frac{1}{2}-\beta}m(T)^{-2}\right\} \cap \{u;\ |u| \leq T^{\epsilon_1}\} \cap \{u;\ H_T(u, \mathcal{C}, f) = 0\}$, then it turns out to be in contradiction to (44) and (45).

$$
\begin{aligned}
&\lesssim l(T)^2 \cdot \sqrt{\frac{k(T)}{l(T)^2}} \cdot T^{-\frac{1}{2}+\beta} m(T) k(T) \\
&= T^{-\frac{1}{2}+\beta} k(T)^{\frac{3}{2}} l(T) m(T) \lesssim T^{-\frac{1}{2}+\beta+\frac{5}{2}\epsilon+\epsilon^*} \to 0
\end{aligned}
$$

as $T \to \infty$. Thus, [A1](iv), (47) and the definition of $\Omega_4(T)$ yield

$$
3 \le A_1 + A_2 x_2^{k(T)} \exp\left\{ \frac{l(T)^2}{2} \left| Cov_{\mathcal{C}}\left[S_T^*(f), S_T^*(f) \right] \right| \right\} < 3
$$

for universally large T, where A_1 and A_2 are certain positive constants less than $3/2$. It is a contradiction, and thus the proof is completed. □

Proof of Lemma 5. Let $\delta_1 := p'\epsilon^* + 2(\bar{l} + p)\epsilon + \epsilon_2 < 1/(4p)$, where $\epsilon^*, \epsilon, \epsilon_2$ are determined by $[A1]$. Then on the event $\{\psi_T > 0\}$ it follows from $[A1]$ that for $u \in \tilde{U}_T$,

$$
\begin{aligned}
|u|^j m(T)^{2(r-1)} M_T(\omega) &\lesssim (1 + l(T)^{p'}) m(T)^{2(r-1)} M_T(\omega) \\
&\lesssim T^{p'\epsilon^* + 2(\bar{l}+p)\epsilon + \epsilon_2} = T^{\delta_1}.
\end{aligned}
$$

Consequently, one has the desired estimate from Lemmas 14 and 15. Note the inequality $-\frac{r}{2} + \beta(r - p - 1) + 1 \le -\frac{p-2}{2} - \frac{1}{2}$ for $r \ge p + 1$, applied to the second case in Lemma 14. □

Proof of Lemma 6. The proof is straightforward. Take $C_2 > 0$ sufficiently large. Denote $Z_{T,j} = Z_{\tilde{I}_j}$. Put

$$
\begin{aligned}
\Phi_1^* = T^{-\frac{r}{2}} \sum_{\substack{j_1,\dots,j_r : 0 \le j_1,\dots,j_r \le N_T \\ maxgap(j_1,\dots,j_r) \le C_2 m(T)}} &\left| \kappa_{\mathcal{C}}\left[\tilde{Z}_{T,j_1}^{(a_1)}, \dots, \tilde{Z}_{T,j_r}^{(a_r)} \right](0) \right. \\
&\left. -\kappa_{\mathcal{C}}\left[Z_{T,j_1}^{(a_1)}, \dots, Z_{T,j_r}^{(a_r)} \right](0) \right|
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_2^* = T^{-\frac{r}{2}} \sum_{\substack{j_1,\dots,j_r : 0 \le j_1,\dots,j_r \le N_T \\ maxgap(j_1,\dots,j_r) > C_2 m(T)}} &\left\{ \left| \kappa_{\mathcal{C}}\left[\tilde{Z}_{T,j_1}^{(a_1)}, \dots, \tilde{Z}_{T,j_r}^{(a_r)} \right](0) \right| \right. \\
&\left. + \left| \kappa_{\mathcal{C}}\left[Z_{T,j_1}^{(a_1)}, \dots, Z_{T,j_r}^{(a_r)} \right](0) \right| \right\}.
\end{aligned}
$$

For $j_1, \dots, j_r \in \{0, 1, \dots, N_T\}$ and a decomposition $I_1 + \cdots + I_l = \{1, \dots, r\}$,

$$
\begin{aligned}
\left| \Pi_{m=1}^l P_{\mathcal{C}}[\tilde{Z}_{(j_\alpha;\alpha\in I_l)}] - \Pi_{m=1}^l P_{\mathcal{C}}[Z_{(j_\alpha;\alpha\in I_l)}] \right| &\le \sum_{m=1}^{l} \Pi_{m'=1}^{m-1} |P_{\mathcal{C}}[\tilde{Z}_{(j_\alpha;\alpha\in I_{m'})}]| \\
&\quad \cdot |P_{\mathcal{C}}[\tilde{Z}_{(j_\alpha;\alpha\in I_m)} - Z_{(j_\alpha;\alpha\in I_m)}]| \\
&\quad \cdot \Pi_{m'=m+1}^{l} |P_{\mathcal{C}}[Z_{(j_\alpha;\alpha\in I_{m'})}]| \\
&\lesssim M_T(\omega) T^{-(p+1-r)\beta}.
\end{aligned}
$$

Here we used the following estimate: for nonnegative random variables $z_1, ..., z_{r'-1}$, $z_{r'}$, $r' \le r \le p$,

$$
\begin{aligned}
P_{\mathcal{C}}[z_1 \cdots z_{r'-1} z_{r'} 1_{\{z_{r'} \ge a\}}] &\le P_{\mathcal{C}}[z_1 \cdots z_{r'-1} z_{r'} 1_{\{z_{r'} \ge a\}} (z_{r'} a^{-1})^{p+1-r}] \\
&\le \left\{ \Pi_{i=1}^{r'-1} P_{\mathcal{C}}[z_i^{p+1}] \cdot (P_{\mathcal{C}}[z_{r'}^{p+1}])^{p+2-r} \right\}^{1/(p+1)} \\
&\quad \times a^{-(p+1-r)}.
\end{aligned}
$$

In view of (40), we see that $\Phi_1^* \lesssim T^{-\frac{r}{2}+1-(p+1-r)\beta} m(T)^{r-1} M_T(\omega)$.

On the other hand, it follows from a similar formula as (41) and the so-called covariance inequality for mixing processes that

$$
\left| \kappa_{\mathcal{C}} \left[Z_{T,j_1}^{(a_1)}, ..., Z_{T,j_r}^{(a_r)} \right](0) \right| \lesssim M_T(\omega)^{r/(p+1)} \left\{ \alpha(\delta C_2' m(T) | \mathcal{C}) \right\}^{1/(p+1)}
$$

for some constant C_2', and a similar estimate for $\tilde{Z}_{T,j}$. Consequently, $\Phi_2^* \lesssim T^{\frac{r}{2}}$ $\alpha(\delta m(T)|\mathcal{C})^{1/(p+1)} M_T(\omega)^{r/(p+1)}$. This completes the proof. □

Proof of Lemma 7. Let $\psi_T(\omega) > 0$. Then, for large T, $H_T(u', \mathcal{C}) \ne 0$ if $|u'| \le |u|$ for $u \in \tilde{U}_T$. Denote $\kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](V) = \kappa_{\mathcal{C}}[u \cdot S_T^*, ..., u \cdot S_T^*](V)$ (r times). For $f(s) = \log P_{\mathcal{C}}[e^{isu \cdot S_T^*}]$, $s \in [0, 1]$,

$$
(\partial_s)^r f(s) = (\partial_{s_1})_0 \cdots (\partial_{s_r})_0 f(s + s_1 + \cdots + s_r) = i^r \kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](su \cdot S_T^*).
$$

Expanding $f(1)$ around zero with Taylor's formula, we have

$$
H_T(u, \mathcal{C}) = \exp\left(\frac{1}{2} \chi_{T,2,\mathcal{C}}(u) + \sum_{r=3}^{p} \frac{i^r}{r!} \kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](0) + R_{p+1}(u) \right),
$$

where

$$
\begin{aligned}
R_{p+1}(u) = \frac{i^{p+1}}{p!} &\int_0^1 (1-s)^p \kappa_{\mathcal{C}}^{*(p+1)}[u^{\otimes(p+1)}](su \cdot S_T^*) ds \\
&- \frac{1}{2} \left(\kappa_{\mathcal{C}}^{*2}[u^{\otimes 2}](0) + \chi_{T,2,\mathcal{C}}(u) \right).
\end{aligned}
$$

Obviously, $H_T(u, \mathcal{C}) = \hat{\Psi}_{T,p,\mathcal{C}}^*(u) + R_{p+1}^*(u)$, where

$$
\begin{aligned}
\hat{\Psi}_{T,p,\mathcal{C}}^*(u) = \exp\left(\frac{1}{2} \chi_{T,2,\mathcal{C}}(u) \right) &+ \exp\left(\frac{1}{2} \chi_{T,2,\mathcal{C}}(u) \right) \\
&\cdot \sum_{j=1}^{p} \sum_{r_1, ..., r_j = 1}^{p-2} 1_{\{r_1 + \cdots + r_j \le p-2\}} \\
&\times (-1)^j i^{r_1 + \cdots + r_j} \frac{\kappa_{\mathcal{C}}^{*(r_1+2)}[u^{\otimes r_1+2}](0) \cdots \kappa_{\mathcal{C}}^{*(r_j+2)}[u^{\otimes r_j+2}](0)}{j!(r_1+2)! \cdots (r_j+2)!}
\end{aligned}
$$

and

$$
\begin{aligned}
R^*_{p+1}(u) = \exp\left(\frac{1}{2}\chi_{T,2,\mathcal{C}}(u)\right) \\
&\cdot \sum_{j=1}^{p} \sum_{r_1,\ldots,r_j=1}^{p-2} 1_{\{r_1+\cdots+r_j \geq p-1\}} \\
&\times (-1)^j i^{r_1+\cdots+r_j} \frac{\kappa_{\mathcal{C}}^{*(r_1+2)}[u^{\otimes r_1+2}](0)\cdots\kappa_{\mathcal{C}}^{*(r_j+2)}[u^{\otimes r_j+2}](0)}{j!(r_1+2)!\cdots(r_j+2)!} \\
&+\exp\left(\frac{1}{2}\chi_{T,2,\mathcal{C}}(u)\right)\cdot\left(\sum_{r=3}^{p}\frac{i^r}{r!}\kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](0)+R_{p+1}(u)\right)^{p+1} \\
&\cdot(p!)^{-1}\int_0^1(1-t)^p\exp\left(t\sum_{r=3}^{p}\frac{i^r}{r!}\kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](0)+tR_{p+1}(u)\right)dt \\
&+\exp\left(\frac{1}{2}\chi_{T,2,\mathcal{C}}(u)\right)\left\{\sum_{j=1}^{p}\sum_{j'=0}^{j-1}(j!)^{-1}\binom{j}{j'}\right. \\
&\left.\times\left(\sum_{r=3}^{p}\frac{i^r}{r!}\kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](0)\right)^{j'}\left(R_{p+1}(u)\right)^{j-j'}\right\}.
\end{aligned}
$$

From the inequality in [A1](i): $L_1 \geq (p+1)(p-1-(p-1)\epsilon+\beta)\left(\log\frac{x_2}{x_1}\right)\left(\log\frac{4}{x_1}\right)^{-1}$ and $-1+\epsilon+\beta<0$, it holds that

$$
T^{\frac{r}{2}}\left(\frac{x_1}{4}\right)^{L_1(p+1)^{-1}(\log\frac{x_2}{x_1})^{-1}\log T} \lesssim T^{-\frac{r}{2}+1+(r-1)\epsilon-(p+1-r)\beta}
$$

for any $r \leq p$; hence, $T^{\frac{r}{2}}\{\alpha(\delta m(T)|\mathcal{C})\}^{1/(p+1)} \lesssim T^{-\frac{r}{2}+1+(r-1)\epsilon-(p+1-r)\beta}$ on $\Omega_1(T)$ for any $r \leq p$ since $k(T)/\log T \geq L_1/\log(x_2/x_1)$ by [A1](i). Therefore, from Lemma 6, it follows that

$$
\begin{aligned}
&1_{\Omega_1(T)}\left|\kappa_{\mathcal{C}}\left[S_T^{*(a_1)},\ldots,S_T^{*(a_r)}\right](0)-\kappa_{\mathcal{C}}\left[S_T^{(a_1)},\ldots,S_T^{(a_r)}\right](0)\right| \\
&\lesssim M_T(\omega)T^{-\frac{r}{2}+1+(r-1)\epsilon-(p+1-r)\beta}
\end{aligned}
$$

for $r \leq p$ under [A1]. It follows from [A1](iv) that

$$
l\epsilon+(r-1)\epsilon+d\epsilon^*+(l+3p-2)\epsilon^*+\epsilon_2<(\bar{l}+p)\epsilon+(p'+d)\epsilon^*+\epsilon_2<\frac{1}{4p},
$$

therefore, that

$$
\begin{aligned}
&-\frac{r}{2}+1+(r-1)\epsilon-(p+1-r)\beta+l\epsilon+(l+3p-2)\epsilon^*+\epsilon_2 \\
&<-\frac{r}{2}+1-(p+1-r)\beta+\frac{1}{4p}-d\epsilon^* \\
&<-\frac{p-2}{2}-d\epsilon^*. \qquad (48)
\end{aligned}
$$

Since $|\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]|\lesssim k(T)\lesssim T^{\epsilon}$ on $\Omega_4(T)$ from [A1](ii), we see from (48) that there exists a constant $\delta_0 \in (0, \frac{1}{2})$ such that

$$\begin{aligned}
&1_{\{\psi_T>0\}}1_{\tilde{U}_T}(u)\left|\partial_u^l\left\{\hat{\Psi}^*_{T,p,\mathcal{C}}(u)-\hat{\Psi}_{T,p,\mathcal{C}}(u)\right\}\right|\lesssim \exp\left(\frac{1}{2}\chi_{T,2,\mathcal{C}}(u)\right)\\
&\quad\cdot\left(1+|\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]|\right)^l l(T)^{l+(3p-2)}\\
&\quad\cdot M_T(\omega)\max_{r:3\le r\le p} T^{-\frac{r}{2}+1+(r-1)\epsilon-(p+1-r)\beta}\\
&\quad\lesssim T^{-(p-2)/2-\delta_0-d\epsilon^*}
\end{aligned}$$

for u ($|u|\le l(T)$) and $l\le \bar{l}$; note that $\chi_{T,2,\mathcal{C}}(u)\le 0$ and we here used [A1](iii).

On the other hand, applying Lemma 5 to the case $\eta=0$, we obtain

$$\begin{aligned}
&1_{\{\psi_T>0\}}1_{\tilde{U}_T}(u)\Bigg|\partial_u^l\Bigg\{\exp\bigg(\frac{1}{2}\chi_{T,2,\mathcal{C}}(u)\bigg)\cdot\sum_{j=1}^{p}\sum_{r_1,\dots,r_j=1}^{p-2}1_{\{r_1+\cdots+r_j\ge p-1\}}(-1)^j i^{r_1+\cdots+r_j}\\
&\quad\cdot\frac{\kappa_{\mathcal{C}}^{*(r_1+2)}[u^{\otimes r_1+2}](0)\cdots\kappa_{\mathcal{C}}^{*(r_j+2)}[u^{\otimes r_j+2}](0)}{j!(r_1+2)!\cdots(r_j+2)!}\Bigg\}\Bigg|\\
&\lesssim T^{-(p-1)/2+\delta_1'}
\end{aligned}$$

for some $\delta_1'<1/2-d\epsilon^*$. In fact, since $p^2\epsilon^*+\bar{l}\epsilon^*+\bar{l}\epsilon<p'\epsilon^*+\bar{l}\epsilon<-d\epsilon^*+2p'\epsilon^*+\bar{l}\epsilon<-d\epsilon^*+\frac{1}{2p}<-d\epsilon^*+\frac{1}{4}$ from [A1](iv), $T^{-(p-1)/2+p\delta_1}l(T)^{p^2+\bar{l}}\left(1+|\mathrm{Var}_{\mathcal{C}}\left[Z_T/\sqrt{T}\right]|\right)^l\lesssim T^{-(p-1)/2+\delta_1'}$ for $\delta_1'=p\delta_1+(p^2+\bar{l})\epsilon^*+\bar{l}\epsilon<1/2-d\epsilon^*$.

Let $a_1, ..., a_l\in\mathbf{R}^d$. For $\mathbf{R}^d$-valued random variables $X_1, ..., X_j$ and V satisfying suitable integrability,

$$\begin{aligned}
&(\partial_u)^l\left\{\kappa_{\mathcal{C}}[u\cdot X_1, ..., u\cdot X_j](\eta u\cdot V)\right\}[a_1, ..., a_l]\\
&\quad=(\partial_{\epsilon_1})_0\cdots(\partial_{\epsilon_l})_0\left\{\kappa_{\mathcal{C}}[u\cdot X_1, ..., u\cdot X_j](\eta u\cdot V)\Big|_{u\leftarrow u+\sum_{i=1}^l\epsilon_i a_i}\right\}\\
&\quad=\sum_{i_1,\dots,i_j=0}^{l}\sum_{s=0}^{l}\sum_{i_1',\dots,i_s'=1}^{l}c(i_1, ..., i_j; s; i_1', ..., i_s')\\
&\qquad\times\kappa_{\mathcal{C}}[a_{i_1}\cdot X_1, ..., a_{i_j}\cdot X_j, \eta a_{i_1'}\cdot V, ..., \eta a_{i_s'}\cdot V](\eta u\cdot V),
\end{aligned}$$

where $a_0=u$. With this representation and Lemma 5, we have

$$1_{\{\psi_T>0\}}1_{\tilde{U}_T(u)}\left|(\partial_u)^l R_{p+1}(u)\right|\lesssim T^{-\frac{p-2}{2}-(\frac{1}{2}-\delta_1)}l(T)^{(p+1)-p'}+M_T l(T)^2T^{a(2)},$$

where $a(r)=-\frac{r}{2}+1+(r-1)\epsilon-(p+1-r)\beta$. In view of Lemma 5, we see that

$$1_{\{\psi_T>0\}}1_{\tilde{U}_T}(u)\left|\partial_u^l\left\{\sum_{r=3}^{p}\frac{i^r}{r!}\kappa_{\mathcal{C}}^*[u^{\otimes r}](0)+R_{p+1}(u)\right\}\right|\lesssim l(T)^{p+1}T^{-\frac{1}{2}+\delta_1}$$

for any $l \leq \bar{l}$. Since $(p+1)\epsilon^* - \frac{1}{2} + \delta_1 < 0$ and hence the derivatives of

$$\exp\left(t\sum_{r=3}^{p}\frac{i^r}{r!}\kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](0) + tR_{p+1}(u)\right)$$

with respect to u are bounded under truncation, we obtain the estimate:

$$\begin{aligned}
&1_{\{\psi_T>0\}}1_{\tilde{U}_T}(u)\Big|(\partial_u)^l\Big\{\exp\Big(\frac{1}{2}\chi_{T,2,\mathcal{C}}(u)\Big)\cdot\left(\sum_{r=3}^{p}\frac{i^r}{r!}\kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](0) + R_{p+1}(u)\right)^{p+1}\\
&\cdot(p!)^{-1}\int_0^1(1-t)^p\exp\left(t\sum_{r=3}^{p}\frac{i^r}{r!}\kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](0) + tR_{p+1}(u)\right)dt\Big\}\Big|\\
&\lesssim l(T)^{\bar{l}+(p+1)^2}T^{\bar{l}\epsilon}T^{(-\frac{1}{2}+\delta_1)(p+1)}\lesssim T^{-p/2-d\epsilon^*}
\end{aligned}$$

with the inequality $1/(2p) - (p+1)/2 + (p+1)/(4p) \leq -p/2$.

Finally,

$$\begin{aligned}
&1_{\{\psi_T>0\}}1_{\tilde{U}_T}(u)\Big|(\partial_u)^l\Big\{\exp\left(\frac{1}{2}\chi_{T,2,\mathcal{C}}(u)\right)\cdot\sum_{j=1}^{p}\sum_{j'=0}^{j-1}(j!)^{-1}\\
&\quad\times\binom{j}{j'}\left(\sum_{r=3}^{p}\frac{i^r}{r!}\kappa_{\mathcal{C}}^{*r}[u^{\otimes r}](0)\right)^{j'}\Big(R_{p+1}(u)\Big)^{j-j'}\Big\}\Big|\\
&\lesssim l(T)^{\bar{l}}T^{\bar{l}\epsilon}\cdot l(T)^{(p+1)(p-1)}\cdot\{T^{-\frac{p-2}{2}-(\frac{1}{2}-\delta_1)}l(T)^{(p+1)-p'} + M_T l(T)^2T^{a(2)}\}\\
&\lesssim T^{-\frac{p-2}{2}-\delta_2-d\epsilon^*}
\end{aligned}$$

for some positive δ_2. In the last part, the following inequalities were used:

$$\begin{aligned}
&\bar{l}\epsilon^* + \bar{l}\epsilon + (p+1)(p-1)\epsilon^* - \frac{p-2}{2} - (\frac{1}{2}-\delta_1) + ((p+1)-p')\epsilon^*\\
&\quad< (\bar{l} + (p+1)^2 - p')\epsilon^* + \bar{l}\epsilon - (\frac{1}{2}-\delta_1) - \frac{p-2}{2}\\
&\quad< \bar{l}\epsilon - \frac{1}{2} - \frac{p-2}{2} + \delta_1 < -\frac{p-2}{2} - \frac{1}{4} < -\frac{p-2}{2} - \frac{1}{8} - d\epsilon^*,
\end{aligned}$$

and

$$\begin{aligned}
&\bar{l}\epsilon^* + \bar{l}\epsilon + (p+1)(p-1)\epsilon^* + \epsilon_2 + 2\epsilon^* + (\epsilon - (p-1)\beta)\\
&\quad< (\bar{l} + (p+1)(p-1) + 2)\epsilon^* + (\bar{l}+1)\epsilon + \epsilon_2 - \frac{p-1}{2}[1 - \frac{2p-1}{2p(p+1)}]\\
&\quad< -\frac{p-2}{2} + \Big[-\frac{1}{2} + \frac{1}{4p} + \frac{(p-1)(2p-1)}{4p(p+1)}\Big]\\
&\quad< -\frac{p-2}{2} - d\epsilon^*.
\end{aligned}$$

since $d\epsilon^* < \frac{1}{4p}$.

After all, we obtained $1_{\{\psi_T>0\}}1_{\tilde{U}_T}(u)\left|(\partial_u)^l R^*_{p+1}(u)\right| \lesssim T^{-\frac{p-2}{2}-\delta_3-d\epsilon^*}$, $l \leq \bar{l}$, for some $\delta_3 > 0$. □

Acknowledgements. I thank Professors Yuji Sakamoto, Masayuki Uchida and Dr. Hiroki Masuda for stimulative discussions and cooperation. I also appreciate Mr. Kazuhiro Mitani's assistance. They extensively helped me to improve the previous version of this manuscript. Discussions with Professors F. Götze, J. L. Jensen, M. Taniguchi, Y. Yajima, Y. Ishikawa and T. Honda were very helpful. I express my gratitude to the associate editor and the referees for valuable comments.

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