Probab. Theory Relat. Fields 116, 457–484 (2000)

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Malliavin calculus, geometric mixing, and expansion of diffusion functionals

Received: 7 September 1997 / Revised version: 17 March 1999

Abstract. Under geometric mixing condition, we presented asymptotic expansion of the distribution of an additive functional of a Markov or an ϵ-Markov process with finite autoregression including Markov type semimartingales and time series models with discrete time parameter. The emphasis is put on the use of the Malliavin calculus in place of the conditional type Cramér condition, whose verification is in most case not easy for continuous time processes without such an infinite dimensional approach. In the second part, by means of the perturbation method and the operational calculus, we proved the geometric mixing property for *non*-symmetric diffusion processes, and presented a sufficient condition which is easily checked in practice. Accordingly, we obtained asymptotic expansion of diffusion functionals and proved the validity of it under mild conditions, e.g., without the strong contractivity condition.

1. Introduction

In the asymptotic statistical theory, after studies of the first-order asymptotics, the asymptotic expansion is a promising tool to investigate the higher-order performance of statistics used for statistical inference, and thorough investigations have been made mainly for independent cases; see e.g. the monograph by Ghosh [6]. As for dependent data, the work of Götze and Hipp [7] was a breakthrough: they gave an asymptotic expansion of the distribution of an additive functional of a discrete-time process under the geometric mixing condition and a conditional type of Cramér condition. To execute their program, checking the conditional type of Cramér condition is not a simple matter, and they successively in [8], presented sufficient conditions for time series models. The reason of the difficulty is that it is nothing but the problem of regularity of the distribution of a random variable, and it is in many cases a difficult problem, unlike in independent observation cases, to prove the regularity of the distribution from a structural definition of the random variable, e.g., a solution of a stochastic difference/differential equation. Here we are aiming at expansions for stochastic processes with continuous-time parameter such as semimartingales. In this case, the regularity part inevitably requires an infinite-dimensional argument, and really, it is possible if we make use of the

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Mathematics Subject Classification (1991): 60H07, 60F05, 60J25, 62E20

Key words and phrases: Malliavin calculus – mixing – asymptotic expansion – ϵ-Markov process – diffusion – semimartingale

Malliavin calculus over the Wiener space or, if necessary, that over more abstract space including the Wiener-Poisson space to treat jump processes.

In the first part, as an underlying process with geometric mixing condition, we will deal with a somewhat abstract ϵ-Markov process driven by another process with independent increments, and present an asymptotic expansion of the distribution of an additive functional defined with those processes; we indeed encounter such functionals in most statistical applications. By considering an ϵ-Markov process, more general than a Markov process, it is possible to treat time series models with discrete time parameter together in a unified framework because the adopted general Malliavin calculus by Bichteler, Gravereaux and Jacod [2] is available even for such processes. The conditional type Cramér condition is replaced by the nondegeneracy condition of the Malliavin covariance of the functional, and it can be verified for practical models appearing in applications; indeed, we can apply the Hörmander condition to diffusion models, and a set of conditions in Bichteler et al. [2] to a stochastic differential equation with jumps.

In order to apply the first part, it is necessary to verify the naive geometric mixing condition. In the second part, we will confine our attention to diffusion processes, and present a sufficient condition which is easily checked by looking at the coefficient vector fields of the stochastic differential equation. It has been known that the geometric mixing condition holds for certain symmetric diffusions, cf. Stroock [14], Doukham [4], Roberts and Tweedie [11]. If one considers a symmetric diffusion process, then by using properties of a compact self-adjoint operator, the mixing condition is obtained because of the existence of the spectral gap. For nonsymmetric diffusions, we cannot follow this plot, but by using the perturbation method and the operational calculus, we can still prove the geometric mixing property. The reader will observe that the Malliavin calculus (or hypoellipticity argument) also works implicitly in the fundamental level of our discussion in the second part.

Finally, combining the first and second parts, with the help of a result at hand on the nondegeneracy of the Malliavin covariance of the diffusion process, we will provide a sufficient condition which is easy to verify since it has replaced the original two technically difficult conditions, i.e., the geometric mixing condition and the conditional type Cramér condition, by an easily checked condition written with a dual generator, and a nondegeneracy condition of the Lie algebra of vector fields.

The organization of the present article is as follows. In Section 2, we will give the definition of the ϵ-Markov model and examples. Section 3 presents fundamentals of the Malliavin calculus for jump processes. In Section 4, under the geometric mixing condition, we will present asymptotic expansion for functionals of ϵ-Markov processes in two cases with different Malliavin operators. Also as examples, we discuss applications to an ARMA(p,q) process and a semimartingale satisfying a stochastic differential equation with jumps. In Section 5, we will confine our attention to diffusion processes, and provide a result on the geometric mixing property under a set of mild, easily verifiable conditions. It should be noted that we there treat general *non*-symmetric diffusion processes, and certain functional analytic techniques are used in the proof. After that, we will present the asymptotic expansion of diffusion functionals by combining the results there and in Section 4. In Section 6, the expansion for a functional having a stochastic expansion will be presented. Most of statistics have such a stochastic expansion; thus it provides us with a basis of higher-order statistical inference for stochastic processes. Finally in Section 7, we will give proofs of our results.

2. ϵ-Markov model

In order to treat generalized Markov chains with discrete time parameter and Markov processes with continuous time parameter in a unified way, we will consider the following ϵ-Markov process.

Given a probability space $(\Omega, \mathscr{F}, P)$, let $Y = (Y_t)_{t\in\mathbf{R}_+} : \Omega \times \mathbf{R}_+ \to \mathbf{R}^{d_2}$ denote a cadlag process (or a separable process), and $X = (X_t)_{t\in\mathbf{R}_+}$ a d_1-dimensional cadlag process with independent increments, i.e., $\mathscr{B}^{X,Y}_{[0,r]}$ is independent of $\mathscr{B}^{dX}_{[r,\infty)}$ for $r \in \mathbf{R}_+$, where

$$\mathscr{B}^{X,Y}_{[0,r]} = \sigma[X_u, Y_u : u \in [0,r]] \vee \mathscr{N}$$

and $\mathscr{B}^{dX}_I = \sigma[X_t - X_s : s, t \in I \cap \mathbf{R}_+] \vee \mathscr{N}$, $I \subset \mathbf{R}$, $\mathscr{N}$ being the σ-field generated by null sets. Define sub σ-fields $\mathscr{B}^Y_I$, $\mathscr{B}_I$ of $\mathscr{F}$ by $\mathscr{B}^Y_I = \sigma[Y_t : t \in I \cap \mathbf{R}_+] \vee \mathscr{N}$ and by $\mathscr{B}_I = \sigma[X_t - X_s, Y_t : s, t \in I \cap \mathbf{R}_+] \vee \mathscr{N}$. Assume that, for some fixed $\epsilon \geq 0$, the process Y is an ϵ-Markov process driven by X; more precisely, we assume that for an $\epsilon \in \mathbf{R}_+$,

$$Y_t \in \mathscr{F}\left(\mathscr{B}^Y_{[s-\epsilon,s]} \vee \mathscr{B}^{dX}_{[s,t]}\right)$$

for $\epsilon \leq s \leq t$. Clearly, $\mathscr{B}^Y_{[s-\epsilon,t]} \subset \mathscr{B}^Y_{[s-\epsilon,s]} \vee \mathscr{B}^{dX}_{[s,t]}$.

In this paper, we are interested in the asymptotic expansion of the distribution of the normalized additive functional $T^{-1/2}Z_T$, where $Z = (Z_t)_{t\in\mathbf{R}_+}$ is an $\mathbf{R}^d$-valued process satisfying $Z_0 \in \mathscr{F}\mathscr{B}_{[0]}$ and

$$Z^s_t := Z_t - Z_s \in \mathscr{F}\mathscr{B}_{[s,t]}$$

for every $s, t \in \mathbf{R}_+$, $0 \leq s \leq t$.

For a sub σ-field $\mathscr{G}$ of $\mathscr{F}$, $B\mathscr{G}$ denotes the set of all bounded $\mathscr{G}$-measurable functions. In order to derive asymptotic expansions, we will consider the situation where the following two conditions hold true:

[A1] There exists a positive constant a such that

$$\|P_{\mathscr{B}^Y_{[s-\epsilon,s]}}[f] - P[f]\|_{L^1(P)} \leq a^{-1}e^{-a(t-s)}\|f\|_\infty$$

for any $s, t \in \mathbf{R}_+$, $s \leq t$, and for any $f \in B\mathscr{B}^Y_{[t,\infty)}$.

[A2] For any $\Delta > 0$, $\sup_{t\in\mathbf{R}_+, 0\leq h\leq\Delta} \|Z^t_{t+h}\|_{L^p(P)} < \infty$ for any $p > 1$, and $P[Z^t_{t+\Delta}] = 0$. Moreover, $Z_0 \in \cap_{p>1} L^p(P)$ and $P[Z_0] = 0$.

Example 1. Let $\{Y_n\}_{n\in\mathbf{Z}_+}$ be an m-Markov chain (non-linear time series model) taking values in $\mathbf{R}^{d_2}$ satisfying the stochastic equation

$$Y_n = S_n(Y_{n-1}, \dots, Y_{n-m}, \xi_n),\ n \geq m\ , \tag{1}$$

where $\{\xi_n\}_{n\geq m}$ is an independent sequence taking values in $\mathbf{R}^{d_1}$ and independent of $\{Y_n\}_{n=0}^{m-1}$. Let $Z_n = \sum_{j=1}^{n} f_j(Y_j, \xi_j)$ and $X_n = \sum_{j=1}^{n} \xi_j$. Clearly, it is possible to embed the process $\{X_n, Y_n, Z_n\}_{n\in\mathbf{Z}_+}$ into a process $\{X_t, Y_t, Z_t\}_{t\in\mathbf{R}_+}$ with continuous time parameter as $X_t = X_{[t]}$, $Y_t = Y_{[t]}$ and $Z_t = Z_{[t]}$. Then Y is an $(m-1)$-Markov process driven by the process X with independent increments.

Example 2. Let us consider a stochastic process $\{Y_t, Z_t\}_{t\in\mathbf{R}_+}$ defined as a strong solution of the following stochastic integral equation with jumps:

$$\begin{aligned} Y_t &= Y_0 + A(Y_-) * t + B(Y_-) * w_t + C(Y_-) * \tilde{\mu}_t \\ Z_t &= Z_0 + A'(Y_-) * t + B'(Y_-) * w_t + C'(Y_-) * \tilde{\mu}_t\ , \end{aligned} \tag{2}$$

where Z_0 is $\sigma[Y_0]$-measurable, $A \in C^\infty(\mathbf{R}^{d_2}; \mathbf{R}^{d_2})$, $B \in C^\infty(\mathbf{R}^{d_2}; \mathbf{R}^{d_2} \otimes \mathbf{R}^m)$, $C \in C^\infty(\mathbf{R}^{d_2} \times E; \mathbf{R}^{d_2})$, and similarly, $A' \in C^\infty(\mathbf{R}^{d_2}; \mathbf{R}^d)$, $B' \in C^\infty(\mathbf{R}^{d_2}; \mathbf{R}^d \otimes \mathbf{R}^m)$, $C' \in C^\infty(\mathbf{R}^{d_2} \times E; \mathbf{R}^d)$, where w is an m-dimensional Wiener process, E is an open set in $\mathbf{R}^b$, and $\tilde{\mu}$ is a compensated Poisson random measure on $\mathbf{R}_+ \times E$ with intensity $dt \otimes \lambda(dx)$, λ being the Lebesgue measure on E. Under usual regularity conditions, (Y_t, Z_t) can be regarded as smooth functionals over the canonical space $\Omega = \{(y_0, w, \mu)\}$, where μ denotes the integer-valued random measure on $\mathbf{R}_+ \times E$. For details, see III.6 and IV.10 of Bichteler et al. [2]. Denote by $\mathscr{F}$ the σ-field generated by the canonical maps on Ω. The process X_t may in this case be taken as $X_t = (w_t, \mu_t(g_i); i \in \mathbf{N})$, where (g_i) is a countable measure determining family over E; see Remark *1*. In this case, Y is a Markov process, i.e., $\epsilon = 0$, driven by X with independent increments.

3. Malliavin calculus

To ensure the regularity of distributions, we will use the nondegeneracy of the Malliavin covariance in place of the conditional type Cramér condition. We here adopted the formulation of the Malliavin calculus by Bichteler et al. [2] in view of semimartingales with jumps.

Let $(\Omega, \mathscr{B}, \Pi)$ be a probability space. A linear operator $\mathscr{L}$ on $\mathscr{D}(\mathscr{L}) \subset \cap_{p>1} L^p(\Pi)$ into $\cap_{p>1} L^p(\Pi)$ is called a *Malliavin operator* if the following conditions are satisfied:

(1) $\mathscr{B}$ is generated by $\mathscr{D}(\mathscr{L})$.
(2) For $f \in C^2_\uparrow(\mathbf{R}^n)$, $n \in \mathbf{N}$, and $F \in \mathscr{D}(\mathscr{L})^n$, $f \circ F \in \mathscr{D}(\mathscr{L})$.
(3) For any $F, G \in \mathscr{D}(\mathscr{L})$, $E^\Pi[F\mathscr{L}G] = E^\Pi[G\mathscr{L}F]$.
(4) For $F \in \mathscr{D}(\mathscr{L})$, $\mathscr{L}(F^2) \geq 2F\mathscr{L}F$. In other words, the bilinear operator Γ on $\mathscr{D}(\mathscr{L}) \times \mathscr{D}(\mathscr{L})$ associated with $\mathscr{L}$ by $\Gamma(F, G) = \mathscr{L}(FG) - F\mathscr{L}G - G\mathscr{L}F$ is nonnegative definite.
(5) For $F = (F^1, \dots, F^n) \in \mathscr{D}(\mathscr{L})^n$, $n \in \mathbf{N}$, and $f \in C^2_\uparrow(\mathbf{R}^n)$,

$$\mathscr{L}(f \circ F) = \sum_{i=1}^{n} \partial_i f \circ F \mathscr{L} F^i + \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j f \circ F \Gamma(F^i, F^j) \ .$$

Fix a Malliavin operator $(\mathscr{L}, \mathscr{D}(\mathscr{L}))$. For $p \geq 2$, define $\|F\|_{D_{2,p}}$ by

$$\|F\|_{D_{2,p}} = \|F\|_p + \|\mathscr{L}F\|_p + \|\Gamma^{\frac{1}{2}}(F, F)\|_p \ .$$

Let $D_{2,p}$ denote the completion of $\mathscr{D}(\mathscr{L})$ with respect to $\|\cdot\|_{D_{2,p}}$. Then $(D_{2,p}, \|\cdot\|_{D_{2,p}})$ is a Banach space, and there are inclusions:

$$\begin{array}{ccc} D_{2,p} & \subset & L_p \\ \cup & & \cup \\ D_{2,q} & \subset & L_q \end{array}$$

for $2 \leq p \leq q$. The existence of a Malliavin operator leads us to the existence of an integration-by-parts setting (IBPS). Let $D_{2,\infty-} = \cap_{p\geq 2} D_{2,p}$. Then by Theorem 8-18 of [2] p. 107, we have the following IBP formula (with truncation).

Proposition 1. (1) *$\mathscr{L}$ is extended uniquely to an operator (say $\mathscr{L}$) on $D_{2,\infty-}$, and the operator $(\mathscr{L}, D_{2,\infty-})$ is a Malliavin operator. In particular, $D_{2,\infty-}$ is an algebra.*
(2) *There exists an IBPS: for $f \in C^2_{\uparrow}(\mathbf{R}^d)$, $F \in D_{2,\infty-}(\mathbf{R}^d) \equiv (D_{2,\infty-})^d$ and $\psi \in D_{2,\infty-}$,*

$$E^{\Pi}\left[\sum_{i=1}^{d} \partial_i f(F) \sigma_F^{i,j} \psi\right] = E^{\Pi}\left[f(F) T_F^j(\psi)\right]$$

for $j = 1, \cdots, d$, where

$$\sigma_F^{i,j} = \Gamma(F^i, F^j) \ ,$$

and

$$T_F^j(\psi) = -2\psi \mathscr{L} F^j - \Gamma(\psi, F^j) \ .$$

(3) *Let $\Delta \equiv \Delta_F = \det \sigma_F$, $\sigma_F = (\sigma_F^{i,j})_{i,j=1}^d$. $\sigma_{[i,i']}$ denotes the (i, i')-cofactor of σ_F. Suppose that $F \in D_{2,\infty-}(\mathbf{R}^d)$ and that $\Delta \cdot \Delta^{-1}\psi = \psi$ a.s., i.e., $\Delta = 0 \Rightarrow \Delta^{-1}\psi = 0$ a.s.: this implicitly means that $\psi = 0$ a.s. on $\{\Delta = 0\}$ since $\Delta^{-1} = \infty$ on it. If $\sigma_F^{i,j} \in D^2_{\infty-}$ and $\Delta^{-1}\psi \in D_{2,\infty-}$, then for $f \in C^2_{\uparrow}(\mathbf{R}^d)$,*

$$E^{\Pi}[\partial_i f(F)\psi] = E^{\Pi}\left[f(F) \mathscr{I}_i^F \psi\right] \ ,$$

where the operator $\mathscr{I}_i^F : \{\psi : \Theta \to \bar{\mathbf{R}}$ such that $\Delta^{-1}\psi \in D_{2,\infty-}\} \to \cap_{p>1} L_p(\Pi)$ is defined by

$$\begin{aligned} \mathscr{I}_i^F \psi &= \sum_{i'=1}^{d} T_F^{i'}(\Delta^{-1}\psi\sigma_{[i,i']}) \\ &= -\sum_{i'=1}^{d} \left\{2\Delta^{-1}\psi\sigma_{[i,i']}\mathscr{L}F^{i'} + \Gamma\left(\Delta^{-1}\psi\sigma_{[i,i']}, F^{i'}\right)\right\} \ . \end{aligned}$$

For $k \in \mathbf{N}$, define $S'_k[F]$ and $S''_k[\psi]$ as follows:
$S'_1[F] := \{\sigma_F^{i,j} : i, j = 1, \dots, d\}$ if $F \in D_{2,\infty-}(\mathbf{R}^d)$;
$S'_k[F] := \{\sigma_F^{i,j}, \mathscr{L}F^i, S'_{k-1}[F], \Gamma(S'_{k-1}[F], F^i) : i, j = 1, \dots, d\}$ if $F \in D_{2,\infty-}(\mathbf{R}^d)$ and $S'_{k-1}[F] \subset D_{2,\infty-}$;
$S''_1[\psi; F] := \{\Delta^{-1}\psi\}$ if $\Delta = 0$ implies $\Delta^{-1}\psi = 0$;
$S''_k[\psi; F] := \{\Delta^{-1}S''_{k-1}[\psi; F], \Delta^{-1}\Gamma(S''_{k-1}[\psi; F], F^i) : i = 1, \dots, d\}$ if $S''_{k-1}[\psi; F] \subset D_{2,\infty-}$ and if $\Delta = 0$ implies $\Delta^{-1}S''_{k-1}[\psi; F] \cup \Delta^{-1}$ $\Gamma(S''_{k-1}[\psi; F], F) = \{0\}$.
Put
$S_1[\psi; F] := S'_1[F] \cup S''_1[\psi; F]$ if $F \in D_{2,\infty-}(\mathbf{R}^d)$ and if $\Delta = 0$ implies $\Delta^{-1}\psi = 0$;
$S_k[\psi; F] := S_{k-1}[\psi; F] \cup S'_k[F] \cup S''_k[\psi; F]$ if $F \in D_{2,\infty-}(\mathbf{R}^d)$, $S'_{k-1}[F] \subset D_{2,\infty-}$ and $S''_{k-1}[\psi; F] \subset D_{2,\infty-}$, and if $\Delta = 0$ implies $\Delta^{-1}S''_{k-1}[\psi; F] \cup \Delta^{-1}\Gamma(S''_{k-1}[\psi; F], F) = \{0\}$. Here we denoted $\Gamma(A, B) = \{\Gamma(a, b) : a \in A, b \in B\}$ for function sets A and B, and denoted Δ_F simply by Δ.

Proposition 2. *Suppose that $F \in D_{2,\infty-}(\mathbf{R}^d)$. If $S_k[\psi; F] \subset D_{2,\infty-}$, then for $f \in C_\uparrow^{k+1}(\mathbf{R}^d)$,*

$$E^\Pi\left[\partial_{i_1}\partial_{i_2}\dots\partial_{i_k} f(F)\psi\right] = E^\Pi\left[f(F)\mathscr{I}_{i_k}^F \dots \mathscr{I}_{i_2}^F \mathscr{I}_{i_1}^F \psi\right] .$$

4. Asymptotic expansion for the functional Z_T

Let τ denote a fixed positive constant satisfying $\tau > \epsilon$. Suppose that for each $T > 0$, $u(j)$ and $v(j)$ are sequences of real numbers such that $\epsilon \le u(1) \le u(1) + \tau \le v(1) \le u(2) \le u(2) + \tau \le v(2) \le \dots$, and that $\sup_{j,T}\{v(j) - u(j)\} < \infty$. Let $I_j = [u(j) - \epsilon, u(j)]$ and $J_j = [v(j) - \epsilon, v(j)]$. Suppose that for each $T \in \mathbf{R}_+$, $n(T) \in \mathbf{N}$ and that $v(n(T)) \le T$. Let $Z_j = Z_{v(j)}^{u(j)}$ for $j = 1, 2, \dots, n(T)$.[1]

The r-th cumulant $\chi_{T,r}(u)$ of $T^{-1/2}Z_T$ is defined by

$$\chi_{T,r}(u) = \left(\frac{d}{d\epsilon}\right)_0^r \log P[\exp(i\epsilon u \cdot T^{-1/2}Z_T)] .$$

Next, define functions $\tilde{P}_{T,r}(u)$ by the formal Taylor expansion:

$$\exp\left(\sum_{r=2}^{\infty} r!^{-1}\epsilon^{r-2}\chi_{T,r}(u)\right) = \exp\left(\frac{1}{2}\chi_{T,2}(u)\right) + \sum_{r=1}^{\infty} \epsilon^r T^{-r/2}\tilde{P}_{T,r}(u) . \tag{3}$$

Let $\hat{\Psi}_{T,k}(u)$ be the k-th partial sum of the right-hand side of (3) with $\epsilon = 1$:

$$\hat{\Psi}_{T,k}(u) = \exp\left(\frac{1}{2}\chi_{T,2}(u)\right) + \sum_{r=1}^{k} T^{-r/2}\tilde{P}_{T,r}(u) .$$

[1] An abusive use of "Z": Z_j is not Z_T at $T = j$.

Finally, for $T > 0$ and $k \in \mathbf{N}$, a signed measure $\Psi_{T,k}$ is defined as the Fourier inversion of $\hat{\Psi}_{T,k}(u)$. In the sequel, we will assume that the second cumulant $\chi_{T,2}(u)$ converges to a negative definite quadratic form $-u'\Sigma u$ as $T \to \infty$. Fix a symmetric matrix Σ^o satisfying $\Sigma < \Sigma^o$.

Theorem 1 below is rather for processes with finite range dependency than for ϵ-Markov processes; Theorem 2 is suitable for them. However, the method used in the proof of Theorem 2 is essentially the same as that of Theorem 1, which is rather simpler than Theorem 2. Another connection is explained in Remark *4* after the proof of Theorem 2 in Section 7.

Let

$$F = f(X_{u_k} - X_{u_{k-1}}, Y_{u_k}; X_{v_l} - X_{v_{l-1}}, Y_{v_l}; 1 \le k \le m, 1 \le l \le n) \ , \tag{4}$$

where $u(j) - \epsilon \le u_0 \le \cdots \le u_m \le u(j)$, $v(j) - \epsilon \le v_0 \le \cdots \le v_n \le v(j)$, $m, n \in \mathbf{N}$, and $f \in C_B^\infty(\mathbf{R}^{(m+n)(d_1+d_2)} \to \mathbf{R})$. Let $(\mathscr{L}_j)_{j=1,2,\cdots,n(T)}$ be a family of Malliavin operators, each $\mathscr{L}_j$ being defined over $(\Omega, \mathscr{B}_{[u(j)-\epsilon,v(j)]}, P)$, and suppose that for every $j = 1, 2, \ldots, n(T)$, $X_t^{(i)} - X_{u(j)-\epsilon}^{(i)}, Y_t^{(i)} \in \mathscr{D}(\mathscr{L}_j)$ for $t \in [u(j)-\epsilon, v(j)]$, hence $F \in \mathscr{D}(\mathscr{L}_j)$, and suppose that $\mathscr{L}_j F = 0$. The measurable function $\psi_j : (\Omega, \mathscr{B}_{[u(j)-\epsilon,v(j)]}) \to ([0,1], \mathbf{B}([0,1]))$ denotes a truncation functional. Put

$$S_{1,j} = \{\Delta_{Z_j}^{-1}\psi_j, \sigma_{Z_j}^{kl}, \mathscr{L}_j Z_{j,k}, \Gamma_{\mathscr{L}_j}(\sigma_{Z_j}^{kl}, Z_{j,m}), \Gamma_{\mathscr{L}_j}(\Delta_{Z_j}^{-1}\psi_j, Z_{j,l})\}$$

corresponding to $\mathscr{L}_j$. Let $\mathscr{E}(M,\gamma) = \{f : \mathbf{R}^d \to \mathbf{R}$, measurable, $|f(x)| \le M(1+|x|)^\gamma \ (x \in \mathbf{R}^d)\}$. $\phi(x;\mu,\Sigma)$ is the density function of the normal distribution with mean μ and covariance matrix Σ. The sequences $\{u(j), v(j)\}$, $\{\mathscr{L}_j\}$ and $\{\psi_j\}$ may depend on T. We will assume

[A3] (i) $\inf_{j,T} P[\psi_j] > 0$;

(ii) $\liminf_{T\to\infty} n(T)/T > 0$;

(iii) $Z_j \in (D_{2,\infty-}^{\mathscr{L}_j})^d$, $S_1[\psi_j; Z_j] \subset D_{2,\infty-}^{\mathscr{L}_j}$, and $\cup_{\substack{j=1,\ldots,n(T)\\ T>0}} S_{1,j}$ is bounded in $L^p(P)$ for any $p > 1$.

Theorem 1. *Let $k \in \mathbf{N}$, and let $M, \gamma, K > 0$. Suppose that Conditions* [A1], [A2] *and* [A3] *are satisfied. Then there exist constants $\delta > 0$ and $c > 0$ such that for $f \in \mathscr{E}(M,\gamma)$,*

$$\left| P\left[f(\frac{Z_T}{\sqrt{T}})\right] - \Psi_{T,k}[f]\right| \le c\omega(f, T^{-K}) + \epsilon_T^{(k)} \ ,$$

where

$$\omega(f,r) = \int_{\mathbf{R}^d} \sup\{|f(x+y) - f(x)| : |y| \le r\}\phi(x;0,\Sigma^o)dx$$

and $\epsilon_T^{(k)} = o(T^{-(k+\delta)/2})$ uniformly in $\mathscr{E}(M,\gamma)$.

4.1. Process with finite autoregression

Suppose that a sequence $\{u(j), v(j)\}$ is given as before. The process considered here is a process with finite autoregression; more precisely, we assume that for each interval $J_j = [v(j) - \epsilon, v(j)]$, there exists a finite number of functionals $\mathscr{Y}_j = \{\mathscr{Y}_{j,k}\}_{k=1,\ldots,M_j}$ such that $\sigma[\mathscr{Y}_j] =: \mathscr{B}'_{J_j} \subset \mathscr{B}_{J_j}$ and $P_{\mathscr{B}_{[0,v(j)]}} = P_{\mathscr{B}'_{J_j}}$ on $B\mathscr{B}_{[v(j),\infty)}$. For each j, let $(L_j, \mathscr{D}(L_j))$ denote a Malliavin operator over $(\Omega, \mathscr{B}_{[u(j)-\epsilon,v(j)]}, P)$. Here we do not assume that $L_j F$ vanishes for functionals F of the form of (4); contrarily, we will assume that for any $f \in C_B^\infty(\mathbf{R}^{(d_1+d_2)m})$ and any $u_0, u_1, \ldots, u_m$ satisfying $u(j) - \epsilon \le u_0 \le u_1 \le \cdots \le u_m \le u(j)$, the functional $F = f(X_{u_k} - X_{u_{k-1}}, Y_{u_k} : 1 \le k \le m) \in D_{2,\infty-}^{L_j}$ and $L_j F = 0$. Let $\sigma_{\mathscr{Z}_j}$ be the Malliavin covariance matrix of $\mathscr{Z}_j = (Z_j, \mathscr{Y}_j)$, and suppose that $Z_{j,l}, \mathscr{Y}_{j,k}, \sigma_{\mathscr{Z}_j}^{pq} \in D_{2,\infty-}^{L_j}$, where $Z_j = (Z_{j,l})$. Suppose $\sup_{j,T} M_j < \infty$.

ψ_j denotes a truncation functional defined on $(\Omega, \mathscr{B}_{[u(j)-\epsilon,v(j)]}, P)$.
As before, let

$$S_{1,j} = \{\Delta_{\mathscr{Z}_j}^{-1}\psi_j, \sigma_{\mathscr{Z}_j}^{kl}, L_j\mathscr{Z}_{j,k}, \Gamma_{L_j}(\sigma_{\mathscr{Z}_j}^{kl}, \mathscr{Z}_{j,m}), \Gamma_{L_j}(\Delta_{\mathscr{Z}_j}^{-1}\psi_j, \mathscr{Z}_{j,l})\}$$

for operator L_j.
[A3$'$] (i) $\inf_{j,T} P[\psi_j] > 0$;
(ii) $\liminf_{T\to\infty} n(T)/T > 0$;
(iii) $\mathscr{Z}_j \in (D_{2,\infty-}^{L_j})^{d+M_j}$, $S_1[\psi_j; \mathscr{Z}_j] \subset D_{2,\infty-}^{L_j}$, and $\cup_{\substack{j=1,\cdots,n(T)\\ T>0}} S_{1,j}$ is bounded in $L^p(P)$ for any $p > 1$.

Theorem 2. *Let $k \in \mathbf{N}$. Suppose that Conditions* [A1], [A2] *and* [A3$'$] *are satisfied. Then the same inequality as Theorem* 1 *holds true.*

Remark 1. We may take $d_1 = \infty$ if necessary. The proofs do not change except for minor modifications even in this case; thus we can treat Poisson random measures as the input process X.

Models in Example 1 and 2 satisfy the finite autoregression condition.

Example 1$'$. (Continuation of Example 1) Assume that the driving process $\xi_t = \tilde{X}_t$ is an $\mathbf{R}^{d_1}$-valued i.i.d. sequence with smooth density w and Y_t is defined by (1) with $m = 1$. Taking $\Omega = \{(y_0, (x_i)_{i\in\mathbf{N}}); y_0 \in \mathbf{R}^{d_2}, x_i \in \mathbf{R}^{d_1}\}$ and $\mathscr{B}_{[j-1,j]} = \sigma[Y_{j-1}, \tilde{X}_j](= \sigma[Y_{j-1}, Y_j, \tilde{X}_j])$, the j-th Malliavin operator L_j over $(\Omega, \mathscr{B}_{[j-1,j]}, P)$ is defined by

$$\mathscr{D}(L_j) = \{f = f(Y_{j-1}(y_0, x_1, \ldots, x_{j-1}), x_j); f \in C_\uparrow^2(\mathbf{R}^{d_1+d_2})\}$$

and

$$L_j f = \frac{1}{2}\rho(x_j)\Delta_{x_j} f + \frac{1}{2}w(x_j)^{-1}\nabla_{x_j}(\rho w) \cdot \nabla_{x_j} f$$

for $f \in \mathscr{D}(L_j)$. ρ is an auxiliary smooth positive function. We may use consecutive noises for x_j, if necessary. As an example, let us consider an ARMA(p,q) process $\{\tilde{Y}_t\}$ which is defined by the equation

$$\phi(B)\tilde{Y}_t = \theta(B)\tilde{X}_t, \quad t \in \mathbf{Z}_+ \ ,$$

where ϕ and θ are polynomials :

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$$

and

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \ ,$$

and B is the backward shift operator: $B\tilde{Y}_t = \tilde{Y}_{t-1}$. It is known that $\tilde{Y}_t$ has a state-space representation as follows (cf. Brockwell and Davis [3], Chapter 12). Let $r = \max\{p, q+1\}$ and $Y_t = (y_{t-r+1}, y_{t-r+2}, \ldots, y_t)'$, and define Y_t so that Y_t satisfies

$$Y_t = \begin{bmatrix} 0 & I_{r-1} \\ \phi_r & \phi_{r-1} \cdots \phi_1 \end{bmatrix} Y_{t-1} + \begin{bmatrix} 0_{r-1} \\ 1 \end{bmatrix} \tilde{X}_t \ ,$$

and

$$\tilde{Y}_t = \left[\theta_{r-1}\ \theta_{r-2} \cdots \theta_0\right] Y_t \ ,$$

where $\phi_j = 0$ for $j > p, \theta_0 = 1$ and $\theta_j = 0$ for $j > q$. The driving process X_t may in this case be taken as $X_t = \sum_{j=1}^{[t]} \tilde{X}_j$. A typical form of Z_t in statistical applications is $Z_t^{t-1} = f_t(Y_t, \tilde{X}_t)$, which is within the present scope. In this example, $\tilde{Y}$ itself is not ϵ-Markov but its functional can be dealt with in our context.

Example 2′. (Continuation of Example 2) The j-th Malliavin operator is defined as follows. Let $1_j = 1_{[u(j),v(j)]\times E}$. The domain $\mathscr{R}_j = \mathscr{D}(L_j)$ is the set of functionals Φ of the form

$$\Phi = F(Y_{u(j)}, w_{t_1} - w_{t_0}, \ldots, w_{t_N} - w_{t_{N-1}}, (1_j\mu)(f_1), \ldots, (1_j\mu)(f_n)) \quad (5)$$

where $u(j) = t_0 \le t_1 \le \cdots t_N \le v(j)$, $f_i \in C^2_{K,v}(\mathbf{R}_+ \times E)$ (continuous functions with compact support, and of class C^2 in the $v \in E$-direction), and $F \in C^2_{\uparrow}(\mathbf{R}^{d_2+Nm+n})$. Clearly, $\mathscr{R}_j$ generates $\mathscr{B}_{[u(j),v(j)]}$. With an auxiliary function $\alpha : E \to \mathbf{R}_+$, we define L_j by

$$L_j\Phi = L_j^{(1)}\Phi + L_j^{(2)}\Phi \ ,$$

where

$$L_j^{(1)}\Phi = \frac{1}{2}\sum_{i=1}^{N} \mathrm{trace}\frac{\partial^2 F}{\partial \mathbf{x}_i^2}(t_i - t_{i-1}) - \frac{1}{2}\sum_{i=1}^{N} \frac{\partial F}{\partial \mathbf{x}_i} \cdot (w_{t_i} - w_{t_{i-1}})$$

and

$$L_j^{(2)}\Phi = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(1_j\mu)\Big(\alpha\Delta_v f_i + (\partial_v\alpha)\cdot\partial_v f_i\Big)$$
$$+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 F}{\partial x_i\partial x_j}(1_j\mu)\Big(\alpha(\partial_v f_i)\cdot(\partial_v f_j)\Big)$$

for $\Phi \in \mathscr{R}_j$ having the form of (5). In this case, the reference variables are given by $\mathscr{Y}_j = Y_{v(j)}$ and $\mathscr{Z}_j = (Z_{v(j)}^{u(j)}, Y_{v(j)})$.

Put $\bar{X}_t = (Y_t, Z_t^{u(j)})$, then (2) is written as

$$\bar{X}_t = \bar{X}_{u(j)} + \bar{A}(\bar{X}_-) * t + \bar{B}(\bar{X}_-) * w_t + \bar{C}(\bar{X}_-) * \tilde{\mu}_t,\ t \in [u(j), v(j)],$$
$$\bar{X}_{u(j)} = (Y_{u(j)}, 0)\ \ .$$

As IV.10 Bichteler et al. [2], let us consider a process U_t^x defined by a stochastic differential equation corresponding to $\bar{X}$ with $\bar{X}_{u(j)} = x$ like (10-4) in [2]. Put $Q_t^x = \det(U_t^x)$ and $t_0 = v(j)$. Assume that there exists an open set S in $\mathbf{R}^{d_2+d}$, $S \cap \{z = 0\} \neq \phi$, on which the mapping $x \mapsto E[|Q_{t_0}^x|^{-p}]$ is locally bounded for any $p > 1$. Then $\bar{X}(t_0, x)$ is nondegenerate uniformly in S in the wide sense. Taking a truncation functional $\psi_j = \Psi(\bar{X}_{u(j)})$ with $\Psi \in C_K^\infty(\mathbf{R}^{d_2+d}; [0,1])$ satisfying supp $\Psi \subset S$, and Int(supp $\Psi)\cap\{z = 0\} \neq \phi$, we can apply Theorem 2 under Condition [A3′](i)–(ii) and the conditions of moments, and hence obtain an asymptotic expansion of $P[f(Z_T/\sqrt{T})]$. For details of this example, see [17].

5. Geometric mixing property of diffusion processes and asymptotic expansion

As seen in the previous section, the geometric mixing condition is a key to obtain asymptotic expansion for functionals of stochastic processes. For a class of symmetric diffusions, this property was proved by using the spectral gap of the compact self-adjoint operator when the elements of the semigroup are of the Hilbert-Schmidt type. See Stroock [14], also Roberts and Tweedie [11]. The aim of this section is to prove that the geometric mixing property holds true for diffusion processes that are not necessarily symmetric.

In this section, we consider a d-dimensional diffusion process X[2] defined as the strong solution of the following stochastic differential equation:

$$dX(t,x) = \sum_{i=1}^{r} V_i(X(t,x)) \circ dw_t^i + V_0(X(t,x))dt$$
$$X(0,x) = x\ \ ,$$

[2] We here use the letter “X” to denote a diffusion process differently from the previous sections, where X stood for a driving process with independent increments.

where $V_i \in C_B^\infty(\mathbf{R}^d;\mathbf{R}^d)$, $V_0 \in C^\infty(\mathbf{R}^d;\mathbf{R}^d)$ with $\nabla V_0 \in C_B^\infty(\mathbf{R}^d;\mathbf{R}^d \otimes \mathbf{R}^d)$, and $w = (w^i)$ is an r-dimensional Wiener process. We assume that
[C1] $Lie[V_1,\ldots,V_r](x) = \mathbf{R}^d$ for all $x \in \mathbf{R}^d$.
Let

$$L = \frac{1}{2}\sum_{i=1}^{r} V_i^2 + V_0 \ .$$

The formal adjoint L^* of L can be written as

$$L^* = \frac{1}{2}\sum_{i=1}^{r} V_i^2 + \tilde{V}_0 + U_0 \ ,$$

where $U_0 \in C_B^\infty(\mathbf{R}^d;\mathbf{R})$ and $\tilde{V}_0 \in C^\infty(\mathbf{R}^d;\mathbf{R}^d)$ is a vector field with $\nabla\tilde{V}_0 \in C_B^\infty(\mathbf{R}^d;\mathbf{R}^d \otimes \mathbf{R}^d)$.

Moreover, we assume
[C2] there exists a function $\rho \in C_B^\infty(\mathbf{R}^d;\mathbf{R})$ such that $\rho > 0$, $\int_{\mathbf{R}^d}\rho(x)dx = 1$ and

$$\limsup_{|x|\to\infty} \rho^{-1}(x)L^*\rho(x) < 0 \ .$$

Let P_t denote the semigroup associated with the operator L. We then have the following theorem:

Theorem 3. *Suppose that Conditions* [C1] *and* [C2] *hold. Then*

(1) *there exists a unique invariant probability measure μ on $\mathbf{R}^d$ corresponding to P_t.*

(2) *μ has a C^∞-density with respect to the Lebesgue measure, and*

$$\sup_{x\in\mathbf{R}^d} \rho(x)^{-1}\frac{d\mu}{dx}(x) < \infty \ .$$

(3) *There exist positive constants λ and C such that*

$$\|P_t f - \int_{\mathbf{R}^d} f d\mu\|_{L^1(\rho dx)} \le Ce^{-\lambda t}\|f\|_{L^1(\rho dx)}$$

for all $f \in C_B(\mathbf{R}^d;\mathbf{R})$.

We are now on the point of combining Theorem 3 with Theorem 2. For a diffusion process X_t satisfying

$$dX_t = \sum_{i=1}^{r} V_i(X_t)\circ dw_t^i + V_0(X_t)dt \ , \tag{6}$$

let Z_t be defined by

$$Z_t = Z_0 + \sum_{i=1}^{r} \int_0^t V_i'(X_s) \circ dw_s^i + \int_0^t V_0'(X_s) ds \ ,$$

where Z_0 is $\sigma[X_0]$-measurable with $Z_0 \in \cap_{p>1} L^p(P)$ and $E[Z_0] = 0$, $V_i' \in C_{\uparrow}^{\infty}(\mathbf{R}^d; \mathbf{R}^{d'})$ and $V_0' \in C_{\uparrow}^{\infty}(\mathbf{R}^d; \mathbf{R}^{d'})$. Moreover, the flow $Z(t, 0)$ is defined by the same equation corresponding to $X(t, x)$. As in Example 2, define the extended diffusion process $\bar{X}(t, x)$ by $\bar{X}(t, x) = (X(t, x), Z(t, 0))$, then it has a representation:

$$d\bar{X}(t, x) = \sum_{i=1}^{r} \bar{V}_i(\bar{X}(t, x)) \circ dw_t^i + \bar{V}_0(\bar{X}(t, x))dt \ .$$

Among several possible sufficient conditions for regularity, the Hörmander condition for the extended process $\bar{X}$ is a practical convenience. For vector fields $V_0, V_1, \dots, V_r$, let $\Sigma_0 = \{V_1, \dots, V_r\}$ and $\Sigma_n = \{[V_\alpha, V]; V \in \Sigma_{n-1}, \alpha = 0, 1, \dots, r\}$ for $n \in \mathbf{N}$. Moreover, $Lie[V_0; V_1, \dots, V_r]$ denotes the linear manifold spanned by $\cup_{n=0}^{\infty} \Sigma_n$. The next theorem uses the following condition:
[C3] There exists an $x \in \mathbf{R}^d$ such that

$$Lie[\bar{V}_0; \bar{V}_1, \dots, \bar{V}_r](x, 0) = \mathbf{R}^{d+d'} \ .$$

By using the relation between the Hörmander condition and the regularity of distributions (cf. Kusuoka-Stroock [10]), we obtain the following theorem.

Theorem 4. *Let X_t be a stationary diffusion process satisfying the stochastic differential equation* (6). *Assume Conditions* [A1] *with $\mathscr{B}_I^X$ for '$\mathscr{B}_I^Y$' (or* [C1], [C2]), [C3] *at an x in the support of the invariant measure and* [A2]. *Then the asymptotic expansion given in Theorem 1 is valid if d is replaced by d'.*

6. Expansion for functionals admitting a stochastic expansion

Estimators for unknown parameter appearing in the statistical inference are not in general a normalized additive functional itself but have a stochastic expansion with the principal part being a normalized additive functional and the higher parts written as functions of the first term and other functionals. When we consider the maximum likelihood estimator, the Bayes estimator, etc., the higher-order terms are a polynomial of normalized additive functionals, while other estimators such as U-statistics need another development of the asymptotic theory. Thus, by the *Delta*-method if necessary, we may without loss of generality consider the expansion corresponding to a sequence of random variables S_T defined by

$$S_T = \bar{Z}_T^{(0)} + \sum_{i=1}^{k} T^{-\frac{i}{2}} Q_i(\bar{Z}_T^{(0)}, \bar{Z}_T^{(1)}) \ ,$$

where Q_i are $\mathbf{R}^{d^{(0)}}$-valued polynomials, $\bar{Z}_T^{(j)} = T^{-1/2} Z_T^{(j)}$, $j = 0, 1$, and $Z_T := (Z_T^{(0)}, Z_T^{(1)})$ is a $d = d^{(0)} + d^{(1)}$-dimensional additive functional satisfying the measurability condition stated in Section 2 for the processes X and Y. Moreover we assume that there exists a finite regressor $\mathscr{Y}_j$ for each interval $J_j = [v(j) - \epsilon, v(j)]$ as Theorem 2. The coefficients of Q_i may depend on T if they are bounded.

Theorem 5. *Let $M, \gamma, K > 0$. Suppose that Conditions* [A1], [A2] *and* [A3′] *hold. Then for any $K \in \mathbf{N}$, there exist smooth functions $q_{j,k,T} : \mathbf{R}^{d^{(0)}} \to \mathbf{R}$ such that $q_{0,k,T} = \phi(\cdot; 0, Cov(\bar{Z}_T^{(0)}))$ and that for some $b > 0$ and $B > 0$,*

$$|q_{j,k,T}(y^{(0)})| \le Be^{-b|y^{(0)}|^2} \quad ,$$

and there exist constants $\delta > 0$ and $c > 0$ such that

$$\left| P[f(S_T)] - \int_{\mathbf{R}^{d^{(0)}}} f(y^{(0)}) \sum_{j=0}^{k} T^{-j/2} q_{j,k,T}(y^{(0)}) dy^{(0)} \right| \le c\omega(f, T^{-K}) + \epsilon_T^{(k)}$$

for any $f \in \mathscr{E}(M, \gamma)$, where $\epsilon^{(k)}$ is a sequence of constants independent of f with $\epsilon^{(k)} = o(T^{-\frac{1}{2}(k+\delta)\wedge K})$.

Remark 2. Sakamoto and Yoshida [12] gave expression to $q_{j,2,T}$, $j = 0, 1, 2$:

$$\begin{aligned}
q_{0,2,T}(y^{(0)}) &= \int_{\mathbf{R}^q} p_{T,2}(y) dy^{(1)}, \\
q_{1,2,T}(y^{(0)}) &= -\partial_a \int_{\mathbf{R}^q} p_{T,1}(y) Q_1^a(y) dy^{(1)}, \\
q_{2,2,T}(y^{(0)}) &= -\partial_a \int_{\mathbf{R}^q} p_{T,0}(y) Q_2^a(y) dy^{(1)} \\
&\quad + \frac{1}{2}\partial_a \partial_b \int_{\mathbf{R}^q} p_{T,0}(y) Q_1^a(y) Q_1^b(y) dy^{(1)} \quad .
\end{aligned} \tag{7}$$

Here $y = (y^{(0)}, y^{(1)})$, $p = d^{(0)}$, $q = d^{(1)}$ and functions $p_{T,j}$, $j = 0, 1, 2$, are defined, with the summation convention, by:

$$\begin{aligned}
p_{T,0}(z) &= \phi(z; 0, \Sigma_T), \\
p_{T,1}(z) &= \phi(z; 0, \Sigma_T)\Big(1 + \frac{1}{6}\lambda^{\alpha\beta\gamma} h_{\alpha\beta\gamma}(z; \Sigma_T)\Big), \\
p_{T,2}(z) &= p_{T,1}(z) + \phi(z; 0, \Sigma_T) \\
&\qquad \Big(\frac{\lambda^{\alpha\beta\gamma\delta}}{24} h_{\alpha\beta\gamma\delta}(z; \Sigma_T) + \frac{\lambda^{\alpha\beta\gamma}\lambda^{\delta\epsilon\sigma}}{72} h_{\alpha\beta\gamma\delta\epsilon\sigma}(z; \Sigma_T)\Big) \quad ,
\end{aligned}$$

where $\Sigma_T = Cov(\bar{Z}_T)$ and the Hermite polynomials $h_{\alpha_1 \cdots \alpha_k}(z; \Sigma_T)$ are defined by

$$h_{\alpha_1 \cdots \alpha_k}(z; \Sigma_T) = (-1)^k \phi(z; 0, \Sigma_T)^{-1} \partial_{\alpha_1} \cdots \partial_{\alpha_k} \phi(z; 0, \Sigma_T) \quad ,$$

and $\lambda^{\alpha_1 \cdots \alpha_k}$ denotes $(\alpha_1 \cdots \alpha_k)$-cumulant of $\bar{Z}_T$. Moreover, it is possible to show that Formulas (7) are valid even when there is a linear relation between the ancillary elements $Z^{(1)}$ if one interprets Formulas (7) with Schwartz distribution theory; thus it extends Theorem 5. Such extension is necessary when we treat the maximum likelihood estimator in the context of the M-estimator, cf. Sakamoto and Yoshida [13]. In [12], they also directly obtained the third order expansion formula for the maximum likelihood estimator for a diffusion process.

Remark 3. The approach adopted in this paper is the “local approach”, which uses the Malliavin calculus over short time intervals. Contrarily, it is also possible to take the “global approach”, which applies the Malliavin calculus directly to functionals defined over a global time interval. The advantage of the global approach was that it can apply in various situations with or without mixing condition or Markovian property; examples are in [15, 16]. However, if those conditions are assumed, the present “local approach” provides a more effective way to the solution and reduces conditions such as the strong contractivity condition as [16].

7. Proofs

Lemma 1. *Suppose that Condition* [A1] *holds true. Then there exists a positive constant a such that*

[A1′] $$\|P_{\mathscr{B}_{[s-\epsilon,s]}}[f]-P[f]\|_{L^1(P)}\le a^{-1}e^{-a(t-s)}\|f\|_\infty$$

for any $s,t\in\mathbf{R}_+$, $s\le t$, and any $f\in B\mathscr{B}_{[t,\infty)}$.

Proof. Since X has independent increments, when $\epsilon\le u\le v$, $P_{\mathscr{B}_{[0,u]}}[C]\in B\mathscr{B}^Y_{[u-\epsilon,u]}$ for every $C\in B\mathscr{B}_{[v,\infty)}$. In particular, for $t\in[\epsilon,\infty)$ and $C\in B\mathscr{B}_{[t,\infty)}$, there exists a measurable $C'\in B\mathscr{B}^Y_{[t-\epsilon,t]}$ such that $\|C'\|_\infty\le\|C\|_\infty$ and that $C'=P_{\mathscr{B}_{[0,t]}}[C]$ a.s. Let $\epsilon\le s\le t-\epsilon$. Then, in the same fashion, we see that $P_{\mathscr{B}_{[0,s]}}[C']\in B\mathscr{B}^Y_{[s-\epsilon,s]}$; hence $P_{\mathscr{B}_{[0,s]}}[C']=P_{\mathscr{B}^Y_{[s-\epsilon,s]}}[C']$ a.s., and it equals $P_{\mathscr{B}_{[s-\epsilon,s]}}[C']$. Therefore, by using Condition [A1], we obtain

$$\left\|P_{\mathscr{B}_{[s-\epsilon,s]}}[C]-P[C]\right\|_{L^1(P)}\le a^{-1}e^{a\epsilon-a(t-s)}\|C\|_\infty.\qquad\square$$

As stated in Remark 3, the approach taken here is the “local approach”. To reduce the estimate of the characteristic function of $T^{-1/2}Z_T$ into those over short time intervals, we will later use the following lemma.

Lemma 2. *Let $(\Omega,\mathscr{F},P)$ be a probability space, and $\{\mathscr{B}_I;\ I\subset\mathbf{R}_+\}$ an increasing family of sub σ-fields of $\mathscr{F}$, i.e., $\mathscr{B}_I\subset\mathscr{B}_J$ if $I\subset J$.*
(1) *Let $u\ge\epsilon$. Suppose that*

$$P_{\mathscr{B}_{[0,u]}}[g]=P_{\mathscr{B}'_{[u-\epsilon,u]}}[g]$$

for any $g\in B\mathscr{B}_{[u,\infty)}$, where $\mathscr{B}'_{[u-\epsilon,u]}$ is a sub σ-field of $\mathscr{B}_{[u-\epsilon,u]}$. Then for $f\in B\mathscr{B}_{[0,u]}$ and $g\in B\mathscr{B}_{[u,\infty)}$,

$$P_{\mathscr{B}_{[u-\epsilon,u]}}[fg]=P_{\mathscr{B}_{[u-\epsilon,u]}}[f]\cdot P_{\mathscr{B}'_{[u-\epsilon,u]}}[g]\ .$$

In particular,

$$P_{\mathscr{B}'_{[u-\epsilon,u]}}[fg]=P_{\mathscr{B}'_{[u-\epsilon,u]}}[f]\cdot P_{\mathscr{B}'_{[u-\epsilon,u]}}[g]\ .$$

(2) *Let* $\epsilon \le u \le v - \epsilon$,*and let* $I = [u-\epsilon, u]$ *and* $J = [v-\epsilon, v]$. *Suppose that* $\mathscr{B}'_{[u-\epsilon,u]} \subset \mathscr{B}_{[u-\epsilon,u]}$ *and* $\mathscr{B}'_{[v-\epsilon,v]} \subset \mathscr{B}_{[v-\epsilon,v]}$ *are sub* σ*-fields, and that*

$$P_{\mathscr{B}_{[0,s]}}[g'] = P_{\mathscr{B}'_{[s-\epsilon,s]}}[g']$$

for all $g' \in B\mathscr{B}_{[s,\infty)}$, $s = u, v$. *Then, for* $f \in B\mathscr{B}_{[0,u]}$, $g \in B\mathscr{B}_{[u,v]}$ *and* $h \in B\mathscr{B}_{[v,\infty)}$,

a) $P_{\mathscr{B}'_I \vee \mathscr{B}'_J}[f] = P_{\mathscr{B}'_I}[f]$ *and* $P_{\mathscr{B}'_I \vee \mathscr{B}'_J}[h] = P_{\mathscr{B}'_J}[h]$;
b) $P_{\mathscr{B}'_I \vee \mathscr{B}'_J}[fh] = P_{\mathscr{B}'_I \vee \mathscr{B}'_J}[f] P_{\mathscr{B}'_I \vee \mathscr{B}'_J}[h] = P_{\mathscr{B}'_I}[f] P_{\mathscr{B}'_J}[h]$;

c) $P_{\mathscr{B}_{[0,u]} \vee \mathscr{B}_{[v,\infty)}}[g] = P_{\mathscr{B}_{[0,u]} \vee \mathscr{B}_{[v,\infty)}}\left[P_{\mathscr{B}'_I \vee \mathscr{B}'_J}[g]\right]$, *or equivalently* $P[fgh] = P[f P_{\mathscr{B}'_I \vee \mathscr{B}'_J}[g] h]$.

Proof. (1) By assumption, one has $P_{\mathscr{B}_{[0,u]}}[fg] = f P_{\mathscr{B}'_{[u-\epsilon,u]}}[g]$. The operator $P_{\mathscr{B}_{[u-\epsilon,u]}}$ yields the result.

(2) For simplicity, we will use P_I for $P_{\mathscr{B}'_I}$, P_J for $P_{\mathscr{B}'_J}$ and $P_{I \vee J}$ for $P_{\mathscr{B}'_I \vee \mathscr{B}'_J}$, respectively.

(a) As for the second part, $P_{I\vee J}[h] = P_{I \vee J}\left[P_{\mathscr{B}_{[0,v]}}[h]\right] = P_J[h]$. Next, for all $i \in B\mathscr{B}'_I$ and $j \in B\mathscr{B}'_J$, (1) implies that $P_I[fij] = i P_I[f] P_I[j] = P_I[ij P_I[f]]$, and hence that $P[ijf] = P[ij P_I[f]]$, and we obtained the first part.

(b) For i, j given above,

$$\begin{aligned} P\left[ij P_{I\vee J}[f] P_{I\vee J}[h]\right] &= P\left[P_{I\vee J}\left[P_{I\vee J}[f] hij\right]\right] \\ &= P\left[ij P_I[f] h\right] \\ &= P\left[P_I[fi] hj\right] \\ &= P\left[P_I[fi] P_I[hj]\right] \\ &= P\left[P_I[fihj]\right] \text{ (by (1))} \\ &= P[ijfh] \ . \end{aligned}$$

Consequently, we have the desired result.

(c) By assumption, we see that $P_I[gh] = P_I[g P_{\mathscr{B}_{[0,v]}}[h]] = P_I[g P_J[h]] = P_I[P_{I\vee J}[g] P_J[h]]$. This together with (1) and (2b) implies that

$$\begin{aligned} P[fgh] &= P[P_I[f] P_I[gh]] \\ &= P[P_I[f] P_{I\vee J}[g] P_J[h]] \\ &= P[P_{I\vee J}[fh] P_{I\vee J}[g]] \\ &= P[fh P_{I\vee J}[g]] \\ &= P\left[fh P_{\mathscr{B}_{[0,u]} \vee \mathscr{B}_{[v,\infty)}}[P_{I\vee J}[g]]\right] \ , \end{aligned}$$

which completes the proof. ◻

Proof of Theorem 1. Let F denote any functional taking the form of (4). Let $\mathscr{B}_j = \mathscr{B}_{I_j} \vee \mathscr{B}_{J_j}$, and $g_j = P_{\mathscr{B}_j}[e^{iu\cdot Z_j}\psi_j]$. We see that $\|\Gamma_{\mathscr{L}_j}(F,F)\|_1 \le 2\|F\|_2\|\mathscr{L}_j F\|_2 = 0$ and hence $|\Gamma_{\mathscr{L}_j}(F, Z_j^k)| \le \Gamma_{\mathscr{L}_j}(F,F)^{1/2}\Gamma_{\mathscr{L}_j}(Z_j^k, Z_j^k)^{1/2} = 0$; therefore, $\Gamma_{\mathscr{L}_j}(F, Z_j^k) = 0$. When $S_1[\psi_j; Z_j] \subset D_{2,\infty-}^{\mathscr{L}_j}$,

$$iu^k P[e^{iu\cdot Z_j}\psi_j F] = P[e^{iu\cdot Z_j}\mathscr{I}_k^{Z_j}(\psi_j F)] \ ,$$

where

$$\begin{aligned}\mathscr{I}_k^{Z_j}(\psi_j F) &= -\sum_{k'=1}^{d}\Big\{2\Delta_{Z_j}^{-1}\psi_j F\sigma_{[k,k']}^{Z_j}\mathscr{L}_j Z_j^{k'} \\ &\qquad + F\Gamma_{\mathscr{L}_j}\left(\Delta_{Z_j}^{-1}\psi_j\sigma_{[k,k']}^{Z_j}, Z_j^{k'}\right)\Big\} \\ &= G_{j,k}F \text{ (say)} \ .\end{aligned}$$

Therefore,

$$|u||P[g_j F]| \le \sum_{k=1}^{d}\|G_{j,k}F\|_{L^1(\Omega,\mathscr{B}_{[u(j)-\epsilon,v(j)]},P)};$$

since the family of F's is dense in $L^p(\Omega,\mathscr{B}_j,P)$, $p>1$,

$$\|g_j\|_q \le |u|^{-1}\sum_{k=1}^{d}\|G_{j,k}\|_q$$

for any $q>1$.

Choose a smooth function $\phi : \mathbf{R}^d \to [0,1]$ so that $\phi(x)=1$ if $|x|\le 1/2$, and $\phi(x)=0$ if $|x|\ge 1$. Let β be a positive constant with $\beta<1/2$. Define a functional Ψ_j depending on T by $\Psi_j = \psi_j\phi(Z_j/T^\beta)$, and let $Z_j^* = Z_j\phi(Z_j/(2T^\beta)) - P[Z_j\phi(Z_j/(2T^\beta))]$. Since $\cup_{j,T}S_{1,j}$ is bounded in $L^p(P)$, $p>1$, we obtain for $g_j = P_{\mathscr{B}_j}[e^{iu\cdot Z_j}\Psi_j]$,

$$\begin{aligned}\sup_j P\left[\left|P_{\mathscr{B}_j}\left[e^{iu\cdot Z_j^*}\right]\right|\right] &\le \sup_j P\left[|g_j|\right] + \sup_j P\left[\left|P_{\mathscr{B}_j}\left[e^{iu\cdot Z_j^*}\left(1-\Psi_j\right)\right]\right|\right] \\ &\le C|u|^{-1} + \sup_j\|1-\Psi_j\|_1 \\ &\le C|u|^{-1} + \sup_j\|1-\psi_j\|_1 + \sup_j P[|Z_j| > T^\beta/2] \\ &\le C|u|^{-1} + 1 - \inf_{j,T}P[\psi_j] + CT^{-\beta} \ ,\end{aligned}$$

where C is a constant independent of u and T. Consequently,

$$\sup_{\substack{j=1,\cdots,n(T)\\ T\ge T_0}} P\left[\left|P_{\mathscr{B}_j}\left[e^{iu\cdot Z_j^*}\right]\right|\right] \le c \tag{8}$$

for $|u|\ge b$, where $c<1$, $b>0$ and $T_0>0$ are some constants.

Fix $e' > 0$ arbitrarily, and let $0 < \nu_1 < e' \wedge 1$. By assumption, we can find $j_1, j_2, \ldots, j_{n'} \in \{1, 2, \ldots, n(T)\}$ such that for large T, $v(j_l) + T^{\nu_1} \le u(j_{l+1})$ for $l = 1, 2, \ldots, n'$, and that $n' \ge BT^{1-\nu_1}$, where B is a positive constant depending only on τ, $\liminf_{T\to\infty} n(T)/T$ and $\tau_1 := \sup_{j,T}\{v(j) - u(j)\}$. Indeed, put $j_1 = 1$ and $j_i = \min\{j; u(j) \ge v(j_{i-1}) + T^{\nu_1}\}$ as far as it can be defined. Then one has

$$n'(T^{\nu_1} + 2\tau_1) \ge n(T)\tau \ ,$$

which yields that for some $B > 0$ and large T, $n' \ge BT^{1-\nu_1}$.

Divide each one of the intervals $[0, u(j_1)], [v(j_1), u(j_2)], \ldots, [v(j_{n'}), T]$ into subintervals with length τ except for the last interval with length at most τ, and call them $I_{0,1}, \ldots, I_{0,k_0}; I_{1,1}, \ldots, I_{1,k_1}; \ldots; I_{n',1}, \ldots, I_{n',k_{n'}}$. Let

$$Z^*_{I_{l,k}} = Z_{I_{l,k}}\phi(Z_{I_{l,k}}/(2T^\beta)) - P[Z_{I_{l,k}}\phi(Z_{I_{l,k}}/(2T^\beta))] \ ,$$

where Z_I denotes Z^s_t for interval $I = [s,t]$. Put $I_{0,0} = [0]$ and define $Z^*_{I_{0,0}}$ similarly for $Z_{I_{0,0}} = Z_0$. Line up the intervals $I_{l,k}$ and $[u(j_l), v(j_l)]$, and call them $T_1, T_2, \ldots, T_S$ from the left. For $k \in \mathbf{Z}_+$, choose any k numbers $s_1, \ldots, s_k$ from $C := \{1, 2, \ldots, S\}$ with replacement. Let

$$C_1 = \{n \in C : T_n = [u(j_l), v(j_l)] \text{ for some } j_l \text{ and } T_n \notin \{T_{s_1}, \ldots, T_{s_k}\}\} \ ,$$

and let $\mathscr{B}'_l = \mathscr{B}_{[\min T_l - \epsilon, \min T_l]} \vee \mathscr{B}_{[\max T_l - \epsilon, \max T_l]}$. We will estimate

$$E[Z^{*(i_1)}_{T_{s_1}} \cdots Z^{*(i_k)}_{T_{s_k}} e^{iu\cdot\tilde{Z}^*_T}] \ ,$$

where $\tilde{Z}^*_T = Z^*_T/\sqrt{T}$, $Z^*_T = \sum_{s=1}^S Z^*_{T_s}$, and $i_1, \ldots, i_k \in \{1, \ldots, d\}$.
For $\mathscr{B}_{[\min T_l - \epsilon, \max T_l]}$-measurable random variables A_l, $l \in C_1 = \{l_1, \ldots, l_{\#C_1}\}$, with $\|A_l\|_\infty \le 1$, we see from Lemma 1 that

$$\begin{aligned}
&|P[\Pi_{l\le l_i} A_l]\Pi_{l\ge l_{i+1}} P[A_l] - P[\Pi_{l\le l_{i-1}} A_l]\Pi_{l\ge l_i} P[A_l]| \\
&\quad = |P[\Pi_{l\le l_{i-1}} A_l\{P_{\mathscr{B}_{[0,\max T_{l_{i-1}}]}}[A_{l_i}] - P[A_{l_i}]\}]\Pi_{l\ge l_{i+1}} P[A_l]| \\
&\quad \le \|P_{\mathscr{B}_{[\max T_{l_{i-1}}-\epsilon, \max T_{l_{i-1}}]}}[A_{l_i}] - P[A_{l_i}]\|_1 \\
&\quad \le a^{-1}e^{-a(T^{\nu_1}-\epsilon)}
\end{aligned}$$

for large T. It follows from Lemma 2 that for some $a > 0$,

$$\begin{aligned}
\left|P[Z^{*(i_1)}_{T_{s_1}} \cdots Z^{*(i_k)}_{T_{s_k}} e^{iu\cdot\tilde{Z}^*_T}]\right| &= \Bigg|P\Big[Z^{*(i_1)}_{T_{s_1}} \cdots Z^{*(i_k)}_{T_{s_k}} \Pi_{l\in C-C_1} e^{iu\cdot\tilde{Z}^*_{T_l}} \\
&\qquad \times \Pi_{l\in C_1} P_{\mathscr{B}'_l}\Big[e^{iu\cdot\tilde{Z}^*_{T_l}}\Big]\Big]\Bigg| \left(\tilde{Z}^*_{T_l} = Z^*_{T_l}/\sqrt{T}\right) \\
&\le 4^k T^{k\beta} P[\Pi_{l\in C_1}|P_{\mathscr{B}'_l}[e^{iu\cdot\tilde{Z}^*_{T_l}}]|] \\
&\le 4^k T^{k\beta}\left\{\Pi_{l\in C_1} P[|P_{\mathscr{B}'_l}[e^{iu\cdot\tilde{Z}^*_{T_l}}]|] + n'a^{-1}e^{-a(T^{\nu_1}-\epsilon)}\right\} \\
&\le 4^k T^{k\beta}(\max\{e^{-b_0|u|^2/T}, c\})^{n'-k} \\
&\qquad + 4^k T^{k\beta} n' a^{-1} e^{-a(T^{\nu_1}-\epsilon)} \\
&\le \delta^{-1} c^{T^{(1-\nu_1)/2}} + \delta^{-1} e^{-T^{e'-\nu_1}} + \delta^{-1} e^{-T^\delta}
\end{aligned}$$

if $|u| > T^{e'}$ and $T > T_0$, where b_0, T_0 and δ are some positive values. Here, in the third inequality, we used Petrov's lemma (Lemma (3.2) of Götze and Hipp [7]).

Put $H_T(u) = P[\exp(iu \cdot T^{-1/2} Z_T^*)]$. From the above inequality, it follows that for every positive c_1, c_2, e and E, there exists a positive constant δ such that

$$|D^\alpha H_T(u)| \leq \delta^{-1} e^{-T^\delta} \tag{9}$$

for $u \in \mathbf{R}^d$, $c_1 T^e \leq |u| \leq c_2 T^E$, and $\alpha \in \mathbf{Z}_+^d$, $|\alpha| \leq k$, where $D^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}$, $D_i = \partial / \partial u^i$, with $\alpha = (\alpha_1, \ldots, \alpha_d)$.

Thus the validity of the asymptotic expansion follows from a continuous version of Theorem 2.8 of Götze and Hipp [7]. In fact, their Condition (2.3) can be immediately checked, and Lemma 1 implies Condition (2.4): for any $e \in \mathscr{B}_{[0,s]}$ and $f \in \mathscr{B}_{[t,\infty)}$ with $\|e\|_\infty \leq 1$ and $\|f\|_\infty \leq 1$,

$$\begin{aligned} |P[ef] - P[e]P[f]| &= \left| P\left[e P_{\mathscr{B}_{[0,s]}} \left[f - P\left[f \right] \right] \right] \right| \\ &\leq \left\| P_{\mathscr{B}_{[0,s]}} \left[f - P\left[f \right] \right] \right\|_{L^1(P)} \\ &= \left\| P_{\mathscr{B}_{[s-\epsilon,s]}} \left[f - P\left[f \right] \right] \right\|_{L^1(P)} \text{ (the proof of Lemma 1)} \\ &\leq a^{-1} \exp\left(-a\left(t - s \right) \right) \quad . \end{aligned}$$

Therefore, it is possible to obtain the same estimate as Lemma (3.33) of Götze and Hipp [7]. Instead of Conditions (2.5) and (2.6) in Götze and Hipp [7], under the present assumptions, we have already had the estimate (9) corresponding to Lemma (3.43) of Götze and Hipp [7]. We then obtain the desired result as they did so from Lemmas (3.33) and (3.43). Jensen [9] gave a good exposition of Götze and Hipp's work. □

Proof of Theorem 2. We assume that $S_1[\psi_j; \mathscr{Z}_j] \subset D_{2,\infty-}^{L_j}$, in particular, $\Delta_{\mathscr{Z}_j}^{-1} \psi_j \in D_{2,\infty-}^{L_j}$, $\Delta_{\mathscr{Z}_j} = \det \sigma_{\mathscr{Z}_j}$. The matrix $(\gamma_{\mathscr{Y}_j}^{mn})$ denotes the inverse matrix of $\sigma_{\mathscr{Y}_j}$. Let $\phi \in D_{2,\infty-}^{L_j}$ and assume that $\phi \det \sigma_{\mathscr{Y}_j}^{-1} \in D_{2,\infty-}^{L_j}$. Then

$$\phi \Gamma_{L_j}(Z_{j,l}, F) = \sum_{m,n=1}^{M_j} \phi \Gamma_{L_j}(Z_{j,l}, \mathscr{Y}_{j,m}) \gamma_{\mathscr{Y}_j}^{mn} \Gamma_{L_j}(\mathscr{Y}_{j,n}, F) \tag{10}$$

for functionals F taking the form of

$$F = f(X_{u_k} - X_{u_{k-1}}, Y_{u_k} : 1 \leq k \leq m_1) g(\mathscr{Y}_j)$$

for $f \in C_B^\infty(\mathbf{R}^{(d_1+d_2)m_1})$ and $g \in C_B^\infty(\mathbf{R}^{M_j})$. The integration-by-parts formula yields

$$\sum_p i u_p P[e^{iu \cdot Z_j} \sigma_{Z_j}^{pq} \phi F] = P[e^{iu \cdot Z_j} \{-2\phi F L_j Z_{j,q} - \Gamma_{L_j}(\phi F, Z_{j,q})\}] \ . \tag{11}$$

Since for $A, B, C \in D^{L_j}_{2,\infty-}$,

$$P[A\Gamma_{L_j}(B, C)] = P[\{-\Gamma_{L_j}(A, B) - 2AL_jB\}C] \ ,$$

we obtain

$$\begin{aligned}\sum_p iu_p P[e^{iu\cdot Z_j}&\Gamma_{L_j}(\mathscr{Y}_{j,k}, Z_{j,p})\gamma^{kl}_{\mathscr{Y}_j}\Gamma_{L_j}(\mathscr{Y}_{j,l}, Z_{j,q})\phi F] \\ &= P[\Gamma_{L_j}(\mathscr{Y}_{j,k}, e^{iu\cdot Z_j})\gamma^{kl}_{\mathscr{Y}_j}\Gamma_{L_j}(\mathscr{Y}_{j,l}, Z_{j,q})\phi F] \\ &= P[e^{iu\cdot Z_j}\{-\Gamma_{L_j}(\mathscr{Y}_{j,k}, \gamma^{kl}_{\mathscr{Y}_j}\Gamma_{L_j}(\mathscr{Y}_{j,l}, Z_{j,q})\phi F) \\ &\quad -2\gamma^{kl}_{\mathscr{Y}_j}\Gamma_{L_j}(\mathscr{Y}_{j,l}, Z_{j,q})\phi F L_j\mathscr{Y}_{j,k}\}] \ . \end{aligned} \tag{12}$$

On $\{\phi > 0\}$, define $\bar{\sigma}_{Z_j}$ by

$$\bar{\sigma}^{pq}_{Z_j} = \sigma^{pq}_{Z_j} - \sum_{k,l}\Gamma_{L_j}(\mathscr{Y}_{j,k}, Z_{j,p})\gamma^{kl}_{\mathscr{Y}_j}\Gamma_{L_j}(\mathscr{Y}_{j,l}, Z_{j,q}) \ .$$

It follows from (11), (12) and (10) that

$$\sum_p iu_p P[e^{iu\cdot Z_j}\bar{\sigma}^{pq}_{Z_j}\phi F] = P[e^{iu\cdot Z_j}\Psi^q_j(\phi)F] \ , \tag{13}$$

where

$$\begin{aligned}\Psi^q_j(\phi) = \sum_{k,l}&\Gamma_{L_j}(\mathscr{Y}_{j,k}, \gamma^{kl}_{\mathscr{Y}_j}\Gamma_{L_j}(\mathscr{Y}_{j,l}, Z_{j,q})\phi) - \Gamma_{L_j}(\phi, Z_{j,q}) \\ &-2\phi L_j Z_{j,q} + 2\sum_{k,l}\gamma^{kl}_{\mathscr{Y}_j}\Gamma(\mathscr{Y}_{j,l}, Z_{j,q})\phi L_j\mathscr{Y}_{j,k} \ .\end{aligned}$$

Since $\det\bar{\sigma}^{-1}_{Z_j} = \det\sigma_{\mathscr{Y}_j}\cdot\det\sigma^{-1}_{\mathscr{Z}_j}$, $\phi' := (\det\bar{\sigma}_{Z_j})^{-1}\bar{\sigma}_{j,[q,s]}\psi_j \in D^{L_j}_{2,\infty-}$, where $\bar{\sigma}_{j,[q,s]}$ is the (q,s)-cofactor of $\bar{\sigma}_{Z_j}$, and $\phi'\det\sigma^{-1}_{\mathscr{Y}_j} \in D^{L_j}_{2,\infty-}$. Substituting ϕ' into ϕ of (13), and summing up, we obtain

$$iu_p P[e^{iu\cdot Z_j}\psi_j F] = P[e^{iu\cdot Z_j}G_{j,p}F] \ ,$$

where

$$G_{j,p} = \sum_q \Psi^q_j((\det\bar{\sigma}_{Z_j})^{-1}\bar{\sigma}_{j,[q,p]}\psi_j) \ .$$

Taking $g_j = P_{\mathscr{B}_{I_j}\vee\mathscr{B}'_{J_j}}[e^{iu\cdot Z_j}\Psi_j]$ and with the help of Lemma 2, it is possible to obtain the result in the same fashion as Theorem 1. □

Remark 4. For L_j, define another operator $(\mathscr{L}_j, \mathscr{D}(\mathscr{L}_j))$ by

$$\mathscr{D}(\mathscr{L}_j) = \{F : F \in \mathscr{D}(L_j), \Gamma_{L_j}(F, F) \in \mathscr{D}(L_j)\}$$

and

$$\mathscr{L}_j F = L_j F - \sum_{k,l=1}^{M_j}\frac{1}{2}\Gamma_{L_j}\left(\Gamma_{L_j}\left(\mathscr{Y}_{j,k}, F\right)\gamma^{kl}_{\mathscr{Y}_j}, \mathscr{Y}_{j,l}\right) - \sum_{k,l=1}^{M_j}\Gamma_{L_j}\left(\mathscr{Y}_{j,k}, F\right)\gamma^{kl}_{\mathscr{Y}_j}L_j\mathscr{Y}_{j,l} \ .$$

Suppose that $\mathscr{Y}_j$ is nondegenerate and $\mathscr{Y}_{j,l}, \sigma^{kl}_{\mathscr{Y}_j} \in \mathscr{D}(L_j)$. Then $(\mathscr{L}_j, \mathscr{D}(\mathscr{L}_j))$ is a Malliavin operator *if* $\mathscr{D}(\mathscr{L}_j)$ generates $\mathscr{B}_{[u(j)-\epsilon, v(j)]}$. It is then also possible to obtain the same result as in Theorem 1 as a corollary of it.

Proof of Theorem 3.
Step 1. Define a stochastic flow $X^*(t,x)$ by the stochastic differential equation

$$dX^*(t,x) = \sum_{i=1}^r V_i(X^*(t,x)) \circ dw_t^i + \tilde{V}_0(X^*(t,x))dt$$
$$X^*(0,x) = x \quad .$$

Under Condition [C1], $X^*(s,x)$ is nondegenerate uniformly in every compact set in $(0,\infty) \times \mathbf{R}^d$.

It follows from Condition [C1] that there exists $p \in C^\infty\left((0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d\right)$ such that

$$P_t f(x) = \int_{\mathbf{R}^d} p(t,x,y) f(y) dy$$

for $f \in C_B(\mathbf{R}^d; \mathbf{R})$. Put

$$P_t^* f(x) = E\left[\exp\left(\int_0^t U_0\left(X^*(s,x)\right) ds\right) f(X^*(t,x))\right]$$

for $f \in C_B(\mathbf{R}^d; \mathbf{R})$. Then, Feynman-Kac formula says that for $u_t(x) = P_t^* f(x)$,

$$\frac{\partial u_t}{\partial t} = L^* u_t, \quad u_0 = f \quad .$$

Since for $f, g \in C_K(\mathbf{R}^d; \mathbf{R})$, with $u_t = P_t^* f$ and $v_t = P_t g$,

$$\int \frac{d}{dt}(u_t v_{T-t}) dx = \int (v_{T-t} L^* u_t - u_t L v_{T-t}) dx = 0 \quad ,$$

$(P_t^* f, g)_{L^2(dx)} = (f, P_t g)_{L^2(dx)}$, and hence we see that

$$P_t^* f(x) = \int_{\mathbf{R}^d} p(t,y,x) f(y) dy$$

for $f \in C_B(\mathbf{R}^d; \mathbf{R})$.

Step 2. Let

$$\epsilon = \left(-\frac{1}{4} \limsup_{|x|\to\infty} \left(\rho^{-1} L^* \rho\right)(x)\right) \wedge 1$$

and let $U_1(x) = -\rho^{-1}(x) L^* \rho(x)$. Then there exists $R > 0$ such that $U_1(x) \geq 2\epsilon$ if $|x| > R$, and $(L^* + U_1)\rho = 0$. The Feynman-Kac formula again yields

$$\rho(x) = E\left[\exp\left(\int_0^t (U_0 + U_1)\left(X^*(s,x)\right) ds\right) \rho\left(X^*(t,x)\right)\right] \quad . \qquad (14)$$

Take a function $\varphi \in C_K^\infty(\mathbf{R}^d)$ satisfying that $0 \le \varphi \le 1$ and that $\varphi(x) = 1$ if $|x| \le R$. Let $U_2(x) = \varphi(x)(U_1(x) - 2\epsilon) \in C_K^\infty(\mathbf{R}^d)$, then

$$\begin{aligned} U_1(x) &= 2\epsilon + (1 - \varphi(x))(U_1(x) - 2\epsilon) + U_2(x) \\ &\ge 2\epsilon + U_2(x) \end{aligned} \tag{15}$$

for $x \in \mathbf{R}^d$, $|x| > R$. Therefore, it follows from (14) and (15) that

$$\rho(x) \ge E\left[\exp\left(2\epsilon t + \int_0^t (U_0 + U_2)\left(X^*(s,x)\right) ds\right) \rho\left(X^*(s,x)\right)\right] . \tag{16}$$

Define $Q_t : C_B(\mathbf{R}^d; \mathbf{R}) \to C_B(\mathbf{R}^d; \mathbf{R})$ by

$$Q_t f(x) = \rho(x)^{-1} E\left[\exp\left(\int_0^t (U_0 + U_2)\left(X^*(s,x)\right) ds\right) (\rho f)\left(X^*(t,x)\right)\right] . \tag{17}$$

Then

$$Q_t \ge 0 \quad \text{and} \quad Q_t 1 \le e^{-2\epsilon t} \tag{18}$$

for $t \in \mathbf{R}_+$.

Put $T_s = \rho^{-1} P_s^* \rho$, $s \in \mathbf{R}_+$. We know that

$$\sup_{\substack{s \in [s_0, s_1] \\ x \in C \\ y \in \mathbf{R}^d}} |\nabla_x^j p(s, y, x)| < \infty \tag{19}$$

for any $0 < s_0 < s_1$ and compact $C \subset \mathbf{R}^d$. [Let $G_x^s = \exp(\int_0^s U_0(X^*(u,x))du)$. Under [C1], for any $k \in \mathbf{N}$, $p(s, y, x)$ can be expressed as

$$\begin{aligned} p(s, y, x) &= E[G_x^s \delta_y(X^*(s,x))] \\ &= E[h_{k,y}(X^*(s,x)) \Psi_k(G_x^s; X^*(s,x))] \ , \end{aligned}$$

where $h_{k,y} : \mathbf{R}^d \to \mathbf{R} \in C_B^k(\mathbf{R}^d)$ with uniformly (in $y \in \mathbf{R}^d$) bounded derivatives up to k-th order, and $\Psi_k(G_x^s; X^*(s,x))$ are certain $L^p(P)$-bounded uniformly in (s, x) over every compact set in $(0, \infty) \times \mathbf{R}^d$. Let $S = [s_0, s_1] \times \mathbf{R}^d \times C$. It is easy to show that $\sup_{(s,y,x) \in S} |y|^i |\nabla_y^j \nabla_x^l \partial_s^m p(s, y, x)| < \infty$ by using $\partial_s p(s, y, x) = L_y p(s, y, x)$.] Consequently, for every bounded set $B \subset C_B(\mathbf{R}^d)$, $f \in T_s(B)$ are equi-continuous on each compact set, and hence $C_s := U_2 \rho^{-1} P_s^* \rho$ is compact since $U_2 \rho^{-1} \in C_K^\infty(\mathbf{R}^d)$. Moreover, $C_s(B)$ is bounded in $C_B^\infty(\mathbf{R}^d)$ equipped with seminorms $\|\nabla^j f\|_\infty$, and supp$f \subset$ suppU_2 for all $f \in C_s(B)$.

Clearly, $\|T_s\|_{op} \le \exp(s(\|U_2\|_\infty - 2\epsilon))$, and $\|C_s\|_{op} \le \|U_2\|_\infty \exp(s(\|U_2\|_\infty - 2\epsilon))$, where $\|\cdot\|_{op}$ is the operator norm on $L(C_B(\mathbf{R}^d) \to C_B(\mathbf{R}^d))$. Again with (19), we see that $C : (0, \infty) \to L(C_B(\mathbf{R}^d) \to C_B(\mathbf{R}^d))$ is continuous with respect to $\|\cdot\|_{op}$.

Let B' be any bounded set in $C_B^2(\mathbf{R}^d)$ such that $\operatorname{supp} f \subset \operatorname{supp} U_2$ for all $f \in B'$. Then, for $f \in B'$ and $0 < s < t$,

$$\begin{aligned}
\|T_s f - T_t f\|_\infty &= \|\rho^{-1} \int (p(s,y,\cdot) - p(t,y,\cdot))\, \rho(y) f(y) dy\|_\infty \\
&= \|\rho^{-1} \int_s^t du \int \frac{\partial p}{\partial u}(u,y,\cdot)\rho(y) f(y) dy\|_\infty \\
&= \|\rho^{-1} \int_s^t du \int p(u,y,\cdot) L^*(\rho f)(y) dy\|_\infty \\
&\le \int_s^t du \|T_u(L^*(\rho f)/\rho)\|_\infty \\
&\le (t-s)\exp(t\|U_2\|_\infty)\|L^*(\rho f)/\rho\|_\infty \ .
\end{aligned}$$

Therefore,

$$\sup_{f\in B'} \|T_s f - T_t f\|_\infty \le C_{B'} \exp(t\|U_2\|_\infty)(t-s) \ . \tag{20}$$

For $n \in \mathbf{N}$ and $s_0, s_1, \ldots, s_n \in (0,\infty)$, define $K_n(s_0, s_1, \ldots, s_n)$ by

$$K_n(s_0, s_1, \ldots, s_n) = T_{s_n} C_{s_{n-1}} \cdots C_{s_0} \ .$$

Then $\|K_n(s_0, s_1, \ldots, s_n)\|_{op} \le c_1^n e^{(n+1)c_2 \max\{s_0,s_1,\ldots,s_n\}}$ for some constants c_1, c_2, and the continuity of C_s and (20) implies that $K_n : (0,\infty)^{n+1} \to L(C_B(\mathbf{R}^d) \to C_B(\mathbf{R}^d))$ is continuous with respect to the operator norm $\|\cdot\|_{op}$. The Riemann sum approximation shows that

$$\tilde{K}_n(s_0) := \int_0^{s_0} ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n K_n(s_0 - s_1, s_1 - s_2, \cdots, s_{n-1} - s_n, s_n)$$

is a compact operator from $C_B(\mathbf{R}^d)$ into $C_B(\mathbf{R}^d)$, and that

$$\|\tilde{K}_n(t)\|_{op} \le \frac{c_1^n e^{(n+1)c_2 t}}{n!} t^n \ .$$

After all, $K_t := \sum_{n=1}^\infty \tilde{K}_n(t)$ is a compact operator from $C_B(\mathbf{R}^d)$ into $C_B(\mathbf{R}^d)$.

From the definition of Q_t, we obtain

$$Q_t = T_t + K_t \ .$$

In fact, for $f \in C_B(\mathbf{R}^d)$,

$$\begin{aligned}
Q_t f(x) &= T_t f(x) + \rho^{-1}(x) \sum_{n=1}^\infty \frac{1}{n!} E\Big[\exp\left(\int_0^t U_0\left(X^*(s,x)\right) ds\right) \\
&\qquad \left(\int_0^t U_2\left(X^*(s,x)\right) ds\right)^n (\rho f)\left(X^*(t,x)\right)\Big] \\
&= T_t f(x) + \sum_{n=1}^\infty \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_n - 1} ds_n \\
&\qquad T_{s_n} C_{s_{n-1}-s_n} \cdots C_{s_1 - s_2} C_{t-s_1} f(x) \\
&= T_t f(x) + K_t f(x) \ .
\end{aligned}$$

Step 3. Hereafter, the operators Q_t, T_t, K_t are regarded as operators on $\mathscr{X} = C_B(\mathbf{R}^d;\mathbf{C})$. Let $T = T_1 = Q_1 - K_1 = \rho^{-1}P_1^*\rho$, then T is a bounded linear operator on $\mathscr{X}$ and $\|T + K_1\|_{op} \le e^{-2\epsilon}$. In the same way as the proof of VIII 8.2, p. 709, of Dunford-Schwartz [5], it is possible to prove

Claim 1. $\sigma(T) \cap \{z \in \mathbf{C}; |z| \ge e^{-\epsilon}\}$ is a finite set, and the dimension of the range $\mathscr{R}(E(z;T))$ of $E(z;T)$ is finite if $z \in \sigma(T)$, $|z| \ge e^{-\epsilon}$.

Claim 2. $\sigma(T) \cap \{z \in \mathbf{C}; |z| \ge 1\} = \{1\}$.
(proof) Since $T^n f = \rho^{-1}P_n^*\rho f$ for $f \in C_B(\mathbf{R}^d)$,

$$\begin{aligned}\int_{\mathbf{R}^d}(T^n f)(x)\rho(x)dx &= \int_{\mathbf{R}^d} P_n^*(\rho f)(x)dx \\ &= \int_{\mathbf{R}^d}\rho(x)f(x)dx \ .\end{aligned}$$

Let $z \in \sigma(T) \cap \{z \in \mathbf{C}; |z| \ge 1\}$. The subspace $\mathscr{X}_z := E(z;T)\mathscr{X}$ of $\mathscr{X}$ is finite-dimensional (Claim 1) and $\mathscr{X}_z$ is invariant by T; therefore, there exists a nonzero vector $f \in \mathscr{X}_z$ for which $Tf = \lambda f$ for some $\lambda \in \mathbf{C}$. By using the Dunford-integral representation of $E(z;T)$, we see that for $f = E(z;T)g$,

$$0 = (\lambda - T)f = (\lambda - T)E(z;T)g = (\lambda - z)E(z;T)g = (\lambda - z)f \ ,$$

and hence $\lambda = z$, after all, $Tf = zf$. Since

$$T(|f|) \ge |Tf| = |z||f| \ , \tag{21}$$

$$\begin{aligned}\int_{\mathbf{R}^d} T(|f|)(x)\rho(x)dx &\ge |z|\int_{\mathbf{R}^d}|f(x)|\rho(x)dx \\ &= |z|\int_{\mathbf{R}^d} T(|f|)(x)\rho(x)dx \ ,\end{aligned}$$

and hence $|z| \le 1$. If $|z| = 1$, then $T(|f|) = |Tf|$, which implies that for some constant $c = c_f \in \mathbf{C}$, $f(x) = c|f(x)|$ for all $x \in \mathbf{R}^d$ since $\mathrm{supp}P_1^*(x,\cdot) = \mathbf{R}^d$.

$$zc|f| = zf = Tf = cT(|f|) = c|Tf| = c|f| \ ;$$

therefore $z = 1$. Thus we obtain

$$\sigma(T) \cap \{z \in \mathbf{C} : |z| \ge 1\} \subset \{1\} \ .$$

If $\sigma(T) \cap \{z \in \mathbf{C}; |z| \ge 1\}$ were void, because of Claim 1, $\sigma(T) \subset \{z \in \mathbf{C}; |z| \le r\}$ for some $r < 1$, and

$$\limsup_{n\to\infty}\|T^n\|_{op}^{1/n} \le r \ .$$

In particular, $\|T^n 1\|_\infty \to 0$ as $n \to \infty$. On the other hand,

$$\int_{\mathbf{R}^d}(T^n 1)(x)\rho(x)dx = \int_{\mathbf{R}^d}\rho(x)dx = 1 \ ,$$

which is a contradiction.

Claim 3. The dimension of $\mathscr{X}_1$ is one.
(proof) Since $\mathscr{X}_1$ is finite dimensional, it follows from the equivalence of norms that there exists a constant C such that

$$\|f\|_\infty \leq C\|f\|_{L^1(\rho dx)},\ f \in \mathscr{X}_1 \ .$$

As

$$\|T^n f\|_\infty \leq C\|T^n|f|\|_{L^1(\rho dx)} = C\||f|\|_{L^1(\rho dx)}$$

for any $f \in \mathscr{X}_1$ and $n \in \mathbf{N}$. Thus we see that

$$\lim_{n\to\infty} \frac{1}{n}\|T^n f\|_\infty = 0 \ .$$

From Theorem 3, VIII. 8, Dunford-Schwartz [5] p. 711 (or the proof of it), and Claim 2, the spectral point 1 is a simple pole. Moreover, Theorem 18, VII. 3, Dunford-Schwartz [5] p. 573, yields that

$$Tf = f \text{ for all } f \in \mathscr{X}_1 \ .$$

Clearly, for any $f \in \mathscr{X}_1$,

$$\|f\|_{L^1(\rho dx)} = \|T^n|f|\|_{L^1(\rho dx)} \geq \|T^n f\|_{L^1(\rho dx)} = \|f\|_{L^1(\rho dx)} \ ,$$

and so $T^n|f| = |T^n f|$, therefore, $f = c_f|f|$ for some constant $c_f \in \mathbf{C}$. This implies that $dim(\mathscr{X}_1) = 1$. Indeed, for any $f, g \in \mathscr{X}_1$ and any $x, y \in \mathbf{R}^d$, $(f(x) + ug(x))/(f(y) + ug(y))$ must be positive for any $u \in \mathbf{R}$, but this means that $f(x)g(y) - f(y)g(x) = 0$.

Step 4. Now we return to the proof of Theorem 3. In view of the last part of the proof of Claim 3, there exists $u \in C_B(\mathbf{R}^d)$ such that $u > 0$, $\int_{\mathbf{R}^d} u(x)\rho(x)dx = 1$ and $Tu = u$. Let $E_1 = E(1; T)$, $\tilde{\mathscr{X}} = \{x \in \mathscr{X};\ E_1 x = 0\}$, and $\tilde{T} = T|_{\tilde{\mathscr{X}}}$; note that $\tilde{T}\tilde{\mathscr{X}} \subset \tilde{\mathscr{X}}$. Put $G(\zeta) = \zeta(1 - g(\zeta))$, where $g(\zeta)$ is an analytic function near $\sigma(T)$, and it equals one near 1 and zero otherwise. By the spectral mapping theorem, $\sigma(T(1 - E_1)) = G(\sigma(T))$. Since $G(1) = 0$, $G(\sigma(T)) \subset \{\zeta; |\zeta| < r\}$ for some $r < 1$. From the fact that for $f \in \tilde{\mathscr{X}}$, $\tilde{T}^n f = T^n f = (T(1 - E_1))^n f$, it holds that

$$\limsup_{n\to\infty} \|\tilde{T}^n\|_{op}^{1/n} < r \ .$$

For $f, g \in C_B(\mathbf{R}^d)$,

$$\begin{aligned}
&\left|\int_{\mathbf{R}^d} (P_n g)(x)f(x)\rho(x)dx - E_1(f)\int_{\mathbf{R}^d} g(x)\rho(x)u(x)dx\right| \\
&\quad = \left|\int_{\mathbf{R}^d} (T^n f)(x)g(x)\rho(x)dx - E_1(f)\int_{\mathbf{R}^d} g(x)\rho(x)u(x)dx\right| \\
&\quad \leq \|T^n - E_1\|_{op}\|f\|_\infty \int_{\mathbf{R}^d} |g(x)\rho(x)|dx \ .
\end{aligned}$$

Define a probability measure μ by $\mu(dx) = \rho(x)u(x)dx$. With $T^n - E_1 = T^n(1-E_1)$, we obtain an estimate

$$\left|\int_{\mathbf{R}^d}(P_n g)(x)f(x)\rho(x)dx - E_1(f)\int_{\mathbf{R}^d} g(x)\mu(dx)\right| \\ \leq Cr^n \|f\|_\infty \|g\|_{L^1(\rho dx)} \ .$$

In particular, substituting $g = 1$ and taking limit, we have

$$E_1(f) = \int_{\mathbf{R}^d} f(x)\rho(x)dx \ .$$

Hence, it follows from the duality that

$$\left\| P_n g - \int_{\mathbf{R}^d} g(x)\mu(dx)\right\|_{L^1(\rho dx)} \leq Cr^n \|g\|_{L^1(\rho dx)} \tag{22}$$

for $g \in C_B(\mathbf{R}^d)$.

For any $t \in \mathbf{R}_+$, $T_t u = u$. [In the same argument, for every positive irrational number α, there exists a $u_\alpha \in C_B(\mathbf{R}^d)$ such that $T_\alpha u_\alpha = u_\alpha$. It is easy to show that $u_\alpha = u$ by using (22) and a similar inequality and the continuity of the semigroup $\{T_t\}$. Then $u_t = u$ also follows from the continuity.] In particular, u is smooth. Finally, the semigroup property $P_t = P_{[t]}P_{t-[t]}$ completes the proof of Theorem 3. □

Proof of Theorem 4. Condition [A3′] follows from Condition [C3] by the same argument as Example 2′; therefore, the assertion is just a corollary of Theorem 2 under Condition [A1]. We shall verify [A1] under Conditions [C1] and [C2] together with the stationarity. First, note that Theorem 3 (3) holds for any bounded measurable function f. Let $f \in B\mathscr{B}^X_{[t,\infty)}$. (The symbol '$X$' here takes the place of 'Y' in the previous sections.) Theorem 3 (2) says that $c := \sup_{x\in\mathbf{R}^d} \rho(x)^{-1} d\mu(x)/dx < \infty$. For $\mathscr{B}^X_{[t]}$-measurable function $P_{\mathscr{B}^X_{[t]}}[f]$, there exists a Borel measurable function $H_t : \mathbf{R}^d \to \mathbf{R}$ such that $P_{\mathscr{B}^X_{[t]}}[f] = H_t(X_t)$ P-a.s. and $\|H_t\|_\infty \leq \|f\|_\infty$. By using the stationarity of X_t and the Markov property $P_{\mathscr{B}^X_{[t]}}[f] = P_{\mathscr{B}^X_{[0,t]}}[f]$ P-a.s., we see that

$$\begin{aligned}
\left\| P_{\mathscr{B}^X_{[s]}}[f] - P[f]\right\|_{L^1(P)} &= \left\| P_{\mathscr{B}^X_{[s]}}[H_t(X_t)] - P[H_t(X_t)]\right\|_{L^1(P)} \\
&= \| P_{t-s}H_t(X_s) - P[H_t(X_t)]\|_{L^1(P)} \\
&= \int_{\mathbf{R}^d} |P_{t-s}H_t(x) - \mu[H_t]|\, d\mu(x) \\
&\leq c\,\| P_{t-s}H_t - \mu[H_t]\|_{L^1(\rho dx)} \\
&\leq cCe^{-\lambda(t-s)}\, \|H_t\|_{L^1(\rho dx)} \\
&\leq cCe^{-\lambda(t-s)}\, \|f\|_\infty \ .
\end{aligned}$$

This completes the proof. □

For convenience of reference, we will give:

Proof of Theorem 5. For $z = (z^{(0)}, z^{(1)})$, define $S_T(z)$ by

$$S_T(z) = z^{(0)} + \sum_{j=1}^{k} T^{-j/2} Q_j(z^{(0)}, z^{(1)}) \ .$$

Let $M_T = \{z \in \mathbf{R}^d | |z| < T^\alpha\}$. Take $\alpha > 0$ sufficiently small so that for some constant C,

$$T^{-j/2} |Q_j^{(i)}|(T^\alpha, T^\alpha) \le C T^{-\alpha}$$

for $j = 1, \ldots, k$, $T > 1$, where $|Q_j^{(i)}|$ denotes the polynomial with coefficients of $Q_j^{(i)}$ replaced by their absolute values. The Bhattacharya-Ghosh map ([1]) is defined by

$$y = \begin{bmatrix} y^{(0)} \\ y^{(1)} \end{bmatrix} = \begin{bmatrix} S_T(z) \\ z^{(1)} \end{bmatrix} \ .$$

Let $f \in \mathscr{E}(M, \gamma)$. Applying Theorem 2 to $f \circ S_T(\cdot) 1_{\{\cdot \in M_T\}}$ and using Condition [A2], we see that, with $d\Psi_{T,k}(z) = p_{k,T} dz$,

$$\begin{aligned} P[f(S_T)] = \int dz f(S_T(z)) 1_{\{z \in M_T\}} p_{k,T}(z) + o(T^{-(k+\delta)/2}) \\ + O(\omega(1_{M_T} f \circ S_T, T^{-K})) \ , \end{aligned} \tag{23}$$

where δ is a positive constant and the small o-term depends on $\mathscr{E}(M, \gamma)$. By definition, there exists a constant C_1 such that $z \in M_T$ implies $|y| \le C_1 T^\alpha$. From the non-degeneracy of the Jacobian, it is easy to see that the mapping $z \to y$ is one-to-one on M_T. Consequently, the first term on the right-hand side of (23) is equal to

$$\int dy f(y^{(0)}) 1_{\{z(y) \in M_T, |y| \le C_1 T^\alpha\}} \left| \tfrac{\partial z}{\partial y}(y) \right| p_{k,T}(z(y)) \tag{24}$$

Put $A_T(z) = y(z) - z$, and let $z_1^* = y - A_T(y)$, $z_2^* = y - A_T(y - A_T(y))$, $z_3^* = y - A_T(y - A_T(y - A_T(y)))$, ... It is then easy to obtain

$$|z_j^* - z(y)| \le T^{-(j+1)/2} \ \times \ \text{(a polynomial of } |z(y)|)$$

and similar estimates for the gradients. Expanding $|\partial z_k^* / \partial y(y)| p_{k,T}(z_k^*(y))$, we have

$$\begin{aligned} \left| \frac{\partial z_k^*}{\partial y}(y) \right| p_{k,T}(z_k^*(y)) = \phi(y; 0, \Sigma_T^Z) \left(1 + \sum_{i=1}^{k} T^{-i/2} q_{i,k,T}^*(y) \right) \\ + T^{-(k+1)/2} R_{k,T}(y) \ , \end{aligned}$$

where $\Sigma_T^Z = Cov(\bar{Z}_T)$, $q^*_{i,k,T}$ are smooth functions of at most polynomial growth order and

$$|R_{k,T}(y)| \leq e^{-c_0|y|^2} \times \text{ (a polynomial of } |y|)$$

for some positive constant c_0. Since for small $c_1 > 0$, $z(y) \in M_T$ if $|y| < c_1 T^\alpha$, it follows from (24) that, taking δ sufficiently small if necessary,

$$\begin{aligned} P[f(S_T)] &= \int dy f(y^{(0)}) \phi(y; 0, \Sigma_T^Z) \left(1 + \sum_{i=1}^k T^{-i/2} q^*_{i,k,T}(y) \right) + o(T^{-(k+\delta)/2}) \\ &\quad + O(\omega(1_{M_T} f \circ S_T, T^{-K})) \\ &= \int dy^{(0)} f(y^{(0)}) \sum_{i=0}^k T^{-i/2} q_{i,k,T}(y^{(0)}) + o(T^{-(k+\delta)/2}) \\ &\quad + O(\omega(1_{M_T} f \circ S_T, T^{-K})) \ , \end{aligned}$$

where $q_{i,k,T}$ are given by

$$\sum_{i=0}^k T^{-i/2} q_{i,k,T}(y^{(0)}) = \int dy^{(1)} \phi(y; 0, \Sigma_T^Z) \left(1 + \sum_{i=1}^k T^{-i/2} q^*_{i,k,T}(y) \right) \ .$$

This completes the proof since one can estimate the last term on the right-hand side with $\omega(1_{M_T}, T^{-K})$ and $\omega(f, T^{-K})$, first taking a different Σ^o if necessary. □

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