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Malliavin calculus and asymptotic expansion for martingales[★]

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Summary. We present an asymptotic expansion of the distribution of a random variable which admits a stochastic expansion around a continuous martingale. The emphasis is put on the use of the Malliavin calculus; the uniform nondegeneracy of the Malliavin covariance under certain truncation plays an essential role as the Cramér condition did in the case of independent observations. Applications to statistics are presented.

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1. Introduction

We consider a sequence of random variables $X_n, n \in N$, which have a stochastic expansion $X_n = M_n + r_n N_n$, where for each $n \in N, M_n$ is the terminal random variable M_{n,T_n} of a continuous martingale $(M_{n,t}, \mathbb{F}_{n,t})_{0 \le t \le T_n}$ with $M_{n,0} = 0, N_n$ is a random variable, and (r_n) is a sequence of positive numbers tending to zero. The martingale central limit theorem says that if the quadratic variation $\langle M_n \rangle_{T_n}$ converges in probability to 1 and if $N_n = O_p(1)$, then the distribution of X_n converges weakly to the standard normal distribution $N(0,1)$. See, e.g., Jacod-Shiryaev [14].

As for refinements of the central limit theorem for martingales, we know several results. Among others, Bolthausen [4] and Haeusler [11] obtained Berry-Esseen type bounds. Liptser-Shiryaev [17] presented the rate of con-

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vergence in the central limit theorem for semi-martingales. Recently, Mykland [20] obtained an asymptotic expansion of the expectation $E[g(M_{n,T_n})]$ for a class of C^2-functions g. There exists an example of X_n for which $X_n = Z + o_p(r_n^m)$ with Z a $N(0,1)$ random variable and m any positive integer, however the distribution of X_n does not admit approximation by any continuous function up to $o(r_n)$. This example suggests the necessity of an assumption of the regularity of X_n. Generally speaking, in the case of independent observations, in order to prove the validity of the asymptotic expansions one usually needs a certain regularity condition for the underlying distribution, such as the Cramér condition; this type of condition then ensures the regularity of the distribution and hence the smoothness assumption on g can be removed (e.g., Bhattacharya-Rao [2]). On the other hand, it is well-known that the Malliavin calculus leads to the regularity of the distribution of a functional with nondegenerate Malliavin covariance. Therefore it seems natural to apply this theory to the asymptotic distribution theory, and the emphasis of this article is put on the use of the Malliavin calculus.

Watanabe [31] introduced the notion of asymptotic expansion for generalized Wiener functionals, and it was applied to heat kernels (Watanabe [31], Uemura [30], Takanobu [27], Takanobu-Watanabe [28]). Kusuoka-Stroock [16] took another approach toward asymptotic expansions for certain Wiener functionals by using the Malliavin calculus. As for statistical estimators, Watanabe's theory was applied in [32, 37, 33, 34, 23] to obtain asymptotic expansions of their distributions. We may regard these results as a refinement of the martingale central limit theorems. However, the situation considered here is different from the one considered in our previous papers in the sense that the limit random variable of a sequence of weakly converging random variables may not exist on the same probability space as the sequence exists on; as a matter of fact this situation is rather usual in central limit theorems. In this sense, our results are principally concerned with distributions, and this fact is reflected by the proof where Berry-Esseen's smothing inequality (or Fourier analysis) plays an important role together with estimations of characteristic functions by means of the Malliavin calculus.

In this paper, we assume that the Malliavin covariance of either X_n or M_n is nondegenerate under truncation by a functional ψ_n; more precisely, we assume a certain regularity condition (Condition [r] stated in Section 3) of characteristic functions, as this is a consequence of the nondegeneracy of the Malliavin covariance in the case of Wiener functionals. Under this condition, we will present an asymptotic expansion of the distribution of X_n in Section 3 (Lemma 1′), and prove the validity of it in Section 4.

Let X be a differentiable, $\mathbb{R}$-valued Wiener functional defined on a Wiener space. Assume that there exists a functional ψ such that

$$\sup_{u\in\mathbb{R}} |u|^j |E[e^{iuX} X^\alpha \psi]| < \infty, \quad \alpha \in Z_+ \ .$$

If $j > 1$, then the function $g(x) = (2\pi)^{-1} \int_R e^{-iux} E[e^{iuX} \psi] du$ is well-defined: in fact, $g(x)$ is a continuous version of $E[\psi | X = x] d\mu^X / dx$, where μ^X is the induced measure of X. The functional ψ is a truncation functional extracting, from the Wiener space, the portion on which the distribution of X is regular. If X is almost regular, we may take ψ nearly equal to one. In this sense, we call g the local density of X on ψ. Under regulatiry conditions, we will present asymptotic expansion of local density $(2\pi)^{-1} \int_R e^{-iux} E[e^{iuX_n} \psi_n] du$ and prove a non-uniform bound for the error term of this expansion (Lemma 1 of Section 3). From this result, one obtains the asymptotic expansion of the mean value $E[f(X_n)]$ for any measurable function f of at most polynomial growth order.

For practical purposes, the (partial) Malliavin calculus seems to be the most effective to verify Condition [r]: with the aid of the (partial) Malliavin calculus, the main results will be stated in Section 2 and proved in Section 5 as corollaries of Lemmas 1 and 1′ of Section 3.

These results are generalizations of those in [35] and implement the theory of higher order statistical inference, especially inference for diffusion type processes. We will present in Section 6 applications of our result to estimation problems for unknown parameter of ergodic diffusions and for diffusion coefficients of diffusion type processes. For example, one can show the uniform nondegeneracy (with certain truncation) of the Malliavin covariance of the functional $\int_0^T f(X_t) dw_t / T^{\frac{1}{2}}$, where $f: \mathbb{R} \to \mathbb{R}$ and X_t is a one dimensional, stationary, ergodic diffusion process satisfying some conditions. Thus it is possible to derive the asymptotic expansion for this functional. It is well-known to statisticians that the asymptotic expansion is an indispensable tool to develop the higher-order statistical inference (Ghosh [9], Pfanzagl [21, 22], Akahira-Takeuchi [1], Taniguchi [29], and vast literature). In spite of its importance, there were no results for semimartingale models from lack of expansion formulas for distributions. Beyond the first-order argument, Mishra-Prakasa Rao [19] presented the Berry-Esseen bound for maximum likelihood estimator for linearly parametrized (but in general nonlinear) diffusion processes. For instance, their result gives $O(T^{\frac{1}{5}})$-error bound for the Ornstein-Uhlenbeck process, and in this case, the $O(T^{\frac{1}{2}})$-bound was obtained by Bose [5]. Our result concerning the second-order asymptotics improves their results for a class of nonlinear diffusion processes as well as the linear diffusions.

The estimation of diffusion coefficients (volatility) is an essentially important problem in economics. Among many papers, the recent crucial work in the first-order was done by Dohnal [6] and Genon = Catalot-Jacod [7]. No results have been known on asymptotic expansion except for the very trivial cases. We here treat an estimator for the linear (but we often met in applications) parameter of the diffusion coefficient of Itô processes, and present an asymptotic expansion as an application of our general result. If the diffusion coefficient is parametrized non-linearly, then reasonable estimators asymptotically have a non-normal distribution even in the first-order, and this case would be difficult to treat, at least from the second-order aspects, for we have not yet had any general higher-order limit theorem for such non-central cases.

The first applications of the Malliavin calculus to statistics were done to derive asymptotic expansions for small diffusions. The second-order expansion was important to go into the second-order inference from already established first-order theory, and the previous second-order results (asymptotic expansion, second-order efficiency, etc.) for small diffusion models are again obtained by using the results here. Another statistical application is the asymptotic expansion of mixture type estimators. One there meets an unusual expansion; unusual because each term consists of a nonlinear function multiplied by normal density, and hence it is no longer a familiar Edgeworth expansion. This result has statistical importance, since from this formula, we can show the inadmissibility of the natural prediction region in the decision theory, as it is referred to as Stein's phenomenon (Takada-Sakamoto-Yoshida [26]).

The method here is a "global" approach in the sense that it applies the Malliavin calculus directly to Wiener functionals. The global approach has the advantage of applicability to various kinds of problems, a few of which were mentioned above. On the other hand, as we recently found it, with the aid of the Malliavin calculus, there is still another method ("local" approach) which provides in a more effective way a solution to expansion of a functional of a process with the geometrically strong mixing property.

2. Main results

For each $n \in \mathbb{N}$, let (W_n, H_n, P_n) denote an r-dimensional Wiener space, and let $D^n_{p,s}$ be the Sobolev space of Wiener functionals on W_n (cf. Ikeda-Watanabe [12]). More generally, $D^n_{p,s}$ may be the Sobolev spaces in the partial Malliavin calculus (Michel [18], Bismut-Michel [3], Kusuoka-Stroock [15]); for definition see Subsection 6.1. For each $n \in \mathbb{N}, D^n_{p,s}$ is equipped with a Sobolev norm, which is denoted by $\|\cdot\|_{p,s}$ without the index n. Let (r_n) be a sequence of positive numbers tending to zero as $n \to \infty$. We consider functionals X_n on (W_n, P_n), $n \in \mathbb{N}$, defined by:

$$X_n = M_n + r_n N_n \ ,$$

where, for each $n \in \mathbb{N}$, M_n is the terminal random variable M_{n,T_n} of a continuous martingale $(M_{n,t}\colon 0 \le t \le T_n)$ defined on W_n with respect to some stochastic basis $(\mathbb{F}_{n,t}\colon 0 \le t \le T_n)$, and N_n is another random variable on W_n. We do not assume that N_n has particular stochastic properties, such as the martingale property. The predictable quadratic variation process of $(M_{n,t}\colon 0 \le t \le T_n)$ is denoted by $\langle M_n \rangle$, and for simplicity we will use the same notation $\langle M_n \rangle$ for $\langle M_n \rangle_{T_n}$.

Let ϕ be the density function of the standard normal distribution. As in [33, 34], the truncation functional ψ_n plays an important role in this article.

We consider the following conditions (the first one is the martingale assumption stated above):
[A1] M_n is the terminal random variable of a continuous martingale vanishing at $t = 0$, and $X_n = M_n + r_n N_n$ for any $n \in \mathbb{N}$.
$[A2]_k$ $M_n, N_n \in D^n_{p,k+1}$ and $\langle M_n \rangle_{T_n} \in D^n_{p,k}$ for any $p > 1$. Moreover, $\sup_n \|M_n\|_{p,k+1} + \sup_n \|r_n^{-1}(\langle M_n \rangle_{T_n} - 1)\|_{p,k} + \sup_n \|N_n\|_{p,k+1} < \infty$ for any $p > 1$.
[A3] The random vector $(M_n, r_n^{-1}(\langle M_n \rangle_{T_n} - 1), N_n)$ converges in distribution to a random vector (Z, ξ, η) on a certain probability space.
$[A3]_+$ The condition [A3] holds and there exists the integrable bounded derivative $\partial_x^2 (E[\xi | Z = z]\phi(z))$.
$[A4]_k$ There exist $\psi_n \in \cap_{p>1} D^n_{p,k}$ satisfying the following conditions: (1) $0 \leq \psi_n \leq 1$; (2) There exists a constant a such that $0 < a < 1/3$ and such that, on $\{w\colon r_n^{-(1-a)} | \langle M_n \rangle_{T_n} - 1| > 1\}$, $D^j \psi_n(w) = 0$ a.s. for all $j \in Z^+$ with $0 \leq j \leq k$; (3) $\psi_n \to^p 1$ as $n \to \infty$; (4) There exists $t > 1$ such that $\sup_n E[|D^j \psi_n|^t_{H_n^{\otimes j}} \sigma_{X_n}^{-p}] < \infty$ for any $p > 1$ and $j \in Z^+$ with $0 \leq j \leq k$.

We then have the following theorem:

Theorem 1 *Suppose that Conditions* $[A1]$, $[A2]_3$, $[A3]_+$ *and* $[A4]_3$ *hold. Set* $\Delta_n = \sup_x |P[X_n \leq x] - \int_{-\infty}^x p_n(z) dz|$, *where*

$$p_n(z) = \phi(z) + \frac{1}{2} r_n \partial_z^2 (E[\xi | Z = z]\phi(z)) - r_n \partial_z (E[\eta | Z = z]\phi(z)) \ .$$

Then there exist a sequence ϵ_n *with* $\epsilon_n = o(r_n)$ *and constants* $C_p (p > 1)$ *such that*

$$\Delta_n \leq C_p \Big(1 + \log^+ (r_n^{-(1-a)/2})\Big) \|1 - \psi_n\|_{L^p} + \epsilon_n \ .$$

Remark 1. (1) If $[A2]_4$ and $[A4]_4$ hold true, then the integrable bounded derivative $\partial_z^2 (E[\xi | Z = z]\phi(z))$ exists, and hence we can replace $[A3]_+$ by [A3].

(2) The condition of the existence of the integrable bounded derivative $\partial_z^2 (E[\xi | Z = z]\phi(z))$ can also be removed if (Z, ξ) is defined on a Wiener space and if Condition [r] is satisfied for $(Z, \xi, 4)$ (the definition is given in Section 3 below).

In case the Malliavin covariance of X_n (or M_n) is bounded from below with large probability, we have the following result.

Theorem 2 *Let* Y_n *denote either* X_n or M_n, *and let* σ_{Y_n} *be the Malliavin covariance of* Y_n. *Assume that Conditions* $[A1]$, $[A2]_3$ and $[A3]_+$ *hold. Suppose that for some positive constant* c, $\lim_{n \to \infty} P(\sigma_{Y_n} < c) = 0$. *Then, for any* $p > 1$, *there exist a constant* C *and a sequence* $\epsilon'_n, \epsilon'_n = o(r_n)$, *such that*

$$\Delta_n \leq C \big(1 + \log^+ (r_n^{-1})\big) P(\sigma_{Y_n} < c)^{\frac{1}{p}} + \epsilon'_n$$

for any $n \in \mathbb{N}$.

The following four theorems are concerning asymptotic expansions of the local density or are obtained through those expansions.

Theorem 3 *Suppose Conditions* [*A1*], [*A2*]$_4$, [*A3*] *and* [*A4*]$_4$ *are satisfied. Then the local density* g_n^0 of X_n on ψ_n *exists and, for any* $\alpha \in \mathbb{Z}_+$, *there exist a sequence* (ϵ_n^α) *with* $\epsilon_n^\alpha = o(r_n)$ as $n \to \infty$, *and a constant* C_p^α *for any* $p > 1$ *such that*

$$\sup_{x\in\mathbb{R}} |x|^\alpha |g_n^0(x) - p_n(x)| \leq C_p^\alpha r_n^{-(1-a)} \|1 - \psi_n\|_{L^p} + \epsilon_n^\alpha$$

for any $n \in \mathbb{N}$, *where* p_n *is the function given in Theorem 1.*

The following theorem gives the asymptotic expansion of $E[f(X_n)]$ for a measurable function f.

Theorem 4 *Suppose Conditions* [*A1*], [*A2*]$_4$, [*A3*] and [*A4*]$_4$ *are satisfied. Then, for any* $\alpha \in \mathbb{Z}_+$, *there exist a sequence* $(\tilde{\epsilon}_n^\alpha)$ *with* $(\tilde{\epsilon}_n^\alpha) = o(r_n)$ as $n \to \infty$, *and a constant* $\tilde{C}_p^\alpha$ *for any* $p > 1$ *such that*

$$\begin{aligned}\left|E[f(X_n)] - \int_{\mathbb{R}} f(x)p_n(x)dx\right| \leq {} & (\tilde{C}_p^\alpha r_n^{-2(1-a)} \|f\|_{L^1(\mathbb{R},d\nu^x)} \\ & + \|f(X_n)\|_{L^{p'}})\|1 - \psi_n\|_{L^p} \\ & + \|f\|_{L^1(\mathbb{R},d\nu^\alpha)} \tilde{\epsilon}_n^\alpha\end{aligned}$$

for any $n \in \mathbb{N}$ and any measurable function $f\colon \mathbb{R} \to \mathbb{R}$ *satisfying* $E[|f(X_n)|] < \infty$ *and* $\int_{\mathbb{R}} |f(x)| p_n(x)dx < \infty$, *where* $p' = p/(p-1)$ *and the measure* ν^α *is defined as* $d\nu^\alpha(x) = (1 + |x|^2)^{-\alpha/2} dx$.

The Malliavin covariance dominates the convergence rate explicitly in the following theorem.

Theorem 5 *Let* Y_n *be either* X_n *or* M_n. *Suppose that Conditions* [*A1*], [*A2*]$_4$, [*A3*] *are satisfied. Moreover, assume:*

[*A4′*] *There exist* $s_n \in D^n_{\infty-,4} = \bigcap_{p>1} D^n_{p,4}$ *satisfying*
(*1*) $\sup_{n\in\mathbb{N}} \|s_n\|_{p,4} < \infty$ *and* $\sup_{n\in\mathbb{N}} E[s_n^{-p}] < \infty$ *for any* $p > 1$;
(*2*) $\lim_{n\to\infty} P(\sigma_{Y_n} \geq s_n) = 1$.
Then, for any $\alpha \in Z_+, p > 1$ and $q' > 2/3$, *there exist a sequence* $(\epsilon'_n) = (\epsilon'^{\,\alpha,p,q'}_n)$ *with* $\epsilon'_n = o(r_n)$ *as* $n \to \infty$, *and a constant* C_p^α *such that*

$$\sup_{x\in\mathbb{R}} |x|^\alpha |g_n^0(x) - p_n(x)| \leq C_p^\alpha r_n^{-q'} P(\sigma_{Y_n} < s_n)^{\frac{1}{p}} + \epsilon'_n$$

for any $n \in N$. *Here* g_n^0 *implicitly depends on a certain choice of the truncation functional* ψ_n *in the proof.*

As a corollary, we have asymptotic expansion of the expectation of funcionals of X_n.

Theorem 6 *Let* Y_n *be either* X_n or M_n. *Suppose that Conditions* [*A1*], [*A2*]$_4$, [*A3*] *and* [*A4′*] *are satisfied. Then, for any* $\alpha \in \mathbb{Z}_+, p > 1$ *and* $q' > 2/3$, *there exist a sequence* $(\tilde{\epsilon}'_n) = (\tilde{\epsilon}'^{\,\alpha,p,q'}_n)$ *with* $\tilde{\epsilon}'_n = o(r_n)$ *as* $n \to \infty$, *and a constant* $\tilde{C}_p^\alpha$ *such that*

$$\left|E[f(X_n)] - \int_{\mathbb{R}} f(x)p_n(x)dx\right| \leq \tilde{C}_p^{\alpha}(\|f(X_n)\|_{L^{p'}} + \|f\|_{L^1(R,dv^{\alpha})}) \cdot (r_n^{-q'}P(\sigma_{Y_n} < s_n)^{\frac{1}{\mathrm{p}}} + \tilde{\epsilon}'_n)$$

for any $n \in \mathbb{N}$ *and any measurable function* f *satisfying* $E[|f(X_n)|] < \infty$ *and* $\int_{\mathbb{R}} |f(x)|p_n(x)dx < \infty$.

Remark 2. To obtain our results, it is not necessary to assume that $M_n, N_n, \langle M_n \rangle$ themselves are smooth Wiener functionals as in $[A2]_4$. In fact, we can prove the same inequality as Theorem 3 under Conditions [A1], [A3] and [A4″]:

[A4″] There exist $M'_n \in D^n_{\infty-,5}, N'_n \in D^n_{\infty-,5}$, $\xi'_n \in D^n_{\infty-,4}$ and $\psi_n \in D^n_{\infty-,4}$ satisfying the following conditions:

(1) $0 \leq \psi_n \leq 1$;
(2) There exists a constant $a, 0 < a < \frac{1}{3}$, such that on $\{r_n^a|\xi'_n| > 1\}$, $D_n^j\psi_n = 0$ a.s. for $0 \leq j \leq 4$; moreover, with $\xi_n = r_n^{-1}(\langle M_n \rangle - 1)$, if $|M_n - M'_n| + |N_n - N'_n| + |\xi_n - \xi'_n| \neq 0$, then $D_n^j\psi_n = 0$ a.s. for $0 \leq j \leq 4$;
(3) $\psi_n \to^p 1$ as $n \to \infty$;
(4) For some $t > 1, \sup_{n \in N} E[|D_n^j\psi_n|^t_{H_n^{\otimes j}}\sigma_{X'_n}^{-p}] < \infty$ for any $p > 1$, where $X'_n = M'_n + r_nN'_n$;
(5) For any $p > 1, \sup_{n\in\mathbb{N}}\|M'_n\|_{p,5} + \sup_{n\in\mathbb{N}}\|N'_n\|_{p,5} + \sup_{n\in\mathbb{N}}\|\xi'_n\|_{p,4} < \infty$.

3. Preliminary lemmas

Our argument suits Wiener functionals on Wiener spaces, however, we will start from a more general setting to clarify necessary assumptions for our proof. There is a simple example of X_n whose distribution converges to the normal distribution but has an atom with mass r_n, hence, it does not admit approximation up to $o(r_n)$ by any continuous function. This example shows that, in order to obtain such approximation, it is necessary to impose some regularity condition on the distribution of X_n. For this purpose, Condition [r] below will be adopted, motivated by the Malliavin calculus. Though, in the later subsection, we will consider Wiener functionals and use integration-by-parts formulas on Wiener spaces to verify it, Condition [r] originally does not depend on a particular form of the integration-by-parts formulas.

In this subsection, we denote by (W_n, P_n) probability spaces indexed by $n \in \mathbb{N}$. Let Y_n be a random variable defined on W_n, and let $Y_{n,u}$ be random variables on W_n with index $u \in \Lambda_n$ for each $n \in N$, where Λ_n is a subset of R. For each $n \in \mathbb{N}, \psi_n$ denotes a random variable on W_n satisfying $0 \leq \psi_n \leq 1$. Let $j \in Z_+$. We say that *Condition* [r] is satisfied for $(Y_n, \psi_n, Y_{n,u}, \Lambda_n, j)$ if $\psi_n Y_{n,u} \in L^1$ for any $u \in \Lambda_n, n \in \mathbb{N}$, and

$$\sup_{\substack{u\in\Lambda_n \\ n\in\mathbb{N}}} |u|^j \left|E\left[e^{iuY_n}\psi_n Y_{n,u}\right]\right| < \infty \quad .$$

Moreover, if $\Lambda_n = \mathbb{R}, Y_n = Y, \psi_n = 1$, $Y_{n,u} = Y'$ for any $u \in \Lambda_n = \mathbb{R}, n \in \mathbb{N}$, we simply say that Condition [r] is satisfied for (Y, Y', j).

We are still considering a sequence of random variables X_n, on W_n, decomposed as: $X_n = M_n + r_n N_n$, where r_n is a sequence of positive numbers tending to zero as $n \to \infty$; for each $n \in \mathbb{N}$, M_n is the terminal random variable M_{n,T_n} of a continuous martingale $(M_{n,t}\colon 0 \le t \le T_n)$ defined on (W_n, P_n) with respect to some filtration $(\mathbb{F}_{n,t}\colon 0 \le t \le T_n)$, $M_{n,0} = 0$, and N_n is another random variable on W_n. $\langle M_n \rangle$ denotes the predictable quadratic variation of M_n, and the terminal value $\langle M_n \rangle_{T_n}$ will be often denoted by $\langle M_n \rangle$ for simplicity.

Hereafter we fix a truncation sequence ψ_n satisfying $0 \le \psi_n \le 1$ a.s. Assume that there exists a constant $a, 0 < a < 1/3$, such that, if $\psi_n(w) > 0$, then $r_n^{-(1-a)} |\langle M_n \rangle_{T_n} - 1| \le 1$ a.s.

Each of the following conditions specifies the limit distribution of X_n.

[C1] (a) $\psi_n \to^p 1$ as $n \to \infty$; (b) the family $(\psi_n r_n^{-1}(\langle M_n \rangle - 1), \psi_n N_n\colon n \in \mathbb{N})$ is uniformly integrable; (c) there exist random variables (Z, ξ, η) on a probability space such that

$$(M_n, r_n^{-1}(\langle M_n \rangle - 1), N_n) \to^d (Z, \xi, \eta)$$

as $n \to \infty$.

[C1]$_+$ $\psi_n \to^p 1$ as $n \to \infty$; For any $p_1, p_2, p_3 \in \mathbb{Z}_+$,

$$\sup_{n \in \mathbb{N}} E\left[\psi_n |M_n|^{p_1} |r_n^{-1}(\langle M_n \rangle_{T_n} - 1)|^{p_2} |N_n|^{p_3}\right] < \infty \ ;$$

There exist random variables (Z, ξ, η) on a probability space such that $(M_n, r_n^{-1}(\langle M_n \rangle_{T_n} - 1), N_n) \to^d (Z, \xi, \eta)$ as $n \to \infty$.

Here the expectation means the one with respect to the probability measure P_n. The martingale central limit theorem holds that Z has the standard normal distribution under Condition [C1] or [C1]$_+$ (Jacod-Shiryaev [14]). However, it does not generally lead to the asymptotic expansion. Therefore, we need certain regularity conditions to go further.

Put

$$P_\alpha(u, z, r) = e^{-iuz - \frac{1}{2}u^2 r}(-i\partial_u)^\alpha e^{iuz + \frac{1}{2}u^2 r}$$

and

$$Q_\alpha(u, z, r) = e^{\frac{1}{2}u^2 r} P_\alpha(u, z, r)$$

for $\alpha \in Z_+, u, z, r \in \mathbb{R}$. Define $B^\alpha_{n,u}, C^\alpha_{n,u}$ as follows:

$$\begin{aligned} B^\alpha_{n,u} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} u^{-2} r_n^{-1} \{ & Q_\beta(u, M_n, 0) \\ & - Q_\beta(u, M_n, \langle M_n \rangle - 1)\} \end{aligned}$$

and

$$\begin{aligned} C^\alpha_{n,u} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} & u^{-1} r_n^{-1} \{(r_n N_n)^{\alpha-\beta} - \delta_{\alpha,\beta} e^{-iur_n N_n}\} \\ & \cdot Q_\beta(u, M_n, \langle M_n \rangle - 1) \ . \end{aligned}$$

With $q = (1-a)/2$, let $\Lambda_n^0 = \{u \in \mathbb{R}\colon |u| \leq r_n^{-q}\}$ and let $\Lambda_n^l = \{u \in \mathbb{R}\colon 1 \leq |u| \leq r_n^{-q}\}$. The following conditions ensure the regularity of X_n.
$[\mathrm{C2}]_3$ Condition [r] is satisfied for

(a) $(X_n, \psi_n, 1, \mathbb{R} - \Lambda_n^0, 3)$;
(b) $(X_n, \psi_n, B_{n,u}^0, \Lambda_n^1, 3)$;
(c) $(X_n, \psi_n, C_{n,u}^0, \Lambda_n^1, 2)$.

$[\mathrm{C2}]_4$ For any $\alpha \in \mathbb{Z}_+$, Condition [r] is satisfied for

(a) $(X_n, \psi_n, X_n^\alpha, \mathbb{R} - \Lambda_n^0, 4)$;
(b) $(X_n, \psi_n, B_{n,u}^\alpha, \Lambda_n^1, 4)$;
(c) $(X_n, \psi_n, C_{n,u}^\alpha, \Lambda_n^1, 3)$.

For convenience of reference, we name the following conditions while they are fully or in part derived from the above conditions.
$[\mathrm{C3}]_3$ There exist integrable bounded derivatives $\partial_z^j(E[\xi|Z=z]\phi(z))$ for $j = 0, 1, 2$, and Condition [r] is satisfied for

(a) $(Z, \xi, 3)$;
(b) $(Z, \eta, 2)$.

$[\mathrm{C3}]_4$ For any $\alpha \in \mathbb{Z}_+$, Condition [r] is satisfied for

(a) $(Z, Z^\alpha \xi, 4)$;
(b) $(Z, Z^\alpha \eta, 3)$.

As shown later, $[\mathrm{C1}]_+ + [\mathrm{C2}]_4(b) \Rightarrow [\mathrm{C3}]_4(a)$, and $[\mathrm{C1}]_+ + [\mathrm{C2}]_4(c) \Rightarrow [\mathrm{C3}]_4(b)$. Furthermore, $[\mathrm{C1}] + [\mathrm{C2}]_3(b) \Rightarrow [\mathrm{C3}]_3(a)$, and $[\mathrm{C1}] + [\mathrm{C2}]_3(c) \Rightarrow [\mathrm{C3}]_3(b)$.

Let $\hat{g}_n^\alpha(u) = E[e^{iuX_n}\psi_n X_n^\alpha]$. Under Condition $[\mathrm{C2}]_4$ (a), the function $\hat{g}_n^\alpha$ is integrable with respect to the Lebesgue measure for each $\alpha \in \mathbb{Z}_+$ and $n \in \mathbb{N}$; we can define g_n^α by

$$g_n^\alpha(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iux}\hat{g}_n^\alpha(u)du \ .$$

Then $g_n^\alpha(x) = x^\alpha g_n^0(x)$ for any $\alpha \in \mathbb{Z}_+$. $g_n^0(x)$ is referred to the local density of X_n on ψ_n.

Let $j \geq 2$ and let κ be any random variable. If, for any $\alpha \in Z_+, Z^\alpha\kappa \in L^1$ and $\sup_{u\in\mathbb{R}}|u|^j|E[e^{iuz}Z^\alpha\kappa]| < \infty$, then a version of the function $y \mapsto y^\beta\partial_y^i(E[Z^\alpha\kappa|Z=y]\phi(y))$, $i \leq j-2$, is continuous, tending to zero as $|y| \to \infty$, and integrable with respect to the Lebesgue measure for any $\alpha, \beta \in \mathbb{Z}_+$.

Put $c_n = E[\psi_n]$, $A(x) = E[\xi|Z=x]$ and $B(x) = E[\eta|Z=x]$. Define $h_n^0(x)$ by

$$h_n^0(x) = c_n\phi(x) + \frac{1}{2}r_n\partial_x^2(A(x)\phi(x)) - r_n\partial_x(B(x)\phi(x)) \ .$$

Under Conditions $[\mathrm{C3}]_4, h_n^0(x)$ is well-defined. Let $h_n^\alpha(x) = x^\alpha h_n^0(x)$ for $\alpha \in \mathbb{Z}_+$. With $\hat{h}_n^\alpha(u) = \int_{\mathbb{R}} e^{iux}h_n^\alpha(x)dx$, Conditions $[\mathrm{C3}]_4$ and integration-by-parts yield

$$\begin{aligned}\hat{h}_n^\alpha(u) &= \int_{\mathbb{R}} e^{iux}x^\alpha[c_n\phi(x) + \frac{1}{2}r_n\partial_x^2(A(x)\phi(x)) - r_n\partial_x(B(x)\phi(x))]dx \\ &= \int_{\mathbb{R}} e^{iux}[c_nx^\alpha + \frac{1}{2}r_nA_\alpha(u,x) + r_nB_\alpha(u,x)]\phi(x)dx \ ,\end{aligned}$$

where

$$A_\alpha(u,x) = A(x)\{(iu)^2x^\alpha + 2iu\alpha x^{\alpha-1} + \alpha(\alpha-1)x^{\alpha-2}\}$$

and

$$B_\alpha(u,x) = B(x)(iux^\alpha + \alpha x^{\alpha-1}) \ .$$

We may from the beginning define $\hat{h}_n^\alpha(u)$ by the second expression above: it is well-defined just under $[C1]_+$.

We have the following preliminary results:

Lemma 1 *Suppose Conditions* $[C1]_+$ *and* $[C2]_4$ *are satisfied. Then, for any* $\alpha \in \mathbb{Z}_+$, *there exist a sequence* (ϵ_n^α) *with* $\epsilon_n^\alpha = o(r_n)$ *as* $n \to \infty$, *and a constant* C_p^α *for any* $p > 1$ *such that*

$$\sup_{x\in\mathbb{R}} |x|^\alpha |g_n^0(x) - h_n^0(x)| \le C_p^\alpha r_n^{-2q}\|1 - \psi_n\|_{L^p} + \epsilon_n^\alpha$$

for any $n \in \mathbb{N}$.

Lemma 1′ *Suppose that Conditions* $[C1]$ *and* $[C2]_3$ *hold, and that there exists an integrable bounded second derivative* $\partial_z^2(E[\xi|Z = z]\phi(z))$. *Then there exist a sequence* $\epsilon_n, \epsilon_n = o(r_n)$, *and positive constants* $C_p(p > 1)$ *such that*

$$\Delta_n \le C_p\left(1 + \log^+(r_n^{-(1-a)/2})\right)\|1 - \psi_n\|_{L^p} + \epsilon_n$$

for any $n \in \mathbb{N}$.

4. Proof of preliminary lemmas

In this subsection, we will prove Lemma 1 and Lemma 1′. First, we decompose $\hat{g}_n^\alpha(u) - \hat{h}_n^\alpha(u)$ into three parts:

$$\hat{g}_n^\alpha(u) - \hat{h}_n^\alpha(u) = J_n^\alpha(u) + K_n^\alpha(u) + L_n^\alpha(u) \ ,$$

where

$$\begin{aligned}J_n^\alpha(u) &= E[\psi_nX_n^\alpha e^{iuX_n}] \\ &\quad - E[\psi_n\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}(r_nN_n)^{\alpha-\beta}Q_\beta(u, M_n, \langle M_n\rangle - 1)e^{iuX_n}] \\ &\quad - \frac{1}{2}r_nE[e^{iuz}A_\alpha(u,Z)];\end{aligned}$$

$$K_n^\alpha(u) = E[\psi_n \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} Q_\beta(u, M_n, \langle M_n\rangle - 1) e^{iuX_n}]$$
$$- E[\psi_n Q_\alpha(u, M_n, \langle M_n\rangle - 1) e^{iuM_n}]$$
$$- r_n E[e^{iuZ} B_\alpha(u, Z)] \ ;$$

and

$$L_n^\alpha(u) = E[\psi_n Q_\alpha(u, M_n, \langle M_n\rangle - 1) e^{iuM_n}] - E[\psi_n (-i\partial_u)^\alpha e^{-\frac{1}{2}u^2}] \ .$$

Define $S_\beta(u, r), \beta \in \mathbb{Z}_+$, by

$$S_\beta(u, r) = e^{-\frac{1}{2}u^2 r} \partial_u^\beta e^{\frac{1}{2}u^2 r} = i^\beta P_\beta(u, 0, r) \ .$$

Then it is easy to show the following lemma.

Lemma 2 *(1)* $S_\beta(u, r) = \sum_{j=0}^{\beta} c_j^\beta u^j r^{(j+\beta)/2}$, *where* $c_{odd}^{even} = 0$ *and* $c_{even}^{odd} = 0$. *In particular,* $c_0^0 = 1; c_0^1 = 0, c_1^1 = 1; c_0^2 = 1, c_1^2 = 0, c_2^2 = 1$.

(*2*) $P_\alpha(u, z, r) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} (-i)^\beta z^{\alpha-\beta} S_\beta(u, r)$.

Lemma 3 *Suppose Condition* $[C1]_+$ *is satisfied. Then, for each* $u \in \mathbb{R}$, $J_n^\alpha(u) = o(r_n)$ *as* $n \to \infty$. *Moreover, if Condition* $[C2]_4$ (*b*) *is satisfied, then* $\int_{\Lambda_n^0} |J_n^\alpha(u)| du = o(r_n)$ *as* $n \to \infty$.

Proof. Let $\xi_n = r_n^{-1}(\langle M_n\rangle - 1)$. Define $j_{1,n}^\alpha(u),\ j_{2,n}^\alpha(u)$ as

$$j_{1,n}^\alpha(u) = \psi_n \sum_{\beta=0}^{\alpha-1} \binom{\alpha}{\beta} (r_n N_n)^{\alpha-\beta} \{Q_\beta(u, M_n, 0) - Q_\beta(u, M_n, r_n\xi_n)\} e^{iuX_n}$$

and

$$j_{2,n}^\alpha(u) = \psi_n \{Q_\alpha(u, M_n, 0) - Q_\alpha(u, M_n, r_n\xi_n)\} e^{iuX_n} \ .$$

Then, from Condition $[\mathrm{C1}]_+$ and continuity of Q_β, one has $r_n^{-1} j_{1,n}^\alpha(u) \to^p 0$ as $n \to \infty$ for each $u \in \mathbb{R}$. On $\{w \colon \psi_n(w) > 0\}$, $|r_n\xi_n| \le r_n^{1-a}$ a.s. Since $0 \le \psi_n \le 1$, the boundedness of moments in Condition $[\mathrm{C1}]_+$ implies the uniform integrability of $(r_n^{-1} j_{1,n}^\alpha(u) \colon n \in \mathbb{N})$ for each $u \in \mathbb{R}$; hence, $r_n^{-1} E[|J_{1,n}^\alpha|] = o(1)$ as $n \to \infty$ for each $u \in \mathbb{R}$. Since $Q_\alpha(u, z, 0) = P_\alpha(u, z, 0) = z^\alpha$, we have, from Lemma 2,

$$r_n^{-1} j_{2,n}^\alpha(u) = \psi_n r_n^{-1} \left\{ M_n^\alpha - \sum_{j=0}^{n} \binom{\alpha}{j} (-i)^j M_n^{\alpha-j} S_j(u, r_n\xi_n) e^{\frac{1}{2}u^2 r_n \xi_n} \right\} e^{iuX_n}$$

$$
\begin{aligned}
&= e^{iuX_n}\psi_n \Bigg\{ M_n^{\alpha} r_n^{-1}\big(1 - e^{\frac{1}{2}u^2 r_n \xi_n}\big) + i\alpha M_n^{\alpha-1} u \xi_n e^{\frac{1}{2}u^2 r_n \xi_n} \\
&\qquad + \frac{1}{2}\alpha(\alpha-1) M_n^{\alpha-2}(\xi_n + u^2 r_n \xi_n^2) e^{\frac{1}{2}u^2 r_n \xi_n} \\
&\qquad - e^{\frac{1}{2}u^2 r_n \xi_n} \sum_{\beta=3}^{\alpha} \binom{\alpha}{\beta} (-i)^{\beta} M_n^{\alpha-\beta} \sum_{j=0}^{\beta} c_j^{\beta} r_n^{\frac{j+\beta}{2}-1} \xi_n^{\frac{j+\beta}{2}} u^j \Bigg\} \ .
\end{aligned}
$$

As $r_n^{-1}(1 - e^{r_n x}) = -x \int_0^1 e^{r_n x s} ds$, it follows, from Condition $[\mathrm{C1}]_+$, that the distribution

$$
L\{r_n^{-1} j_{2,n}^{\alpha}(u)\} \Rightarrow L\left\{ e^{iuZ}\left[-\frac{1}{2}u^2 Z^{\alpha}\xi + i\alpha u Z^{\alpha-1}\xi + \frac{\alpha(\alpha-1)}{2} Z^{\alpha-2}\xi \right] \right\} \ .
$$

Again by Condition $[\mathrm{C1}]_+$ and implied uniform integrability, we obtain

$$
r_n^{-1} E[j_{2,n}^{\alpha}(u)] \to \frac{1}{2} E[e^{iuZ} A_{\alpha}(u, Z)]
$$

for each $u \in \mathbb{R}$. Obviously

$$
J_n^{\alpha}(u) = E[j_{1,n}^{\alpha}(u) + j_{2,n}^{\alpha}(u)] - \frac{1}{2} r_n E[e^{iuZ} A_{\alpha}(u, Z)] \ ; \tag{1}
$$

therefore $J_n^{\alpha}(u) = o(r_n)$ as $n \to \infty$ for each $u \in \mathbb{R}$.

Under Condition $[\mathrm{C2}]_4(b)$, there exists a constant C_1 independent of $n \in \mathbb{N}$ and $u \in \mathbb{R}$ such that

$$
\begin{aligned}
|r_n^{-1} E[j_{1,n}^{\alpha}(u) + j_{2,n}^{\alpha}(u)]| 1_{\Lambda_n^1}(u) &= |u^2 E[e^{iuX_n}\psi_n B_{n,u}^{\alpha}]| 1_{\Lambda_n^1}(u) \\
&\le C_1(1 + u^2)^{-1}
\end{aligned} \tag{2}
$$

for any $n \in \mathbb{N}$ and any $u \in \mathbb{R}$. It is also possible to replace Λ_n^1 by Λ_n^0 in the above inequality under Condition $[\mathrm{C1}]_+$. With (1), (2) and the fact that $J_n^{\alpha}(u) = o(r_n)$, we see that

$$
|E[e^{iuZ} A_{\alpha}(u, Z)]| \le 2C_1(1 + |u|^2)^{-1} \tag{3}
$$

for any $u \in \mathbb{R}$. Therefore, by dominated convergence theorem, we obtain

$$
\int_{\Lambda_n^0} r_n^{-1} |J_n^{\alpha}(u)| du = o(1)
$$

as $n \to \infty$. $\Diamond$

By induction with (3), we see that $[\mathrm{C1}]_+ + [\mathrm{C2}]_4(b) \Rightarrow [\mathrm{C3}]_4(a)$: it is sufficient to note the inequality

$$\sup_{|u|\ge 1}|u|^4\left|E[e^{iuZ}Z^\alpha\xi]\right| \le 2C_1 + 2\alpha\sup_{|u|\ge 1}|u|^4\left|E[e^{iuZ}\xi Z^{\alpha-1}]\right| + \alpha(\alpha-1)\sup_{|u|\ge 1}|u|^4|E[e^{iuZ}\xi Z^{\alpha-2}]| \ .$$

Similarly, we see that $[\mathrm{C1}] + [\mathrm{C2}]_3(b) \Rightarrow [\mathrm{C3}]_3(a)$.

Lemma 4 *Suppose Condition* $[C1]_+$ *is satisfied. Then, for each* $u \in \mathbb{R}$, $K_n^\alpha(u) = o(r_n)$ *as* $n \to \infty$. *Moreover, if Condition* $[C2]_4(c)$ *is satisfied, then* $\int_{\Lambda_n^0}|K_n^\alpha(u)|du = o(r_n)$ *as* $n \to \infty$.

Proof. Let

$$k_{1,n}^\alpha(u) = \psi_n Q_\alpha(u, M_n, r_n\xi_n)(e^{iuX_n} - e^{iuM_n}) + \psi_n\alpha r_n N_n Q_{\alpha-1}(u, M_n, r_n\xi_n)e^{iuX_n}$$

and let

$$k_{2,n}^\alpha(u) = \psi_n\sum_{\beta=0}^{\alpha-2}\binom{\alpha}{\beta}(r_nN_n)^{\alpha-\beta}Q_\beta(u, M_n, r_n\xi_n)e^{iuX_n} \ .$$

Then $K_n^\alpha(u) = E[k_{1,n}^\alpha(u) + k_{2,n}^\alpha(u)] - r_nE[e^{iuZ}B_\alpha(u,Z)]$. From Condition $[\mathrm{C1}]_+$ and the property of ψ_n, one has $r_n^{-1}E[k_{2,n}^\alpha] \to 0$ as $n \to \infty$ for each $u \in \mathbb{R}$. In view of Lemma 2, we see, from Condition $[\mathrm{C1}]_+$, that

$$\begin{aligned} & L\{M_n, \xi_n, N_n, Q_{\alpha-1}(u, M_n, r_n\xi_n), Q_\alpha(u, M_n, r_n\xi_n)\} \\ \Rightarrow\ & L\{Z, \xi, \eta, Z^{\alpha-1}, Z^\alpha\} \ ; \end{aligned}$$

hence

$$L\{r_n^{-1}k_{1,n}^\alpha(u)\} \Rightarrow L\{e^{iuZ}(iuZ^\alpha\eta + \alpha Z^{\alpha-1}\eta)\} \ .$$

The uniform integrability implies $r_n^{-1}E[k_{1,n}^\alpha(u)] - E[e^{iuZ}B_\alpha(u,Z)] \to 0$; therefore $K_n^\alpha(u) = o(r_n)$ as $n \to \infty$ for each $u \in \mathbb{R}$.

Since

$$r_n^{-1}K_n^\alpha(u) = uE[e^{iuX_n}\psi_n C_{n,u}^\alpha] - E[e^{iuZ}B_\alpha(u,Z)] \ ,$$

it follows, from Conditions $[\mathrm{C2}]_4(c)$ and $[\mathrm{C1}]_+$, as in the proof of Lemma 3, that $|E[e^{iuZ}B_\alpha(u,Z)]| \le C_2(1+|u|^2)^{-1}$, and hence that

$$|r_n^{-1}K_n^\alpha(u)1_{\Lambda_n^0}(u)| \le 2C_2(1+u^2)^{-1}$$

for any $n \in \mathbb{N}$ and any $u \in \mathbb{R}$, where C_2 is a constant independent of $n \in \mathbb{N}$ and $u \in \mathbb{R}$. Hence, we have $\int_{\Lambda_n^0}r_n^{-1}|K_n^\alpha(u)|du \to 0$ as $n \to \infty$. ◊

By the argument above, we see by induction that $[\mathrm{C1}]_+ + [\mathrm{C2}]_4(c) \Rightarrow [\mathrm{C3}]_4(b)$, and similarly that $[\mathrm{C1}] + [\mathrm{C2}]_3(c) \Rightarrow [\mathrm{C3}]_3(b)$.

Lemma 5 *Suppose Condition* $[\mathrm{C1}]_+$ *holds. Then, for any* $p > 1$, *there exists a constant* $C_p = C_p(\alpha)$ *independent of* $n \in \mathbb{N}$ *and* $u \in \mathbb{R}$ *such that*

$$|L_n^\alpha(u)|1_{\Lambda_n^0}(u) \leq C_p(|u|+1)\|1-\psi_n\|_{L^p}$$

for any $n \in \mathbb{N}$ *and* $u \in \mathbb{R}$. *Moreover,*

$$\int_{\Lambda_n^0} |L_n^\alpha(u)|du \leq C_p r_n^{-2q}\|1-\psi_n\|_{L^p} \ .$$

Proof. We extend $(M_{n,t})_{0\leq t\leq T_n}$ as $M_{n,t} = M_{n,T_n}$ for $t \geq T_n$, the filtrations also extended in a similar way. Define stopping times τ_n as

$$\tau_n = \inf\{t \geq 0\colon r_n^{-(1-a)}(\langle M_n\rangle_t - 1) > 1\} \ ;$$

obviously, on $\{w\colon \psi_n(w) > 0\}$, a.s. $T_n \leq \tau_n$. By Itô's formula,

$$e^{iuM_{n,T_n\wedge\tau_n}} Q_\alpha(u, M_{n,T_n\wedge\tau_n}, \langle M_n\rangle_{T_n\wedge\tau_n} - 1) = (-i\partial_u)^\alpha e^{-\frac{1}{2}u^2} + \int_0^{T_n\wedge\tau_n} U(u, M_{n,t}, \langle M_n\rangle_t - 1)dM_{n,t} \ ,$$

where

$$U(u,z,r) = (-i\partial_u)^\alpha[iue^{iuz+\frac{1}{2}u^2r}] \ .$$

If $t \leq \tau_n$ and $u \in \Lambda_n^0$, then $u^2(\langle M_n\rangle_t - 1) \leq u^2 r_n^{1-a} \cdot r_n^{-(1-a)}(\langle M_n\rangle_{\tau_n} - 1) \leq 1$. By Lemma 2, we see that

$$|P_\alpha(u,z,r)e^{iuz+\frac{1}{2}u^2r}| \leq \sum_{\beta=0}^{\alpha}\sum_{j=0}^{\beta}\binom{\alpha}{\beta}|c_j^\beta| \sup_{x\leq 1}(|x|^{\frac{1}{2}j}e^{\frac{1}{2}x})|z|^{\alpha-\beta}|r|^{\beta/2}$$

if $u^2r \leq 1$. Therefore, with some constant $c(\alpha)$,

$$\sup_{\substack{u\in\Lambda_n^0 \\ t\leq T_n\wedge\tau_n}} |P_\alpha(u, M_{n,t}, \langle M_n\rangle_t - 1)e^{iuM_{n,t}+\frac{1}{2}u^2(\langle M_n\rangle_t - 1)}| \leq c(\alpha)\sum_{\beta=0}^{\alpha} M_{n,T_n\wedge\tau_n}^{*\alpha-\beta}(\langle M_n\rangle_{T_n\wedge\tau_n} + 1)^{\beta/2} \ .$$

Here, for a process X, $X_t^* = \sup_{0\leq s\leq t}|X_s|$. Since

$$U(u,z,r) = iuP_\alpha(u,z,r)e^{iuz+\frac{1}{2}u^2r} + \alpha P_{\alpha-1}(u,z,r)e^{iuz+\frac{1}{2}u^2r} \ ,$$

by using the Burkholder-Davis-Gundy inequality and the inequality $\langle M_n\rangle_{T_n\wedge\tau_n} \leq 1 + r_n^{1-a}$, we can obtain

$$\left\| \int_0^{T_n\wedge\tau_n} U(u, M_{n,t}, \langle M_n\rangle_t - 1)dM_{n,t} \right\|_{L^{p'}} \leq C_p(|u|+1)$$

for any $u \in \Lambda_n^0, n \in \mathbb{N}$, where $p' = p/(p-1)$ and C_p is a constant independent of u, n. Consequently,

$$\begin{aligned}|L_n^\alpha(u)|1_{\Lambda_n^0}(u) &= |E[\psi_n \int_0^{T_n\wedge\tau_n} U(u, M_{n,t}, \langle M_n\rangle_t - 1)dM_{n,t}]|1_{\Lambda_n^0}(u)\\ &= |E[(\psi_n - 1)\int_0^{T_n\wedge\tau_n} U(u, M_{n,t}, \langle M_n\rangle_t - 1)dM_{n,t}]|1_{\Lambda_n^0}(u)\\ &\le C_p\|1-\psi_n\|_{L^p}(|u|+1) \ ;\end{aligned}$$

hence

$$\int_{\Lambda_n^0} |L_n^\alpha(u)|du \le C_p r_n^{-2q}\|1-\psi_n\|_{L^p}$$

for some C_p. ◊

Proof of Lemma 1. Conditions $[\mathrm{C3}]_4$ implies that for each $\alpha \in \mathbb{Z}_+$, there exists a constant C_3 independent of $u \in \{v \in \mathbb{R}\colon |v| \ge 1\}$ and $n \in \mathbb{N}$ such that

$$\begin{aligned}|\hat{h}_n^\alpha(u)| = |c_n E[e^{iuZ}Z^\alpha] &+ \frac{1}{2}r_n\{(iu)^2E[e^{iuZ}Z^\alpha\xi]\\ &+ 2iu\alpha E[e^{iuZ}Z^{\alpha-1}\xi] + \alpha(\alpha-1)E[e^{iuZ}Z^{\alpha-2}\xi]\}\\ &+ r_n\{iuE[e^{iuZ}Z^\alpha\eta] + \alpha E[e^{iuZ}Z^{\alpha-1}\eta]\}|\\ \le (2\pi)^{\frac{1}{2}}c_n&|H_\alpha(u)|\phi(u) + C_3 r_n u^{-2}\end{aligned}$$

for any $u \in \{v \in \mathbb{R}\colon |v| \ge 1\}$ and $n \in \mathbb{N}$; hence,

$$\int_{\mathbb{R}-\Lambda_n^0} |\hat{h}_n^\alpha(u)|du = o(r_n) \ . \tag{4}$$

By Fourier inversion formula, we have

$$\begin{aligned}\sup_{x\in\mathbb{R}}|g_n^\alpha(x) - h_n^\alpha(x)| &= \sup_{x\in\mathbb{R}}\frac{1}{2\pi}\left|\int_{\mathbb{R}} e^{-iux}(\hat{g}_n^\alpha(u) - \hat{h}_n^\alpha(u))du\right|\\ &\le \frac{1}{2\pi}\int_{\mathbb{R}-\Lambda_n^0}(|\hat{g}_n^\alpha(u)| + |\hat{h}_n^\alpha(u)|)du\\ &\quad + \frac{1}{2\pi}\int_{\Lambda_n^0}|\hat{g}_n^\alpha(u) - \hat{h}_n^\alpha(u)|du \ .\end{aligned} \tag{5}$$

It follows from Lemmas 3, 4 and 5 that

$$\frac{1}{2\pi}\int_{\Lambda_n^0}|\hat{g}_n^\alpha(u) - \hat{h}_n^\alpha(u)|du \le C_p r_n^{-2q}\|1-\psi_n\|_{L^p} + o(r_n) \ . \tag{6}$$

From Condition $[\mathrm{C2}]_4(a)$, one has, for some constant C_4 independent of $n \in \mathbb{N}$,

$$\int_{\mathbb{R}-\Lambda_n^0}|\hat{g}_n^\alpha(u)|du \le C_4 r_n^{3q} = o(r_n) \tag{7}$$

as $n \to \infty$ since $a < 1/3$ by definition. Consequently, it follows, from (5), (6), (4) and (7), that

$$\sup_{x\in\mathbb{R}} |g_n^\alpha(x) - h_n^\alpha(x)| \le C_p r_n^{-2q} \|1-\psi_n\|_{L^p} + o(r_n)$$

This completes the proof. ◊

For the proof of Lemma 1′, we use the following lemmas, which can be proved in a similar fashion as Lemmas 3, 4 and 5. For details, see [35]. Put $J_n = J_n^0, K_n = K_n^0$ and $L_n = L_n^0$.

Lemma 3′ *Suppose that Conditions* [*C1*] *and* $[C2]_3(b)$ *are satisfied. Then*

$$\int_{\Lambda_n^0} r_n^{-1} |u|^{-1} |J_n(u)| du \to 0$$

as $n \to \infty$.

Lemma 4′ *Suppose that Conditions* [*C1*] *and* $[C2]_3(c)$ *are satisfied. Then*

$$\int_{\Lambda_n^0} r_n^{-1} |u|^{-1} |K_n(u)| du = o(1)$$

as $n \to \infty$.

Lemma 5′ *For any* $p > 1$, *there exists a constant* C_p *such that for any* $n \in \mathbb{N}$,

$$\int_{\Lambda_n^0} |u|^{-1} |L_n(u)| du \le C_p \Big(1 + \log^+ (r_n^{-(1-a)/2})\Big) \|1 - \psi_n\|_{L_p} \quad .$$

Lemma 6′ *(1) Suppose Condition* $[C2]_3(a)$ *is satisfied. Then*

$$\int_{\mathbb{R}-\Lambda_n^0} |u|^{-1} |E[\psi_n e^{iuX_n}]| du = o(r_n) \quad .$$

(2) Under Condition $[C3]_3(a)$,

$$\int_{\mathbb{R}-\Lambda_n^0} |u|^{-1} r_n \left| E\left[e^{iuZ} \left(-\frac{1}{2} u^2 \right) \xi \right] \right| du = o(r_n)$$

and

$$E\left[e^{iuZ} \left(-\frac{1}{2} u^2 \right) \xi \right] = \int_{\mathbb{R}} \frac{1}{2} e^{iuz} \partial_z^2 (E[\xi | Z = z] \phi(z)) dz \quad .$$

(3) Under Condition $[C3]_3(b)$,

$$\int_{\mathbb{R}-\Lambda_n^0} |u|^{-1} r_n |iuE[e^{iuZ}\eta]| du = o(r_n)$$

and

$$E[iu\eta e^{iuZ}] = \int_{\mathbb{R}} e^{iuz} \partial_z (-E[\eta | Z = z] \phi(z)) dz \quad .$$

We will now prove Lemma 1′.

Proof of Lemma 1′. Define G_n: $\mathbb{R} \to \mathbb{R}_+$ by

$$G_n(x) = \int_{-\infty}^{x} E[\psi_n | X_n = y] \mu^{X_n}(dy) \ ,$$

where μ^{X_n} is the distribution of X_n, and define H_n: $\mathbb{R} \to \mathbb{R}$ by

$$H_n(x) = \int_{-\infty}^{x} \Big[E[\psi_n]\phi(z) + \frac{1}{2} r_n \partial_z^2 (E[\xi | Z = z]\phi(z)) - r_n \partial_z (E[\eta | Z = z]\phi(z)) \Big] dz \ .$$

Then, from Lemma 6′, we see

$$\hat{G}_n(u) = \int_{\mathbb{R}} e^{iux} dG_n(x)$$

and

$$\hat{H}_n(u) = \int_{\mathbb{R}} e^{iuz} dH_n(z) \ .$$

Since

$$\begin{aligned} \lim_{x \to -\infty} |x| G_n(x) &\le \lim_{x \to -\infty} \int_{-\infty}^{x} |y| dG_n(y) \\ &= \lim_{x \to -\infty} E[1_{\{X_n \le x\}} \psi_n |X_n|] \\ &= 0 \ , \end{aligned}$$

the integration-by-parts yields

$$\int_{-\infty}^{0} G_n(x) dx = \int_{-\infty}^{0} |x| dG_n(x) < \infty \ ;$$

in the same fashion,

$$\int_{0}^{\infty} (E[\psi_n] - G_n(x)) dx = \int_{0}^{\infty} |x| dG_n(x) < \infty \ .$$

Hence, by $[C3]_3$ one has

$$\int_R |G_n(x) - H_n(x)| dx < \infty \ .$$

Clearly, $G_n(-\infty) = H_n(-\infty) = 0$ and $G_n(\infty) = H_n(\infty) = E[\psi_n]$. Thus, by applying the smoothing lemma (e.g., Shimizu [25]) to G_n and H_n, we obtain, for $\alpha > 1$,

$$\begin{aligned} &\sup_{x \in \mathbb{R}} |G_n(x) - H_n(x)| \\ &\quad \le \pi^{-1} \int_{|u| \le r_n^{-\alpha}} |u|^{-1} |\hat{G}_n(u) - \hat{H}_n(u)| du + 24\pi^{-1} \sup_x |H_n'(x)| r_n^{\alpha} \\ &\quad =: \Delta_n' \ . \end{aligned}$$

Since

$$\int_{\mathbb{R}-\Lambda_n^0} |u|^{-1}(|\hat{G}_n(u)| + |\hat{H}_n(u)|)du = o(r_n)$$

from Lemma 6′, we obtain from Lemmas 3′–5′

$$\begin{aligned}\Delta_n' &\le \pi^{-1}\int_{\Lambda_n^0} |u|^{-1}(|J_n(u)| + |K_n(u)| + |L_n(u)|)du \\ &\quad + \pi^{-1}\int_{\mathbb{R}-\Lambda_n^0} |u|^{-1}(|\hat{G}_n(u)| + |\hat{H}_n(u)|)du \\ &\quad + 24\pi^{-1}\sup_x |H_n'(x)| r_n^\alpha \\ &= C_p\Big(1 + \log^+(r_n^{-(1-a)/2})\Big)\|1 - \psi_n\|_{L^p} + o(r_n) \ .\end{aligned}$$

Since

$$\begin{aligned}&\sup_x |G_n(x) - E[1_{(-\infty,x]}(X_n)]| \\ &\quad = \sup_x |E[\psi_n 1_{(-\infty,x]}(X_n)] - E[1_{(-\infty,x]}(X_n)]| \\ &\quad \le \|1 - \psi_n\|_{L^1}\end{aligned}$$

and

$$\sup_x \left| \int_{(-\infty,x]} E[\psi_n]\phi(z)dz - \int_{(-\infty,x]} \phi(z)dz \right| \le \|1 - \psi_n\|_{L'} \ ,$$

we have finished the proof. ◊

5. Proof of Theorems

Proof of Theorem 3. We will verify Conditions $[C1]_+$ and $[C2]_4$ of Lemma 1. $[C1]_+$ is obvious from $[A2]_4$, [A3] and $[A4]_4(3)$.

Conditions $[A2]_4$ and $[A4]_4(4)$ with the integration-by-parts formula under truncation imply that

$$(iu)^4 E[e^{iuX_n}\psi_n X_n^\alpha] = E[e^{iuX_n}\Psi_4^{X_n}(\cdot;\psi_n X_n^\alpha)]$$

for some integrable functional $\Psi_4^{X_n}(\cdot;\psi_n X_n^\alpha)$; hence one has

$$|u|^4 |E[e^{iuX_n}\psi_n X_n^\alpha]| \le C_4, \tag{8}$$

where C_4 is a constant independent of $u \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus Condition $[C2]_4(a)$ has been verified.

When $\alpha = 0, \partial_r Q_\alpha(u, z, r) = (1/2)u^2 e^{u^2 r/2}$; when $\alpha \ge 1$, from Lemma 2, one has

$$\begin{aligned}\partial_r Q_\alpha(u, M_n, r_n\xi_n) &= \frac{1}{2}u^2 e^{\frac{1}{2}u^2 r_n \xi_n} \sum_{0\le j\le\beta\le\alpha} c_{\alpha,\beta,j} u^j r_n^{(j+\beta)/2} M_n^{\alpha-\beta} \xi_n^{(j+\beta)/2} \\ &\quad + e^{\frac{1}{2}u^2 r_n \xi_n} \sum_{0\le j\le\beta\le\alpha} \frac{1}{2}(j+\beta) c_{\alpha,\beta,j} u^j r_n^{(j-2+\beta)/2} \\ &\quad \cdot M_n^{\alpha-\beta} \xi_n^{(j-2+\beta)/2} \\ &= u^2 e^{\frac{1}{2}u^2 r_n \xi_n} \sum_{k,l=0}^{\alpha} b^{\alpha,k,l}\left(\frac{1}{u}, u r_n^{\frac{1}{2}}, r_n^{\frac{1}{2}}\right) M_n^k \xi_n^l \ , \end{aligned} \tag{9}$$

where

$$c_{\alpha,\beta,j} = \binom{\alpha}{\beta}(-i)^\beta c_j^\beta$$

and $b^{\alpha,k,l}(x, y, z)$ are polynomials in x, y, z. In view of Condition $[\mathrm{A4}]_4(2)$, we have

$$\begin{aligned} &|D_n^j \psi_n|_{H_n^{\otimes j}} |D_n^k e^{\frac{1}{2}u^2 r_n \xi_n s}|_{H_n^{\otimes k}} \\ &\quad \le |D_n^j \psi_n|_{H_n^{\otimes j}} e^{\frac{1}{2}u^2 r_n \xi_n s} \sum_{\substack{k_1+\ldots+k_m=k \\ m\le k}} c_{k_1,\ldots,k_m}^m \left(\frac{1}{2} s u^2 r_n\right)^m \\ &\qquad \left| D_n^{k_1}\xi_n \otimes \ldots\ldots \otimes D_n^{k_m}\xi_n \right|_{H_n^{\otimes k}} \\ &\quad \le e^{\frac{1}{2}} |D_n^j \psi_n|_{H_n^{\otimes j}} \sum_{\substack{k_1+\ldots+k_m=k \\ m\le k}} c_{k_1,\ldots,k_m}^m |D_n^{k_1}\xi_n|_{H_n^{\otimes k_1}} \ldots |D_n^{k_m}\xi_n|_{H_n^{\otimes k_m}} \end{aligned} \tag{10}$$

if $k \in \mathbb{Z}_+, u \in \Lambda_n^1$ and $s \in [0, 1]$. Here $D_n^k e^{\frac{1}{2}u^2 r_n \xi_n s}$ reads $e^{\frac{1}{2}u^2 r_n \xi_n s} P(k)$, $P(k)$ being a tensor polynomial obtained by the formal differential rule, and the latter is well defined without multiplication of the truncation functional ψ_n or its derivative. From (9) and (10), and approximating sequence argument using tame functions $\{\exp(-\xi_n^2/K) \colon K \in \mathbb{N}\}$ if necessary, we see that

$$\psi_n u^{-2} r_n^{-1}\{Q_\beta(u, M_n, 0) - Q_\beta(u, M_n, r_n\xi_n)\} \in D_{p,4}^n$$

for any $p > 1$, and that for $j \le 4$,

$$\begin{aligned} &|D_n^j[\psi_n u^{-2} r_n^{-1}\{Q_\beta(u, M_n, 0) - Q_\beta(u, M_n, r_n\xi_n)\}]|_{H_n^{\otimes j}} \\ &\quad \le \int_0^1 ds |D_n^j[u^{-2}\psi_n \xi_n \partial_r Q_\beta(u, M_n, r_n\xi_n s)]|_{H_n^{\otimes j}} \\ &\quad \le \sum_{i\le j} |D_n^i \psi_n|_{H_n^{\otimes i}} K_\beta^i(|D_n^k M_n|_{H_k^{\otimes k}}, |D_n^l \xi_n|_{H_n^{\otimes l}}; k, l \le j) \end{aligned}$$

for any $u \in \Lambda_n^1$ and $n \in \mathbb{N}$, where $K_\beta^i(x_k, y_l; k, l \le j)$ are polynomials in $x_k, y_l, k, l \le j$, independent of u, n. With $[\mathrm{A4}]_4(4)$, this shows that Condition

$[\mathrm{C2}]_4(b)$ holds true, which is a consequence of the integration-by-parts formula in the (partial) Malliavin calculus. See the notes after Theorem 7 below. In the same fashion, Condition $[\mathrm{C2}]_4(c)$ can be verified; thus we obtained the inequality of Lemma 1. Since

$$\sup_{x\in\mathbb{R}}|x|^{\alpha}|p_n(x)-h_n^0(x)|\le\sup_{x\in\mathbb{R}}|x|^{\alpha}\phi(x)|1-E[\psi_n]|\le C(\alpha)\|1-\psi_n\|_{L_p}\ ,$$

the proof completed. ◊

Proof of Theorem 4. It is sufficient to prove the inequality for bounded f. Obviously, we have

$$|E[f(X_n)]-E[f(X_n)\psi_n]|\le\|f(X_n)\|_{L^{p'}}\|1-\psi_n\|_{L^p}\ ,$$

$E[f(X_n)\psi_n]=\int_{\mathbb{R}}f(x)g_n^0(x)dx$, and

$$\Big|\int_{\mathbb{R}}f(x)g_n^0(x)dx-\int_{\mathbb{R}}f(x)p_n(x)dx\Big|\le\int_{\mathbb{R}}|f(x)|(1+|x|^2)^{-\frac{1}{2}\alpha}dx$$
$$\cdot\sup_{x\in\mathbb{R}}|(1+|x|^2)^{\frac{1}{2}\alpha}(g_n^0(x)-p_n(x))|\ .$$

Hence, we obtain the result from Theorem 3. ◊

Proof of Theorem 5. We may assume $q'<1$; put $q=q'/2$. Let $\varphi\colon\mathbb{R}\to[0,1]$ be a smooth function satisfying $\varphi(x)=1$ if $|x|\le\frac{1}{2}$ and $\varphi(x)=0$ if $|x|\ge1$. Define ψ_n by

$$\psi_n=\varphi\left(\frac{s_n}{2\sigma_{Y_n}}\right)\varphi\left(\frac{4\sigma_{r_nN_n}}{s_n}\right)\varphi\left(\left(r_n^{-(1-a)}(\langle M_n\rangle-1)\right)^2\right)\ .$$

then $[\mathrm{A4}]_4(1)$ and $[\mathrm{A4}]_4(2)$ are trivial; [it is easy to show $[\mathrm{A4}]_4(3)$ by using [A4′] and [A1]]. If for some $j\le4, D_n^j\psi_n\ne0$, then $s_n/2<\sigma_{Y_n}$ and $\sigma_{r_nN_n}<s_n/4$; hence $\sigma_{X_n}>s_n/25$; therefore $[\mathrm{A4}]_4(4)$ follows from [A4′](1). Thus we have the inequality of Theorem 3. Clearly, [A4′](1), [A1] and Markov's inequality imply that

$$\begin{aligned}\|1-\psi_n\|_{L_p}^p&\le P(\psi_n<1)\\&\le P(s_n>\sigma_{Y_n})+P\left(\frac{r_n^2\sigma_{N_n}}{s_n}>\frac{1}{8}\right)\\&\quad+P\left(\left(r_n^{-(1-a)}(\langle M_n\rangle-1)\right)^2>\frac{1}{2}\right)\\&\le P(s_n>\sigma_{Y_n})+o(r_n^m)\end{aligned}$$

for any $m\in\mathbb{N}$, which completes the proof. ◊

It is easy to prove Theorem 6 like Theorem 4. Finally we prove Theorems 1 and 2.

Proof of Theorem 1. We will verify that Conditions [C1], $[C2]_3, [C3]_3$ of Lemma 1′ are satisfied. [C1] is easy to check. In order to verify $[C2]_3$, it is sufficient to show that for any $i \leq 3, i+j \leq 6$ and for some $q > 1$, the L^q norm of $(\sigma_{X_n})^{-j} D^i(\psi_n Y_{n,u})$ is bounded uniformly in u (in $\mathbb{R} - \Lambda_n^0$ or in Λ_n^1) and in n, for $Y_{n,u} = 1, B_{n,u}^0, C_{n,u}^0$. In view of Assumption $[A4]_3(2)$, we can show this fact by using Assumption $[A4]_3(4)$. Furthermore, it is possible to prove Condition [r] is satisfied for $(X_n, \psi_n, X_n^\alpha r_n^{-1}(\langle M_n \rangle - 1), \mathbb{R}, 3)$ and for $(X_n, \psi_n, X_n^\alpha N_n, \mathbb{R}, 3)$ for any $\alpha \in \mathbb{Z}_+$. In particular, there exists a constant $C_\alpha < \infty$ such that

$$\sup_{n \in \mathbb{N}, u \in \mathbb{R}} |u|^3 |E[e^{iuX_n} \psi_n X_n^\alpha r_n^{-1}(\langle M_n \rangle - 1)]| < C_\alpha$$

and

$$\sup_{n \in \mathbb{N}, u \in \mathbb{R}} |u|^3 |E[e^{iuX_n} \psi_n X_n^\alpha N_n]| < C_\alpha \quad .$$

Hence, by $[A2]_3, [A3]_+$, one has

$$\sup_{u \in \mathbb{R}} |u|^3 |E[e^{iuZ} Z^\alpha \xi]| \leq C_\alpha$$

and

$$\sup_{u \in \mathbb{R}} |u|^3 |E[e^{iuZ} Z^\alpha \eta]| \leq C_\alpha \quad .$$

Therefore, there exist continuous, bounded, integrable versions of $\partial_z^j(E[Z^\alpha \xi | Z = z]\phi(z)), \partial_z^j(E[Z^\alpha \eta | Z = z]\phi(z))$ for $j = 0, 1$. Thus Condition $[C3]_3$ follows from this fact and Condition $[A3]_+$. ◊

Proof of Theorem 2. We will reduce this case to Theorem 1. We may assume $1/3 < q < 1/2$. Let $q = (1-a)/2$; then $0 < a < 1/3$. Fix any a_1 so that $0 < a_1 < 1$. Let φ: $\mathbb{R}_+ \to [0,1]$ be an increasing smooth function such that $\varphi(x) = 0$ if $x \leq 1/2$, and $\varphi(x) = 1$ if $x \geq 2/3$. For $v = 3c/2$, let

$$\psi_n = \varphi([1 + |r_n^{-(1-a)}(\langle M_n \rangle_{T_n} - 1)|^2]^{-1}) \cdot \varphi([1 + \sigma_{r_n^{a_1} N_n}]^{-1}) \varphi(v^{-1} \sigma_{Y_n}) \quad .$$

Then it is not difficult to verify that the Condition $[A4]_3$ is satisfied. In fact, if $D^j \psi_n \neq 0$ for some j, then $\sigma_{X_n}^{1/2} \geq \sigma_{M_n}^{1/2} - \sigma_{r_n N_n}^{1/2} > (v/2)^{1/2} - r_n^{1-a_1}$ when $Y_n = M_n$. Clearly $\sigma_{X_n} \geq v/2$ when $Y_n = X_n$. From this fact, $[A4]_3(4)$ follows immediately. Other conditions are easy to verify. Thus one has the estimate for the distribution function of X_n in Theorem 1. From the inequality

$$\|1 - \psi_n\|_{L^p}^p \leq P\left(|r_n^{-(1-a)}(\langle M_n \rangle_{T_n} - 1)|^2 > \frac{1}{2}\right) + P\left(r_n^{2a_1} \sigma_{N_n} > \frac{1}{2}\right) + P(\sigma_{Y_n} < c) \quad ,$$

we obtain the result. ◊

Remark 3. Suppose X_n has the form $X_n = s_n^{-1} M_n$ with a positive random variable s_n converging in probability to 1. If we set $N_n = r_n^{-1}(s_n^{-1} - 1) M_n$, then the asymptotic expansion of the distribution function of X_n is given by Lemma 1′ with

$$p_n(z) = \phi(z) + \frac{1}{2} r_n \partial_z^2 (E[\xi | Z = z] \phi(z)) + r_n \partial_z (E[\eta' | Z = z] z \phi(z)) \ ,$$

where the random vector (Z, ξ, η') is the weak limit of $(M_n, r_n^{-1}(\langle M_n \rangle - 1), r_n^{-1}(s_n - 1))$; in this situation $\eta = -\eta' Z$, and Theorem 2.2 of Mykland [20] originally treated this case.

Remark 4. Here is a simple example suggesting the necessity of the condition on the nondegeneracy of the Malliavin covariance. Suppose M is a $N(0, 1)$-random variable of 1-dimension. Take smooth functions φ_n on $\mathbb{R}$ so that $\varphi_n(x) = x$ if $|x| \geq 2r_n$ and $\varphi_n(x) = 0$ if $|x| \leq r_n$. Then $\varphi_n(M)$ has a decomposition $\varphi_n(M) = M + (\varphi_n(M) - M)$; the second term on the right-hand side is of $o_p(r_n^m)$, for any $m > 0$, so this is a problem treated here. The distribution function of $\varphi_n(M)$ has a jump of order r_n at the origin. Therefore no functions written by an integration of a density p_n can approximate this distribution function up to $o(r_n)$. Since $\varphi_n(M)$ does not satisfy the conditions of theorems here, it is not a counter-example; however, it suggests the necessity of certain regularity conditions of distributions.

6. Applications to statistics

We present examples of applications of the results in Section 2. The first one is a refinement of the central limit theorem for a functional of an ergodic diffusion process. The second example gives an application to statistics, and the asymptotic expansion of the distribution of the maximum likelihood estimator will be presented. Finally, we will mention the asymptotic expansion for an estimator of the diffusion coefficient (volatility) of an Itô process defined on a finite time interval.

6.1 Asymptotic expansion of a functional of an ergodic diffusion

We will treat a one-dimensional, stationary, ergodic diffusion process $X = (X_t\colon t \in \mathbb{R}_+)$ defined by the stochastic differential equation:

$$dX_t = \beta(X_t) dt + dw_t \ , \tag{11}$$

where β is a given $\mathbb{R}$-valued function. The probability measure v denotes the invariant measure of X. The martingale central limit theorem holds that if $v(f^2) < \infty$, then

$$M_T := \frac{1}{\sqrt{T}} \int_0^T f(X_t) dw_t \Rightarrow N(0, v(f^2)) \tag{12}$$

as $T \to \infty$. Martingale central limit theorems were extensively used in the first-order asymptotic theory on statistical inference for semimartingales, but one needs more to go further into higher-order problems; asymptotic expansion is then one of the promising methods. In this subsection, we will present an asymptotic expansion of the distribution of M_T. In order to apply the results given in Section 2, we will focus our attention to verifying the boundedness of $D_{p,s}$-norms and the uniform nondegeneracy of the Malliavin covariance.

We are now considering a stationary, ergodic diffusion process defined by (11) with the stationary distribution v given by

$$v(dx) = \frac{n(x)}{\int_{-\infty}^{\infty} n(u)du} dx$$

where $n(x) = e^{\int_0^x 2\beta(v)dv}$ (cf. Gihman-Skorohod [8]). In order to obtain the asymptotic expansion of the distribution of the normalized martingale (12), we assume several conditions stated below. $C_{\uparrow}^r(\mathbb{R})$ stands for the set of C^r-functions with derivatives of at most polynomial growth order.

$(C1)$ $X = (X_t\colon t \in \mathbb{R}_+)$ is a stationary, ergodic diffusion process with stationary distribution $v(dx)$.

$(C2)_r$ $\beta \in C_{\uparrow}^r(\mathbb{R})$ and satisfies $\sup_{x\in\mathbb{R}} \beta'(x) < 0$.

$(C3)_r$ $f \in C_{\uparrow}^r(\mathbb{R})$ and $v(f^2) = 1$.

For continuous function g: $\mathbb{R} \to \mathbb{R}$, define G_g: $\mathbb{R} \to \mathbb{R}$ by:

$$G_g(x) = -\int_0^x dyn(y)^{-1} \int_y^{\infty} 2n(u)g(u)du \ ,$$

if $\int_0^{\infty} n(u)|g(u)|du < \infty$.

Theorem 7 *Suppose that Conditions* $(C1), (C2)_4, (C3)_4$ *hold true. Then*

$$\sup_{x\in\mathbb{R}} |P(M_T \leq x) - Q_T(x)| = o\left(\frac{1}{\sqrt{T}}\right)$$

as $T \to \infty$, *where*

$$Q_T(x) = \Phi(x) + \frac{\sigma_{12}}{2\sqrt{T}}(1 - x^2)\phi(x)$$

with σ_{12} *given by*

$$\sigma_{12} = -\int_{\mathbb{R}} f(x)\partial G_{f^2-1}(x)v(dx) \ .$$

Before the proof of this theorem, we need notation and several lemmas. For later use, we remind of several notations used in the partial Malliavin calculus.

Let $(W^{(i)}, B^{(i)}, P^{(i)}), i = 1, 2$, be probability spaces, and let $(W^{(2)}, B^{(2)}, P^{(2)})$ be a Wiener space, i.e., $W^{(2)} = \{w\colon \mathbb{R}_+ \to \mathbb{R}^r$ *continuous*, $w(0) = 0\}$, $B^{(2)} = \mathbb{B}(W^{(2)})$ and $P^{(2)}$ is a Wiener measure on $B^{(2)}$. H denotes the Cameron-

Martin subspace of $W^{(2)}$. Let (W, B, P) be the product space of $(W^{(1)}, B^{(1)}, P^{(1)})$ and $(W^{(2)}, B^{(2)}, P^{(2)})$. Given a separable Hilbert space $E, S(E)$ denotes the linear space spanned by E-valued smooth functionals $F\colon W \to E$ such that for some $n \in \mathbb{Z}_+$, there exists a measurable function $f : W^{(1)} \times \mathbb{R}^n \to \mathbb{R}$ satisfying that for any $\alpha \in \mathbb{Z}^n_+$, for some constant c_α, $|\partial^\alpha_\xi f(w^{(1)}, \xi)| \le c_\alpha(1 + |\xi|^{c_\alpha})$ for any $w^{(1)} \in W^{(1)}, \xi \in \mathbb{R}^n$, and that $F(w^{(1)}, w^{(2)}) = f(w^{(1)}, \xi(w^{(2)}))\mathrm{e}$ for any $w \in W$, where $\mathrm{e} \in \mathrm{E}$ and $\xi(\cdot) \in (W^{(2)'})^n$. $S(E)$ is dense in $L^p(W, P; E)$ for any $p \ge 1$. The differential operator in the direction of each $h \in H$ is defined by $D_h F(w^{(1)}, w^{(2)}) = (\partial/\partial t)_{t=0} F(w^{(1)}, w^{(2)} + th)$ for each $F \in S(E)$. We denote by $D_{p,s}(E)$ the completion of $S(E)$ with respect to the norm $\|\cdot\|_{p,s} = \left(\sum_{j=0}^{s} \|D^j \cdot\|^p_{L^p(P)}\right)^{1/p}$. Then D is uniquely extended to an continuous operator, denoted by D again, from $D_{p,s+1}(E)$ into $D_{p,s}(H \otimes E)$. The integration-by-parts formula has the same form as in the usual Malliavin calculus over the classical Wiener spaces.

In the sequel, we regard the diffusion process $(X_t\colon t \in \mathbb{R}_+)$ as a solution to the stochastic differential equation

$$\begin{cases} dX_t(w) = \beta(X_t(w))dt + dw^{(2)} \\ X_0(w) = w^{(1)} \end{cases} \tag{13}$$

constructed on the Wiener space (W, B, P) with $W^{(1)} = \mathbb{R}, P^{(1)} = \nu$, and $r = 1$. Hereafter, $w^{(2)}$ will be denoted by w to simplify the notation.

For separable real Hilbert spaces $H_1, H_2, L(H_1, H_2; \mathbb{R})$ denotes the set of all bilinear forms $v\colon H_1 \times H_2 \to \mathbb{R}$ that is continuous in the sense that there exists a constant c such that $|v(h_1, h_2)| \le c|h_1|_{H_1}|h_2|_{H_2}$ for any $h_1 \in H_1, h_2 \in H_2$. $L(H_1; H_2)$ is the set of continuous linear operators from H_1 into H_2. Clearly, there is a one-to-one correspondence between $L(H_1, H_2; \mathbb{R})$ and $L(H_1; H_2)$. For a pair of bases $\{h_{1,i}\}, \{h_{2,j}\}$ of H_1, H_2, respectively, the Hilbert-Schmidt norm $|v|_{H_1 \otimes H_2}$ of a bilinear form $v\colon H_1 \times H_2 \to \mathbb{R}$ is defined as $|v|_{H_1 \otimes H_2} = \left(\sum_{ij} v(h_{1,i}, h_{2,j})^2\right)^{\frac{1}{2}}$. This norm is independent of the choice of the bases $\{h_{1,i}\}$ and $\{h_{2,j}\}$. $H_1 \otimes H_2$ denotes the set of bilinear forms v satisfying that $|v|_{H_1 \otimes H_2} < \infty$, and is a Hilbert space equipped with the Hilbertian norm $|v|_{H_1 \otimes H_2}$. It is clear that $H_1 \otimes H_2 \subset L(H_1, H_2; \mathbb{R})$.

Let $F \in L^2(\mathbb{R}_+ \to E \otimes \mathbb{R}^r, ds)$. Suppose that the linear operator $L\colon H \to E$ is defined by

$$L[h] = \int_0^\infty F_s \cdot \dot{h}_s ds, \quad h \in H$$

Then $L\colon H \to E$ is a continuous linear operator, and if $L\colon H \times E \to \mathbb{R}$ denotes the corresponding continuous bilinear form then, $L \in H \otimes E$ and

$$|L|_{H \otimes E} = \sqrt{\int_0^\infty |F_s|^2_{E \otimes \mathbb{R}^r} ds} \ . \tag{14}$$

Lemma 7 *Let* $(X_t\colon t \in \mathbb{R}_+)$ *satisfy the stochastic differential equation (11). Suppose that the following conditons are satisfied:*

(1) $\sup_{t\in\mathbb{R}_+}\|X_t\|_p<\infty$ *for any* $p\geq 1$.
(2) $c:=-\sup_{x\in\mathbb{R}}\beta'(x)>0$.
(3) β *is j-times differentiable, and* $\sup_{t\in\mathbb{R}_+}\|\beta^{(i)}(X_t)\|_p<\infty$ *for every* $p\geq 1$ *and* i $(0\leq i\leq j)$.
Then $\sup_{t\in\mathbb{R}_+}\|X_t\|_{p,j}<\infty$ *for all* $p\geq 1$.

Proof. From (11), we see that for each $h\in H, D_hX_t$ satisfies

$$\begin{cases} dD_hX_t=\beta'(X_t)D_hX_tdt+\dot{h}_tdt\ , \\ D_hX_0=0\ . \end{cases}$$

Therefore,

$$D_hX_t=\int_0^t\alpha_s^t\dot{h}_sds \tag{15}$$

for each $h\in H$, where $\alpha_s^t=\exp\left(\int_s^t\beta'(X_\tau)d\tau\right)$. By (15) and by definition of c,

$$|DX_t|_H^2=\int_0^t(\alpha_s^t)^2ds\leq\int_0^t e^{-2c(t-s)}ds\leq\frac{1}{2c}<\infty\ ,$$

and hence, together with Assumption (1), this inequality implies that $\sup_{t\in\mathbb{R}}\|X_t\|_{p,1}<\infty$ for any $p\geq 1$.

From (15), one has

$$DD_hX_t=\int_0^t\alpha_s^t\int_s^t\beta''(X_\tau)DX_\tau d\tau\cdot\dot{h}_sds\ . \tag{16}$$

From (16) and (14), applying Jensen's inequality to the sub-stochastic kernel $2ce^{-2c(t-s)}1_{[0,t]}(s)$ and using it again, we have, for any $p\geq 1$,

$$\begin{aligned}|D^2X_t|_{H\otimes H}^{2p}&\leq\left[\int_0^t e^{-2c(t-s)}\left\{\int_s^t|\beta''(X_\tau)||DX_\tau|_Hd\tau\right\}^2ds\right]^p\\ &\leq\left\{\frac{1}{2c}\right\}^{p-1}\int_0^t e^{-2c(t-s)}(t-s)^{2p-1}\left\{\int_s^t|\beta''(X_\tau)|^{2p}|DX_\tau|_H^{2p}d\tau\right\}ds\ .\end{aligned}$$

Therefore,

$$E\left[|D^2X_t|_{H\otimes H}^{2p}\right]\leq\sup_{\tau\in\mathbb{R}_+}E\left[|\beta''(X_\tau)|^{2p}|DX_\tau|_H^{2p}\right]\cdot\left(\frac{1}{2c}\right)^{3p}\int_0^\infty e^{-u}u^{2p}du$$

for any $t\in\mathbb{R}_+$. This means that $\sup_{t\in\mathbb{R}_+}\|X_t\|_{p,2}<\infty$ for any $p\geq 1$.

In a similar fashion, by induction, we obtain the desired result. ◊.

Remark 5. To prove the boundedness of $D_{p,s}$- norms of X_t, we used Condition (2) of Lemma 7. However, as seen in the proof, for this purpose, it suffices to assume that $\sup_{t\in\mathbb{R}_+}E\left[\int_0^t(\alpha_s^t)^p(t-s)^qds\right]<\infty$ for any $p\geq 2$ and $q\in\mathbb{Z}_+$.

Lemma 8 *Let* $M_T = \frac{1}{\sqrt{T}} \int_0^T f(X_t) dw_t$. *Suppose that the following two conditions are satisfied:*

(1) $\sup_{t \in \mathbb{R}_+} \|X_t\|_{p,j} < \infty$ *for any* $p > 1$.

(2) $\sup_{t \in \mathbb{R}_+} \|f^{(i)}(X_t)\|_p < \infty$ *for any* $p > 1$ *and* i $(0 \leq i \leq j)$.

Then $\sup_{T \in \mathbb{R}_+} \|M_T\|_{p,j} < \infty$ *for any* $p > 1$.

Proof. Since

$$D_h M_T = \int_0^T \frac{1}{\sqrt{T}} f'(X_t) D_h X_t dw_t + \int_0^T \frac{1}{\sqrt{T}} f(X_t) \dot{h}_t dt \tag{17}$$

for any $h \in H$, it follows from (14) that

$$|DM_T|_H^2 \leq 2 \left\{ \left| \int_0^T \frac{1}{\sqrt{T}} f'(X_t) DX_t dw_t \right|_H^2 + \int_0^T \frac{1}{T} f(X_t)^2 dt \right\} .$$

The Burkholder inequality for Hilbert space valued square-integrable progressively measurable processes on $W \times \mathbb{R}_+$ then implies that for some constant c'_p,

$$\sup_{T \in \mathbb{R}_+} E\left[|DM_T|_H^{2p} \right] \leq c'_p \left\{ \sup_{t \in \mathbb{R}_+} E\left[|f'(X_t)|^{2p} |DX_t|_H^{2p} \right] + \sup_{t \in \mathbb{R}_+} E\left[|f(X_t)|^{2p} \right] \right\} .$$

Consequently, $\sup_{T \in \mathbb{R}_+} \|M_T\|_{p,1} < \infty$ for any $p \geq 1$.

Differentiating (17), and by using (14) once again, one has

$$\begin{aligned} |D^2 M_T|_{H \otimes H}^2 \leq 4 \Bigg\{ & \left| \int_0^T \frac{1}{\sqrt{T}} f''(X_t) DX_t \otimes DX_t dw_t \right|_{H \otimes H}^2 \\ & + \left| \int_0^T \frac{1}{\sqrt{T}} f'(X_t) D^2 X_t dw_t \right|_{H \otimes H}^2 \\ & + 2 \int_0^T \left| \frac{1}{\sqrt{T}} f'(X_t) DX_t \right|_H^2 dt \Bigg\} . \end{aligned}$$

Consequently

$$\begin{aligned} \sup_{T \in \mathbb{R}_+} E\left[|D^2 M_T|_{H \otimes H}^{2P} \right] \leq c''_p \Bigg\{ & \sup_{t \in \mathbb{R}_+} E\left[|f''(X_t)|^{2p} |DX_t|_H^{4p} \right] \\ & + \sup_{t \in \mathbb{R}_+} E\left[|f'(X_t)|^{2p} |D^2 X_t|_{H \otimes H}^{2p} \right] \\ & + \sup_{t \in \mathbb{R}_+} E\left[|f'(X_t)|^{2p} |DX_t|_H^{2p} \right] \Bigg\} , \end{aligned}$$

which means that $\sup_{T \in \mathbb{R}_+} \|M_T\|_{p,2} < \infty$ for any $p \geq 1$.

In the same way, we can show the general case. ◊

The differential operator L denotes the generator of the diffusion X: $L = \frac{1}{2}\partial^2 + \beta\partial$. It is then clear that $LG_g = g$. Set $\xi_T = \sqrt{T}(\langle M_{T,\cdot}\rangle_T - 1)$.

Lemma 9 *Suppose that the following three conditions are satisfied:*

(1) $f\colon \mathbb{R} \to \mathbb{R}$ *is continuous, and*

$$\int_0^\infty n(u)|f^2(u) - 1|du < \infty \ .$$

(2) $\sup_{t\in\mathbb{R}_+}\|X_t\|_{p,j-1} < \infty$ *for any* $p > 1$.

(3) $\sup_{t\in\mathbb{R}_+}\|G^{(i)}_{f^2-1}(X_t)\|_p < \infty$ *for any* $p > 1$ *and* i $(0 \le i \le j)$.
Then $\sup_{T>\delta}\|\xi_T\|_{p,j-1} < \infty$ *for any* $p > 1$ *and* $\delta > 0$.

Proof. By using Itô's formula, we see that

$$\begin{aligned}\xi_T &= \frac{1}{\sqrt{T}}\int_0^T \left[f(X_t)^2 - 1\right]dt \\ &= \frac{1}{\sqrt{T}}G_{f^2-1}(X_T) - \frac{1}{\sqrt{T}}G_{f^2-1}(X_0) \quad - \frac{1}{\sqrt{T}}\int_0^T \partial G_{f^2-1}(X_t)dw_t \ . \end{aligned} \tag{18}$$

It follows from the chain rule and the assumptions that

$$\sup_{T>\delta}\left\|T^{-\frac{1}{2}}G_{f^2-1}(X_T)\right\|_{p,j-1} + \sup_{T>\delta}\left\|T^{-\frac{1}{2}}G_{f^2-1}(X_0)\right\|_{p,j-1} < \infty$$

for any $p > 1$. Moreover, by means of Lemma 8, we see that

$$\sup_{T>\delta}\left\|\frac{1}{\sqrt{T}}\int_0^T \partial G_{f^2-1}(X_t)dw_t\right\|_{p,j-1} < \infty$$

for any $p > 1$. These inequalities together with (18) complete the proof. ◊

Lemma 10 *Let* $r \in \mathbb{Z}_+$. *Suppose that Conditions* $(C2)_r$ *and* $(C3)_r$ *are satisfied. Moreover, suppose that* $\int_{-\infty}^\infty f(u)n(u)du = 0$. *Then* $G_f \in C^{r+2}_{\uparrow}(\mathbb{R})$.

Proof. First, from Condition $(C2)_r$, it is easy to see that

$$n(x) \le e^{2\beta(0)x - cx^2}$$

for all $x \in \mathbb{R}$. Since $\int_0^\infty |f(u)|n(u)du < \infty, G_f$ is well-defined and $G_f \in C^{r+2}(\mathbb{R})$. Let $1 \le j \le r+2$. By Leibniz's rule,

$$\begin{aligned}\partial^j G_f &= -\left(\partial^{j-1}n(y)^{-1}\right)\int_x^\infty 2n(u)f(u)du \\ &\quad - \sum_{i=1}^{j-1}\binom{j-1}{i}(\partial^{j-1-i}n(y)^{-1})\partial^{i-1}(2n(x)f(x)) \\ &= P_1(x)n(x)^{-1}\int_x^\infty 2n(u)f(u)du + P_2(x) \ , \end{aligned} \tag{19}$$

where P_1 is a polynomial of $\beta, \beta', \cdots, \beta^{(j-2)}$, and P_2 is a polynomial of $\beta, \beta', \cdots, \beta^{(j-3)}$ and $f, f', \cdots, f^{(j-2)}$. On the other hand, it follows that $q(u) := -2\beta(u) - mu^{-1}$ is positive and increases as u increases for large u, for any $m \in \mathbb{Z}_+$. For large x,

$$\begin{aligned}\int_x^\infty n(u)u^m du &\le \int_x^\infty \frac{q(u)}{q(x)} n(u)u^m du \\ &= \int_x^\infty \frac{-(n(u)u^m)'}{q(x)} du \\ &= n(x)\frac{x^m}{q(x)} \quad .\end{aligned}$$

Therefore $n(x)^{-1} \int_x^\infty n(u)u^m du = O(x^{m-1})$ as $x \to \infty$ for any $m \in \mathbb{Z}_+$. The same argument can be applied to a similar functional in the case where $x \to -\infty$. If $\int_{\mathbb{R}} f(x)n(x)dx = 0$, then $\int_x^\infty f(u)n(u)du = -\int_{-\infty}^x f(u)n(u)du$, so that with the aid of (19), we see $G_f \in C_\uparrow^{r+2}(\mathbb{R})$. $\Diamond$

Let $g = \left(\frac{1}{2}\partial + \beta\right)f$. Then $\partial G_g = f$ and

$$\begin{aligned}M_T &= \frac{1}{\sqrt{T}}\int_0^T f(X_t)dw_t \\ &= \frac{1}{\sqrt{T}}G_g(X_T) - \frac{1}{\sqrt{T}}G_g(X_0) - \frac{1}{\sqrt{T}}\int_0^T g(X_t)dt \quad . \qquad (20)\end{aligned}$$

[G_g is of at most polynomial growth order due to Lemma 10.] Instead of the nondegeneracy of the Malliavin covariance of M_T, we will consider that of $\bar{M}_T$ defined by

$$\bar{M}_T = \frac{1}{\sqrt{T}}\int_0^T g(X_t)dt \quad .$$

From (15), for $h \in H$,

$$\begin{aligned}D_h\bar{M}_T &= \frac{1}{\sqrt{T}}\int_0^T g'(X_t)\left(\int_0^t \alpha_s^t \dot{h}_s ds\right)dt \\ &= \int_0^T ds\dot{h}_s\left(\frac{1}{\sqrt{T}}\int_s^T g'(X_t)\alpha_s^t dt\right) \quad .\end{aligned}$$

Therefore the Malliavin covariance of $\sigma_{\bar{M}_T}$ is given by

$$\sigma_{\bar{M}_T} = \frac{1}{T}\int_0^T\left[\int_s^T g'(X_t)\alpha_s^t dt\right]^2 ds \quad . \qquad (21)$$

Lemma 11 *Suppose that Conditions* $(C1), (C2)_2, (C3)_3$ *hold true. Then there exists a positive constant c such that* $P(\sigma_{\bar{M}_T} < c) = O\left(\frac{1}{T}\right)$ *as* $T \to \infty$.

Proof. f, f', β have at most polynomial growth order, and $\sup_x \beta'(x) < 0$. It then follows from integration-by-parts formula that

$$\int_{-\infty}^{\infty} g(x)n(x)dx = \int_{-\infty}^{\infty} f(x)\left(-\frac{1}{2}\partial + \beta(x)\right)n(x)dx = 0 \ .$$

Therefore if g were a constant, then $g = 0$, and hence $\left(\frac{1}{2}\partial + \beta\right)f = 0$. If $f \neq 0$, then $f(x) = f(0)n(x)^{-1}$, which contradicts the assumption that f has at most polynomial growth order. Consequently $f \equiv 0$. Therefore, g is not a constant.

There exists $x_0 \in \mathbb{R}$ such that $g'(x_0) \neq 0$. Fix $\Delta > 0$. We will take Δ sufficiently small later. Let $S_0 = \inf\{t \in \mathbb{R}_+ \colon X_t = x_0\}$. Next take any point $x_1 \in B(x_0, \Delta)^c$, and let $S_0' = \inf\{t > S_0 \colon X_t = x_1\}$. Moreover we define stopping times $S_i, S_i', i \in \mathbb{N}$, inductively by $S_i = \inf\{t > S_{i-1}' \colon X_t = x_0\}$ and by $S_i' = \inf\{t > S_i \colon X_t = x_1\}$. There exists a positive constant $\delta = \delta(\Delta)$ such that if $\sup_{S_i \leq \tau \leq S_i + \Delta} |X_\tau - x_0| \leq \Delta$, then

$$\inf_{s \in [S_i, S_i + \Delta]} (\alpha_s^{S_i + \Delta})^2 \geq \delta \ .$$

Define $A_i(s)$ by

$$A_i(s) = \frac{\int_s^{S_i + \Delta} g'(X_t)\alpha_s^t dt}{\alpha_s^{S_i + \Delta}}$$

Let

$$M = M(\Delta) = \max\left\{\sup_{x \in B(x_0, \Delta)} |g'(x)|, \sup_{x \in B(x_0, \Delta)} |g''(x)|, \sup_{x \in B(x_0, \Delta)} |\beta'(x)|\right\} \ .$$

It is then easy to show that

$$A_i(s) = l_i(s) + r_i(s)$$

for $s \in [S_i, S_i + \Delta]$, where $l_i(s) = (S_i + \Delta - s)g'(x_0)$ and

$$\sup_{s \in [S_i, S_i + \Delta]} |r_i(s)| \leq \Delta^2 q(M, \Delta, e^{M\Delta})$$

if $\sup_{S_i \leq \tau \leq S_i + \Delta} |X_\tau - x_0| \leq \Delta$, where q is a polynomial defined on $\mathbb{R}^3$. Choose $\Delta > 0$ so that $\sup_{s \in [S_i, S_i + \Delta]} |r_i(s)| \leq \frac{1}{6}|g'(x_0)|\Delta$. For any $\mu \in \mathbb{R}$,

$$\sqrt{\int_{S_i}^{S_i + \Delta} (l_i(s) + r_i(s) - \mu)^2 ds} \geq \sqrt{\int_{S_i}^{S_i + \Delta} (l_i(s) - \mu)^2 ds} - \sqrt{\int_{S_i}^{S_i + \Delta} r_i^2(s) ds}$$

$$\begin{aligned} &\geq \sqrt{\int_{S_i}^{S_i+\Delta}\left(l_i(s)-\frac{\Delta}{2}g'(x_0)\right)^2 ds} \\ &\qquad - \sqrt{\int_{S_i}^{S_i+\Delta} r_i^2(s)ds} \\ &\geq |g'(x_0)|\sqrt{\frac{\Delta^3}{12}} - |g'(x_0)|\sqrt{\frac{\Delta^3}{36}} \\ &\qquad = \frac{\sqrt{3}-1}{6}|g'(x_0)|\Delta^{\frac{3}{2}} \quad . \end{aligned} \tag{22}$$

Let

$$c_i = \int_{S_i+\Delta}^{T} g'(X_t)\alpha_{S_i+\Delta}^t dt \quad .$$

By using (22), we see that if $S_i + \Delta \leq T$ and if $\sup_{\tau\in[S_i,S_i+\Delta]}|X_\tau - x_0| \leq \Delta$, then

$$\begin{aligned} \int_{S_i}^{S_i+\Delta}\left[\int_s^T g'(X_t)\alpha_s^t dt\right]^2 ds &= \int_{S_i}^{S_i+\Delta}\left(\alpha_s^{S_i+\Delta}\right)^2[A_i(s)+c_i]^2 ds \\ &\geq \delta\left(\frac{\sqrt{3}-1}{6}\right)^2|g'(x_0)|^2\Delta^3 \\ &=: \delta_1 \quad . \end{aligned}$$

Therefore, by (21)

$$\begin{aligned} \sigma_{\bar{M}_T} &= \frac{1}{T}\int_0^T\left[\int_s^T g'(X_t)\alpha_s^t dt\right]^2 ds \\ &\geq \sum_{i:S_i+\Delta\leq T}\frac{1}{T}\int_{S_i}^{S_i+\Delta}\left[\int_s^T g'(X_t)\alpha_s^t dt\right]^2 ds \\ &\qquad \cdot 1_{\{\sup_{S_i\leq\tau\leq S_i+\Delta}|X_\tau - x_0|\leq\Delta\}} \\ &\geq \sum_{i=0}^{\infty}\frac{\delta_1}{T}1_{\{i:S_i+\Delta\leq T\}}1_{\{\sup_{S_i\leq\tau\leq S_i+\Delta}|X_\tau-x_0|\leq\Delta\}} \quad . \end{aligned} \tag{23}$$

Let $\mu = P_{x_0}(\sup_{0\leq\tau\leq\Delta}|X_\tau - x_0| \leq \Delta)$, and let

$$\xi_i^T = \frac{1}{\sqrt{T}}1_{\{S_i+\Delta\leq T\}}\left[1_{\{\sup_{S_i\leq\tau\leq S_i+\Delta}|X_\tau-x_0|\leq\Delta\}} - \mu\right] \quad .$$

By the support theorem, it is easily seen that $\mu > 0$. Let $\mathbb{F}_t$ be the filtration generated by X_0 and $(w_s\colon s \leq t)$, and set

$$m_n^T = \sum_{i=0}^{n}\xi_i^T \quad .$$

It is then easy to show that $(m_n^T, \mathbb{F}_{S_{n+1}})_{n\in\mathbb{Z}_+}$ is a martingale for each $T \in \mathbb{R}_+$ (cf. Jacod-Shiryaev [14], p. 4, 1.17). With

$$v := E_{x_0}\left(\left[1_{\{\sup_{0\le\tau\le\Delta}|X_\tau - x_0|\le\Delta\}} - \mu\right]^2\right) \ ,$$

we also see that

$$\sum_{i=0}^{n} E\left[(\xi_i^T)^2|\mathbb{F}_{S_i}\right] = \frac{v}{T}\sum_{i=0}^{n} 1_{\{S_i+\Delta\le T\}} \ . \tag{24}$$

Define positive random variables $\tau_i, i \in \mathbb{Z}_+$, as $\tau_0 = S_0, \tau_i = S_i - S_{i-1} (i \in \mathbb{N})$. It is well-known that $\tau_i (i \in \mathbb{Z}_+)$ are mutually independent, square-integrable random variables, and that $\tau_i (i \in \mathbb{N})$ is an i.i.d. sequence with positive finite mean value γ_1. [In fact, it is possible to express the moments of those stopping times explicitly and to estimate them (cf. Gihman-Skorohod [8], [36]).]

By definition,

$$\begin{aligned}
&P\left(\sum_{i=0}^{\infty}\frac{\gamma_1}{T}1_{\{S_i+\Delta\le T\}}1_{\{\sup_{S_i\le\tau\le S_i+\Delta}|X_\tau - x_0|\le\Delta\}} < \frac{\mu}{4}\right)\\
&\quad = P\left(m_\infty^T < \sqrt{T}\frac{\mu}{\gamma_1}\left(\frac{1}{4} - \sum_{i=0}^{\infty}\frac{\gamma_1}{T}1_{\{S_i+\Delta\le T\}}\right)\right)\\
&\quad \le P\left(\sum_{i=0}^{\infty}\frac{\gamma_1}{T}1_{\{S_i+\Delta\le T\}} < \frac{1}{2}\right) + P\left(m_\infty^T < -\frac{\sqrt{T}\mu}{4\gamma_1}\right) \ .
\end{aligned} \tag{25}$$

The first term on the right-hand side (25) is not greater than

$$\begin{aligned}
&P\left(\tau_0 + \tau_1 + \cdots + \tau_{\left[\frac{T}{2\gamma_1}\right]+1} > T - \Delta\right)\\
&\quad\le \left(T - \Delta - \left\{E[\tau_0] + \left(\left[\frac{T}{2\gamma_1}\right] + 1\right)\gamma_1\right\}\right)^{-2}\\
&\qquad \times \left(var(\tau_0) + \left(\left[\frac{T}{2\gamma_1}\right] + 1\right)var(\tau_1)\right)\\
&\quad = O\left(\frac{1}{T}\right) \ .
\end{aligned}$$

By (24) and Lenglart's inequality (cf. Jacod-Shiryaev [14], p. 35) under usual extension of processes $m_n^T, \sum_{i=0}^{n} 1_{\{S_i+\Delta\le T\}}$, and filtration $(\mathbb{F}_{S_{n+1}})$, we see that the second term on the right-hand side of (25) is not greater than

$$\frac{24\gamma_1 v}{T\mu^2} + P\left(\frac{\gamma_1}{T}\sum_{i=0}^{\infty}1_{\{S_i+\Delta\le T\}} \ge \frac{3}{2}\right) \ ;$$

the last term can be estimated in the same way as above. Therefore we obtain

$$P\left(\sigma_{\bar{M}_T} < \frac{\delta_1 \mu}{4\gamma_1}\right) = O\left(\frac{1}{T}\right)$$

as $T \to \infty$. ◊

The following lemma is easy to show by the martingale central limit theorem.

Lemma 12 *As* $T \to \infty$,

$$\left(M_T, \sqrt{T}(\langle M_{T,\cdot}\rangle_T - 1)\right) \to^d N(0, \Sigma) \ ,$$

where $\Sigma = (\sigma_{ij})_{i,j=1}^2$ *is given by* $\sigma_{11} = 1, \sigma_{12} = \sigma_{21} = -\int_{\mathbb{R}} f \partial G_{f^2-1} dv$ *and* $\sigma_{22} = \int_{\mathbb{R}} (\partial G_{f^2-1})^2 dv$.

Proof of Theorem 7. It suffices to verify the assumptions of Theorem 2. [A1] is trivial. We see that the inequality $\sup_{t\in\mathbb{R}_+} \|X_t\|_{p,4} < \infty$ follows from $(C2)_4$ and Lemma 7; $\sup_{T\in\mathbb{R}_+} \|M_T\|_{p,4} < \infty$ from $(C3)_4$ and Lemma 8; similarly, $\sup_{T>\delta} \|\xi_T\|_{p,3} < \infty$ from Lemmas 9 and 10. Thus we obtain $[A2]_3$. Lemma 12 implies $[A3]_+$. In view of the inequality

$$P\left(\sigma_{M_T}^{\frac{1}{2}} < \frac{c^{\frac{1}{2}}}{2}\right) \leq P\left(\sigma_{A_T}^{\frac{1}{2}} > \frac{c^{\frac{1}{2}}}{2}\right) + P\left(\sigma_{\bar{M}_T}^{\frac{1}{2}} < c^{\frac{1}{2}}\right) \ ,$$

where $A_T = T^{-\frac{1}{2}}[G_g(X_T) - G_g(X_0)]$, we have $P(\sigma_{M_T} < c/4) = O(T^{-1})$, and obtain the desired result. ◊

6.2 Asymptotic expansion of the maximum likelihood estimator for an ergodic diffusion

In this subsection, we consider the following stochastic differential equation like the previous subsection but with unknown parameter:

$$dx_t = b(x_t, \theta)dt + dw_t$$

Let θ_0 be the true value of the unknown parameter θ and we abbreviate θ_0 in functions of θ when they are evaluated at θ_0. We assume that x_t is stationary and x_0 obeys the stationary distribution $v = v_{\theta_0}$, and that $\sup_{x\in\mathbb{R}} \partial b(x, \theta_0) < 0$. Furthermore, we assume for simplicity that b is smooth and $|\delta^l \partial^j b(x, \theta)| \leq C_{j,l}(1 + |x|^{C_{j,l}})$ for any $x \in \mathbb{R}$ and θ, where $\delta = \partial/\partial\theta$, and often denoted by dot. Under a usual identifiability condition, the statistical experiment induced by this diffusion model is entirely separated, and hence there exists a consistent estimator. By using this consistent estimator, it is easy to show the existence of the consistent maximum likelihood estimator $\hat{\theta}_T$ for which there exists a sequence of events A_T such that $P(A_T) \to 1$ and $\dot{l}_T(\hat{\theta}_T) = 0$ on A_T, where $l_T(\theta)$ is the log-likelihood function:

$$l_T(\theta) = \int_0^T b(x_t, \theta) dx_t - \frac{1}{2} \int_0^T b(x_t, \theta)^2 dt \ .$$

Moreover, on some condition of global nondegeneracy of an information amount, the unique existence of $\hat{\theta}_T$ is ensured and $P(|\hat{\theta}_T - \theta_0| > T^{-\rho}) = o(T^{-\frac{1}{2}})$ for some $\rho > 0$ (e.g., [24]).

Then it is not difficult to show by the well-known *Delta*-method that the asymptotic expansion of the distribution function of $\sqrt{IT}(\hat{\theta}_T - \theta_0)$ ($I := v(\dot{b}^2)$, the Fisher information amount as θ_0) coincides up to $o(T^{-\frac{1}{2}})$ with the asymptotic expansion of the distribution function of X_T defined by

$$X_T = M_T + \frac{1}{\sqrt{T}} N_T \ ,$$

where $M_T = (IT)^{-\frac{1}{2}} \int_0^T \dot{b}(x_t, \theta_0) dw_t$, and $N_T = I^{-1} M_T Z_{2,T} - \frac{1}{2} I^{-1.5} L M_T^2$ with $Z_{2,T} = \sqrt{T}\left(\frac{\ddot{l}_T}{T} + I\right)$ and $L = -\lim_{T\to\infty} \frac{\delta^3 l_T}{T} = 3 \int_{\mathbb{R}} \dot{b}(x) \ddot{b}(x) v(dx)$.

Let $k = \partial G_{(\dot{b}(\cdot,\theta_0))^2 - I}$ for $\beta = b(\cdot, \theta_0)$. Then $\xi_T = \sqrt{T}(\langle M_{T,\cdot}\rangle_T - 1) \equiv^a -\frac{1}{I\sqrt{T}} \int_0^T k(x_t) dw_t$; and $Z_{2,T} \equiv^a \frac{1}{\sqrt{T}} \int_0^T [\ddot{b} + k](x_t, \theta_0) dw_t$, where $\equiv^a$ stands for the asymptotic equivalence. In particular, $(M_T, \xi_T, Z_{2,T}) \to^d (Z, \xi, Z_2)$, where the random vector (Z, ξ, Z_2) has the 3-dimensional normal distribution $N_3(0, \Sigma)$ with $\Sigma_{11} = 1, \Sigma_{12} = -I^{-1.5} v(\dot{b}k)$, $\Sigma_{13} = I^{-0.5} v(\dot{b}[\ddot{b} + k])$, $\Sigma_{22} = I^{-2} v(k^2)$, $\Sigma_{23} = -I^{-1} v(k[\ddot{b} + k])$ and $\Sigma_{33} = v([\ddot{b} + k]^2)$.

Since $L\{Z, \xi, \eta\} = L\{Z, \xi, I^{-1} Z Z_2 - \frac{1}{2} I^{-1.5} L Z^2\}$, it follows from the above fact, that $E[\xi | Z = x] = \Sigma_{12} x$ and that $E[\eta | Z = x] = (\frac{\Sigma_{13}}{I} - \frac{L}{2I^{1.5}}) x^2$. Let $A = \frac{\Sigma_{12}}{2} = -v(\dot{b}k)/(2I^{1.5})$ and $B = \frac{\Sigma_{12}}{2} + \frac{\Sigma_{13}}{I} - \frac{L}{2I^{1.5}} = -\{v(\dot{b}\ddot{b}) - v(\dot{b}k)\}/(2I^{1.5})$. We can still use the proof (Section 6.1) of the nondegeneracy of the first term M_T. Thus we obtain

Theorem 8 *The distribution function of $\sqrt{IT}(\hat{\theta}_T - \theta_0)$ has the asymptotic expansion*

$$P\left(\sqrt{IT}\left(\hat{\theta}_T - \theta_0\right) \le x\right) \sim \Phi(x) + \frac{1}{\sqrt{T}}(A - Bx^2)\phi(x) + o\left(\frac{1}{\sqrt{T}}\right) \ .$$

This expansion holds uniformly in $x \in \mathbb{R}$.

For a scalar parameter α, the Amari-Chentsov affine α-connection is expressed with coefficients $\Gamma^{\alpha}_{ijk,T} = E[\{\delta_i \delta_j l_T + \frac{1-\alpha}{2} \delta_i l_T \delta_j l_T\} \delta_k l_T]$ for coordinates $\theta = (\theta^i)$ in multi-parameter case. Returning to our one-parameter case, let $\Gamma^{\left(-\frac{1}{3}\right)} = \lim_{T\to\infty} \Gamma^{\left(-\frac{1}{3}\right)}_{111,T}/T$. Indeed, by simple calculus with Itô's formula and ergodicity, we see that this limit exists and that $\Gamma^{\left(-\frac{1}{3}\right)} = v(\dot{b}\ddot{b}) - v(\dot{b}k)$. From this relation, one has $B = -\Gamma^{\left(-\frac{1}{3}\right)}/(2I^{1.5})$. Denote by $\hat{\theta}^*_T$ a second order mean-unbiased maximum likelihood estimator of θ, then Theorem 8 provides the asymptotic expansion:

$$P(\sqrt{IT}(\hat{\theta}^*_T - \theta_0) \le x) \sim \Phi(x) + \frac{\Gamma^{\left(-\frac{1}{3}\right)}}{2I^{1.5}\sqrt{T}}(x^2 - 1)\phi(x) + o\left(\frac{1}{\sqrt{T}}\right) \ .$$

This expression was familiar one in independent observation cases. Grigelionis [10] calculates α-connections for Markov statistical models.

It is also possible to derive asymptotic expansions for M-estimators ([24]).

6.3 Estimator for diffusion coefficient

Let us consider a semimartingale X_t having the following decomposition with unknown parameter θ:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sqrt{\theta}\sigma_s dw_s, \quad t \in [0,1] \ ,$$

where b_t, σ_t are Itô processes defined on a one-dimensional Wiener space (W, H, P) in the partial Malliavin calculus: $W = \mathbb{R} \times W^{(2)}$, $\mathbb{B} = \mathbb{B}^1 \times \mathbb{B}^{(2)}$, and $P = L\{X_0\} \otimes P^{(2)}$, where $(W^{(2)}, \mathbb{B}^{(2)}, H, \ P^{(2)})$ is a one-dimensional Wiener space. $(\tilde{F}_t)_{0 \le t \le 1}$ is the filtration generated by X_0 and w: $\tilde{F}_t^0 = \sigma(X_0, w_s\colon 0 \le s \le t)$ and $\tilde{F}_t = \cap_{u>t}\tilde{F}_u^0$. b_t may depend on unknown parameters (θ, ϑ). We assume that b_t, σ_t are adapted to the filtration $(\tilde{F}_t)_{0 \le t \le 1}$. Assume also that b_t has the decomposition $b_t = b_0 + \int_0^t b_s^{[1]} dw_s + \int_0^t b_s^{[0]} ds$, and that $b_t^{[1]}$ has a decomposition $b_t^{[1]} = b_0^{[1]} + \int_0^t b_s^{[1,1]} dw_s + \int_0^t b_s^{[1,0]} ds$. Similarly, assume that $\sigma_t, \sigma^{[1]}$ and $\sigma^{[0]}$ have a corresponding decomposition. Suppose that $\sup_{t \in [0,1]} \|f_t\|_{p,3} < \infty$ for $f_t = b_t$, $b_t^{[1]}$, $b_t^{[0]}$, $b_t^{[1,0]}$, $b_t^{[1,1]}$, σ_t, $\sigma_t^{[1]}$, $\sigma_t^{[0]}$, $\sigma_t^{[1,0]}$, $\sigma_t^{[1,1]}$, $\sigma_t^{[0,0]}$, $\sigma_t^{[0,1]}$, and for $p > 1$.

Based on the data set $\{X_{t_i}, \sigma_{t_i}\colon i = 0, 1, \cdots, n\}$ with $t_i = i/n$, a natural quasi-likelihood estimator of the unknown parameter θ is given by

$$\hat{\theta}_n = \sum_{i=1}^n \left(\frac{X_{t_i} - X_{t_{i-1}}}{\sigma_{t_{i-1}}} \right)^2$$

(cf. Genon = Catalot-Jacod [7]). We assume that $\sup_{t \in [0,1]} \|1/\sigma_t\|_{L^p} < \infty$ for each $p > 1$. Let θ denote the true value of the unknown parameter. Then $\mathscr{X}_n := \frac{\sqrt{n}}{\sqrt{2}\theta}(\hat{\theta}_n - \theta)$ asymptotically has the standard normal distribution. Let $H_1(x) = x, H_2(x) = (x^2 - 1)/\sqrt{2}$, and $H_3(x) = (x^3 - 3x)/\sqrt{6}$. Denote $\Delta w_i = w_{t_i} - w_{t_{i-1}}$. The following lemma gives the stochastic expansion of $\mathscr{X}_n$, from which the asymptotic expansion of the distribution function of $\mathscr{X}_n$ will be presented later.

Lemma 13 *$\mathscr{X}_n$ has the stochastic expansion: $\mathscr{X}_n = M_n + \frac{1}{\sqrt{n}} N_n$, where*

$$M_n = \sum_{i=1}^n \frac{1}{\sqrt{n}} H_2(\sqrt{n}\Delta w_i) \ ;$$

and

$$
\begin{aligned}
N_n &= \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{3}\sigma_{t_{i-1}}^{[1]}}{\sigma_{t_{i-1}}} H_3(\sqrt{n}\Delta w_i) \\
&\quad + \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \cdot \left(\frac{\sqrt{2}\sigma_{t_{i-1}}^{[1]}}{\sigma_{t_{i-1}}} + \frac{\sqrt{2} b_{t_{i-1}}}{\sqrt{\theta}\sigma_{t_{i-1}}} \right) H_1(\sqrt{n}\Delta w_i) \\
&\quad + \sum_{i=1}^{n} \frac{1}{n} F_{t_{i-1}} + R_n
\end{aligned}
$$

with $\|R_n\|_{L^p} = O\left(\frac{1}{\sqrt{n}}\right)$ *for any* $p > 1$, *and with* F_t *given by*

$$
F_t = \frac{\sqrt{2}\sigma_t^{[0]}}{2\sigma_t} + \frac{\left(\sigma_t^{[1]}\right)^2}{2\sqrt{2}\sigma_t^2} + \frac{\sqrt{2} b_t^{[1]}}{2\sqrt{\theta}\sigma_t} + \frac{\sqrt{2} b_t^2}{2\theta\sigma_t^2} \quad .
$$

Proof. First, we see that $\mathscr{X}_n = \Psi + \Phi_2 + \Phi_3 + \Phi_4$, where

$$
\begin{aligned}
\Psi &= \sum_{i=1}^{n} \frac{\sqrt{n}}{\sqrt{2}} \left\{ \frac{1}{\sigma_{t_{i-1}}^2} \left(\int_{t_{i-1}}^{t_i} \sigma_t dw_t \right)^2 - \frac{1}{n} \right\}, \\
\Phi_2 &= \sum_{i=1}^{n} \frac{\sqrt{2n}}{\sqrt{\theta}\sigma_{t_{i-1}}^2} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} b_s ds \sigma_t dw_t, \\
\Phi_3 &= \sum_{i=1}^{n} \frac{\sqrt{2n}}{\sqrt{\theta}\sigma_{t_{i-1}}^2} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \sigma_s dw_s b_t dt \quad ,
\end{aligned}
$$

and

$$
\Phi_4 = \sum_{i=1}^{n} \frac{\sqrt{2n}}{\theta\sigma_{t_{i-1}}^2} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} b_s ds b_t dt \quad .
$$

We say that $T_n = R(n^{-k})$ if $\|T_n\|_p = O(n^{-k})$ for any $p > 1$. By assumption and by Burkholder's inequality for martingales with discrete parameter,

$$
\begin{aligned}
\sqrt{n}\Phi_3 &= \sum_{i=1}^{n} \frac{\sqrt{2}n}{\sqrt{\theta}\ \sigma_{t_{i-1}}^2} \left\{ \int_{t_{i-1}}^{t_i} \sigma_{t_{i-1}} b_{t_{i-1}} \int_{t_{i-1}}^{t} dw_s dt \right. \\
&\quad + \int_{t_{i-1}}^{t_i} b_{t_{i-1}} \int_{t_{i-1}}^{t} \left(\int_{t_{i-1}}^{s} \sigma_u^{[1]} dw_u \right) dw_s dt + \int_{t_{i-1}}^{t_i} \sigma_{t_{i-1}} \int_{t_{i-1}}^{t} b_s^{[1]} ds \cdot dt \\
&\quad \left. + \int_{t_{i-1}}^{t_i} \sigma_{t_{i-1}} \left[\int_{t_{i-1}}^{t} \int_{t_{i-1}}^{v} b_s^{[1]} dw_s dw_v + \int_{t_{i-1}}^{t} \int_{t_{i-1}}^{v} dw_s \cdot b_v^{[1]} dw_v \right] dt \right\} + R\left(n^{-0.5}\right) \\
&= \sum_{i=1}^{n} \frac{\sqrt{2}n}{\sqrt{\theta}\sigma_{t_{i-1}}} \left\{ b_{t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} dw_s dt \right. \\
&\quad \left. + b_{t_{i-1}}^{[1]} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} ds dt \right\} + R\left(n^{-0.5}\right) \quad , \qquad (26)
\end{aligned}
$$

and similarly,

$$\sqrt{n}\Phi_2 = \sum_{i=1}^{n} \frac{\sqrt{2}nb_{t_{i-1}}}{\sqrt{\theta}\sigma_{t_{i-1}}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} ds dw_t + R\left(n^{-0.5}\right) \quad . \tag{27}$$

From (27) and (26), one has

$$\begin{aligned}\sqrt{n}\Phi_2 + \sqrt{n}\Phi_3 = &\sum_{i=1}^{n} \frac{\sqrt{2}b_{t_{i-1}}}{\sqrt{\theta}\sigma_{t_{i-1}}}(w_{t_i} - w_{t_{i-1}}) \\ &+ \sum_{i=1}^{n} \frac{\sqrt{2}b^{[1]}_{t_{i-1}}}{2\sqrt{\theta}\sigma_{t_{i-1}}} \cdot \frac{1}{n} + R\left(n^{-0.5}\right) \quad .\end{aligned} \tag{28}$$

Obviously,

$$\sqrt{n}\Phi_4 = \sum_{i=1}^{n} \frac{\sqrt{2}b^2_{t_{i-1}}}{2\theta\sigma^2_{t_{i-1}}} \cdot \frac{1}{n} + R\left(n^{-0.5}\right) \quad . \tag{29}$$

On the other hand, Ψ is decomposed as follows:

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 \quad , \tag{30}$$

where

$$\Psi_1 = \sum_{i=1}^{n} \frac{\sqrt{n}}{\sqrt{2}} \left((\Delta w_i)^2 - \frac{1}{n} \right) \quad ,$$

$$\Psi_2 = \sum_{i=1}^{n} \frac{\sqrt{2n}}{\sigma_{t_{i-1}}} \Delta w_i \int_{t_{i-1}}^{t_i} (\sigma_t - \sigma_{t_{i-1}}) dw_t \quad ,$$

and

$$\Psi_3 = \sum_{i=1}^{n} \frac{\sqrt{n}}{\sqrt{2}\sigma^2_{t_{i-1}}} \left(\int_{t_{i-1}}^{t_i} (\sigma_t - \sigma_{t_{i-1}}) dw_t \right)^2 \quad .$$

By repeated use of Burkholder's inequality, we have

$$\begin{aligned}\sqrt{n}\Psi_3 &= \sum_{i=1}^{n} \frac{n}{\sqrt{2}\sigma^2_{t_{i-1}}} \left(\int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \sigma^{[1]}_s dw_s dw_t \right)^2 + R\left(n^{-0.5}\right) \\ &= \sum_{i=1}^{n} \frac{n}{\sqrt{2}\sigma^2_{t_{i-1}}} \int_{t_{i-1}}^{t_i} \left(\int_{t_{i-1}}^{t} \sigma^{[1]}_s dw_s \right)^2 dt + R\left(n^{-0.5}\right) \\ &= \sum_{i=1}^{n} \frac{n}{\sqrt{2}\sigma^2_{t_{i-1}}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} \left(\sigma^{[1]}_s \right)^2 ds dt + R\left(n^{-0.5}\right) \\ &= \sum_{i=1}^{n} \frac{1}{2\sqrt{2}} \cdot \frac{\left(\sigma^{[1]}_{t_{i-1}} \right)^2}{\sigma^2_{t_{i-1}}} \cdot \frac{1}{n} + R\left(n^{-0.5}\right) \quad .\end{aligned} \tag{31}$$

Moreover, in a similar fashion, by using Itô's formula, we obtain

$$\sqrt{n}\Psi_2 = \sum_{i=1}^{n} \frac{\sqrt{2}\sigma_{t_{i-1}}^{[1]}}{\sigma_{t_{i-1}}} \cdot n\Delta w_i \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t} dw_s dw_t + \sum_{i=1}^{n} \frac{\sqrt{2}\sigma_{t_{i-1}}^{[0]}}{2\sigma_{t_{i-1}}} \cdot \frac{1}{n} + R\left(n^{-0.5}\right) \quad . \tag{32}$$

From (29), (28), (31), and (32), we obtain the desired expansion. ◊

Let $\bar{\tilde{F}}_{n,t} = \tilde{F}_{\frac{[nt]}{n}}$. We will use the following notation later:

$$\bar{M}_{j,n}(t) := \sum_{i=1}^{[nt]} \frac{1}{\sqrt{n}} G_j(t_{i-1}) H_j(\sqrt{n}\Delta w_i), \quad j = 1, 2, 3 \ ,$$

where

$$G_1(t) = \frac{\sqrt{2}\sigma_t^{[1]}}{\sigma_t} + \frac{\sqrt{2}b_t}{\sqrt{\theta}\sigma_t}, \quad G_2(t) = 1 \ ,$$

$$G_3(t) = \frac{\sqrt{3}\sigma_t^{[1]}}{\sigma_t} \cdot \ \bar{F}_n(t) = \sum_{i=1}^{[nt]} \frac{1}{n} F_{t_{i-1}} \ .$$

$$\bar{M}_n^{\xi}(t) = \sum_{i=1}^{[nt]} 4n\sqrt{n} \int_{t_{i-1}}^{t_i} (t_i - u)(w_u - w_{t_i-1}) dw_u \ .$$

It is easy to show that $\xi_n = \bar{M}_n^{\xi}(1)$ [here

$$M_{n,t} := \sum_{i=1}^{n} 2\sqrt{n} \int_{t\wedge t_{i-1}}^{t\wedge t_i} \int_{t_{i-1}}^{s} dw_u dw_s$$

and ξ_n is defined with the original $(\tilde{F}_t)$-bracket $\langle M_{n,\cdot}\rangle$]. Then the predictable quadratic covariations with respect to the filtration $\left(\bar{\tilde{F}}_{n,t}\right)_{0\le t\le 1}$ are given by

$$\langle \bar{M}_{j,n}, \bar{M}_{k,n} \rangle_t = \delta_{j,k} \sum_{i=1}^{[nt]} \frac{1}{n} G_j(t_{i-1}) G_k(t_{i-1}) =: \bar{A}_{j,k,n}(t) \ ,$$

for $\bar{w}_{n,t} = w_{\frac{[nt]}{n}}$,

$$\langle \bar{w}_n, \bar{w}_n \rangle_t = \frac{[nt]}{n} =: \bar{C}_n(t) \ ,$$

$$\langle \bar{w}_n, \bar{M}_{j,n} \rangle_t = \delta_{1,j} \sum_{i=1}^{[nt]} \frac{1}{n} G_1(t_{i-1}) =: \bar{B}_{j,n}(t) \ ,$$

$$\bar{D}_{j,n}(t) = \langle \bar{M}_{j,n}, \bar{M}_n^{\xi} \rangle_t \ ,$$

$$\bar{E}_n(t) = \langle \bar{w}_n, \bar{M}_n^{\xi} \rangle_t, \quad \bar{G}_n(t) = \langle \bar{M}_n^{\xi} \rangle_t \ .$$

Lemma 14 *The following relations hold for the predictable quadratic covariations* $\bar{E}_n(t), \bar{D}_{j,n}(t)$ and $\bar{G}_n(t)$: $\bar{E}_n(t) = 0, \bar{D}_{1,n}(t) = 0$, $\bar{D}_{2,n}(t) = \frac{2\sqrt{2}}{3} \cdot \frac{[nt]}{n}$, $\bar{D}_{3,n}(t) = 0$ *and* $\bar{G}_n(t) = \frac{4}{3} \cdot \frac{[nt]}{n}$.

Proof. First,

$$\langle \bar{w}_n, \bar{M}_n^{\xi} \rangle_t = \sum_{i=1}^{[nt]} 4n\sqrt{n} \int_{t_{i-1}}^{t_i} E[(t_i - u)(w_u - w_{t_i-1})]du = 0 \ .$$

Next,

$$\langle \bar{M}_{j,n}, \bar{M}_n^{\xi} \rangle_t = 4n\sqrt{n} \sum_{i=1}^{[nt]} E\Bigg[\int_{t_{i-1}}^{t_i} (t_i - u)(w_u - w_{t_{i-1}}) \cdot \frac{1}{\sqrt{n}} G_j(t_{i-1}) \sqrt{j!} \sqrt{n}^j I_{j-1,i}(u) du | \tilde{F}_{t_{i-1}} \Bigg] \ ,$$

where $I_{0,i}(u) = 1$ and

$$I_{j-1,i}(u_{j-1}) = \int_{t_{i-1}}^{u_{j-1}} \cdots \int_{t_{i-1}}^{u_1} dw_{u_1} \cdots dw_{u_{j-1}} \ .$$

It is then not difficult to obtain $\bar{D}_{j,n}(t)$; and similarly $\bar{G}_n(t)$. ◊

We need the following notation.

$$\begin{aligned} A_{j,k}(t) &= \delta_{j,k} \int_0^t G_j(s) G_k(s) ds \ , \\ B_j(t) &= \delta_{1,j} \int_0^t G_1(s) ds, \quad C(t) = t \ , \\ D_j(t) &= \delta_{2,j} \frac{2\sqrt{2}}{3} t \ , \\ E(t) &= 0, \quad G(t) = \frac{4}{3} t, \quad H(t) = \int_0^t F_s ds \ . \end{aligned}$$

In order to identify the limit distributions, we shall adopt the method used in Genon-Catalot and Jacod [7] for first order asymptotics of estimators, while it may probably be possible to simplify to some extent the proof of Lemma 15 and the first part of that of Lemma 16 if we use the latest theorem presented by Jacod [13]. Define τ_n by:

$$\tau_n = (X_0, \bar{w}_n, \bar{F}_n, (\bar{A}_{j,k,n}), (\bar{B}_{j,n}), \bar{C}_n, (\bar{D}_{j,n}), \bar{E}_n, \bar{G}_n, (\bar{M}_{j,n}), \bar{M}_n^{\xi}) \ .$$

τ_n is a random element taking values in

$$\bar{\Omega} = D([0,1] \to \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{3\times 3} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R})$$

(X_0 is embedded in the first argument as constant functions). Denote the canonical variable on $\bar{\Omega}$ by

$$\tau = (\bar{x}, \bar{w}, \bar{F}, (\bar{A}_{j,k}), (\bar{B}_j), \bar{C}, (\bar{D}_j), \bar{E}, \bar{G}, (\bar{M}_j), \bar{M}^{\xi}) \ .$$

Clearly, $\tau_n^1 := (X_0, \bar{w}_n, \bar{F}_n, (\bar{A}_{j,k,n}), (\bar{B}_{j,n}), \bar{C}_n, (\bar{D}_{j,n}), \bar{E}_n, \bar{G}_n)$ converges in distribution to $(X_0, w, H, (A_{j,k}), (B_j), C, (D_j), E, G)$; hence $(\tau_n^1 \colon n \in \mathbb{N})$ is C-tight. Since $(\bar{A}_{j,j,n} \colon n \in \mathbb{N})$ is C-tight, $(\bar{M}_{j,n} \colon n \in \mathbb{N})$ is tight. It is not difficult to show that for any positive ϵ,

$$P(\sup_{t \in [0,1]} |\Delta \bar{M}_{j,n}(t)| > \epsilon) = O\left(\frac{1}{\sqrt{n}}\right) .$$

Therefore, $(\bar{M}_{j,n} \colon n \in \mathbb{N})$ is C-tight (Jacod-Shiryaev [14], p. 315, Proposition 3.26). Similarly, $(\bar{M}_n^{\xi} \colon n \in \mathbb{N})$ is also C-tight. Consequently, $(\tau_n \colon n \in \mathbb{N})$ is C-tight (Jacod-Shiryaev [14], p. 317, Corollary 3.33).

We shall show the uniqueness of the weak limit point and identify this limit. Without loss of generality, we may assume that the sequence $(\tau_n \colon n \in \mathbb{N})$ converges weakly to a probability distribution $\bar{P}$ on $\bar{\Omega}$. Let $(\tilde{F}_t)_{0 \le t \le 1}$ be the filtration generated by τ.

Lemma 15 *On the stochastic basis* $(\bar{\Omega}, \bar{\tilde{F}}_1, (\bar{\tilde{F}}_t)_{0 \le t \le 1}, \bar{P})$, *the canonical processes* $\bar{w}, \bar{M}_j, \bar{M}^{\xi}$ *are continuous square-integrable martingales with predictable quadratic covariations:* $\langle \bar{M}_j, \bar{M}_k \rangle_t = \bar{A}_{j,k}(t)$, $\langle \bar{w}, \bar{M}_j \rangle_t = \bar{B}_j(t)$, $\langle \bar{w}, \bar{w} \rangle_t = \bar{C}(t)$, $\langle \bar{M}_j, \bar{M}^{\xi} \rangle_t = \bar{D}_j(t)$, $\langle \bar{w}, \bar{M}^{\xi} \rangle_t = \bar{E}(t)$, *and* $\langle \bar{M}^{\xi}, \bar{M}^{\xi} \rangle_t = \bar{G}(t)$.

Proof. Put $m(\tau) = \bar{M}_j(\tau)\bar{M}_k(\tau) - \bar{A}_{j,k}(\tau)$. The mapping $\bar{\Omega} \ni \tau \mapsto (\tau, m(\tau)) \in \bar{\Omega}'$ is continuous with respect to the Skorohod topology, where $\bar{\Omega}' = D([0,1] \to \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R})$. Therefore $Y_n := (\tau_n, m(\tau_n)) \to^d Y := (\tau, m(\tau))$ (under $\bar{P}$). Since $(m(\tau_n)_t \colon t \in [0,1], n \in \mathbb{N})$ is uniformly integrable, it follows from Lemma 14 and Proposition 1.12 of Jacod-Shiryaev [14], p. 484, that $m(\tau)$ is a martingale with respect to the filtration generated by Y, hence by τ. (Let $M = m \circ p_1$ with $p_1 \colon (\tau, m') \mapsto \tau$. Then $M_t \circ Y^n = m(\tau_n)_t$, and put $M^n = m(\tau_n)$.) This means that $\langle \bar{M}_j, \bar{M}_k \rangle_t = \bar{A}_{j,k}(t)$. In a similar way, we obtain other martingale properties and the quadratic covariations. ◊

Lemma 16 (1) $\bar{P}$ *is uniquely determined.*

(2) $E[\xi | Z = z] = \bar{D}_2(1)z$ *and* $E[\eta | Z = z] = E^P[H(1)]$.

Proof. Enlarge the stochastic basis $(\bar{\Omega}, \bar{\tilde{F}}_1, (\bar{\tilde{F}}_t)_{0 \le t \le 1}, \bar{P})$ to $(\bar{\Omega}^1, \bar{\tilde{F}}_1^1, (\bar{\tilde{F}}_t^1)_{0 \le t \le 1}, \bar{P}^1)$ on which there exist mutually independent Wiener processes $\bar{Z}_i$ $(i = 1, 3, \xi)$ independent of $(\bar{x}, \bar{w}, \bar{M}_j, \bar{M}^{\xi})$. Let $(\bar{\tilde{F}}_t^2)_{0 \le t \le 1}$ be the filtration generated by $(\bar{x}_0, \bar{w})$. Note that

$$\begin{aligned} &L\{(\bar{x}, \bar{w}, \bar{F}, (\bar{A}_{j,k}), (\bar{B}_j), \bar{C}, (\bar{D}_j), \bar{E}, \bar{G}) | \bar{P}\} \\ &= L\{(X_0, w, H, (A_{j,k}), (B_j), C, (D_j), E, G) | P\} \ , \end{aligned}$$

and that $\bar{B}_1$ is absolutely continuous $\bar{P}$-a.s. for which an adapted (with respect to $\bar{\tilde{F}}_t^2$) derivative $d\bar{B}_1/dt$ is well-defined $\bar{P}$-a.s. on $\bar{\Omega}$. Let $\bar{M}_1' = \bar{M}_1 - d\bar{B}_1/dt \cdot \bar{w}$ and let $\bar{M}_{\xi}' = \bar{M}^{\xi} - d\bar{D}_2/dt \cdot \bar{M}_2$. Then $\bar{w}, \bar{M}_1', \bar{M}_2, \bar{M}_3, \bar{M}_{\xi}'$ are mutuallly orthogonal $(\tilde{F}_t)_{0 \le t \le 1}$-martingales. Next, for $i = 1, 3, \xi$, let

$$\bar{M}_i'' = 1_{\left\{\frac{d\langle \bar{M}_i'\rangle}{dt}\neq 0\right\}}\left(\frac{d\langle \bar{M}_i'\rangle}{dt}\right)^{-\frac{1}{2}}\cdot \bar{M}_i' + 1_{\left\{\frac{d\langle \bar{M}_i'\rangle}{dt}=0\right\}}\cdot \bar{Z}_i$$

with $\bar{M}_3' = \bar{M}_3$. Clearly, $(\bar{w}, \bar{M}_1'', \bar{M}_2, \bar{M}_3'', \bar{M}_\xi'')$ are independent $(\bar{\bar{F}}_t^1)$-Wiener processes. Furthermore, by these Wiener processes, $M_1, \bar{M}_3$ and $\bar{M}^\xi$ are expressed as:

$$\bar{M}_1 = \frac{d\bar{B}_1}{dt}\cdot \bar{w} + \left(\frac{d\langle \bar{M}_1'\rangle}{dt}\right)^{\frac{1}{2}}\cdot \bar{M}_1'' \ ,$$

$$\bar{M}_3 = \left(\frac{d\langle \bar{M}_3\rangle}{dt}\right)^{\frac{1}{2}}\cdot \bar{M}_3'' \ ,$$

and

$$\bar{M}^\xi = \frac{d\bar{D}_2}{dt}\cdot \bar{M}_2 + \left(\frac{d\langle \bar{M}_\xi'\rangle}{dt}\right)^{\frac{1}{2}}\cdot \bar{M}_\xi'' \ .$$

From the facts that $(\bar{M}_1'', \bar{M}_2, \bar{M}_3'', \bar{M}_\xi'')$ are independent of $\bar{\bar{F}}_1^2$, and that $\frac{d\bar{B}_1}{dt}, \frac{d\langle \bar{M}_1'\rangle}{dt}, \frac{d\langle \bar{M}_3\rangle}{dt}, \frac{d\bar{D}_2}{dt}$ and $\frac{d\langle \bar{M}_\xi'\rangle}{dt}$ are $\bar{\bar{F}}_1^2$-measurable by assumption, we see that conditionally on $\bar{\bar{F}}_1^2$, the processes $(\frac{d\langle \bar{M}_1'\rangle}{dt})^{\frac{1}{2}}\cdot\ \bar{M}_1''$, $(\frac{d\langle \bar{M}_3\rangle}{dt})^{\frac{1}{2}}\cdot\ \bar{M}_3''$, $\frac{d\bar{D}_2}{dt}\cdot\ \bar{M}_2$ and $(\frac{d\langle \bar{M}_\xi'\rangle}{dt})^{\frac{1}{2}}\cdot\ \bar{M}_\xi''$ are independent continuous Gaussian processes with quadratic variations $\langle \bar{M}_1'\rangle, \langle \bar{M}_3\rangle$, $\int_0^\cdot \left(\frac{d\bar{D}_2}{dt}\right)^2 dt$ and $\langle \bar{M}_\xi'\rangle$, respectively. Therefore, $\bar{P}$ is uniquely determined.

As seen above, $\bar{M}_2$ is independent of $(\bar{x}, \bar{w}, \bar{M}_1, \bar{M}_3, \bar{F})$ under $\bar{P}$. We may take $Z = \bar{M}_2(1), \xi = \bar{M}^\xi(1)$ and $\eta = \bar{M}_3(1) + \bar{M}_1(1) + \bar{F}(1)$. Therefore,

$$E[\eta|Z=z] = E^{\bar{P}}[\bar{F}(1)] = E^P[H(1)] \ ;$$

and

$$\begin{aligned} E[\xi|Z=z] &= E^{\bar{P}}[\bar{M}^\xi(1)|\bar{M}_2(1)=z] \\ &= \bar{D}_2(1)z. \quad \diamondsuit \end{aligned}$$

Define an orthonormal basis $(h_k)\subset H$ such that $\dot{h}_k(t) = \sqrt{n}1_{(t_{k-1},t_k]}(t)$ for $k=1,...,n$. The H-derivative of $H_j(\sqrt{n}(w_{t_i}-w_{t_{i-1}}))$ is given by $\sqrt{j}H_{j-1}(\sqrt{n}(w_{t_i}-w_{t_{i-1}}))\delta_{k,i}$. Consequently, the H-derivatives of M_n and N_n are uniformly bounded with respect to each $\|\cdot\|_{p,s}$-norm. It is easy to verify the nondegeneracy of the Malliavin covariance of M_n under a suitable truncation. Thus we can apply Theorem 2 to $\mathscr{X}_n$. Put $\alpha = \frac{2\sqrt{2}}{3}$ and $\beta = E^P[H(1)]$.

Theorem 9 *Let θ denote the true value of the unknown parameter. Then*

$$\sup_{x\in\mathbb{R}}\left|P\left(\frac{\sqrt{n}}{\sqrt{2}\theta}(\hat{\theta}_n-\theta)\le x\right) - P_n(x)\right| = o\left(\frac{1}{\sqrt{n}}\right) \ ,$$

where

$$P_n(x) = \Phi(x) + \frac{1}{\sqrt{n}}\left(\frac{1}{2}\alpha(1-x^2) - \beta\right)\phi(x) \ .$$

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