Nonsynchronous covariation process and limit theorems

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Abstract: An asymptotic distribution theory of the nonsynchronous covariation process for continuous semimartingales is presented. Two continuous semimartingales are sampled at stopping times in a nonsynchronous manner. Those sampling times possibly depend on the history of the stochastic processes and themselves. The nonsynchronous covariation process converges to the usual quadratic covariation of the semimartingales as the maximum size of the sampling intervals tends to zero. We deal with the case where the limiting variation process of the normalized approximation error is random and prove the convergence to mixed normality, or convergence to a conditional Gaussian martingale. A class of consistent estimators for the asymptotic variation process is proposed based on kernels, which will be useful for statistical applications to high-frequency data analysis in finance. As an illustrative example, a Poisson sampling scheme with random change point is discussed.

Key words: discrete sampling; high-frequency data; martingale central limit theorem; nonsynchronicity; quadratic variation; realized volatility; stable convergence; semimartingale

1 Introduction

Suppose that X and Y are two Itô semimartingales. As obviously known, the simple quadratic form of increments $U(\mathcal{I})_t = \sum_i (X_{s_i} - X_{s_{i-1}})(Y_{s_i} - Y_{s_{i-1}})$ converges in probability to the quadratic covariation $[X,Y]_t$ when s_i are deterministic and $\max\{s_i - s_{i-1}\} \to 0$ along a sequence of partitions $\mathcal{I} = (s_i)$ of the interval $[0,t]$. It is also known that $b_n^{-1}(U(\mathcal{I})_t - [X,Y])$ converges stably to a mixture of Gaussian martingales as $n \to \infty$ for some deterministic saling constants b_n when the sequence $\mathcal{I}$ satisfying certain regularity conditions; for example, the simplest case is $\mathcal{I} = (it/n)_i$.

Two natural questions arise about the weak convergence of such a quadratic form. The first one is "do the same weak convergence takes place when $\mathcal{I}$ consists of stopping times?" The second one is "when the increments $X(s_i) - X(s_{i-1})$ and $Y(t_j) - Y(t_{j-1})$ are given for two different partitions

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$\mathcal{I} = (s_i)$ and $\mathcal{J} = (t_j)$ of $[0, t]$, is it possible to construct a quadratic form $U(\mathcal{I}, \mathcal{J})$ of these increments that performs like $U(\mathcal{I})$, apart from the diagoanl quadratic form?"

As it is a simple matter, the first one is negative in general without putting conditions on the stopping times. Though $U(\mathcal{I})$ seems to be a quadratic form of increments, it is not a real quadratic form because its kernel $\sum_i 1_{(s_{i-1},s_i]}^{\otimes 2}$ is possibly a function of X and Y and then $U(\mathcal{I})$ can possess completely different nature from the quadratic. Really, to discriminate between real quadratic forms and fake ones, we need a strong predictability condition for stopping times. The second question is more serious. It requires a new type functional of nonsynchronous increments. Even if we confine our attention to quadratic forms, the construction of the kernel is not clear. Indeed, the synchronization techniques fail to give a correct kernel; see Hayashi and Yoshida (2005b) for details. The aim of this article is to answer those basic questions. We will consider a quadratic functional of nonsynchronous increments, which was essentially introduced by the authors in the previous work, and prove the stable convergence to a random mixture of Gaussian martingales under standardization, as well as the convergence to the quadratic variation.

Estimation of the covariance structure of the diffusion type process under sampling is one of the fundamental problems in the theory of statistical inference for stochastic processes. This problem had been investigated theoretically by many mathematical statisticians in each setting; there is already a long history of studies but among them we can list (Dohnal (1987), Prakasa Rao (1983), Prakasa Rao (1988), Yoshida (1992), Kessler (1997), Bibby and Sørensen (1995), Genon-Catalot and Jacod (1993)), and also refer the reader to the book by Prakasa Rao (Prakasa Rao (1999)) for more references. The sampling procedures treated so far were the synchronous scheme in the sense that the components of the process are observed on a single sequence of sampling times commonly for all components. The statistic considered there was related to $U(\mathcal{I})$ by the local triviality of the stochastic differential equation and the synchronicity. The synchronous scheme fits well into the standard formulation of stochastic analysis.

Theoretical study of nonsynchronicity seems to have almost been left. Malliavin and Mancino (2002) proposed a Fourier analytic approach to this problem. It is importance work to give theoretical consideration to nonsynchronicity.

The current authors presented a nonsynchronous quadratic form in Hayashi and Yoshida (2005b). [1] This quadratic form is, if it is regarded as a statistical estimator, free from any tuning parameter because it involves no interpolation and no cut-off number of an infinite series. Computation is easy since the number of terms in the summation is of the same order as the number of the increments. Also, it is the maximul likelihood estimator in a basic setting and hence attains high efficiency; see Hoshikawa, Kanatani, Nagai, and Nishiyama (2008). The quadratic type functional we will investigate includes the nonsynchronous covariance estimator. Thus, the limit theorems presented below can apply estimation of covariance structure based on nonsynchronous data.

Our study is aiming at limit theorems which give an essential extension of the theory of statistical inference for stochastic processes, on the stream described above. Though it is just an application, our study would contribute to the recent trend, or revival with new objects, of the covariance estimation problem quite often discussed today as high-frequency data analysis in finance. We will give some comments on this matter in Section 9.

Let us go back to our primary questions. In this article, we will define a quadratic variation process and investigate the asymptotics from probabilistic aspects. To describe dependency of the random sampling schemes, we will associate them with certain point processes, and show the asymptotic

[1] The authors tackled the covariance estimation problem by use of intraday, high-frequency data, where two asset prices are recorded irregularly and nonsynchronously. Such a setup has been known to be problematic; e.g., Epps (1979). According to Google, our estimator is referred to as the Hayashi-Yoshida covariance estimator.

mixed normality, namely, a convergence of the normalized estimation error of the nonsynchronous covariation process to a conditional Gaussian martingale. It should be noted that our treatment of random sampling schemes is new even in the synchronous case of $X = Y$ and $\mathcal{I} = \mathcal{J}$. In Hayashi and Yoshida (2008a), the authors previously proved a CLT for the same statistic when the sampling schemes are independent of the processes X and Y.

Starting with local martingales as the underlying processes, in Section 3, we will give a stochastic-integral representation to the approximation error. Since the quadratic covariation of the representing martingale still involves stochastic integrals, it is optional in this sense. So we consider an approximation by a completely predictable object as Condition [B2]. Once the convergence of the quadratic covariation or the predictable approximation is assumed, it gives us the limit theorem (Propositions 3.1 and 3.3) without any restrictive condition. It works if the sampling scheme is trivial such as hitting times of a simple, particular structure,[2] however it is far from a general solution to the problem.

As the first step to a solution, we should repeat a simple fact that the object defined as a sum of quadratics to estimate quadratic covariation is not necessarily a real quadratic form of increments if the kernel has dependency on the processes. The essence of the question is whether it is possible to construct a framework in which the real quadratic forms can comprehensively be treated. In order to give a general solution, a strong predictability condition was introduced by Hayashi and Yoshida (2006). Section 5 asserts in Proposition 5.1 that the strong predictability condition [A2] ensures Condition [B2]. The advantage of Condition [B2] is that it reduces the verification of the convergence of the quadratic variation of the statistic to that of the empirical distribution function of the sampling times, and so it becomes a basis of practical applications; this reduction is discussed in Section 4. The strong predictability condition is meaningful even in this sense while it was so in that it gives a natural perspective to the quadratic variation as a real quadratic form.

The main results of this article will be presented in Section 6 for semimartingales as well as local martingales. The reader can jump to this section directly if he/she wishes to avoid technicalities at the first reading.

Section 7 introduces the empirical nonsynchronous covariation process and proves limit theorems. Section 8 will be devoted to statistical aspects. We will discuss studentization and a kernel estimator for the random asymptotic variance. An illustrative example with random change point will be presented. Comments on financial applications will be provided in Section 9. Most of the proofs will be put in Sections 10–14.

2 Observation point processes and the nonsynchronous covariance process

Given a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$, we consider two continuous local martingales $X = (X_t)_{t\in\mathbb{R}_+}$ and $Y = (Y_t)_{t\in\mathbb{R}_+}$, and two sequences of stopping times $\left(S^i\right)_{i\in\mathbb{Z}_+}$ and $\left(T^j\right)_{j\in\mathbb{Z}_+}$ that are increasing a.s., $S^i \uparrow \infty$ and $T^j \uparrow \infty$, and $S^0 = 0$, $T^0 = 0$.

We will regard the sampling scheme as a point process. According to this idea, we will use the

[2] For example, we can consider a continuous martingale sampled when its quadratic variation crosses points on a grid. A Brownian motion observed when it hits grid points is also an example. More generally, it is easy to treat the hitting times at grid points for a strong Markov process if we have sufficient knowledge of the distribution of the intervals between those stopping times.

following symbols throughout the paper to describe random intervals:

$$\begin{aligned}
&I^i = \left[S^{i-1}, S^i\right), \quad J^j = \left[T^{j-1}, T^j\right), \\
&I^i_t = 1_{[S^{i-1},S^i)}(t), \quad J^j_t = 1_{[T^{j-1},T^j)}(t), \\
&I^i(t) = \left[S^{i-1} \wedge t, S^i \wedge t\right), \quad J^j(t) = \left[T^{j-1} \wedge t, T^j \wedge t\right), \\
&r_n(t) = \sup_{i\in\mathbb{N}} |I^i(t)| \vee \sup_{j\in\mathbb{N}} |J^j(t)|.
\end{aligned}$$

Here, $|\cdot|$ denotes the Lebesgue measure, and $\mathbb{N} = \{1, 2, ...\}$. In the preceding paper, the processes X and Y were implicitly assumed to be observable at some fixed terminal time T. This difference is not essential because it causes no difference up to the first order asymptotic results. It is also possible to remove the assumption that both stochastic processes are observed at $t = 0$, while we will not pursuit this version here by the same reason.

We will refer to $\left(I^i\right)_{i\in\mathbb{N}}$ and $\left(J^j\right)_{j\in\mathbb{N}}$, or equivalently to $\left(S^i\right)_{i\in\mathbb{Z}_+}$ and $\left(T^j\right)_{j\in\mathbb{Z}_+}$, as the *sampling designs* (or simply the *designs*) for X and Y. Also, the sampling designs stopped at time t, $\left(I^i(t)\right)_{i\in\mathbb{N}}$ and $\left(J^j(t)\right)_{j\in\mathbb{N}}$, may be referred to as the *random partitions* of $[0, t)$. For simplicity, when we say "pair (i, j) overlaps" it will mean either $I^i(t) \cap J^j(t) \neq \varnothing$ (i.e., the two intervals I^i and J^j overlap by time t), or $I^i \cap J^j \neq \varnothing$ (i.e., by any time), depending on the situation.

For processes V and W, $V \cdot W$ denotes the integral (either stochastic or ordinary) of V with respect to W so far as it exists. When the integrator W is continuous, it is always true that $V_- \cdot W = V \cdot W$, where $V_{t-} := \lim_{s\uparrow t} V_s$. For a stochastic process V and an interval I, let $V(I) = \int 1_I(t-)dV_t$. Write $I\,(t) = I \cap [0, t)$ for interval I, then $V(I)_t = V(I\,(t))$ by definition.

The quantity of interest is the quadratic covariation $[X, Y]$, and as its sample counterpart, we will investigate the following quantity:

Definition 2.1 (Hayashi and Yoshida (2005b), Hayashi and Yoshida (2006)) *The* ***nonsynchronous covariation process*** *of X and Y associated with sampling designs $\mathcal{I} = \left(I^i\right)_{i\in\mathbb{N}}$ and $\mathcal{J} = \left(J^j\right)_{j\in\mathbb{N}}$ is the process*

$$\{X, Y; \mathcal{I}, \mathcal{J}\}_t = \sum_{i,j=1}^{\infty} X(I^i)_t Y(J^j)_t 1_{\{I^i(t)\cap J^j(t)\neq\varnothing\}},\ t \in \mathbb{R}_+.$$

The process $\{X, Y; \mathcal{I}, \mathcal{J}\}$ is not observable from data-analytical point of view. See Section 7 for a statistic corresponding to this process. We will write it simply as $\{X, Y\}_t$ if there is no fear of confusion over sampling designs.

It was proved in Hayashi and Yoshida (2005b) and Hayashi and Kusuoka (2008) that for each $t \in \mathbb{R}_+$, $\{X, Y\}_t \xrightarrow{p} [X, Y]_t$ as $n \to \infty$ provided $r_n(t) \xrightarrow{p} 0$. In light of this result,the authors emphasize that $\{X, Y\}_t$ is regarded as a generalization, in the context of nonsynchronous sampling schemes, of the standard definition of the quadratic covariation process for semimartingales in stochastic analysis. For Itô processes X and Y, we can obtain the same consistency result; see the above papers for details.

3 Stable convergence of the estimation error

The estimation error of $\{X, Y\}$ is given by

(3.1) $$M^n_t = \{X, Y\}_t - [X, Y]_t = \textstyle\sum_{i,j} L^{ij}_t K^{ij}_t,$$

where $K^{ij}_t = 1_{\{I^i(t)\cap J^j(t)\neq\varnothing\}}$ and $L^{ij}_t = \left(I^i_- \cdot X\right)_- \cdot \left(J^j_- \cdot Y\right)_t + \left(J^j_- \cdot Y\right)_- \cdot \left(I^i_- \cdot X\right)_t$. Here helpful is the equality $\sum_{i,j}(I^i_- J^j_-) \cdot [X,Y] = [X,Y]$, as well as the definition of the quadratic covariation or Itô's formula.

We also introduce the symbols $R_\vee(i,j) := S^{i-1} \vee T^{j-1}$ and $R^\wedge(i,j) := S^i \wedge T^j$. L^{ij} is a continuous local martingale such that it equals 0 for $t \le R_\vee(i,j)$, starts varying at $t = R_\vee(i,j)$, and stays at the value $L^{ij}_{S^i\vee T^j}$ after $t = S^i \vee T^j$. It can vary, regardless of whether the pair (i,j) overlaps.

Lemma 3.1 *Suppose that X and Y are continuous. Then $L^{ij}_- \cdot K^{ij} \equiv 0$.*

Proof. Recall that L^{ij} is continuous in t. K^{ij}_t is a step function starting from 0 at $t = 0$ and jumps to $+1$ at $t = R_\vee(i,j)$ when the pair (i,j) overlaps. So, $L^{ij}_- \cdot K^{ij}_t = L^{ij}_{R_\vee(i,j)\wedge t} K^{ij}_t$. However, $L^{ij}_t = 0$ for $t \le R_\vee(i,j)$. □

Now, the integration-by-parts to (3.1) together with Lemma 3.1 yields

$$M^n_t = \sum_{i,j} K^{ij}_- \cdot L^{ij}_t, \tag{3.2}$$

in particular, M^n_t is a continuous local martingale with

$$\mathfrak{V}^n_t := [M^n, M^n]_t = \sum_{i,j,i',j'} \left(K^{ij}_- K^{i'j'}_-\right) \cdot \left[L^{ij}, L^{i'j'}\right]_t. \tag{3.3}$$

Let

$$\mathfrak{V}^n_{X,t} = \sum_{i,j} K^{ij}_- \cdot \left\{X\left(I^i\right) \cdot [X,Y]\left(J^j\right)\right\}_t + \sum_{i,j} K^{ij}_- \cdot \left\{Y\left(J^j\right) \cdot [X,X]\left(I^i\right)\right\}_t$$

and

$$\mathfrak{V}^n_{Y,t} = \sum_{i,j} K^{ij}_- \cdot \left\{X\left(I^i\right) \cdot [Y,Y]\left(J^j\right)\right\}_t + \sum_{i,j} K^{ij}_- \cdot \left\{Y\left(J^j\right) \cdot [X,Y]\left(I^i\right)\right\}_t.$$

In view of the standard martingale central limit theorem, we formally state the following condition. (b_n) denotes a sequence of positive numbers tending to 0 as $n \to \infty$.

[A1*] There exists an $\mathbf{F}$-adapted, nondecreasing, continuous process $(V_t)_{t\in\mathbb{R}_+}$ such that $b_n^{-1}\mathfrak{V}^n_t \to^p V_t$ as $n \to \infty$ for every t.

[B1] $b_n^{-\frac{1}{2}}\mathfrak{V}^n_{X,t} \to^p 0$ and $b_n^{-\frac{1}{2}}\mathfrak{V}^n_{Y,t} \to^p 0$ as $n \to \infty$ for every t.

We denote by $\mathbb{C}(\mathbb{R}_+)$ the space of continuous functions on $\mathbb{R}_+$ equipped with the locally uniform topology, and by $\mathbb{D}(\mathbb{R}_+)$ the space of càdlàg functions on $\mathbb{R}_+$ equipped with the Skorokhod topology. A sequence of random elements X^n defined on a probability space $(\Omega, \mathcal{F}, P)$ is said to *converge stably in law* to a random element X defined on an appropriate extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ of $(\Omega, \mathcal{F}, P)$ if $E\left[Yg\left(X^n\right)\right] \to E\left[Yg\left(X\right)\right]$ for any $\mathcal{F}$-measurable and bounded random variable Y and any bounded and continuous function g. We then write $X^n \to^{d_s} X$. A sequence (X^n) of stochastic processes is said to converge to a process X *uniformly on compacts in probability* (abbreviated *ucp*) if, for each $t > 0$, $\sup_{0\le s\le t}|X^n_s - X_s| \xrightarrow{p} 0$ as $n \to \infty$.

We consider the condition

[**W**] There exists an **F**-predictable process w such that $V_\cdot = \int_0^\cdot w_s^2 ds$.

An aim of this paper is to prove the statement

[**SC**] $b_n^{-\frac{1}{2}} M^n \to^{d_s} M$ in $\mathbb{C}(\mathbb{R}_+)$ as $n \to \infty$, where $M = \int_0^\cdot w_s d\widetilde{W}_s$ and $\widetilde{W}$ is a one-dimensional Wiener process (defined on an extension of $\mathcal{B}$) independent of $\mathcal{F}$.

Proposition 3.1 *Suppose that [A1*], [B1] and [W] are fulfilled. Then [SC] holds.*

Proof. Notice that $[M^n, X] = \mathfrak{V}^n_{X,\cdot}$ and $[M^n, Y] = \mathfrak{V}^n_{Y,\cdot}$. Since $[M^n, N] = 0$ for any bounded martingale N on $\mathcal{B}$, we obtain the stable convergence of $b_n^{-1/2} M^n$ from Jacod (1997). □

Each expression of $\mathfrak{V}^n$, $\mathfrak{V}^n_{X,\cdot}$ and $\mathfrak{V}^n_{Y,\cdot}$ is rather abstract; it may be of little help for explicitly calculating the quadratic variation/covariation and identifying the limiting distribution of M^n. From this regard, it is natural in the following to pursue more concrete appearance especially of $\mathfrak{V}^n$.

Let

$$(3.4) \qquad V_t^{iji'j'} = \left\{ \left(I_-^i I_-^{i'} \right) \cdot [X,X] \right\}_t \left\{ \left(J_-^j J_-^{j'} \right) \cdot [Y,Y] \right\}_t + \left\{ \left(I_-^i J_-^{j'} \right) \cdot [X,Y] \right\}_t \left\{ \left(I_-^{i'} J_-^{j} \right) \cdot [X,Y] \right\}_t$$

and set $V^{ij} := V^{ijij}$.[3] $V^{iji'j'}$ is designed to approximate $[L^{ij}, L^{i'j'}]$ when the interval lengths $|I^i|$, $|I^{i'}|$, $|J^j|$, and $|J^{j'}|$ are sufficiently small, as will be stated in [B2] below; the approximation will turn out to be valid in Section 4 under certain conditions.

Throughout the rest of the discussion in this section, we will postulate the following hypothesis. A sufficient condition for [B2] will be provided in Section 5.

[**B2**] For every $t \in \mathbb{R}_+$,

$$(3.5) \qquad b_n^{-1} \sum_{i,j,i',j'} \left(K_-^{ij} K_-^{i'j'} \right) \cdot \left[L^{ij}, L^{i'j'} \right]_t = b_n^{-1} \sum_{i,j,i',j'} \left(K_-^{ij} K_-^{i'j'} \right) \cdot V_t^{iji'j'} + o_P(1)$$

as $n \to \infty$.

Be reminded that the left-hand side of (3.5) equals to $b_n^{-1} \mathfrak{V}^n_t$.

Denote $[X] = [X,X]$ and $[Y] = [Y,Y]$ as usual. Let

$$\begin{aligned} \bar{V}_t^n &= \sum_{i,j} [X] \left(I^i(t) \right) [Y] \left(J^j(t) \right) K_t^{ij} + \sum_i [X,Y] \left(I^i(t) \right)^2 \\ &\quad + \sum_j [X,Y] \left(J^j(t) \right)^2 - \sum_{i,j} [X,Y] \left(\left(I^i \cap J^j \right)(t) \right)^2 . \end{aligned}$$

The following proposition will be used for identifying the limit of the quadratic variation. It enables us to work with the more tractable process $\bar{V}^n$ than $b_n^{-1} \sum_{i,j,i',j'} (K_-^{ij} K_-^{i'j'}) \cdot V^{iji'j'}$ in [B2]. See Section 10 for a proof.

Proposition 3.2 *Suppose [B2] holds. Then* $\mathfrak{V}^n_t = \bar{V}^n_t + o_p(b_n)$ *as* $n \to \infty$.

[3]Since $[X,X]$ is now continuous, $I_-^i I_-^{i'}$ can be replaced by $I^i I^{i'}$ in the first factor on the right-hand side of (3.4). It is also the case with other factors.

We modify [A1*].

[**A1**] There exists an **F**-adapted, nondecreasing, continuous process $(V_t)_{t\in\mathbb{R}_+}$ such that $b_n^{-1}\bar{V}_t^n \to^p V_t$ as $n\to\infty$ for every t.

By Proposition 3.2, we can rephrase Proposition 3.1 as follows.

Proposition 3.3 *Suppose that* $[A1]$, $[B1]$, $[B2]$, *and* $[W]$ *for* V *in* $[A1]$ *are satisfied. Then* $[SC]$ *holds true.*

Since the variance process $\bar{V}^n$ is much more convenient to handle than V^n, Proposition 3.3 essentially improves Proposition 3.1. However, Proposition 3.3 is on the way to our goal.

First, it is preferable to describe the limiting energy process V_t in light of the sampling scheme itself. In Section 4, we introduce certain sampling measures to do it, following Hayashi and Yoshida (2006).

Second, Condition [B2] is still technical. Indeed, this condition avoids one of the key steps to the answer. The HY estimator, or any quadratic type estimator, is really quadratic only when the random kernel of the "quadratic form" satisfies a kind of predictability condition. Otherwise, limit theorems will fail. The authors introduced a strong predictability condition to give a central limit theorem in Hayashi and Yoshida (2006) by verifying [B2] under mild regularity conditions of the processes. Though the mixed normal limit theorem is the aim of this paper, it will turn out in Section 5 that the same strong predictability condition serves well for our purpose.

It still remains to check the asymptotic orthogonality condition (ii) of [A1*] and $[A1]$ in a practical setting. However, we will show that it is the same kind question as solving [B2], and no additional difficulty occurs to do with it.

4 Convergence of the sampling measures and a representation of V_t

In Hayashi and Yoshida (2005a), the authors introduced empirical distribution functions of the sampling times given by

$$H_n^1(t)=\sum_i |I^i(t)|^2,\ H_n^2(t)=\sum_j |J^j(t)|^2,$$
$$H_n^{1\cap 2}(t)=\sum_{i,j}|\left(I^i\cap J^j\right)(t)|^2,\ H_n^{1*2}(t)=\sum_{i,j}|I^i(t)||J^j(t)|K_t^{ij},$$

where $|\cdot|$ is the Lebesgue measure. Clearly, the four functions are $(\mathcal{F}_t)$-adapted, non-decreasing, piecewise-quadratic continuous functions, whose graphs contain 'kinks' at the observation stopping times.

[**A1′**] There exists a possibly random, nondecreasing, functions $H^1, H^2, H^{1\cap 2}$ and H^{1*2} on $[0,T]$, such that each $H^k=\int_0^t h_s^k ds$ for some density h^k, and that $b_n^{-1}H_n^k(t)\xrightarrow{p} H^k(t)$ as $n\to\infty$ for every $t\in\mathbb{R}_+$ and $k=1,2,1\cap 2,1*2$.

Then, an extension of Theorem 2.2 of Hayashi and Yoshida (2005a) is given as follows.

Proposition 4.1 *Suppose that* $[A1']$, $[B1]$ *and* $[B2]$ *are fulfilled, and that each* $[X]$, $[Y]$ *and* $[X,Y]$ *is absolutely continuous with a locally bounded derivative. Then* $[SC]$ *is valid with* w_s *given by*

$$w_s \quad = \quad \sqrt{[X]'_s[Y]'_sh_s^{1*2}+\left([X,Y]'_s\right)^2\left(h_s^1+h_s^2-h_s^{1\cap 2}\right)}. \tag{4.1}$$

Remark In the case of perfect synchronicity ($I^i \equiv J^j$), $\{X, Y\}$ is nothing more than the *realized covariance* based on all the data. In this case, since $H^1 \equiv H^2 \equiv H^{1\cap 2} \equiv H^{1*2}$ $(=: H)$, the limiting variation process reduces to

$$V_{\cdot} = \int_0^{\cdot} \left\{ [X]'_t \, [Y]'_t + \left([X, Y]'_t\right)^2 \right\} H(dt).$$

Proof of Proposition 4.1. Lemma 4.1 below identifies the limiting variation process V in the condition [A1] by (4.1). Thus Proposition 3.3 implies the assertion. □

Lemma 4.1 *Suppose that $[X]$, $[Y]$, and $[X, Y]$ are continuously differentiable. Then, $[A1']$ implies that*

(i) $b_n^{-1} \sum_i [X, Y] \left(I^i(t)\right)^2 \xrightarrow{p} \int_0^t \left([X, Y]'_s\right)^2 H^1(ds)$,

(ii) $b_n^{-1} \sum_j [X, Y] \left(J^j(t)\right)^2 \xrightarrow{p} \int_0^t \left([X, Y]'_s\right)^2 H^2(ds)$,

(iii) $b_n^{-1} \sum_j [X, Y] \left(\left(I^i \cap J^j\right)(t)\right)^2 \xrightarrow{p} \int_0^t \left([X, Y]'_s\right)^2 H^{1\cap 2}(ds)$,

(iv) $b_n^{-1} \sum_{i,j} [X] \left(I^i(t)\right) [Y] \left(J^j(t)\right) K_t^{ij} \xrightarrow{p} \int_0^t [X]'_s \, [Y]'_s \, H^{1*2}(ds)$.

as $n \to \infty$ for every t.

Proof is in Section 11

5 Strong predictability and Condition [B2]

We presented a basic version of limit theorems as Propositions 3.1, 3.3 and 4.1. They hold without additional conditions for such sampling schemes as the ones given by certain hitting times of the processes.

In Propositions 3.3 and 4.1, we assumed Condition [B2] in Section 3. Under more general sampling schemes, however, Condition [B2] is still technical. In this section, we are going to introduce a more tractable condition on the sampling scheme to ensure [B2]. Such a condition is called *strong predictability* of the sampling times. It was introduced in Hayashi and Yoshida (2006), and a motivation of it is that the future sampling time is determined with delay in practical situations such as the delay caused while a customer is asking the agent to trade in a financial market. Let ξ and ξ' be constants satisfying $\frac{4}{5} \vee \xi < \xi' < 1$.

We need the strong predictability condition introduced by Hayashi and Yoshida (2006).

[**A2**] For every $n, i \in \mathbb{N}$, S^i *and* T^i are $\mathbf{G}^{(n)}$-stopping times, where $\mathbf{G}^{(n)} = (\mathcal{G}_t^{(n)})_{t \in \mathbb{R}_+}$ is the filtration given by $\mathcal{G}_t^{(n)} = \mathcal{F}_{(t - b_n^{\xi}) \vee 0}$ for $t \in \mathbb{R}_+$.

For real-valued functions x on $\mathbb{R}_+$, the *modulus of continuity* on $[0, T]$ is denoted by $w(x; \delta, T) = \sup\{|x(t) - x(s)|\,;\ s, t \in [0, T],\ |s - t| \le \delta\}$ for $T, \delta > 0$. Write $H_t^* = \sup_{s \in [0,t]} |H_s|$ for a process H.

[**A3**] $[X]$, $[Y]$, $[X, Y]$ are absolutely continuous, and for the density processes $f = [X]'$, $[Y]'$ and $[X, Y]'$, $w(f; h, t) = O_p(h^{\frac{1}{2} - \lambda})$ as $h \to 0$ for every $t, \lambda \in (0, \infty)$, and $|f_0| < \infty$ a.s.

[**A4**] $r_n(t) = o_p(b_n^{\xi'})$ for every $t \in \mathbb{R}_+$.

The following is the key statement to the main result stated in Section 6.

Proposition 5.1 *[B2] holds true under [A2], [A3] and [A4].*

We give a proof of Propostion 5.1 in Section 12.

6 Limit theorems for semimartingales: main results

Up to the previous sections we focused for the case where X and Y are continuous local martingales. In this section, we consider two continuous semimartingales and present the main results of this article. Suppose that $X = A^X + M^X$ and $Y = A^Y + M^Y$ are continuous semimartingales, where A^X and A^Y are finite variation parts, M^X and M^Y are continuous local martingale parts. Even in this case, the nonsynchronous covariation process of X and Y associated with $\left(I^i\right)$ and $\left(J^j\right)$ is defined exactly by Definition 2.1. To go ahead, we need two additional conditions below.

[A5] A^X and A^Y are absolutely continuous, and $w(f; h, t) = O_P(h^{\frac{1}{2}-\lambda})$ as $h \to 0$ for every $t \in \mathbb{R}_+$ and some $\lambda \in (0, 1/4)$, for the density processes $f = (A^X)'$ and $(A^Y)'$.

[A6] As $n \to \infty$,

$$b_n^{-1} \sum_i \left|I^i(T)\right|^2 + b_n^{-1} \sum_j \left|J^j(T)\right|^2 = O_p(1). \tag{6.1}$$

Remark 6.1. Condition [A5] is slightly stronger than (C4′) of Hayashi and Yoshida (2004). Condition [A1] does not imply [A6]. Indeed, it is possible to make a sampling scheme that includes $[n^{7/10}]$ intervals of $n^{-4/5}$ in length and $[X]$ and $[Y]$ do not increase on the union of those intervals, and also [A1] holds. However [A6] breaks in this case. On the other hand, [A1′] implies [A6]. The Poisson sampling scheme considered in Hayashi and Yoshida (2004) is an example.

Here is our main result.

Theorem 6.1 *Suppose that X and Y are continuous semimartingales.*

(a) *If [A1]-[A6] and [W] are satisfied, then [SC] holds.*

(b) *If [A1′], [A2]-[A5] are satisfied, then [SC] holds for w given by (4.1).*

As a particular case, we have

Theorem 6.2 *Suppose that X and Y are continuous local martingales.*

(a) *If [A1]-[A4] and [W] are satisfied, then [SC] holds.*

(b) *If [A1′], [A2]-[A4] are satisfied, then [SC] holds for w given by (4.1).*

Theorems 6.1 and 6.2 are proved in Section 13.

7 Empirical nonsynchronous covariation process

The quantity $\{X, Y\}_t$ is not always observable in the statistical context, which is certainly a distracting feature from a viewpoint of practical applications. The argument here is the way how to amend such a minor flaw pertaining to the previous construction in Definition 2.1.

Definition 7.1 *The* ***empirical nonsynchronous covariation process*** *of X and Y associated with sampling designs $\mathcal{I} = (I^i)_{i\in\mathbb{N}}$ and $\mathcal{J} = (J^j)_{j\in\mathbb{N}}$ is the process*

$$\overline{\{X, Y\}}_t = \sum_{\substack{i,j=1 \\ S^i \vee T^j \le t}}^{\infty} X(I^i)Y(J^j)1_{\{I^i \cap J^j \neq \varnothing\}} \qquad t \in \mathbb{R}_+.$$

Obviously,

$$\overline{\{X, Y\}}_t \quad = \quad \sum_{\substack{i,j=1 \\ S^i \vee T^j \le t}}^{\infty} X(I^i)_t Y(J^j)_t 1_{\{I^i(t) \cap J^j(t) \neq \varnothing\}}.$$

It is the piecewise constant, càdlàg version of the nonsynchronous covariation process and $\overline{\{X, Y\}}_t = \{X, Y\}_t$ at $t \in (S^i) \cap (T^j)$. Otherwise they do not coincide in general, however the difference is negligible.

Suppose that X and Y are continuous semimartingales given in Section 6.

For $\overline{M}^n = \overline{\{X, Y\}} - [X, Y]$, we will show

[**SCE**] $b_n^{-\frac{1}{2}} \overline{M}^n \to^{d_s} M$ in $\mathbb{D}(\mathbb{R}_+)$ as $n \to \infty$, where $M = \int_0^{\cdot} w_s d\widetilde{W}_s$ and $\widetilde{W}$ is a one-dimensional Wiener process (defined on an extension of $\mathcal{B}$) independent of $\mathcal{F}$.

We obtain the following results corresponding to Theorems 6.1 and 6.2, respectively. See Section 14 for proof.

Theorem 7.1 *Suppose that X and Y are continuous semimartingales.*

(a) *If* [A1]-[A6] *and* [W] *are satisfied, then* [SCE] *holds.*

(b) *If* [A1′], [A2]-[A5] *are satisfied, then* [SCE] *holds for w given by (4.1).*

Theorem 7.2 *Suppose that X and Y are continuous local martingales.*

(a) *If* [A1]-[A4] *and* [W] *are satisfied, then* [SCE] *holds.*

(b) *If* [A1′], [A2]-[A4] *are satisfied, then* [SCE] *holds for w given by (4.1).*

8 Statistical application and example

8.1 Stochastic differential equation

Suppose that X^1 and X^2 are Itô semimartingales described by the stochastic differential equation

$$dX_t^k = \mu_t^k dt + \sigma_t^k dW_t^k \qquad (k = 1, 2) \tag{8.1}$$

where μ_t^k are $\mathbf{F}$-adapted processes, σ_t^k are strictly positive, $(\mathcal{F}_t)$-adapted, continuous processes, $k=1,2$. The $\mathbf{F}$-adapted Brownian motions W_t^k are correlated with $d\left[W^1,W^2\right]_t=\rho_t dt$, where ρ_t is an $\mathbf{F}$-adapted process. This is a stochastic volatility model in the finance literature. We continue to use the same symbols $\mathcal{I}=\left(I^i\right)_{i\in\mathbb{N}}$ and $\mathcal{J}=\left(J^j\right)_{j\in\mathbb{N}}$ as the sampling designs associated with X^1 and X^2, respectively.

[A3′] **(i)** For every $\lambda>0$ and $t\in\mathbb{R}_+$, $w(f;h,t)=O_p(h^{\frac{1}{2}-\lambda})$ as $h\downarrow 0$ for $f=\sigma^1$, σ^2 and ρ.

(ii) For some $\lambda\in(0,1/4)$ and any $t\in\mathbb{R}_+$, $w(f;h,t)=O_p(h^{\frac{1}{2}-\lambda})$ as $h\downarrow 0$ for $f=\mu^k$, $k=1,2$.

Now, define the distribution functions associated with the sampling designs $\mathcal{I}$ and $\mathcal{J}$ by

$$\overline{H}_n^1(t)=\sum_{i:\,S^i\le t}|I^i|^2,\quad \overline{H}_n^2(t)=\sum_{j:\,T^j\le t}|J^j|^2,$$
$$\overline{H}_n^{1\cap 2}(t)=\sum_{\substack{i,j:\\ S^i\vee T^j\le t}}|I^i\cap J^j|^2,\quad \overline{H}_n^{1*2}(t)=\sum_{\substack{i,j:\\ S^i\vee T^j\le t}}|I^i||J^j|K^{ij},$$

where $K^{ij}=1_{\{I^i\cap J^j\neq\varnothing\}}$. They are all observable at any t.

[A1″] There exists a possibly random, nondecreasing, functions $H^1,H^2,H^{1\cap 2}$ and H^{1*2} on $[0,T]$, such that each $H^k(t)=\int_0^t h_s^k ds$ for some density h^k, and that $b_n^{-1}\overline{H}_n^k(t)\xrightarrow{p}H^k(t)$ as $n\to\infty$ for every $t\in\mathbb{R}_+$ and $k=1,2,1\cap 2,1*2$.

The equivalence between [A1′] and [A1″] can be proved. Indeed,

$$\overline{H}_n^k(s)\le H_n^k(s)\le \overline{H}_n^k(s)+2r_n(t)^2,\quad \text{for all } s\in[0,t]\,,\ k=1,2,1\cap 2,1*2.$$

We take on the case $k=1*2$ only, for all the others are straightforward. The first inequality is obvious by construction. Moreover, according to a similar argument adopted in the proof for Lemma 14.1, for any s,t with $s\le t$,

$$H_n^{1*2}(s)-\overline{H}_n^{1*2}(s)\le\left|I^{i_t}(t)\right|\left|J\left(I^{i_t}\right)(t)\right|\vee\left|I\left(J^{j_t}\right)(t)\right|\left|J^{j_t}(t)\right|\le 2r_n\left(t\right)^2$$

so that

$$\sup_{s\in[0,t]}\left|H_n^{1*2}(s)-\overline{H}_n^{1*2}(s)\right|\le 2r_n\left(t\right)^2.$$

Hence, the second inequality also holds. Since $b_n^{-1}r_n(t)^2\xrightarrow{p}0$ under [A4] for example, we have ascertained that the convergence of $b_n^{-1}\overline{H}_n^{1*2}$ is equivalent to that of $b_n^{-1}H_n^{1*2}$.

Then by the application of Theorem 7.1 we have

Theorem 8.1 *Suppose that X and Y are continuous semimartingales. Suppose that either [A1′] or [A1″], and also [A2], [A3′] and [A4] are satisfied. Then, for $M^n=\{X,Y\}-\int_0^\cdot \rho_s\sigma_s^1\sigma_s^2 ds$ and $\overline{M}^n=\overline{\{X,Y\}}-\int_0^\cdot \rho_s\sigma_s^1\sigma_s^2 ds$, [SC] and [SCE] hold for w given by*

$$w_s=\sqrt{\left(\sigma_s^1\sigma_s^2\right)^2h_s^{1*2}+\left(\rho_s\sigma_s^1\sigma_s^2\right)^2\left(h_s^1+h_s^2-h_s^{1\cap 2}\right)}. \tag{8.2}$$

We shall briefly discuss studentization. Consider w_s given in (8.2). In our context, w_t is not observable since it contains unknown quantities such as $\rho_s\sigma_s^1\sigma_s^2$; nor is $\int_0^\cdot w_s^2 ds$. Suppose we have a statistic $\widehat{\int_0^t w_s^2 ds}$ such that

$$\widehat{\int_0^t w_s^2 ds} \quad \xrightarrow{p} \quad \int_0^t w_s^2 ds$$

as $n \to \infty$.

Then, the stable convergence stated in Theorem 8.1 implies that, for every $t > 0$,

$$\frac{b_n^{-\frac{1}{2}}\left(\overline{\{X^1,X^2\}}_t - \int_0^t \rho_s\sigma_s^1\sigma_s^2 ds\right)}{\sqrt{\widehat{\int_0^t w_s^2 ds}}} \quad \xrightarrow{\mathcal{L}} \quad N(0,1)$$

as $n \to \infty$ whenever $\int_0^t w_s^2 ds > 0$ a.s.

8.2 Construction of $\widehat{\int_0^t w_s^2 ds}$: Kernel-based approach

Let $K(u)$ be a kernel function such that $\int_{-\infty}^\infty K(u)\,du = 1$ and $K(u) \geq 0$ for all u. K is assumed to be absolutely continuous with derivative K' satisfying $\int_{-\infty}^\infty |K'(s)|\,ds < \infty$. For $h > 0$, let $K_h(u) = h^{-1}K(h^{-1}u)$. For every $t \in \mathbb{R}_+$, let

$$\partial\widetilde{\{X^1,X^2\}}_t^h = \int_{-\infty}^\infty \overline{\{X^1,X^2\}}_s K_h'(t-s)\,ds,$$
$$\partial\widetilde{\{X^k\}}_t^h = \partial\widetilde{\{X^k,X^k\}}_t^h, \qquad k = 1,2.$$

Moreover, let

$$\begin{aligned}\widehat{\int_0^t w_s^2 ds} &= \int_0^t \partial\widetilde{\{X^1\}}_s^h \partial\widetilde{\{X^2\}}_s^h b_n^{-1}\overline{H}_n^{1*2}(ds) \\ &\quad + \int_0^t \partial\widetilde{\{X^1,X^2\}}_s^h b_n^{-1}\left(\overline{H}_n^1 + \overline{H}_n^2 - \overline{H}_n^{1\cap 2}\right)(ds).\end{aligned}$$

This quantity is observable.

Proposition 8.1 *Under the assumptions in Theorem 8.1,*

$$\widehat{\int_0^\cdot w_s^2 ds} \quad \xrightarrow{ucp} \quad \int_0^\cdot w_s^2 ds$$

as $n \to \infty$, provided that $b_n^{\frac{1}{2}}h^{-1} \to 0$.

Proof. Let $\overline{M}_t^n = \overline{\{X^1,X^2\}}_t - \left[X^1,X^2\right]_t$, then clearly it satisfies $b_n^{-\frac{1}{2}}\sup_{s\in[0,t]}\left|\overline{M}_s^n\right| = O_p(1)$. Thus, by the integration-by-parts formula,

$$\begin{aligned}\partial\widetilde{\{X^1,X^2\}}_t^h &= \int_{-\infty}^\infty \left(\left[X^1,X^2\right]_s + \overline{M}_s^n\right) K_h'(t-s)\,ds \\ &= -\left[X^1,X^2\right]_s K_h(t-s)\Big|_{s=-\infty}^\infty + \int_{-\infty}^\infty \left[X^1,X^2\right]_s' K_h(t-s)\,ds \\ &\quad + O_p(b_n^{\frac{1}{2}})\int_{-\infty}^\infty \left|K_h'(t-s)\right| ds \\ &= \rho_t\sigma_t^1\sigma_t^2 + w(\rho\sigma^1\sigma^2; h, t) + O_p(b_n^{\frac{1}{2}}h^{-1})\end{aligned}$$

uniformly in $t \in [0,t']$ for any $t' > 0$. This can be realized by choosing h such that $b_n^{\frac{1}{2}}h^{-1} \to 0$. Similarly,

$$\widetilde{\partial\{X^k\}}_t^h = (\sigma_t^k)^2 + w((\sigma^k t)^2; h, t) + O_p(b_n^{\frac{1}{2}}h^{-1})$$

uniformly in $t \in [0,t']$ for any $t' > 0$. Therefore, it is immediate to obtain the assertion of the proposition. □

A special case of the kernel-based approach is the following naïve one . For any $s > 0$ and $h > 0$, we may use

$$\partial\overline{\{X^1, X^2\}}_s^h = \frac{1}{h}\left(\overline{\{X^1, X^2\}}_s - \overline{\{X^1, X^2\}}_{s-h}\right),$$
$$\partial\overline{\{X^k\}}_s^h = \partial\overline{\{X^k, X^k\}}_s^h, \qquad k = 1, 2,$$

provided $b_n^{\frac{1}{2}}h^{-1} \to 0$. We will refer the reader to Hayashi and Yoshida (2008b).

8.3 Poisson sampling with a random change point

As an illustration, we shall discuss a Poisson sampling with a random change point. It is a simple model for stock prices, for instance, whose trading intensities vary at random times such as the times when they hit a threshold price like 10,000 yen.

More precisely, suppose that $(\mathcal{F}'_t)$-adapted processes $\rho, \mu^k, \sigma^k, W^k, \tau^k, k = 1, 2$, are given on a stochastic basis $(\Omega', \mathcal{F}', (\mathcal{F}'_t), P')$. The processes X^k are defined by (8.1). Let $\tau^k (k = 1, 2)$ be $(\mathcal{F}'_t)$-stopping times. On an auxiliary probability space $(\Omega'', \mathcal{F}'', P'')$, there are random variables $\left(\underline{S}^i\right), \left(\underline{T}^j\right), (\overline{S}^i), (\overline{T}^j)$ that are mutually independent Poisson arrival times with intensity $\underline{\lambda}^k = n\underline{p}^k$, $\underline{p}^k \in (0, \infty)$, $k = 1, 2$, respectively for $\left(\underline{S}^i\right), \left(\underline{T}^j\right)$ and with $\overline{\lambda}^k = n\overline{p}^k$, $\overline{p}^k \in (0, \infty)$, $k = 1, 2$, respectively for $(\overline{S}^i), (\overline{T}^j)$.

We construct the product stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ by

$$\Omega = \Omega' \times \Omega'', \quad \mathcal{F} = \mathcal{F}' \times \mathcal{F}'', \quad \mathcal{F}_t = \mathcal{F}'_t \times \mathcal{F}'' \quad P = P' \times P''.$$

On the new basis the random elements aforementioned can be extended in the usual way. That is, $W^k(\omega', \omega'') = W^k(\omega')$, $\underline{S}^i(\omega', \omega'') = \underline{S}^i(\omega'')$, $(\omega', \omega'') \in \Omega$, and so forth.

The sampling design $\mathcal{I} = (S^i)$ for X will be made of $\left(\underline{S}^i\right)$ and $(\overline{S}^i)$ as follows. Set $\tau_n^1 = \tau^1 + \frac{1}{\sqrt{n}}$. Define S^i sequentially by

$$S^1 = \inf_{l\in\mathbb{N}}\left\{\underline{S}^l_{\{\underline{S}^l < \tau_n^1\}}, \tau_n^1 + \overline{S}^1\right\},$$
$$S^i = \inf_{l,m\in\mathbb{N}}\left\{\underline{S}^l_{\{S^{i-1} < \underline{S}^l < \tau_n^1\}}, \left(\tau_n^1 + \overline{S}^m\right)_{\{S^{i-1} < \tau_n^1 + \overline{S}^m\}}\right\}, \quad i \geq 2.$$

Here, for a stopping time T with respect to a filtration $(\mathcal{F}_t)$ and a set $A \in \mathcal{F}_T$, we define T_A by $T_A(\omega) = T(\omega)$ if $\omega \in A$; $T_A(\omega) = +\infty$ otherwise. $\left(T^j\right)$ is defined by the same way from $\underline{T}^l$ and $\overline{T}^l$.

In the present situation, the filtration $\mathbf{G}^{(n)}$ consists of

$$\mathcal{G}_t^{(n)} = \mathcal{F}'_{\left(t-n^{-\xi}\right)_+} \times \mathcal{F}'', \quad t \in \mathbb{R}_+.$$

Lemma 8.1 *S^i and T^j are $\mathbf{G}^{(n)}$-stopping times.*

Proof. $\underline{S}^l$ and $\overline{S}^l$ are $\mathbf{G}^{(n)}$-stopping times. Since τ^1 is an $(\mathcal{F}_t)$-stopping time, $\tau^1_n = \tau^1 + \frac{1}{\sqrt{n}}$ is a $\mathbf{G}^{(n)}$-stopping time, hence, $\tau^1_n + \overline{S}^m$ is a $\mathbf{G}^{(n)}$-stopping time as well. Moreover, since $\left\{\underline{S}^l < \tau^1_n\right\} \in \mathcal{G}^{(n)}_{\underline{S}^l}$, $\underline{S}^l_{\left\{\underline{S}^l < \tau^1_n\right\}}$ is a $\mathbf{G}^{(n)}$-stopping time. Therefore, S^1 is a $\mathbf{G}^{(n)}$-stopping time as well.

Suppose for now that S^{i-1} is a $\mathbf{G}^{(n)}$-stopping time. Then it is true that $\left\{S^{i-1} < \tau^1_n + \overline{S}^m\right\} \in \mathcal{G}^{(n)}_{\tau^1_n + \overline{S}^m}$. At the same time, $\left\{S^{i-1} < \underline{S}^l < \tau^1_n\right\} = \left\{S^{i-1} < \underline{S}^l\right\} \cap \left\{\underline{S}^l < \tau^1_n\right\} \in \mathcal{G}^{(n)}_{\underline{S}^l}$. These facts implies that S^i is a $\mathbf{G}^{(n)}$-stopping time, as asserted. $\square$

Consequently, we have

$$\begin{aligned}
nH^1_n(t) &\xrightarrow{p} \frac{2}{\underline{p}^1}\left(\tau^1 \wedge t\right) + \frac{2}{\overline{p}^1}\left(t - \tau^1 \wedge t\right) =: \int_0^t h^1_s ds, \\
nH^2_n(t) &\xrightarrow{p} \frac{2}{\underline{p}^2}\left(\tau^2 \wedge t\right) + \frac{2}{\overline{p}^2}\left(t - \tau^2 \wedge t\right) =: \int_0^t h^2_s ds, \\
nH^{1\cap 2}_n(t) &\xrightarrow{p} \frac{2}{\underline{p}^1 + \underline{p}^2}\left(\tau^1 \wedge \tau^2 \wedge t\right) + \frac{2}{\underline{p}^1 + \overline{p}^2}\left(\tau^2 \wedge t - \tau^1 \wedge \tau^2 \wedge t\right) \\
&+ \frac{2}{\overline{p}^1 + \underline{p}^2}\left(\tau^1 \wedge t - \tau^1 \wedge \tau^2 \wedge t\right) + \frac{2}{\overline{p}^1 + \overline{p}^2}\left(t - \left(\tau^1 \vee \tau^2\right) \wedge t\right) =: \int_0^t h^{1\cap 2}_s ds, \\
nH^{1*2}_n(t) &\xrightarrow{p} \left(\frac{2}{\underline{p}^1} + \frac{2}{\underline{p}^2}\right)\left(\tau^1 \wedge \tau^2 \wedge t\right) + \left(\frac{2}{\underline{p}^1} + \frac{2}{\overline{p}^2}\right)\left(\tau^2 \wedge t - \tau^1 \wedge \tau^2 \wedge t\right) \\
&+ \left(\frac{2}{\overline{p}^1} + \frac{2}{\underline{p}^2}\right)\left(\tau^1 \wedge t - \tau^1 \wedge \tau^2 \wedge t\right) + \left(\frac{2}{\overline{p}^1} + \frac{2}{\overline{p}^2}\right)\left(t - \tau \wedge t\right) =: \int_0^t h^{1*2}_s ds,
\end{aligned}$$

as $n \to \infty$ for every t. Then, under [A3′],

$$b_n^{-1/2} M^n \xrightarrow{d_s} \int_0^\cdot w_s d\widetilde{W}_s,$$

where

$$w_s = \begin{cases}
\sqrt{\left(\sigma^1_s\sigma^2_s\right)^2\left(\frac{2}{\underline{p}^1} + \frac{2}{\underline{p}^2}\right) + \left(\rho_s\sigma^1_s\sigma^2_s\right)^2\left(\frac{2}{\underline{p}^1} + \frac{2}{\underline{p}^2} - \frac{2}{\underline{p}^1 + \underline{p}^2}\right)} & (s \le \tau^1 \wedge \tau^2) \\
\sqrt{\left(\sigma^1_s\sigma^2_s\right)^2\left(\frac{2}{\underline{p}^1} + \frac{2}{\overline{p}^2}\right) + \left(\rho_s\sigma^1_s\sigma^2_s\right)^2\left(\frac{2}{\underline{p}^1} + \frac{2}{\overline{p}^2} - \frac{2}{\underline{p}^1 + \overline{p}^2}\right)} 1_{\{\tau^1 \le \tau^2\}} & \\
+ \sqrt{\left(\sigma^1_s\sigma^2_s\right)^2\left(\frac{2}{\overline{p}^1} + \frac{2}{\underline{p}^2}\right) + \left(\rho_s\sigma^1_s\sigma^2_s\right)^2\left(\frac{2}{\overline{p}^1} + \frac{2}{\underline{p}^2} - \frac{2}{\overline{p}^1 + \underline{p}^2}\right)} 1_{\{\tau^1 > \tau^2\}} & (\tau^1 \wedge \tau^2 < s \le \tau^1 \vee \tau^2) \\
\sqrt{\left(\sigma^1_s\sigma^2_s\right)^2\left(\frac{2}{\overline{p}^1} + \frac{2}{\overline{p}^2}\right) + \left(\rho_s\sigma^1_s\sigma^2_s\right)^2\left(\frac{2}{\overline{p}^1} + \frac{2}{\overline{p}^2} - \frac{2}{\overline{p}^1 + \overline{p}^2}\right)} & (\tau^1 \vee \tau^2 < s)
\end{cases}$$

and $\widetilde{W}$ is an independent Brownian motion. An example of such τ^k's are boundary hitting times $\tau^1 = \inf\left\{t > 0 : X_t > K^1\right\}$, $\tau^2 = \inf\left\{t > 0 : Y_t > K^2\right\}$, $K^1, K^2 \in (0, \infty)$.

9 Comments on application to finance

The application to finance is not the primary object of this paper, however we give some comments in this section. Since the last decade, intraday financial time-series, so-called high-frequency data, have been becoming increasingly available both in coverage and information contents. The use of high-frequency data has been expected to improve dramatically financial risk managements; one of such applications includes the estimation of variance-covariance structure of the financial markets, which is an essential routine operation for all the financial institutions.

In the literature, it is standard to use *realized volatility* (or *realized variance*) for estimating integrated variance when asset returns are regarded to be sampled from diffusion-type processes.

Likewise, when the integrated covariance is of interest, the use of *realized covariance* is fairly common. Nevertheless, the standard realized covariance estimator has a fundamental flaw in its structure when it is applied to multivariate *tick-by-tick* data, where time-series are recorded irregularly, in a nonsynchronous manner. The realized covariance estimators that have been used commonly involve an interpolation of irregularly sampled data to generate artificial data on a certain equi-spaced grid to apply a standard method for synchronized data. In Hayashi and Yoshida (2005b), we proved that such a naïve method inevitably causes estimation bias, which had been known empirically as the Epps effect when the defining regular interval size is small relative to the frequency of observations. In the same paper, the authors proposed how to circumvent such bias by proposing a new estimator, which is nowadays called the "Hayashi-Yoshida estimator," and showed that the estimator is consistent as the mesh size of observation intervals tends to zero in probability. This paper has been motivated by the quest for a limit distribution of the estimation error.

In the literature, asymptotic distribution theories for realized volatility type quantities have been developed; additionally to the literature given in Introduction, e.g., Jacod (1994), Jacod and Protter (1998), Barndorff-Nielsen and Shephard (2004), and Mykland and Zhang (2006). Differently from them, in this paper we we have dealt with random sampling schemes that are dependent on the underlying processes and it is far from straightforward. Rather it has demanded us to put forth the new set of ideas and technical tools. That is, we cannot simply conduct analysis to condition on the sampling times all the way up to the infinite future at a time and regarding them all as deterministic, as most of the existing results with random but independent sampling schemes do.

To endorese our point, the readers may recall the fact that even in the univariate case there is a striking scarcity of studies which take such dependency into account. Let alone, our treatment of the bivariate case together with nonsynchronicity is new.

In this paper, we did not include discussions on the microstructure noise. It is common in the literature so far to apply a pre-averaging to get back to a classical synchronous sampling. However, the goal of this article lies in developing a new methodology to treat the nonsynchronicity itself. Recently, Robert and Rosenbaum (2008) gave a new insight into the HY-estimator under microstructure noise. See also Ubukata and Oya (2008).

10 Proof of Proposition 3.2

For computational ease, we introduce the following two point processes

$$\begin{aligned}
\underline{K}^{ij}_t &= 1_{[R_\vee(i,j)\wedge R^\wedge(i,j),\infty)}(t) - 1_{[R^\wedge(i,j),\infty)}(t) = 1_{[R_\vee(i,j)\wedge R^\wedge(i,j),R^\wedge(i,j))}(t),\\
\overline{K}^{ij}_t &= \left(1_{(R_\vee(i,j),\infty)}\cdot 1_{[R^\wedge(i,j),\infty)}\right)_t \qquad \text{(stochastic integral)}
\end{aligned}$$

which give orthogonal decomposition of K^{ij}.

Lemma 10.1 **(a)** *$K_t^{ij} = 1_{\{R_\vee(i,j)<R^\wedge(i,j),\, R_\vee(i,j)\le t\}}$, $\underline{K}_t^{ij} = 1_{\{R_\vee(i,j)\le t<R^\wedge(i,j)\}}$ and $\overline{K}_t^{ij} = 1_{\{R_\vee(i,j)<R^\wedge(i,j),\, R^\wedge(i,j)\le t\}}$.*
(b) *K^{ij}, $\underline{K}^{ij}$, and $\overline{K}^{ij}$ are $(\mathcal{F}_t)$-adapted processes with*

$$K^{ij} = \underline{K}^{ij} + \overline{K}^{ij} \text{ and } \underline{K}^{ij}\overline{K}^{ij} \equiv 0. \tag{10.1}$$

In addition,

$$K^{ij} \equiv 0 \iff \underline{K}^{ij} \equiv 0 \iff \overline{K}^{ij} \equiv 0. \tag{10.2}$$

Proof. Easy and omitted. □

Proof of Propostion 3.2. In light of (10.1), we decompose the target quantity as

$$\begin{aligned}
&\sum_{i,j,i',j'} \left(K_-^{ij}K_-^{i'j'}\right)\cdot V_t^{iji'j'} \\
&\equiv \sum_{i,j,i',j'} \left(\underline{K}_-^{ij}\underline{K}_-^{i'j'}\right)\cdot V_t^{iji'j'} + \sum_{i,j,i',j'} \left(\overline{K}_-^{ij}\overline{K}_-^{i'j'}\right)\cdot V_t^{iji'j'} + 2\sum_{i,j,i',j'} \left(\underline{K}_-^{ij}\overline{K}_-^{i'j'}\right)\cdot V_t^{iji'j'} \\
&=: \mathbb{I} + \mathbb{II} + \mathbb{III}.
\end{aligned}$$

(a) Consider $\mathbb{I}$ first. Recall that $\underline{K}^{ij}$ identifies the pair (i,j) uniquely, i.e.,

$$\underline{K}^{ij}\underline{K}^{i'j'} \not\equiv 0 \;\Rightarrow\; \left[i = i' \text{ and } j = j'\right].$$

So,

$$\mathbb{I} = \sum_{i,j} \underline{K}_-^{ij}\cdot V_t^{ij} = \sum_{i,j} 1_{[R_\vee(i,j)\wedge R^\wedge(i,j),R^\wedge(i,j))}\cdot V_t^{ij} = \sum_{i,j}\left\{V^{ij}_{R^\wedge(i,j)\wedge t} - V^{ij}_{R_\vee(i,j)\wedge R^\wedge(i,j)\wedge t}\right\}.$$

Because

$$V^{ij} = \left\{I_-^i\cdot[X]\right\}\left\{J_-^j\cdot[Y]\right\} + \left\{\left(I_-^iJ_-^j\right)\cdot[X,Y]\right\}^2, \tag{10.3}$$

one has

$$V^{ij}_{R^\wedge(i,j)\wedge t} = [X]\left(I^i\left(R^\wedge(i,j)\wedge t\right)\right)[Y]\left(J^j\left(R^\wedge(i,j)\wedge t\right)\right) + [X,Y]\left(\left(I^i\cap J^j\right)(t)\right)^2. \tag{10.4}$$

On the other hand,

$$\begin{aligned}
V^{ij}_{R_\vee(i,j)\wedge R^\wedge(i,j)\wedge t} &= [X]\left(I^i\left(R_\vee(i,j)\wedge R^\wedge(i,j)\wedge t\right)\right)[Y]\left(J^j\left(R_\vee(i,j)\wedge R^\wedge(i,j)\wedge t\right)\right) \\
&\quad + \left\{[X,Y]\left(\left(I^i\cap J^j\right)\left(R_\vee(i,j)\wedge R^\wedge(i,j)\wedge t\right)\right)\right\}^2 = 0.
\end{aligned}$$

Thus it follows that

$$\mathbb{I} = \sum_{i,j}[X]\left(I^i\left(R^\wedge(i,j)\wedge t\right)\right)[Y]\left(J^j\left(R^\wedge(i,j)\wedge t\right)\right) + \sum_{i,j}\left\{[X,Y]\left(\left(I^i\cap J^j\right)(t)\right)\right\}^2.$$

(b) Next consider $\mathbb{II}$. We decompose it as

$$\sum_{i,j,i',j'} = \sum_{\substack{i,i',j,j':\\ i=i',j=j'}} + \sum_{\substack{i,i',j,j':\\ i=i',j\neq j'}} + \sum_{\substack{i,i',j,j':\\ i\neq i',j=j'}} + \sum_{\substack{i,i',j,j':\\ i\neq i',j\neq j'}}. \tag{10.5}$$

The following argument is motivated by Hayashi and Yoshida (2005b).

Case 1: $i = i'$ and $j = j'$. Recall (10.3). When a pair (i, j) does not overlap, i.e., $K^{ij} \equiv 0$, so $\overline{K}^{ij} \equiv 0$. Therefore $\overline{K}^{ij}_{-} \cdot V^{ij} \equiv 0$. When (i, j) overlaps, $\overline{K}^{ij}_{-} \cdot V^{ij}_t = V^{ij}_t - V^{ij}_{R^\wedge(i,j)\wedge t}$. However, the second term becomes (10.4). To put together,

$$\sum_{i,j} \overline{K}^{ij}_{-} \cdot V^{ij}_t = \sum_{i,j} [X]\left(I^i(t)\right)[Y]\left(J^j(t)\right)K^{ij}_t$$
$$- \sum_{i,j} [X]\left(I^i\left(R^\wedge(i,j)\wedge t\right)\right)[Y]\left(J^j\left(R^\wedge(i,j)\wedge t\right)\right).$$

Case 2: $i = i'$ and $j \neq j'$. According to (3.4),

$$V^{ijij'} = [X, Y]\left(I^i \cap J^j\right)[X, Y]\left(I^i \cap J^{j'}\right),$$

which *stops* varying for $t \geq R^\wedge(i, j) \vee R^\wedge(i, j')$. Note that when either pair (i, j) or (i, j') does not overlap, $\left(\overline{K}^{ij}_{-}\overline{K}^{ij'}_{-}\right) \cdot V^{ijij'} \equiv 0$. For two pairs (i, j) and (i, j'), $j < j'$, that overlap at the same time,

$$\left(\overline{K}^{ij}_{-}\overline{K}^{ij'}_{-}\right) \cdot V^{ijij'}_t = \overline{K}^{ij'}_{-} \cdot V^{ijij'}_t = V^{ijij'}_t - V^{ijij'}_{R^\wedge(i,j')\wedge t} \equiv 0.$$

Therefore, $\sum_{\substack{i,i',j,j':\\ i=i',j\neq j'}} \left(\overline{K}^{ij}_{-}\overline{K}^{i'j'}_{-}\right) \cdot V^{iji'j'} \equiv 0$.

Case 3: $i \neq i'$ and $j = j'$. By symmetry, $\sum_{\substack{i,i',j,j':\\ i\neq i',j=j'}} \left(\overline{K}^{ij}_{-}\overline{K}^{i'j'}_{-}\right) \cdot V^{iji'j'} \equiv 0$.

Case 4: $i \neq i'$ and $j \neq j'$. According to (3.4), $V^{iji'j'} = [X, Y]\left(I^i \cap J^{j'}\right)[X, Y]\left(I^{i'} \cap J^j\right)$. Hence, for $V^{iji'j'} \not\equiv 0$, both pairs (i, j') and (i', j) must overlap at the same time, i.e., it must be that $K^{ij'} \not\equiv 0$ and $K^{i'j} \not\equiv 0$. In order that $\left(\overline{K}^{ij}_{-}\overline{K}^{i'j'}_{-}\right) \cdot V^{iji'j'} \not\equiv 0$, it is necessary that $K^{ij} \not\equiv 0$ and $K^{i'j'} \not\equiv 0$. However, these two conditions are incompatible (i.e., (i, j), (i', j), (i, j'), and (i', j) cannot respectively overlap at the same time). Consequently, it follows that

$$\sum_{\substack{i,i',j,j':\\ i\neq i',j\neq j'}} \left(\overline{K}^{ij}_{-}\overline{K}^{i'j'}_{-}\right) \cdot V^{iji'j'} \equiv 0.$$

To put the four sub-cases together, we obtain

$$\mathbb{III} = \sum_{i,j} [X]\left(I^i(t)\right)[Y]\left(J^j(t)\right)K^{ij}_t - \sum_{i,j} [X]\left(I^i\left(R^\wedge(i,j)\wedge t\right)\right)[Y]\left(J^j\left(R^\wedge(i,j)\wedge t\right)\right).$$

(c) Consider $\mathbb{IIII}$. We again decompose it as (10.5) in (B).

Case 1: $i = i'$ and $j = j'$. Recall that $\underline{K}^{ij} = 1_{[R_\vee(i,j)\wedge R^\wedge(i,j), R^\wedge(i,j))}$ and $\overline{K}^{ij} = 1_{(R_\vee(i,j),\infty)} \cdot 1_{[R^\wedge(i,j),\infty)}$. They are orthogonal when $i = i'$ and $j = j'$, i.e.,

$$\underline{K}^{ij}\overline{K}^{i'j'} \equiv 0 \text{ for } i = i' \text{ and } j = j'.$$

Hence, Case 1 ($i = i'$ and $j = j'$) has no contribution.

Case 2 $i = i'$ and $j \neq j'$. Evidently,

$$\underline{K}^{ij}\overline{K}^{i'j'} \equiv 0 \text{ for } i = i' \text{ and } j < j'.$$

Suppose $i = i'$ and $j > j'$. Then, $\underline{K}^{ij}\overline{K}^{ij'} \equiv \underline{K}^{ij}$ so far as $K^{ij'} \not\equiv 0$ (or (i, j') overlaps). Note that

$$V^{ijij'} = [X, Y] \left(I^i \cap J^j\right) [X, Y] \left(I^i \cap J^{j'}\right).$$

So, if (i, j') overlaps, then

$$\begin{aligned}\left(\underline{K}_-^{ij}\overline{K}_-^{ij'}\right) \cdot V_t^{ijij'} &= \underline{K}_-^{ij} \cdot V_t^{ijij'} = V_{R^\wedge(i,j)\wedge t}^{ijij'} - V_{R_\vee(i,j)\wedge R^\wedge(i,j)\wedge t}^{ijij'} \\ &= [X, Y] \left(\left(I^i \cap J^j\right)(t)\right) [X, Y] \left(\left(I^i \cap J^{j'}\right)(t)\right),\end{aligned}$$

noting that (sup $\left(I^i \cap J^{j'}\right) \le$) sup $\left(I^i \cap J^j\right) \le R^\wedge(i, j)$ and $R_\vee(i, j) \wedge R^\wedge(i, j) \le \inf\left(I^i \cap J^j\right)$. Clearly, the last expression includes the case when (i, j') does not overlap because then both l.h.s. and r.h.s. are trivially zero. It follows that

$$\begin{aligned}\sum_{\substack{i,i',j,j':\\ i=i',j\neq j'}} \left(\underline{K}_-^{ij}\overline{K}_-^{i'j'}\right) \cdot V^{iji'j'} &= \sum_{\substack{i,j,j':\\ j>j'}} [X, Y] \left(I^i \cap J^j\right) [X, Y] \left(I^i \cap J^{j'}\right) \\ &= \frac{1}{2} \sum_{\substack{i,j,j':\\ j\neq j'}} [X, Y] \left(I^i \cap J^j\right) [X, Y] \left(I^i \cap J^{j'}\right) \\ &= \frac{1}{2} \sum_i \sum_j [X, Y] \left(I^i \cap J^j\right) \left(\sum_{j'} [X, Y] \left(I^i \cap J^{j'}\right) - [X, Y] \left(I^i \cap J^j\right)\right) \\ &= \frac{1}{2} \sum_i [X, Y] \left(I^i\right)^2 - \frac{1}{2} \sum_{i,j} [X, Y] \left(I^i \cap J^j\right)^2.\end{aligned}$$

Case 3: $i \neq i'$ and $j = j'$. By symmetry,

$$\sum_{\substack{i,i',j,j':\\ i\neq i',j=j'}} \left(\underline{K}_-^{ij}\overline{K}_-^{i'j'}\right) \cdot V^{iji'j'} = \frac{1}{2} \sum_j [X, Y] \left(J^j\right)^2 - \frac{1}{2} \sum_{i,j} [X, Y] \left(I^i \cap J^j\right)^2.$$

Case 4 $i \neq i'$ and $j \neq j'$. Note that in this case

$$V^{iji'j'} = [X, Y] \left(I^i \cap J^{j'}\right) [X, Y] \left(I^{i'} \cap J^j\right).$$

Now, that $I^i \cap J^j = \varnothing$ or $I^{i'} \cap J^{j'} = \varnothing$ implies that $\underline{K}_-^{ij}\overline{K}_-^{i'j'} \equiv 0$, which entails that $\left(\underline{K}_-^{ij}\overline{K}_-^{i'j'}\right) \cdot V^{iji'j'} \equiv 0$. On the other hand, due to the geometric relationships among the four distinct intervals I^i, $I^{i'}$, J^j, and $J^{j'}$, that $I^i \cap J^j \neq \varnothing$ and $I^{i'} \cap J^{j'} \neq \varnothing$ implies that $I^i \cap J^{j'} = \varnothing$ or $I^{i'} \cap J^j = \varnothing$, and hence $V^{iji'j'} \equiv 0$, which induces that $\left(\underline{K}_-^{ij}\overline{K}_-^{i'j'}\right) \cdot V^{iji'j'} \equiv 0$. After all, in this case,

$$\sum_{\substack{i,i',j,j':\\ i\neq i',j\neq j'}} \left(\underline{K}_-^{ij}\overline{K}_-^{i'j'}\right) \cdot V^{iji'j'} \equiv 0.$$

Gathering the four sub-cases together, we have

$$\mathbb{IIII} = \sum_i [X, Y] \left(I^i(t)\right)^2 + \sum_j [X, Y] \left(J^j(t)\right)^2 - 2 \sum_{i,j} [X, Y] \left(\left(I^i \cap J^j\right)(t)\right)^2.$$

(d) Finally, we put the three components in (A)-(C) together to obtain $\mathbb{I} + \mathbb{III} + \mathbb{IIII} = \bar{V}_t^n$. □

11 Proof of Lemma 4.1

By the mean value theorem, we can find a (random) time point $\xi^i \in I^i(t)$ to show that, under [A1′],

$$\begin{aligned} b_n^{-1}\sum_i [X,Y]\left(I^i(t)\right)^2 &= b_n^{-1}\sum_i\left(\int_0^t [X,Y]'_s I^i_s ds\right)^2 = b_n^{-1}\sum_i\left([X,Y]'_{\xi^i}\right)^2\left|I^i(t)\right|^2 \\ &\xrightarrow{p} \int_0^t \left([X,Y]'_s\right)^2 H^1(ds) \end{aligned}$$

as $n\to\infty$, for every t. We obtained (i); (ii) and (iii) can be shown similarly.

Let us prove (iv). We have

$$b_n^{-1}\sum_{i,j}[X]\left(I^i(t)\right)[Y]\left(J^j(t)\right)K^{ij}_t = b_n^{-1}\sum_{i,j=1}^{\infty}\sum_{l=1}^{L(n)}\int_0^t\int_0^t [X]'_s[Y]'_u I^i_s J^i_u 1_{A_l}(s,u)K^{ij}_t dsdu, \tag{11.1}$$

where $A_l := [0,a_l)\times[0,a_l)\setminus[0,a_{l-1})\times[0,a_{l-1})$, with $0=a_0<\cdots<a_l<\cdots<a_{L(n)}=T$ arbitrary deterministic grid points such that $\delta_a^{(n)} := \max_{1\le l\le L(n)}|a_l-a_{l-1}|\to 0$ as $n\to\infty$. Moreover, the r.h.s. on (11.1) equals to

$$\begin{aligned} & b_n^{-1}\sum_l\int_0^t\int_0^t [X]'_s[Y]'_u\left(\sum_{i,j} I^i_s J^i_u 1_{A_l}(s,u)K^{ij}_t\right)dsdu \\ &\overset{(\mathsf{A})}{\simeq} b_n^{-1}\sum_l [X]'_{a_{l-1}}[Y]'_{a_{l-1}}\left(\int_0^t\int_0^t\sum_{i,j} I^i_s J^i_u 1_{A_l}(s,u)K^{ij}_t dsdu\right) \\ &= b_n^{-1}\sum_l [X]'_{a_{l-1}}[Y]'_{a_{l-1}}\left(H_n^{1*2}(a_l\wedge t)-H_n^{1*2}(a_{l-1}\wedge t)\right) \\ &= b_n^{-1}\int_0^t\left(\sum_l [X]'_{a_{l-1}}[Y]'_{a_{l-1}}1_{[a_{l-1},a_l)}(s)\right)H_n^{1*2}(ds) \\ &\overset{(\mathsf{B})}{\simeq} b_n^{-1}\int_0^t [X]'_s[Y]'_s H_n^{1*2}(ds) \xrightarrow{p} \int_0^t [X]'_s[Y]'_s H^{1*2}(ds), \end{aligned}$$

as $n\to\infty$, for every t, under (a-1'), where '$\simeq$' means that the difference goes to zero in probability.

It remains to validate the approximations (A) and (B). We refer the reader to Hayashi and Yoshida (2006) for the proof. □

12 Proof of Proposition 5.1

This section is devoted to the proof of Proposition 5.1. We need two types of modifications of sampling times as stated below. We write $\bar r_n = b_n^{\xi'}$ and for given $T>0$, prepare a sequence of stopping times $\hat S^i$ and $\hat T^j$ defined by

$$\hat S^i = S^i\wedge\inf\left\{t;\ \max_{i'}\{S^{i'}\wedge t - S^{i'-1}\wedge t\}\ge \bar r_n\right\}\wedge(T+1)$$

and

$$\hat T^i = T^j\wedge\inf\left\{t;\ \max_{j'}\{T^{j'}\wedge t - T^{j'-1}\wedge t\}\ge \bar r_n\right\}\wedge(T+1)$$

Then $\hat{S}^i$ and $\hat{T}^j$ are $\mathbf{G}^{(n)}$-stopping times depending on n. Let $\hat{I}^i = [\hat{S}^{i-1}, \hat{S}^{i-1})$ and $\hat{J}^i = [\hat{T}^{i-1}, \hat{T}^{i-1})$. Let $\hat{\mathcal{I}} = (\hat{I}^i)$ and $\hat{\mathcal{J}} = (\hat{J}^j)$. Then for arbitrary sequence (υ_n) of $\mathbf{F}$-stopping times satisfying $\upsilon_n \leq T$ and $P[\upsilon_n < T] \to 0$ as $n \to \infty$, we have $P[\{X^{\upsilon_n}, Y^{\upsilon_n}; \hat{\mathcal{I}}, \hat{\mathcal{J}}\}_t = \{X, Y; \mathcal{I}, \mathcal{J}\}_t \text{ for all } t \in [0,T]] \to 1$, according to [A4]. Thus, we may assume that $\max\{|I^i|, |J^j|; i, j\} \leq \bar{r}_n$ in what follows and also that X and Y satisfy properties characterized by $\upsilon_n = T$.

We take ξ_0 so that $\xi < \xi_0 < \xi'$. Let $\tilde{\mathcal{G}}^{(n)}_t = \mathcal{F}_{(t-b_n^{\xi_0})_+}$ and $\tilde{\mathbf{G}}^{(n)} = (\tilde{\mathcal{G}}^{(n)}_t)_{t\in\mathbb{R}_+}$. We shall prepare a lemma to go to the second modification of stopping times.

Lemma 12.1 *Suppose that* $\max\{|J^j|; i, j\} \leq \bar{r}_n$ *and that* $b_n^{\xi} - \bar{r}_n > b_n^{\xi_0}$. $M^i := \sup_{j\in\mathbb{Z}_+: T^j \leq S^i} T^j$ *is a* $\tilde{\mathbf{G}}^{(n)}$*-stopping time for each* I^i.

Proof. Fix I^i and let

$$\mathcal{T}_j = \begin{cases} (S^i - \bar{r}_n)_+ & \text{on} \quad \{T^j > S^i\} \\ T^j & \text{on} \quad \{T^j \leq S^i\}. \end{cases}$$

Then $(S^i - \bar{r}_n)_+$ and $(T^j - \bar{r}_n)_+$ are $\tilde{\mathcal{G}}^{(n)}$-stopping times. Indeed, for $t \in \mathbb{R}_+$,

$$\{(S^i - \bar{r}_n)_+ \leq t\} = \{S^i \leq t + \bar{r}_n\} \in \mathcal{F}_{(t+\bar{r}_n - b_n^{\xi})_+} \subset \mathcal{F}_{(t-b_n^{\xi_0})_+} = \tilde{\mathcal{G}}^{(n)}_t.$$

Therefore $\{T^j > S^i\} \in \tilde{\mathbf{G}}^{(n)}_{S^i}$, and hence

$$\begin{aligned} \{(S^i - \bar{r}_n)_+ \leq t,\, T^j > S^i\} &= \{(S^i - \bar{r}_n)_+ \leq t,\, (T^j - \bar{r}_n)_+ > (S^i - \bar{r}_n)_+ > 0\} \\ &\quad \cup \{T^j > S^i,\, S^i \leq \bar{r}_n\} \in \tilde{\mathcal{G}}^{(n)}_t \end{aligned}$$

because $\{(T^j - \bar{r}_n)_+ > (S^i - \bar{r}_n)_+ > 0\} \in \tilde{\mathcal{G}}^{(n)}_{(S^i - \bar{r}_n)_+}$ and $\{T^j > S^i,\, S^i \leq \bar{r}_n\} \in \mathcal{G}^{(n)}_{\bar{r}_n} = \mathcal{F}_0 \subset \tilde{\mathcal{G}}^{(n)}_t$ for $t \in \mathbb{R}_+$. Moreover, $\{T^j \leq t,\, T^j \leq S^i\} \in \mathcal{G}^{(n)}_t \subset \tilde{\mathcal{G}}^{(n)}_t$. After all, $\{\mathcal{T}_j \leq t\} \in \tilde{\mathcal{G}}^{(n)}_t$, consequently all $\mathcal{T}_j$ are $\tilde{\mathbf{G}}^{(n)}$-stopping times.

Since $\sup_j |J^j| \leq \bar{r}_n$ and $T^0 = 0$, there is a $T^j \in [(S^i - \bar{r}_n)_+, S^i]$. Therefore, we see $M^i = \sup_{j\in\mathbb{Z}_+: T^j \leq S^i} T^j = \sup_{j\in\mathbb{R}_+} \mathcal{T}_j$ is a $\tilde{\mathbf{G}}^{(n)}$-stopping time. □

We will apply the reduction used in Hayashi and Yoshida (2004) to every realization of (I^i) and (J^j). That is, we combine J^j's into one for $J^j \subset I^i$, for each $i \geq 1$ (do nothing if there is no such J^j), then relabel the index j from left to right. Denote the newly created design by $(\widetilde{J}^j)$, with the associated stopping times $\widetilde{T}^j$. We refer to the operation as $\mathcal{J}$-reduction; $\mathcal{I}$-reduction can be made in the same manner. We refer to the joint operation as $(\mathcal{I}, \mathcal{J})$-reduction.

Consider sufficiently large n. For each I^i, $N^i := \min_{j\in\mathbb{Z}_+: T^j \geq S^{i-1}} T^j$ is a stopping time with respect to $\mathbf{G}^{(n)}$, in force to $\tilde{\mathbf{G}}^{(n)}$. According to Lemma 12.1, M^i are also $\tilde{\mathbf{G}}^{(n)}$-stopping times. While some of N^i, M^i ($i \in \mathbb{N}$) have the same values, we line those times up to obtain $\tilde{T}^j$ again. Routinely, it turns out that $\tilde{T}^j$ are $\tilde{\mathbf{G}}^{(n)}$-stopping times; indeed, $\tilde{T}^0 = 0$ and $\tilde{T}^i = \inf_l \{N^l_{\{N^l > \tilde{T}^{i-1}\}},\, M^l_{\{M^l > \tilde{T}^{i-1}\}}\}$ for $i \in \mathbb{N}$.

Due to the bilineality, $\{X, Y; \mathcal{I}, \mathcal{J}\}_t = \{X, Y; \widetilde{\mathcal{I}}, \widetilde{\mathcal{J}}\}_t$ for $\widetilde{\mathcal{I}} = (\widetilde{I}^i)$ and $\widetilde{\mathcal{J}} = (\widetilde{J}^j)$. It should be noted that $r_n(t)$ is invariant under those reductions. Let $\widetilde{K}^{ij}_t := 1_{\{\widetilde{I}^i(t) \cap \widetilde{J}^j(t) \neq \varnothing\}}$. An advantage of the reduction is that

$$\sum_j \widetilde{K}^{ij}_t \leq 3 \quad \text{and} \quad \sum_i \widetilde{K}^{ij}_t \leq 3. \tag{12.1}$$

Moreover, since for each $\widetilde{I}^i$ (or $\widetilde{J}^j$), one can always find an interval I^i or an interval J^j that covers it,

$$\sum_i \left|\widetilde{I}^i(T)\right|^2 \vee \sum_j \left|\widetilde{J}^j(T)\right|^2 \le \sum_i \left|I^i(T)\right|^2 + \sum_j \left|J^j(T)\right|^2$$

Hence, the conditions [A4] and [A6] imposed for the original designs (I, J) will remain valid for $(\widetilde{I}^i, \widetilde{J}^j)$.

The above argument ensures that if we take ξ_0 close to ξ, all the conditions related to ξ are still fulfilled for ξ_0. Thus, we may assume throughout the proof that $(\mathcal{I}, \mathcal{J})$-reduction operation is already carried out. We will continue to use $\mathcal{I} = (I^i)$ and $\mathcal{J} = (J^j)$ to express those after reduction, as well as ξ in place of ξ_0. Hence (12.1) is assumed to hold for K^{ij}_t from the beginning. Moreover, $r_n(t) \le \bar{r}_n$ by the first modification just before Lemma 12.1. According to the above discussion, we may also assume that $4/5 < \xi < \xi' < 1$ in the sequel.

Set $\beta = \xi - \frac{2}{3}$, and $\alpha = \xi' - \frac{2}{3}$. Let $\gamma \in (0, \frac{10}{9}(\xi - \frac{4}{5}))$, $\epsilon_1 \in (0, \frac{1}{2})$ and $c_n = b_n^{-\frac{3}{4}\gamma}$. Define υ_n by

$$\begin{aligned}\upsilon_n &= \inf\left\{t; |[X]'_t| > c_n\right\} \wedge \inf\left\{t; |[Y]'_t| > c_n\right\} \wedge \inf\left\{t; |[X, Y]'_t| > c_n\right\} \\ &\quad \wedge \inf\left\{t; \sup_{\substack{(r,s):\ s\in[0,t] \\ r\in[(s-b_n^{\xi'})_+, s]}} \frac{|X_s - X_r| + |Y_s - Y_r|}{(s-r)^{1/2-\epsilon_1}} > 1\right\} \wedge T. \end{aligned} \tag{12.2}$$

By construction and from [A3], each υ_n is an $\mathbf{F}$-stopping time and $P[\upsilon_n = T] \to 1$ as $n \to \infty$. Of course, once the localization by υ_n is applied to X and Y, they will depend on n thereafter; however the properties assumed for the original X and Y are unchanged by this stopping, so we will not write "n" on them each time explicitly.

As noted before, we take a sufficiently large, deterministic number n_0 and only consider n such that $n \ge n_0$. In what follows, for arbitrarily given $\varepsilon \in \left(0, \frac{3}{8}\gamma\right)$, we can assume the inequality

$$w\left(X; r_n(T)\right) + w\left(Y; r_n(T)\right) \le b_n^{\frac{1}{2}\xi' - \varepsilon} \tag{12.3}$$

for all n. It is because of the stopping by υ_n and the fact that $r_n(T) \le b_n^{\xi'}$ for all n.

The proof for Proposition 5.1 essentially starts with the following lemma. Lemma 12.2 (i) will be used by Lemma 12.3 (i), which will in turn be used by Lemma 12.4 (i); in the meantime, Lemma 12.2 (ii) will be used by Lemma 12.3 (ii), which will in turn be used by Lemma 12.4 (ii). Lemmas 12.4 (i) as well as 12.5 (i) will be invoked from Lemma 12.6, while Lemmas 12.4(ii) as well as 12.5 (ii) from Lemma 12.7. Both Lemmas 12.6 and 12.7 constitute the main body of the proof of Proposition 5.1.

For notational simplicity, we introduce the symbols $R_\wedge(i, j) := S^{i-1} \wedge T^{j-1}$ and $R^\vee(i, j) := S^i \vee T^j$, in addition to $R_\vee(i, j) = S^{i-1} \vee T^{j-1}$ and $R^\wedge(i, j) = S^i \wedge T^j$ already defined. Notice that they all are $\mathbf{G}^{(n)}$-stopping times with obvious relationships such as $R_\wedge(i, j) \le R_\vee(i, j)$ and $R^\wedge(i, j) \le R^\vee(i, j)$.

By convention, given a class $\mathcal{C}$ of subsets of Ω and a set $A \subset \Omega$, we denote $\mathcal{C} \cap A := \{C \cap A; C \in \mathcal{C}\}$. We may suppose $0 < b_n < 1$ hereafter.

Lemma 12.2 *Suppose* [A2] *and let* $s, t \in [0, T]$ *and* $i, i', i'', j, j' \ge 1$.

(i) *For* $j \le j'$,

$$\mathcal{G}^{(n)}_{s\vee t} \cap \left\{s \vee t < R^\vee(i, j')\right\} \cap \left\{I^i(t) \cap J^j(t) \ne \varnothing\right\} \cap \left\{I^i(s) \cap J^{j'}(s) \ne \varnothing\right\} \subset \mathcal{F}_{R_\wedge(i,j)}.$$

(ii) *For $i \geq i', i''$,*

$$\mathcal{G}^{(n)}_{s\vee t} \cap \{s \vee t < R^{\vee}(i, j \vee j')\} \cap \{I^i(t) \cap J^j(t) \neq \varnothing\} \cap \left\{I^{i'}(t) \cap J^j(t) \neq \varnothing\right\}$$
$$\cap \left\{I^i(s) \cap J^{j'}(s) \neq \varnothing\right\} \cap \left\{I^{i''}(s) \cap J^{j'}(s) \neq \varnothing\right\} \subset \mathcal{F}_{R_\wedge(i'\wedge i'', j\wedge j')}.$$

Proof. We will use repeatedly the simple facts that for any **F**-stopping times σ and τ, $\mathcal{F}_\sigma \cap \{\sigma \leq \tau\} \subset \mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$, in particular $\{\sigma \leq \tau\} \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$ and that $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$ if $\sigma \leq \tau$.

(i): Suppose $j \leq j'$. It suffices to show that $A \cap B \cap C \cap D \in \mathcal{F}_u$ for $A \in \mathcal{G}^{(n)}_{s\vee t}$,

$$\begin{aligned} B &= \{s \vee t < R^\vee(i, j')\} \\ C &= \left\{I^i(t) \cap J^j(t) \neq \varnothing, I^i(s) \cap J^{j'}(s) \neq \varnothing\right\} \\ D &= \{R_\wedge(i, j) \leq u\}, \end{aligned}$$

$u \in \mathbb{R}_+$. We have

$$C \cap D = C \cap D \cap \{R^\vee(i, j') < u + 3b_n^{\xi'}\} \tag{12.4}$$

due to the first reduction we mentioned before because the two pairs (i, j) and (i, j') respectively overlap at the same time on C. Since $A \in \mathcal{G}^{(n)}_{s\vee t}$ and $C \in \mathcal{G}^{(n)}_{s\vee t}$, we have $(A \cap C) \cap B \in \mathcal{G}^{(n)}_{R^\vee(i,j')}$. Thus,

$$A \cap B \cap C \cap \left\{R^\vee(i, j') < u + 3b_n^{\xi'}\right\} \in \mathcal{G}^{(n)}_{u+3b_n^{\xi'}},$$

however, $\mathcal{G}^{(n)}_{u+3b_n^{\xi'}} = \mathcal{F}_{u+b_n^{2/3}(3b_n^\alpha - b_n^\beta)} \subset \mathcal{F}_u$ because $\alpha > \beta$ and $0 < b_n < 1$. This together with the fact that $\{R_\wedge(i, j) \leq u\} \in \mathcal{F}_u$ implies $A \cap B \cap C \cap D \in \mathcal{F}_u$ for any u.
(ii): Consideration similar to (ii) can be made. When four pairs (i, j), (i', j), (i, j'), (i'', j'), $(i \geq i', i'')$ respectively overlap at the same time, the total length of the associated combined interval $\left(I^i \cup I^{i'} \cup I^{i''} \cup J^j \cup J^{j'}\right)$ must be confined as $R^\vee(i, j \vee j') - R_\wedge(i' \wedge i'', j \wedge j') \leq 4b_n^{\xi'}$,; note that $R_\wedge(i' \wedge i'', j \wedge j') = S^{i'-1} \wedge S^{i''-1} \wedge T^{j-1} \wedge T^{j'-1}$. This leads to an identity similar to (12.4), from which one can prove (ii) in the same fashion as (i). □

Remark 12.1. It can also be shown that

$$\mathcal{G}^{(n)}_t \cap \{t < S^i\} \subset \mathcal{F}_{S^{i-1}},\ \mathcal{G}^{(n)}_t \cap \{t < T^j\} \subset \mathcal{F}_{T^{j-1}},$$
$$\mathcal{G}^{(n)}_t \cap \{t < R^\vee(i, j)\} \cap \{I^i(t) \cap J^j(t) \neq \varnothing\} \subset \mathcal{F}_{R_\wedge(i,j)}.$$

Let $H^{ij}_t := I^i_t + J^j_t - I^i_t J^j_t = \left(I^i \cup J^j\right)_t$, which is the indicator of the union of the intervals I^i and J^j up to time t. Let $\Xi^{ii'j}_t = K^{ij}_t K^{i'j}_t J^j_t$ and $\Lambda^{ij}_t = K^{ij}_t H^{ij}_t$.

Lemma 12.3 *Suppose that* [*A*2] *is satisfied. Let* (Z_t) *and* (Z'_t) *be* $\mathbf{G}^{(n)}$*-progressively measurable processes. Let* $s, t > 0$*, and* $i, i', i'', j, j' \geq 1$*. Then*

(i) $\Lambda^{ij}_t \Lambda^{ij'}_s Z_t Z'_s \in \mathcal{F}_{R_\wedge(i,j)}$ *for* $j \leq j'$*.*

(ii) $\Xi^{ii'j}_t \Xi^{ii''j'}_s Z_t Z'_s \in \mathcal{F}_{R_\wedge(i'\wedge i'', j\wedge j')}$ *for* $i \geq i'$ *and* $i \geq i''$*.*

Proof. (i): Note that on $\left\{I^i(t) \cap J^j(t) = \varnothing\right\}$, $K_t^{ij} = 0$ in particular $\Lambda_t^{ij} = 0$, and also that

$$\{s \vee t \geq R^\vee(i,j')\} \subset \{H_s^{ij'} = 0\} \cup \{H_t^{ij} = 0\} \subset \{\Lambda_t^{ij}\Lambda_s^{ij'} = 0\}$$

for $j \leq j'$. For any s, t and Borel measurable set B,

$$\begin{aligned}
&\left\{\Lambda_t^{ij}\Lambda_s^{ij'} Z_t Z_s' \in B\right\} \\
&= \left[\{0 \in B\} \cap A(i,j,j',s,t)^c\right] \cup \left[\left\{\Lambda_t^{ij}\Lambda_s^{ij'} Z_t Z_s' \in B\right\} \cap A(i,j,j',s,t)\right] \\
&\in \mathcal{F}_{R_\wedge(i,j)}
\end{aligned}$$

by Lemma 12.2(i) because $\Lambda_t^{ij}\Lambda_s^{ij'} Z_t Z_s'$ is $\mathcal{G}_{s\vee t}^{(n)}$-measurable by construction.

(ii): A similar argument to (i) can apply with Lemma 12.2(ii) instead of (i). □

Remark 12.2. (i) implies that

$$\Lambda_t^{ij} Z_t = H_t^{ij} K_t^{ij} Z_t \in \mathcal{F}_{R_\wedge(i,j)}$$

by taking $s = t$, $j = j'$. (ii) implies that

$$\Xi_t^{ii'j} Z_t = J_t^j K_t^{ij} K_t^{i'j} Z_t \in \mathcal{F}_{R_\wedge(i',j)} \text{ for } i \geq i',$$

by taking $s = t$, $i' = i''$, and $j = j'$. By argument similar to the proof of Lemma 12.3, it can be shown that

$$\begin{aligned}
&I_t^i Z_t \in \mathcal{F}_{S^{i-1}}, \quad J_t^j Z_t \in \mathcal{F}_{T^{j-1}}, \quad I_t^i J_t^j Z_t \in \mathcal{F}_{R_\wedge(i,j)}, \\
&I_t^i J_t^{j'} K_t^{ij} K_t^{ij'} Z_t \in \mathcal{F}_{R_\wedge(i,j)} \ (j \leq j').
\end{aligned}$$

For $i \neq i'$ and $j < j'$, $\Lambda_t^{ij}\Lambda_s^{i'j'} Z_t Z_s' \in \mathcal{F}_{R_\wedge(i',j')}$. A similar result holds for the statement (ii).

For an $\mathbf{F}$-adapted process Z, we write $\widetilde{Z}_t = Z_{(t-b_n^\xi)_+}$. Then $\widetilde{Z}_t$ is clearly $\mathbf{G}^{(n)}$-adapted. Let $X^{ii'} = \left(I_-^i \cdot X\right)_- \cdot \left(I_-^{i'} \cdot X\right)$ for every i and i'. We notice that

$$X^{ii'} = 0 \text{ for } i > i'. \tag{12.5}$$

Lemma 12.4 *Suppose that [A2] and [A3] hold. Let $s, t \in \mathbb{R}_+$.*

(i) *For $i, i', j, j' \geq 1$ with $i \neq i'$ and $j \neq j'$,*

$$E\left[\Lambda_t^{ij}\Lambda_s^{i'j'}\widetilde{[X,Y]'}_t\widetilde{[X,Y]'}_s L_t^{ij} L_s^{i'j'}\right] = 0.$$

(ii) *For all $i, i', k, k', j, l \geq 1$ with $i \neq k$,*

$$E\left[\Xi_t^{ii'j}\Xi_s^{kk'l}\widetilde{[Y]'}_t\widetilde{[Y]'}_s X_t^{i'i} X_s^{k'k}\right] = 0.$$

Proof. (i): Note that, for the overlapping pairs (i,j) and (i',j'), $i<i'$ implies $j\le j'$ while $j<j'$ implies $i\le i'$, hence we can suppose $i<i'$ and $j<j'$ without loss of generality.

We first claim that, for every t, i and j,

$$E\left[L_t^{ij}\middle|\mathcal{F}_{R_\vee(i,j)}\right]=0.$$

In fact, $L_t^{ij}=1_{\{t>R_\vee(i,j)\}}L_t^{ij}$ because $L_t^{ij}=0$ for $t\le R_\vee(i,j)=S^{i-1}\vee T^{j-1}$ by definition, and hence the optional sampling theorem implies that

$$E\left[L_t^{ij}\middle|\mathcal{F}_{R_\vee(i,j)}\right]=1_{\{t>R_\vee(i,j)\}}E\left[L_t^{ij}\middle|\mathcal{F}_{R_\vee(i,j)}\right]=1_{\{t>R_\vee(i,j)\}}L_{t\wedge R_\vee(i,j)}^{ij}=0.$$

For $i<i'$ and $j<j'$, Lemma 12.3 (i) implies that $\Lambda_t^{ij}\widetilde{[X,Y]'}_t$ is $\mathcal{F}_{R_\wedge(i,j)}\subset\mathcal{F}_{R_\vee(i',j')}$-measurable and $\Lambda_s^{i'j'}\widetilde{[X,Y]'}_s$ is $\mathcal{F}_{R_\wedge(i',j')}\subset\mathcal{F}_{R_\vee(i',j')}$-measurable, for any t,s. Moreover, because L_t^{ij} is $\mathcal{F}_{R^\vee(i,j)}$-measurable (notice that L_t^{ij} stops at $t=R^\vee(i,j)$), it is $\mathcal{F}_{R_\vee(i',j')}$-measurable for any t. It follows that

$$\begin{aligned}&E\left[\Lambda_t^{ij}\Lambda_s^{i'j'}\widetilde{[X,Y]'}_t\widetilde{[X,Y]'}_sL_t^{ij}L_s^{i'j'}\middle|\mathcal{F}_{R_\wedge(i,j)}\right]\\&=E\left[\Lambda_t^{ij}\Lambda_s^{i'j'}\widetilde{[X,Y]'}_t\widetilde{[X,Y]'}_sL_t^{ij}E\left\{L_s^{i'j'}\middle|\mathcal{F}_{R_\vee(i',j')}\right\}\middle|\mathcal{F}_{R_\wedge(i,j)}\right]=0.\end{aligned}$$

(ii): We may assume that $i\ge i'$, $k\ge k'$ due to (12.5), and also that $i>k$ by symmetry. Lemma 12.3 (ii) or Remark 12 implies that $\Xi_t^{ii'j}\widetilde{[Y]'}_t$ is $\mathcal{F}_{S^{i'-1}}$-measurable; in the same way $\Xi_s^{kk'l}\widetilde{[Y]'}_s$ is $\mathcal{F}_{S^{k'-1}}$-measurable. Since $X_s^{k'k}$ is measurable with respect to $\mathcal{F}_{S^k}\subset\mathcal{F}_{S^{i-1}}$,

$$E\left[\Xi_t^{ii'j}\Xi_s^{kk'l}\widetilde{[Y]'}_t\widetilde{[Y]'}_sX_s^{k'k}\;E\left\{X_t^{i'i}\middle|\mathcal{F}_{S^{i-1}}\right\}\middle|\mathcal{F}_{S^{(i'\wedge k')-1}}\right].$$

The optional sampling theorem provides

$$E\left[X_t^{i'i}\middle|\mathcal{F}_{S^{i-1}}\right]=X_{t\wedge S^{i-1}}^{i'i}=0,$$

which concludes the proof. □

Lemma 12.5 *Suppose that* [A2] *and* [A3] *are satisfied.*

(i) *For every i and j, $\sup_{t\in[0,T]}|L_t^{ij}|\le c_nb_n^{\xi'}$ for all n.*

(ii) *For every i and i', $\sup_{t\in[0,T]}|X_t^{i'i}|\le c_nb_n^{\xi'}$ for all n.*

Proof. Since $L^{ij}=(I_-^i\cdot X)\times(J_-^j\cdot Y)-(I_-^iJ_-^j)\cdot[X,Y]$, we have

$$|L_t^{ij}|\le w\left(X;r_n(t)\right)w\left(Y;r_n(t)\right)+w\left([X,Y];r_n(t)\right).$$

By (12.3), $w(X;r_n(T))\le b_n^{\frac12\xi'-\varepsilon}$ and $w\left(Y;r_n(T)\right)\le b_n^{\frac12\xi'-\varepsilon}$ for all n. where $\varepsilon\in\left(0,\frac38\gamma\right)$. On the other hand, from (12.2),

$$w\left([X,Y];r_n(T)\right)\le c_nr_n(T)\le c_nb_n^{\xi'}.$$

Therefore we obtained (i). In the same fashion, from the inequality

$$|X_t^{i'i}| \leq w\left(X; r_n(t)\right)^2 + w\left([X]; r_n(t)\right),$$

we deduce (ii). $\square$

For the main body of the proof for Proposition 5.1, let us consider the gap in (3.5) without scaling and decompose it as

$$\begin{aligned} & \sum_{i,j,i',j'} \left(K_-^{ij} K_-^{i'j'}\right) \cdot \left[L^{ij}, L^{i'j'}\right]_t - \sum_{i,j,i',j'} \left(K_-^{ij} K_-^{i'j'}\right) \cdot V_t^{iji'j'} \\ (12.6) \qquad = \; & \Delta_{1,t} + \Delta_{2,t} + \Delta_{3,t}, \end{aligned}$$

where

$$\begin{aligned} \Delta_{1,t} &= \sum_{i,i',j,j'} \left(K_-^{ij} K_-^{i'j'}\right) \cdot \left[\left\{\left(I_-^i \cdot X\right)\left(I_-^{i'} \cdot X\right)\right\} \cdot \left\{\left(J_-^j J_-^{j'}\right) \cdot [Y]\right\}\right]_t \\ &\quad - \sum_{i,i',j,j'} \left(K_-^{ij} K_-^{i'j'}\right) \cdot \left[\left\{\left(I_-^i I_-^{i'}\right) \cdot [X]\right\} \cdot \left\{\left(J_-^j J_-^{j'}\right) \cdot [Y]\right\}\right]_t, \end{aligned}$$

$$\begin{aligned} \Delta_{2,t} &= \sum_{i,i',j,j'} \left(K_-^{ij} K_-^{ij'}\right) \cdot \left[\left\{\left(J_-^j \cdot Y\right)\left(J_-^{j'} \cdot Y\right)\right\} \cdot \left\{\left(I_-^i I_-^{i'}\right) \cdot [X]\right\}\right]_t \\ &\quad - \sum_{i,i',j,j'} \left(K_-^{ij} K_-^{ij'}\right) \cdot \left[\left\{\left(J_-^j J_-^{j'}\right) \cdot [Y]\right\} \cdot \left\{\left(I_-^i I_-^{i'}\right) \cdot [X]\right\}\right]_t \end{aligned}$$

and

$$\begin{aligned} \Delta_{3,t} &= \sum_{i,j,i',j'} \left(K_-^{ij} K_-^{i'j'}\right) \cdot \left[\left\{\left(I_-^i \cdot X\right)\left(J_-^{j'} \cdot Y\right)\right\} \cdot \left\{\left(J_-^j I_-^{i'}\right) \cdot [Y, X]\right\}\right]_t \\ &\quad - \sum_{i,j,i',j'} \left(K_-^{ij} K_-^{i'j'}\right) \cdot \left[\left\{\left(I_-^i J_-^{j'}\right) \cdot [X, Y]\right\} \cdot \left\{\left(J_-^j I_-^{i'}\right) \cdot [Y, X]\right\}\right]_t \\ &\quad + \sum_{i,j,i',j'} \left(K_-^{ij} K_-^{i'j'}\right) \cdot \left[\left\{\left(J_-^j \cdot Y\right)\left(I_-^{i'} \cdot X\right)\right\} \cdot \left\{\left(I_-^i J_-^{j'}\right) \cdot [X, Y]\right\}\right]_t \\ &\quad - \sum_{i,j,i',j'} \left(K_-^{ij} K_-^{i'j'}\right) \cdot \left[\left\{\left(J_-^j I_-^{i'}\right) \cdot [Y, X]\right\} \cdot \left\{\left(I_-^i J_-^{j'}\right) \cdot [X, Y]\right\}\right]_t. \end{aligned}$$

First, we show that $b_n^{-1}\Delta_{1,t}$ is asymptotically negligible. Let $I(J^j)_t = \sum_i K_t^{ij} I_t^i$ and $J(I^i)_t = \sum_j K^{ij} J_t^j$. Throughout the discussions, for sequences (x_n) and (y_n), $x_n \lesssim y_n$ means that there exists a constant $C \in [0, \infty)$ such that $x_n \leq C y_n$ for large n.

Lemma 12.6 *Under* [*A*2], [*A*3] *and* [*A*4], *it holds that* $b_n^{-1}\Delta_{1,t} \to^p 0$ *and* $b_n^{-1}\Delta_{2,t} \to^p 0$ *as* $n \to \infty$.

Proof. We note that $J_-^j J_-^{j'} \equiv 0$ whenever $j \neq j'$ and that $X^{ii'} = 0$ for $i > i'$, to rewrite $\Delta_{1,t}$ as

$$\Delta_{1,t} = 2\left\{\sum_{i \geq i', j} \left(K_-^{ij} K_-^{i'j} J_-^j\right) X^{i'i} [Y]' \cdot s\right\}_t.$$

Let $\mathcal{R}_t = [Y]'_t - \widetilde{[Y]'}_t$. We have $(\Delta_{1,T})^2 = 4(\mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV})$, where

$$\mathbf{I} = \sum_{i \geq i', j} \sum_{k \geq k', l} \int_0^T \int_0^T \Xi_t^{ii'j} \Xi_s^{kk'l} X_t^{i'i} X_s^{k'k} \widetilde{[Y]'}_t \widetilde{[Y]'}_s dt ds,$$

$$\mathbf{II} = \sum_{i \geq i', j} \sum_{k \geq k', l} \int_0^T \int_0^T \Xi_t^{ii'j} \Xi_s^{kk'l} X_t^{i'i} X_s^{k'k} \widetilde{[Y]'}_t \mathcal{R}_s dt ds,$$

$$\mathbf{III} = \sum_{i \geq i', j} \sum_{k \geq k', l} \int_0^T \int_0^T \Xi_t^{ii'j} \Xi_s^{kk'l} X_t^{i'i} X_s^{k'k} \widetilde{[Y]'}_s \mathcal{R}_t dt ds,$$

$$\mathbf{IV} = \sum_{i \geq i', j} \sum_{k \geq k', l} \int_0^T \int_0^T \Xi_t^{ii'j} \Xi_s^{kk'l} X_t^{i'i} X_s^{k'k} \mathcal{R}_t \mathcal{R}_s dt ds.$$

From [A3],

$$\sup_{t \in [0,T]} |\mathcal{R}_t| = O_p\left(b_n^{\xi\left(\frac{1}{2} - \lambda\right)}\right)$$

for any $\lambda > 0$. On the other hand,

$$\begin{aligned} \sum_{i \geq i', j} \int_0^T \Xi_t^{ii'j} dt &\leq \sum_j \left| J^j(T) \right| \sum_{i'} K_T^{i'j} \sum_{i' \leq i} K_T^{ij} \\ &\leq 9 \sum_j \left| J^j(T) \right| \leq 9T, \end{aligned}$$

thanks to (12.1). Consequently,

$$\begin{aligned} |\mathbf{II}| &\leq \sum_{i \geq i', j} \sum_{k \geq k', l} \int_0^T \int_0^T \Xi_t^{ii'j} \Xi_s^{kk'l} \left| X_t^{i'i} \right| \left| X_s^{k'k} \right| \widetilde{[Y]'}_t \left| \mathcal{R}_s \right| dt ds \\ &\leq \left(\max_{i,i'} (X^{i'i})_T^* \right)^2 \mathcal{R}_T^* (\widetilde{[Y]'})_T^* \left(\sum_{i \geq i', j} \int_0^T \Xi_t^{ii'j} dt \right)^2 \\ &\leq \left(c_n b_n^{\xi'} \right)^2 \cdot O_p\left(b_n^{\xi\left(\frac{1}{2} - \lambda\right)} \right) \cdot c_n \cdot 81T^2. \end{aligned}$$

Since $0 < \gamma < \frac{10}{9}(\xi - \frac{4}{5})$ and since $\lambda > 0$ can be taken arbitrarily small,

$$b_n^{-2} |\mathbf{II}| = O_p\left(b_n^{-2 + 2\xi' + \xi\left(\frac{1}{2} - \lambda\right) - \frac{9}{4}\gamma} \right) \leq O_p\left(b_n^{2(\xi' - \xi)} \right) = o_p(1).$$

In a similar manner, we can show that $b_n^{-1}\mathbf{III} = o_P(1)$ and $b_n^{-1}\mathbf{IV} = o_P(1)$.

Next, we will evaluate $E[\mathbf{I}]$. In light of Lemma 12.4 (ii), the terms contribute to the sum only when $i = k$. Thus, from (12.2) together with the aid of Lemma 12.5 (ii), we have,

$$|E[\mathbf{I}]| \leq c_n^2 b_n^{2\xi'} \cdot c_n^2 \cdot E\left[\sum_{i \geq i', j} \sum_{k', l} \int_0^T \int_0^T \Xi_t^{ii'j} \Xi_s^{i,k',l} dt ds \right].$$

Now,

$$\sum_{i\geq i',j}\sum_{k',l}\int_0^T\int_0^T \Xi_t^{ii'j}\Xi_s^{ik'l}dtds=\sum_i\left(\sum_{i',j:i'\leq i}\int_0^T \Xi_t^{ii'j}dt\right)^2,$$

however, for each i,

$$\sum_{i',j:i'\leq i}\int_0^T \Xi_t^{ii'j}dt=\sum_j\int_0^T K_t^{ij}J_t^j\left(\sum_{i':i'\leq i}K_t^{i'j}\right)dt\leq 3\left|J\left(I^i\right)(T)\right|.$$

Hence,

$$\begin{aligned}\sum_{i\geq i',j}\sum_{k',l}\int_0^T\int_0^T \Xi_t^{ii'j}\Xi_s^{ik'l}dtds&\leq 9\sum_i\left|J\left(I^i\right)(T)\right|^2\\&\leq 9\max_i\left|J\left(I^i\right)(T)\right|\sum_i\left|J\left(I^i\right)(T)\right|\\&\leq 9\left(3r_n(T)\right)(3T)=81r_n(T)T\lesssim b_n^{\xi'}.\end{aligned}$$

Thus it follows that

$$b_n^{-2}\left|E\left[\mathbf{I}\right]\right|\lesssim b_n^{-2}\cdot c_n^2b_n^{2\xi'}\cdot c_n^2\cdot b_n^{\xi'}=b_n^{3(\alpha-\gamma)}\to 0$$

as $n\to\infty$.

After all, we conclude that $b_n^{-1}\Delta_{1,T}=o_p(1)$. By symmetry, we also obtain $b_n^{-1}\Delta_{2,T}=o_p(1)$. $\square$

As the last step for Proposition 5.1, we are going to show that $b_n^{-1}\Delta_{3,t}$ is asymptotically negligible. The expression of $\Delta_{3,t}$ can be simplified as below.

Lemma 12.7 $\Delta_{3,t}=2\sum_{i,j}\left(K_-^{ij}H_-^{ij}L^{ij}\right)\cdot[X,Y]_t$.

Proof. By use of associativity and linearity of integration as well as integration by parts, one has

$$\Delta_{3,\cdot}=\sum_{i,j,i',j'}\left[\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'}+\underline{K}_-^{ij'}L^{i'j}\right)\right]\cdot[X,Y].$$

The summation breaks down to the four cases.

Case 1: $i\neq i',j\neq j'$. Whenever $i<i'$ and $j>j'$, both (i,j) and (i',j') cannot overlap at the same time, hence $K_-^{ij}K_-^{i'j'}\equiv 0$. The case of $i>i'$ and $j<j'$ is similar.

When $i<i'$ and $j<j'$ (and when both (i,j) and (i',j') respectively overlap at the same time), if (i',j) overlaps, then trivially $\underline{K}_-^{ij'}\equiv 0$; moreover, $K_-^{i'j'}\underline{K}_-^{i'j}\equiv 0$ because $K_{t-}^{i'j'}=0$ for $t\leq R_\vee(i',j')=T^{j'-1}$ but $\underline{K}_{t-}^{i'j}\neq 0$ for $R_\vee(i',j)<t\leq R^\wedge(i',j)=T^j\leq T^{j'-1}$. The case when (i,j') overlaps instead can be dealt with similarly.

The case of $i>i'$ and $j>j'$ can be shown by symmetry.

It follows that

$$\sum_{\substack{i,j,i',j':\\ i\neq i',j\neq j'}}\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'}+\underline{K}_-^{ij'}L^{i'j}\right)\equiv 0.$$

Case 2: $i = i', j \neq j'$. When $j < j'$ (and when both (i,j) and (i,j') respectively overlap at the same time), $K_-^{ij'}\underline{K}_-^{ij} \equiv 0$ because $K_{t-}^{ij'} = 0$ for $t \leq R_\vee(i,j') = T^{j'-1}$ but $\underline{K}_{t-}^{ij} \neq 0$ for $R_\vee(i,j) < t \leq R^\wedge(i,j) = T^j \leq T^{j'-1}$, therefore

$$\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'} + \underline{K}_-^{ij'}L^{i'j}\right) = \left(K_-^{ij}K_-^{ij'}\right)\left(\underline{K}_-^{ij}L^{ij'} + \underline{K}_-^{ij'}L^{ij}\right) = K_-^{ij}K_-^{ij'}\underline{K}_-^{ij'}L^{ij}.$$

When $j > j'$, by symmetry,

$$\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'} + \underline{K}_-^{ij'}L^{i'j}\right) = K_-^{ij}K_-^{ij'}\underline{K}_-^{ij}L^{ij'}.$$

It follows that

$$\begin{aligned}\sum_{\substack{i,j,i',j':\\ i=i',j\neq j'}}\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'} + \underline{K}_-^{ij'}L^{i'j}\right) &= \sum_{\substack{i,j,i',j':\\ i=i',j<j'}} K_-^{ij}K_-^{ij'}\underline{K}_-^{ij'}L^{ij} + \sum_{\substack{i,j,i',j':\\ i=i',j>j'}} K_-^{ij}K_-^{ij'}\underline{K}_-^{ij}L^{ij'}\\ &= 2\sum_{\substack{i,j,i',j':\\ i=i',j<j'}} K_-^{ij}K_-^{ij'}\underline{K}_-^{ij'}L^{ij} = 2\sum_{\substack{i,j,j'\\ j<j'}} K_-^{ij}\underline{K}_-^{ij'}L^{ij}\end{aligned}$$

by symmetry and by the fact that $K^{ij}\underline{K}^{ij} \equiv \underline{K}^{ij}$ (Lemma 10.1).

Case 3: $i \neq i', j = j'$. Similarly to the above case, when $i < i'$, $K_-^{i'j}\underline{K}_-^{ij} \equiv 0$ so that

$$\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'} + \underline{K}_-^{ij'}L^{i'j}\right) = \left(K_-^{ij}K_-^{i'j}\right)\left(\underline{K}_-^{i'j}L^{ij} + \underline{K}_-^{ij}L^{i'j}\right) = K_-^{ij}K_-^{i'j}\underline{K}_-^{i'j}L^{ij},$$

while for $i > i'$,

$$\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'} + \underline{K}_-^{ij'}L^{i'j}\right) = K_-^{ij}K_-^{i'j}\underline{K}_-^{ij}L^{i'j}.$$

It follows that

$$\sum_{\substack{i,j,i',j':\\ i\neq i',j=j'}}\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'} + \underline{K}_-^{ij'}L^{i'j}\right) = 2\sum_{\substack{i,j,i',j':\\ i<i',j=j'}} K_-^{ij}K_-^{i'j}\underline{K}_-^{i'j}L^{ij} = 2\sum_{\substack{i,j,i':\\ i<i'}} K_-^{ij}\underline{K}_-^{i'j}L^{ij}.$$

Case 4: $i = i', j = j'$. Evidently,

$$\sum_{\substack{i,j,i',j':\\ i=i',j=j'}}\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'} + \underline{K}_-^{ij'}L^{i'j}\right) = 2\sum_{\substack{i,j,i',j':\\ i=i',j=j'}} K_-^{ij}\underline{K}_-^{ij}L^{ij} = 2\sum_{i,j}\underline{K}_-^{ij}L^{ij}.$$

Putting all the four cases together,

$$\frac{1}{2}\sum_{i,j,i',j'}\left(K_-^{ij}K_-^{i'j'}\right)\left(\underline{K}_-^{i'j}L^{ij'} + \underline{K}_-^{ij'}L^{i'j}\right) = \sum_{\substack{i,j,j'\\ j<j'}} K_-^{ij}\underline{K}_-^{ij'}L^{ij} + \sum_{\substack{i,j,i':\\ i<i'}} K_-^{ij}\underline{K}_-^{i'j}L^{ij} + \sum_{i,j}\underline{K}_-^{ij}L^{ij}.$$

Because

$$K_-^{ij}\underline{K}_-^{ij'}L^{ij} \equiv 0 \text{ for } j' < j \text{ and } K_-^{ij}\underline{K}_-^{i'j}L^{ij} \equiv 0 \text{ for } i' < i,$$

the r.h.s. equals to

$$
\begin{aligned}
&\sum_{\substack{i,j,j':\\ j\neq j'}} K_{-}^{ij}\underline{K}_{-}^{ij'}L^{ij} + \sum_{\substack{i,j,i':\\ i\neq i'}} K_{-}^{ij}\underline{K}_{-}^{i'j}L^{ij} + \sum_{i,j}\underline{K}_{-}^{ij}L^{ij} \\
&= \sum_{i,j,j'} K_{-}^{ij}\underline{K}_{-}^{ij'}L^{ij} + \sum_{i,j,i'} K_{-}^{ij}\underline{K}_{-}^{i'j}L^{ij} - \sum_{i,j}\underline{K}_{-}^{ij}L^{ij} \\
&= \sum_{i,j} K_{-}^{ij}L^{ij}I_{-}^{i} + \sum_{i,j} K_{-}^{ij}L^{ij}J_{-}^{j} - \sum_{i,j}\underline{K}_{-}^{ij}L^{ij} \\
&= \sum_{i,j} K_{-}^{ij}\left(I_{-}^{i} + J_{-}^{j} - I_{-}^{i}J_{-}^{j}\right)L^{ij} = \sum_{i,j} K_{-}^{ij}H_{-}^{ij}L^{ij},
\end{aligned}
$$

therefore, the assertion is obtained. □

Lemma 12.8 *Under [A2], [A3] and [A4],* $b_n^{-1}\Delta_{3,t} \to^p 0$ *as* $n \to \infty$.

Proof. Let $\mathcal{R}_t = [X,Y]_t' - \widetilde{[X,Y]'}_t$. We apply Lemma 12.7 to get $(\Delta_{3,T})^2 = 4(\mathbf{I}+\mathbf{II}+\mathbf{III}+\mathbf{IV})$, where

$$
\begin{aligned}
\mathbf{I} &= \sum_{i,j}\sum_{i',j'}\int_0^T\int_0^T \Lambda_t^{ij}\Lambda_s^{i'j'}\widetilde{[X,Y]'}_t\widetilde{[X,Y]'}_s L_t^{ij}L_s^{i'j'}dtds \\
\mathbf{II} &= \sum_{i,j}\sum_{i',j'}\int_0^T\int_0^T \Lambda_t^{ij}\Lambda_s^{i'j'}\widetilde{[X,Y]'}_t\mathcal{R}_s L_t^{ij}L_s^{i'j'}dtds \\
\mathbf{III} &= \sum_{i,j}\sum_{i',j'}\int_0^T\int_0^T \Lambda_t^{ij}\Lambda_s^{i'j'}\mathcal{R}_t\widetilde{[X,Y]'}_s L_t^{ij}L_s^{i'j'}dtds \\
\mathbf{IV} &= \sum_{i,j}\sum_{i',j'}\int_0^T\int_0^T \Lambda_t^{ij}\Lambda_s^{i'j'}\mathcal{R}_t\mathcal{R}_s L_t^{ij}L_s^{i'j'}dtds.
\end{aligned}
$$

From Lemma 12.5 (i), we have $\sup_{t\in[0,T]}|L_t^{ij}| \leq c_n b_n^{\xi'}$ for all n. Also,

$$
\sup_{t\in[0,T]}|\mathcal{R}_t| \leq w([X,Y]'; b_n^{\xi}) = O_p(b_n^{\xi(\frac{1}{2}-\lambda)})
$$

for any $\lambda > 0$ by [A3](ii). Since $H_t^{ij} \leq I_t^i + J_t^i$,

$$
\begin{aligned}
\sum_{i,j}\int_0^T \Lambda_t^{ij}dt &\leq \sum_i \int_0^\tau I_t^i \sum_j K_t^{ij}dt + \sum_j \int_0^\tau J_t^j \sum_i K_t^{ij}dt \\
&\leq 3\sum_i \left|I^i(\tau)\right| + 3\sum_j \left|J^j(\tau)\right| \leq 6T,
\end{aligned}
$$

Consequently, we conclude that

$$
|\mathbf{II}| \leq \left(c_n b_n^{\xi'}\right)^2 \cdot O_p\left(b_n^{\xi(\frac{1}{2}-\lambda)}\right) \cdot c_n \cdot 36T^2,
$$

so that $b_n^{-2}\mathbf{II} = o_p(1)$. Likewise, we obtain $b_n^{-2}\mathbf{III} = o_p(1)$ and $b_n^{-2}\mathbf{IV} = o_p(1)$.

By the uniform boundedness $\widetilde{[X,Y]}' \le c_n$ and Lemma 12.5 (i), we obtain

$$E\left[\mathbf{I}^2\right] \le c_n^2 b_n^{2\xi'} \cdot c_n^2 \cdot E\left[\sum_{i,j}\sum_{i',j'}\int_0^T\int_0^T \Lambda_t^{ij}\Lambda_s^{i'j'}dtds\right]$$

Now, in light of Lemma 12.4 (i), the case $(i \ne i', j \ne j')$ will make no contribution to the sum. Therefore,

$$\sum_{i,j}\sum_{i',j'}\int_0^T\int_0^T \Lambda_t^{ij}\Lambda_s^{i'j'}dtds \le \sum_i\sum_{j,j'} + \sum_j\sum_{i,i'} + \sum_i\sum_j =: \mathbb{A}_1 + \mathbb{A}_2 + \mathbb{A}_3.$$

Since

$$\mathbb{A}_1 = \sum_i\left(\sum_j\int_0^T \Lambda_t^{ij}dt\right)^2$$

and

$$\begin{aligned}\sum_j\int_0^T \Lambda_t^{ij}dt &\le \int_0^T I_t^i\left(\sum_j K_t^{ij}\right)dt + \int_0^T\left(\sum_j J_t^j K_t^{ij}\right)dt \\ &\le 3\left|I^i(T)\right| + \left|J\left(I^i\right)(T)\right| \le 4\left|J\left(I^i\right)(T)\right|,\end{aligned}$$

we see

$$\begin{aligned}\mathbb{A}_1 &\le 16\sum_i\left|J\left(I^i\right)(T)\right|^2 \le 16\max_i\left|J\left(I^i\right)(T)\right|\sum_i\left|J\left(I^i\right)(T)\right| \\ &\le 16\left(3r_n(T)\right)(3T) = 144r_n(T)T \lesssim b_n^{\xi'}.\end{aligned}$$

By symmetry, $\mathbb{A}_2 \lesssim b_n^{\frac{2}{3}+\alpha}$. Finally, we will consider

$$\mathbb{A}_3 = \sum_{i,j}\left(\int_0^T \Lambda_t^{ij}dt\right)^2.$$

Since

$$\begin{aligned}\left(\int_0^T \Lambda_t^{ij}dt\right)^2 &\le \left(\int_0^T I_t^i K_t^{ij}dt + \int_0^T J_t^j K_t^{ij}dt\right)^2 \\ &\le 2\left(K_T^{ij}\left|I^i(T)\right|\right)^2 + 2\left(K_T^{ij}\left|J^j(T)\right|\right)^2,\end{aligned}$$

one has

$$\begin{aligned}\mathbb{A}_3 &\le 2\sum_i\left|I^i(T)\right|^2\left(\sum_j K_T^{ij}\right) + 2\sum_j\left|J^j(T)\right|^2\left(\sum_i K_T^{ij}\right) \\ &\le 6r_n(T)\left(\sum_i\left|I^i(T)\right| + \sum_j\left|J^j(T)\right|\right) \le 12r_n(T)T \lesssim b_n^{\xi'}.\end{aligned}$$

Putting all together, we obtain $E\left[\mathbb{A}_1+\mathbb{A}_2+\mathbb{A}_3\right] \lesssim b_n^{\xi'}$, and as a result,

$$b_n^{-2} E\left[\mathbf{I}^2\right] \lesssim b_n^{-2} \cdot c_n^2 b_n^{2\xi'} \cdot c_n^2 \cdot b_n^{\xi'} = b_n^{3(\alpha-\gamma)} = o(1)$$

as $n \to \infty$. Lemma 12.8 has been proved. □

Proof of Proposition 5.1. The desired result follows from the decomposition (12.6) and Lemmas 12.6 and 12.8. □

13 Proof of Theorems 6.1 and 6.2

This section is devoted to the proof of Theorems 6.1 and 6.2. With

$$B_{1,t} = \sum_{i,j} A^X(I^i)_t M^Y\left(J^j\right)_t K_t^{ij}, \qquad B_{2,t} = \sum_{i,j} A^Y(J^j)_t M^X\left(X^i\right)_t K_t^{ij},$$
$$B_{3,t} = \sum_{i,j} A^X(I^i)_t A^Y\left(J^j\right)_t K_t^{ij},$$

we have the decomposition

$$\{X,Y\}_t = \{M^X, M^Y\} + B_{1,t} + B_{2,t} + B_{3,t}.$$

The limiting distribution of the first term has been found in the previous sections. We now claim that the rest are, after scaled, asymptotically negligible.

Lemma 13.1 *Suppose that [A2] − [A6] are satisfied. Then* $b_n^{-1/2} B_{l,T}^* \to^p 0$ *as* $n \to \infty$ *for* $l = 1, 2, 3$.

Proof. In this time, in place of (12.2), we will use the random times υ_n defined by

$$\begin{aligned}\upsilon_n &= \inf\left\{t; |[X]'_t| > c_n\right\} \wedge \inf\left\{t; |[Y]'_t| > c_n\right\} \wedge \inf\left\{t; |[X,Y]'_t| > c_n\right\} \\ &\quad \wedge \inf\left\{ t; \sup_{\substack{s\in[0,t] \\ (r,s):\ r\in[(s-b_n^{\xi'})_+, s]}} \frac{|M_s^X - M_r^X| + |M_s^Y - M_r^Y|}{(s-r)^{1/2-\epsilon_1}} > 1 \right\} \\ &\quad \wedge \inf\left\{t; |(A^X)'_t| > d_n\right\} \wedge \inf\left\{t; |(A^Y)'_t| > d_n\right\} \wedge T\end{aligned}$$

with $d_n = b_n^{-\zeta/2}$ for some $\zeta \in (\frac{1}{3}, \frac{1}{3} + 2\alpha - \frac{3}{4}\gamma) \subset (\frac{1}{3}, 1)$. As mentioned in Section 5, we can assume that X and Y are stopped by υ_n; this υ_n here is not greater than υ_n in (12.2), however it does not matter since $P[\upsilon_n = T] \to 1$. Though n will not be written explicitly on the processes but they depend on n after localization. Further, we can assume that the sampling designs have been modified by $(\mathcal{I}, \mathcal{J})$-reduction.

We only consider $B_{1,T}$. The other cases can be shown in the same way. For notational simplicity, we drop X and Y from A^X and M^Y. Since $K_t^{ij} = 1$ for $t \geq R_\vee(i,j)$ in case the pair (i,j) overlaps, and $K_t^{ij} = 0$ otherwise, while the process $A(I^i)M(J^j)$ starts to vary at and beyond $t = R_\vee(i,j)$, one has

$$\begin{aligned} B_{1,t} &= \sum_{i,j} K_-^{ij} \cdot \left\{A(I^i)_t M\left(J^j\right)_t\right\} \\ &= \sum_{i,j} K_-^{ij} \cdot \left\{A(I^i)_- \cdot M\left(J^j\right)\right\}_t + \sum_{i,j} K_-^{ij} \cdot \left\{M\left(J^j\right)_- \cdot A(I^i)\right\}_t \\ &=: \mathbb{I}_t + \mathbb{II}_t. \end{aligned}$$

The process $\mathbb{I}_t$ is clearly a continuous local martingale with the quadratic variation

$$[\,\mathbb{I}\,]_t = \sum_{i,j}\sum_{i',j'}\left\{K_-^{ij}K_-^{i'j'}\left(I_-^i\cdot A\right)_-\left(I_-^{i'}\cdot A\right)_-\left(J_-^jJ_-^{j'}\right)[M]'\cdot s\right\}_t$$

due to [A3]. Each summand vanishes whenever $j\neq j'$, and also $\left|\left(I_-^i\cdot A\right)_t\right|\leq \||A'\||_\infty\left|I^i(t)\right|$ by [A5], hence

$$[\,\mathbb{I}\,]_t \leq \left\|[M]'\right\|_\infty \left\|A'\right\|_\infty^2\sum_i\left|I^i(t)\right|\sum_j K_\tau^{ij}\left|J^j(t)\right|\sum_{i'}K_\tau^{i'j}\left|I^{i'}(t)\right|.$$

Since $\sum_i\left|I^i(t)\right|\leq t$, $\sum_j K_t^{ij}\left|J^j(t)\right|\leq 3r_n(t)$ and $\sum_{i'}K_t^{i'j}\left|I^{i'}(t)\right|\leq 3r_n(t)$, as well as $\left\|[M]'\right\|_\infty\leq c_n$ by localization already done, one has

$$[\,\mathbb{I}\,]_t\leq 9c_n\left\||A'\|\right\|_\infty^2 r_n(t)^2,$$

therefore $b_n^{-1}[\,\mathbb{I}\,]_t\leq 9t\times c_n\times O_p(1)\times b_n^{\frac{1}{3}+2\alpha}=O_p\left(b_n^{2(\alpha-\gamma)+\frac{5}{4}\gamma+\frac{1}{3}}\right)=o_p(1)$. The Lenglart inequality implies that $b_n^{-1/2}\sup_{0\leq t\leq T}|\mathbb{I}_t|\to^p 0$ as desired.

Next we consider $\mathbb{III}$. Since $\left\{b_n^{-1/2}\mathbb{III}\right\}_{n\geq 1}$ is C-tight (cf. Definition VI.3.25 of Jacod and Shiryaev (2000)) by Lemma 13.2 below, it suffices to show that $b^{-1/2}\mathbb{III}_t=o_p(1)$ for every t to conclude its uniform convergence. We rewrite $\mathbb{III}$ as

$$\begin{aligned}\mathbb{III}_t &= \sum_{i,j}A'_{T^{j-1}}\int_0^t K_s^{ij}I_s^i\left(J_-^j\cdot M\right)_s ds\\ &\quad+\sum_{i,j}\int_0^t K_s^{ij}I_s^i\left(J_-^j\cdot M\right)_s\left(A'_s-A'_{T^{j-1}}\right)ds\\ &= :\mathbb{III}_{1,t}+\mathbb{III}_{2,t}.\end{aligned}$$

First we claim that $b_n^{-1/2}\mathbb{III}_{1,t}=o_p(1)$ as $n\to\infty$. If $j<j'$, then clearly $A'_{T^{j-1}}A'_{T^{j'-1}}$ is $\mathcal{F}_{T^{j'-1}}$-measurable; besides, it can be verified easily that $K_s^{ij}K_u^{i'j'}I_s^iI_u^{i'}$ is $\mathcal{F}_{T^{j'-1}}$-measurable due to a variant of Lemma 12.3(i). Therefore

$$\begin{aligned}E\left[\mathbb{III}_{1,t}\right] &= 2E\Bigg[\sum_{i,i',j<j'}A'_{T^{j-1}}A'_{T^{j'-1}}\int_0^t\int_0^t K_s^{ij}K_u^{i'j'}I_s^iI_u^{i'}\left(J_-^j\cdot M\right)_s E\left[\left(J_-^{j'}\cdot M\right)_u\Big|\mathcal{F}_{T^{j'-1}}\right]dsdu\Bigg]\\ &\quad+E\Bigg[\sum_{i,i',j}\left(A'_{T^{j-1}}\right)^2\int_0^t\int_0^t K_s^{ij}K_u^{i'j}I_s^iI_u^{i'}E\left[\left(J_-^j\cdot[M]\right)_s^2\Big|\mathcal{F}_{T^{j-1}}\right]dsdu\Bigg]\\ &= E\Bigg[\sum_{i,i',j}\left(A'_{T^{j-1}}\right)^2\int_0^t\int_0^t K_s^{ij}K_u^{i'j}I_s^iI_u^{i'}\left(J_-^j[M]'\cdot v\right)_s dsdu\Bigg]\\ &\leq 4d_n^2\times c_n\times E\sum_j\left|J_-^j(T)\right|\left(\sum_i K_\tau^{i'j}\left|I^i(T)\right|\right)^2\\ &\leq 4d_n^2\times c_n\times E\left[9Tr_n(T)^2\right]\lesssim b_n^{\frac{4}{3}+2\alpha-\frac{3}{4}\gamma-\zeta}=o(b_n),\end{aligned}$$

and hence $b_n^{-1/2}\mathbb{III}_{1,t} \to^p 0$ for every t.

Next regarding $\mathbb{III}_2$,

$$\begin{aligned}
|\mathbb{III}_{2,t}| &= \left| \sum_{i,j} \int_0^t K_s^{ij} \left(J_-^j \cdot M \right)_s I_s^i \left(A'_s - A'_{T^{j-1}} \right) ds \right| \\
&\leq \int_0^t \sum_{i,j} K_s^{ij} I_s^i \left| \left(J_-^j \cdot M \right)_s \right| \left| K_s^{ij} I_s^i \left(A'_s - A'_{T^{j-1}} \right) \right| ds \\
&\leq w\left(M; r_n(t)\right) w\left(A'; 2r_n(t)\right) \sum_{i,j} \int_0^t K_s^{ij} I_s^i ds \\
&\leq 3T\, w\left(M; r_n(T)\right) w\left(A'; 2r_n(T)\right),
\end{aligned}$$

for every $t \leq T$, where $I\left(J^j\right)_t = \sum_i K_t^{ij} I_t^i$ as before, and we used under the reduced design $\left(J^j\right)$, $\sum_j K_t^{ij} \leq 3$. Because $w\left(M; r_n(t)\right) = O_p\left(r_n(t)^{1/2-\kappa}\right)$ for any $\kappa \in (0, 1/2)$, and from [A5], $w\left(A'; r_n(t)\right) = O_p(r_n(t)^{1/2-\lambda})$ for some $\lambda \in (0, 1/4)$, we have

$$b_n^{-\frac{1}{2}}\mathbb{III}_{2,t} = b_n^{-\frac{1}{2}} O_p\left(r_n(t)^{1-(\kappa+\lambda)}\right) = o_p\left(b_n^{-\frac{1}{2}+\{1-(\kappa+\lambda)\}\left(\frac{2}{3}+\alpha\right)}\right)$$

with [A4]. Noting that $\{1-(\kappa+\lambda)\}\alpha > 0$ and that one can always make $-\frac{2}{3}(\kappa+\lambda) > -\frac{1}{6}$ by choosing κ arbitrarily small, to obtain $b_n^{-1/2}\mathbb{III}_{2,t} = o_p(1)$ for every t. Consequently we obtained $b_n^{-1/2}\sup_{t\in[0,T]}|\mathbb{III}_t| \to^p 0$. Combining it with the previous result completes the proof. □

Lemma 13.2 *$\{b_n^{-1/2}\mathbb{III}\}_{n\geq 1}$ is C-tight.*

Proof. Rewrite $\mathbb{III}_t$ as

$$\mathbb{III}_t = \sum_j M\left(J^j\right)_- \cdot \left\{ I\left(J^j\right)_- \cdot A \right\}_t.$$

For $C_1 > 0$, let $\mu = \inf\{t; [M]'_t > C_1\}$ and let

$$\overline{\mathbb{III}}_t = \textstyle\sum_j M^\mu\left(J^j\right)_- \cdot \left\{ I\left(J^j\right)_- \cdot A \right\}_t.$$

Then for $s < t$,

$$\begin{aligned}
b_n^{-1/2}\left|\overline{\mathbb{III}}_t - \overline{\mathbb{III}}_s\right| &\leq b_n^{-1/2} \sum_j \int_s^t \left| M^\mu\left(J^j\right) \right|_u \left|A'\right|_u I\left(J^j\right)_u du \\
&\leq \sum_j \left\{ \int_s^t \left(b_n^{-1/2} M^\mu\left(J^j\right) \right)_u^2 I\left(J^j\right)_u du \right\}^{\frac{1}{2}} \left\{ \int_s^t \left|A'\right|_u^2 I\left(J^j\right)_u du \right\}^{\frac{1}{2}} \\
&\leq \left\{ \sum_j \int_s^t \left(b_n^{-1/2} M^\mu\left(J^j\right) \right)_u^2 I\left(J^j\right)_u du \right\}^{\frac{1}{2}} \left\{ \sum_j \int_s^t \left|A'\right|_u^2 I\left(J^j\right)_u du \right\}^{\frac{1}{2}} \\
&\leq \Theta_n^{\frac{1}{2}}(T) \times \sqrt{3T}\left\|A'\right\|_\infty^2 (t-s)^{\frac{1}{2}},
\end{aligned}$$

where

$$\Theta_n(\cdot) = \sum_j \int_0^{T\wedge\cdot} \left(b_n^{-1/2} M^\mu \left(J^j\right)\right)_u^2 I\left(J^j\right)_u du.$$

For $C_2 > 0$, let $\theta = \inf\{t; H_n^2(t) > C_2\}$, which is $\mathbf{G}^{(n)}$-stopping time since H_n^2 is $\mathbf{G}^{(n)}$-adapted. Since $1_{\{u<\theta\}} I(J^j)$ is $\mathcal{F}_{T^{j-1}}$-measurable,

$$\begin{aligned}
E\left[\Theta_n(\theta)\right] &= \sum_j E\left[\int_0^T E\left[\left(b_n^{-1/2} M^\mu \left(J^j\right)\right)_{u\wedge\theta}^2 \middle| \mathcal{F}_{T^{j-1}}\right] 1_{\{u<\theta\}} I\left(J^j\right)_u du\right] \\
&= \sum_j E\Big[\int_0^T b_n^{-1}\left[M^\mu\right]\left(J^j\right)_{u\wedge\nu} 1_{\{u<\theta\}} I\left(J^j\right)_u du\Big] \\
&\le E\left[\|[M^\mu]'\|_\infty b_n^{-1} \sum_j |J^j(\theta)|\, |I(J^j)(\theta)|\right] \\
&\le 5C_1C_2,
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
b_n^{-1} \sum_j |J^j(\theta)|\, |I(J^j)(\theta)| &\le b_n^{-1} \sum_j |J^j(\theta)| \sum_{\ell=j-2}^{j+2} |J^\ell(\theta)| \\
&\le 5b_n^{-1} \sum_j |J^j(\theta)|^2 = 5H_n^2(\theta) \le 5C_2.
\end{aligned}$$

Since $\lim_{C_2\to\infty} P[\theta < T] = 0$, it follows from [A6] that the family $\{\Theta_n(T)\}_n$ is tight.

For each $\epsilon > 0$,

$$\begin{aligned}
\sup_n P\left[w(b_n^{-1/2}\mathbb{III}; \delta, T) \ge \epsilon\right] &\le \sup_n P\left[\sup_{s,t;|s-t|\le\delta}\left[\Theta_n(T)^{\frac{1}{2}}\sqrt{3T}\,\left\|\left(A'\right)^\nu\right\|_\infty^2 (t-s)^{\frac{1}{2}}\right] \ge \epsilon\right] \\
&\le \sup_n P\left[\Theta_n(T)^{\frac{1}{2}}\sqrt{3T}\,\left\|A'\right\|_\infty^2 \ge \epsilon\delta^{-\frac{1}{2}}\right] \\
&\to 0
\end{aligned}$$

as $\delta \downarrow 0$ because of the local boundendness of A'. Since $\lim_{C_1\to\infty} P[\mu < T] = 0$, obviously $\sup_n P\left[w(b_n^{-1/2}\mathbb{III}; \delta, T) \ge \epsilon\right] \to 0$ as $\delta \downarrow 0$. Consequently $\{b_n^{-1/2}\mathbb{III}\}_{n\ge1}$ is C-tight. $\square$

Proof of Theorem 6.1. In both cases (a) and (b), [A2]-[A6] holds, hence Lemma 13.1 ensures the behavior of $\{X, Y\}$ is the same as $\{M^X, M^Y\}$ in the first order; that $[X, Y] = [M^X, M^Y]$ is a trivial notice. Eventually, we will consider

$$M_t^n = \{M^X, M^Y\}_t - [M^X, M^Y]_t = \sum_{i,j} L_t^{ij} K_t^{ij}$$

in place of (3.1), but in the present situation for

$$L_t^{ij} = \left(I_-^i \cdot M^X\right)_- \cdot \left(J_-^j \cdot M^Y\right)_t + \left(J_-^j \cdot M^Y\right)_- \cdot \left(I_-^i \cdot M^X\right)_t.$$

Condition [B2] holds under the assumptions according to Proposition 5.1; note that $V_t^{iji'j'}$ is unchanged and [A3] still holds even if (M^X, M^Y) replaces (X, Y). Therefore, once Condition [B1] is verified for (M^X, M^Y) in place of (X, Y), (a) follows from Proposition 3.3, and (b) follows from Proposition 4.1.

After all, what we have to show is that $b_n^{-\frac{1}{2}}\mathfrak{V}^n_{X,t} \to^p 0$ and $b_n^{-\frac{1}{2}}\mathfrak{V}^n_{Y,t} \to^p 0$ as $n \to \infty$ for every t, where for instance $\mathfrak{V}^n_{X,t}$ is now given by

$$\mathfrak{V}^n_{X,t} \;=\; \sum_{i,j} K^{ij}_{-} \cdot \left\{M^X\left(I^i\right) \cdot [X,Y]\left(J^j\right)\right\}_t + \sum_{i,j} K^{ij}_{-} \cdot \left\{M^Y\left(J^j\right) \cdot [X,X]\left(I^i\right)\right\}_t$$

Since, thanks to [A3], $[X,X]$ and $[X,Y]$ satisfy the property in [A5], exactly the same argument made for $\mathbb{III}_t$ in the proof for Lemma 13.1 to give $b_n^{-\frac{1}{2}}\mathfrak{V}^n_{X,t} \to^p 0$. The convergence $b_n^{-\frac{1}{2}}\mathfrak{V}^n_{Y,t} \to^p 0$ is verified in the same fashion. $\square$

14 Proof of Theorems 7.1 and 7.2

Theorems 7.1 and 7.2 can be obtained by applying the following lemmas.

Lemma 14.1 *Suppose that* [A3] *and* [A4] *are satisfied. Then*

$$b_n^{-\frac{1}{2}}\left(\overline{\{M^X, M^Y\}} - \left\{M^X, M^Y\right\}\right) \quad \overset{ucp}{\to} \quad 0$$

as $n \to \infty$.

Proof. We recall the following standard notation:

$$I\left(J^j\right)_t = \sum_i K^{ij}_t I^i_t, \qquad J\left(I^i\right)_t = \sum_j K^{ij}_t J^j_t.$$

According to the notation system introduced earlier, $I\left(J^j\right)(t)$ denotes the aggregate interval truncated by time t, hence $X\left(I\left(J^j\right)(t)\right)$ will mean the increment of X over it.

For any t, there exists a unique pair (i,j) such that $t \in \left[S^{i-1}, S^i\right)$ and $t \in \left[T^{j-1}, T^j\right)$. Call such indices i_t and j_t in what follows. By definition, $i_s - 1 = \max\left\{i; S^i \le s\right\}$ and $j_s - 1 = \max\left\{j; T^j \le s\right\}$ for arbitrary time s. We see that $\left|I\left(J^{j_s}\right)(s)\right| \le 2r_n(s)$ and $\left|J\left(I^{i_s}\right)(s)\right| \le 2r_n(s)$ for any s. Set $\Delta_t = \overline{\{M^X, M^Y\}}_t - \left\{M^X, M^Y\right\}_t$.

Consider the case where $S^{i_s-1} < T^{j_s-1}$. For $i \le i_s - 1$,

$$\left[I^i \cap J^j \neq \varnothing\right] \quad \Longrightarrow \quad \left[\sup J^j \le T^{j_s-1} \le s\right] \quad \Longrightarrow \quad \left[I^i(s) \cap J^j(s) \neq \varnothing\right],$$

Such pairs are included in the summation in both $\overline{\{M^X, M^Y\}}$ and $\left\{M^X, M^Y\right\}$. Consequently, when the gap between the two quantities has to be evaluated, only the remaining overlapping pairs (i,j) with $i = i_s$ are to be taken into account. Thus, for any s,t with $s \le t$,

$$\begin{aligned}|\Delta_s| &\le \left|X\left(I^{i_s}(s)\right)\right|\left|Y\left(J\left(I^{i_s}\right)(s)\right)\right| \\ &\le w\left(M^X; r_n(t), t\right) \cdot w\left(M^Y; 2r_n(t), t\right).\end{aligned}$$

The case $S^{i_s-1} > T^{j_s-1}$ and the case $S^{i_s-1} = T^{j_s-1}$ are similarly treated and we have

$$|\Delta_s| \le w\left(M^X; 2r_n(t), t\right) \cdot w\left(M^Y; 2r_n(t), t\right).$$

in any case.

Since M^X and M^Y are continuous local martingales, from to the martingale representation as Brownian motion together with [A3], it follows that for any $t > 0$ and any $\varepsilon > 0$, $w\left(M^X; h, t\right) \leq h^{\frac{1}{2}-\varepsilon}$ as $h \downarrow 0$. The same inequality is true for M^Y as well. From [A4], we conclude that

$$b_n^{-\frac{1}{2}} \sup_{s\in[0,t]} |\Delta_s| = o_p\left(b_n^{\xi'(1-2\varepsilon)-\frac{1}{2}}\right) = o_p(1),$$

taking small ε. $\square$

Set

$$\overline{B}_{1,t} = \sum_{\substack{i,j=1\\ S^i\vee T^j\leq t}}^{\infty} A^X(I^i)M^Y\left(J^j\right)K^{ij}, \quad \overline{B}_{2,t} = \sum_{\substack{i,j=1\\ S^i\vee T^j\leq t}}^{\infty} A^Y(J^j)M^X\left(X^i\right)K^{ij},$$

$$\overline{B}_{3,t} = \sum_{\substack{i,j=1\\ S^i\vee T^j\leq t}}^{\infty} A^X(I^i)A^Y\left(J^j\right)K^{ij},$$

where $K^{ij} = 1_{\{I^i\cap J^j\neq\varnothing\}}$, then we obtain the discrete version of decomposition

$$\overline{\{X,Y\}}_t = \overline{\{M^X, M^Y\}}_t + \overline{B}_{1,t} + \overline{B}_{2,t} + \overline{B}_{3,t}.$$

Lemma 14.2 *Suppose that* [A2] − [A6] *are satisfied. Then* $b_n^{-1/2}\overline{B}^*_{l,T} \to^p 0$ *as* $n \to \infty$ *for* $l = 1, 2, 3$.

Proof. The uniform difference between $\overline{B}_l$ and B_1, after scaling by $b_n^{-1/2}$, can also be shown to be negligible. But the negligibility of $B^*_{l,t}$ is already given by Lemma 13.1. $\square$

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