Potentially good reduction loci
of Shimura varieties

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ABSTRACT. In this paper, we give a notion of the potentially good reduction locus of a Shimura variety. It consists of the points which should be related with motives having potentially good reductions in some sense. We show the existence of such locus for a Shimura variety of preabelian type. Further, we construct a partition of the adic space associated to a Shimura variety of preabelian type, which is expected to describe degenerations of motives. Using this partition, we prove that the cohomology of the potentially good reduction locus is isomorphic to the cohomology of a Shimura variety up to non-supercuspidal parts.

1 Introduction

Let (G, X) be a Shimura datum, and $\mathrm{Sh}_K(G, X)$ the Shimura variety attached to (G, X) and a compact open subgroup K of $G(\mathbb{A}^\infty)$. It is known to be defined over a number field E, called the reflex field, which is canonically determined by (G, X). We fix a prime number p and a place v of E above p, and write E_v for the completion of E at v. The main theme of this article is the "potentially good reduction locus" of $\mathrm{Sh}_K(G, X)_{E_v} = \mathrm{Sh}_K(G, X) \otimes_E E_v$.

To explain what this locus is, let us first assume that (G, X) is of PEL type, in which case $\mathrm{Sh}_K(G, X)$ parametrizes abelian varieties with additional PEL structures. We denote by $\mathcal{A}$ the universal abelian scheme over $\mathrm{Sh}_K(G, X)$. If moreover the PEL datum is unramified at p and $K = K_{p,0}K^p$ where $K_{p,0}$ is hyperspecial, by extending the moduli problem to $\mathcal{O}_{E_v}$, we can obtain a good integral model $\mathscr{S}_{K^p}$ of $\mathrm{Sh}_K(G, X)$ over $\mathcal{O}_{E_v}$ (see [Kot92b]). This model is quite important in the study of the ℓ-adic cohomology of Shimura varieties; see [Kot92a] for instance. Let us denote by $\mathscr{S}_{K^p}^\wedge$ the formal completion of $\mathscr{S}_{K^p}$ along its special fiber, and by $\mathscr{S}_{K^p}^{\wedge\mathrm{rig}}$ the rigid generic fiber of it. Then, $\mathscr{S}_{K^p}^{\wedge\mathrm{rig}}$ is naturally identified with a quasi-compact rigid-analytic open subset of $\mathrm{Sh}_K(G, X)_{E_v}$. For a finite extension F of E_v, an F-valued point x of $\mathrm{Sh}_K(G, X)_{E_v}$ lies in $\mathscr{S}_{K^p}^{\wedge\mathrm{rig}}$ if and only if the abelian variety $\mathcal{A}_x$ over F has (potentially) good reduction. In this sense, $\mathscr{S}_{K^p}^{\wedge\mathrm{rig}}$ can be considered as the locus over which $\mathcal{A}$ has (potentially) good reduction. By this reason, we will write

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$\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}} = \mathscr{S}_{K^p}^{\wedge \mathrm{rig}}$, and call it the potentially good reduction locus. We also have a rigid-analytic open subspace $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}}$ of $\mathrm{Sh}_K(G, X)_{E_v}$ for a compact open subgroup K whose p-part is smaller than $K_{p,0}$, by taking the inverse image. The ℓ-adic cohomology of $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}}$ can be computed by using the cohomology of the nearby cycle complex, provided that $\mathrm{Sh}_K(G, X)_{E_v}$ has a suitable integral model over $\mathcal{O}_{E_v}$.

In this paper, we will introduce the notion of the potentially good reduction locus for a general Shimura variety. The rough idea is as follows. Let us fix a prime number ℓ. In the PEL type case, let $\mathcal{L}$ be the ℓ-adic automorphic étale sheaf on $\mathrm{Sh}_K(G, X)$ attached to the standard representation of G. Then, for a finite extension F of E_v and an F-valued point x of $\mathrm{Sh}_K(G, X)$, the stalk $\mathcal{L}_{\overline{x}}$ can be identified with the rational ℓ-adic Tate module $V_\ell \mathcal{A}_{\overline{x}}$. By the Neron-Ogg-Shafarevich criterion, we conclude that $\mathcal{A}_x$ has potentially good reduction if and only if $\mathcal{L}_{\overline{x}}$ is potentially unramified (resp. potentially crystalline) when $\ell \neq p$ (resp. $\ell = p$). This observation urges us to define in the general case that an F-valued point x of $\mathrm{Sh}_K(G, X)_{E_v}$ is of potentially good reduction if $\mathcal{L}_{\overline{x}}$ is potentially unramified/crystalline for every automorphic étale sheaf $\mathcal{L}$. Actually in the paper, we look at the torsor over x obtained as the pull-back of $\varprojlim_{K' \subset K} \mathrm{Sh}_{K'}(G, X)_{E_v} \to \mathrm{Sh}_K(G, X)_{E_v}$, which is more concise but essentially equivalent to the above way by the Tannakian duality. Our potentially good reduction locus is defined as a quasi-compact open subset of the adic space $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$ attached to $\mathrm{Sh}_K(G, X)_{E_v}$, whose F-valued points consist of those of potentially good reduction for every F. It is unique, if exists. We will show the existence of the potentially good reduction locus $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}}$ when the Shimura datum (G, X) is of preabelian type. Recall that (G, X) is said to be of preabelian type if there exists a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$. This class contains almost all Shimura data in practice. As in [Del79, Introduction] and [Mil05, §9], a Shimura variety is believed to have a moduli interpretation by motives, if the weight homomorphism for (G, X) is defined over the rational number field. The subset $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}}$ is expected to parametrize motives with potentially good reduction at v.

We are also interested in what happens outside the locus $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}}$. In the PEL type case, degenerations of abelian varieties occur; if a Shimura variety parametrizes motives, then degenerations of motives should occur. Based on this observation, we will construct a partition of $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$ into finitely many locally closed constructible subsets labeled by conjugacy classes of certain kind of adelic parabolic subgroups of G, so that the piece corresponding to G equals $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}}$. It is closely related to the theory of integral toroidal compactifications. Actually, in the PEL type case, we may also use the integral toroidal compactification developed in [Lan13] to construct our partition (see [IM13, §7]); there should be some more cases to which the method in [IM13, §7] can be applied (for example, [MP12]). However, our argument here is almost totally rigid-geometric, and requires only the existence of the integral toroidal compactifications of the Siegel modular varieties with hyperspecial level at p ([FC90]) as an input from the integral theory. Note also

that our partition is independent of any choice, unlike the toroidal compactification that depends on the choice of a cone decomposition.

By using the partition above, we can compare the ℓ-adic cohomology of the tower $\{\mathrm{Sh}_K(G,X)^{\mathrm{pg}}_{E_v}\}_K$ and that of $\{\mathrm{Sh}_K(G,X)\}_K$. We assume that (G,V) is of preabelian type and satisfies the condition SV6 in [Mil05, p. 311]. Let ℓ be a prime number different from p, and $\mathcal{L}$ an ℓ-adic automorphic étale sheaf on $\mathrm{Sh}_K(G,X)$ corresponding to an algebraic representation of G^c over $\overline{\mathbb{Q}}_\ell$, where G^c is the quotient of G defined in [Mil90, p. 347]. The statement is as follows:

Theorem 1.1 (Theorem 6.1) *In the kernel and the cokernel of the natural map*

$$\varinjlim_K H^i_c(\mathrm{Sh}_K(G,X)^{\mathrm{pg}}_{\overline{E}_v}, \mathcal{L}) \to \varinjlim_K H^i_c(\mathrm{Sh}_K(G,X)_{\overline{E}_v}, \mathcal{L}),$$

no irreducible supercuspidal representation of $G(\mathbb{Q}_{p'})$ appears as a subquotient for any prime number p'.

Recall that an irreducible smooth representation of $G(\mathbb{Q}_{p'})$ is said to be supercuspidal if it does not appear as a subquotient of the parabolically induced representations from any proper parabolic subgroup. Loosely speaking, this theorem is a consequence of the observation that the partition of the complement $\mathrm{Sh}_K(G,X)^{\mathrm{ad}}_{E_v} \setminus \mathrm{Sh}_K(G,X)^{\mathrm{pg}}_{E_v}$ is "geometrically induced" from proper parabolic subgroups of $G(\mathbb{Q}_{p'})$. It will be worth noting that our method is totally geometric, so that it is also valid in the torsion coefficient case. See Theorem 6.12 for an analogue of Theorem 1.1 with the $\overline{\mathbb{F}}_\ell$-coefficients.

We have already mentioned that in the PEL type case the ℓ-adic cohomology $H^i_c(\mathrm{Sh}_K(G,X)^{\mathrm{pg}}_{\overline{E}_v}, \mathcal{L})$ can be computed as the cohomology of a nearby cycle complex. Hence, in this case the theorem above says that the nearby cycle cohomology is isomorphic to the compactly supported cohomology up to non-supercuspidal representations. This result is useful, since it connects the cohomology of Shimura varieties and that of Rapoport-Zink spaces; see Section 7.3 for a simple example in this direction. Recently, during the preparation of this article, Lan and Stroh obtained a stronger result that the nearby cycle cohomology is isomorphic to the compactly supported cohomology in the cases where reasonable integral toroidal compactifications exist (see [LS15]). However, we have decided to include our weaker result in this paper, since the argument is totally different.

We sketch the outline of this paper. In Section 2, we consider Galois representations of a p-adic field with values in a general connected reductive group G. Under some condition, we attach a parabolic subgroup of G to such a representation. In Section 3, we give some preliminary results on adic spaces and semi-abelian schemes. In Section 4, we recall some notation and results on Shimura varieties. In Section 5, we construct a partition of the adic space associated to a Shimura variety of preabelian type by using results obtained in Section 2. The potentially good reduction locus is introduced here, as a piece of the constructed partition. In Section 6, we prove the theorem comparing the cohomology of potentially good reduction loci

with that of Shimura varieties. In Section 7, we specialize our results to Shimura varieties of PEL type, and discuss a simple application.

Notation   Put $\widehat{\mathbb{Z}} = \prod_{\text{prime } p} \mathbb{Z}_p$ and $\mathbb{A}^\infty = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. For a prime p, put $\widehat{\mathbb{Z}}^p = \prod_{\text{prime } p' \neq p} \mathbb{Z}_{p'}$ and $\mathbb{A}^{\infty,p} = \widehat{\mathbb{Z}}^p \otimes_{\mathbb{Z}} \mathbb{Q}$. More generally, for a finite set of primes S, we put $\mathbb{A}_S = \prod_{\ell \in S} \mathbb{Q}_\ell$, $\widehat{\mathbb{Z}}^S = \prod_{\text{prime } p \notin S} \mathbb{Z}_p$ and $\mathbb{A}^{\infty,S} = \widehat{\mathbb{Z}}^S \otimes_{\mathbb{Z}} \mathbb{Q}$.

For a scheme X over a field F and an extension field L of F, we write X_L for the base change of X to L. Similar notation will be used for adic spaces.

For an algebraic group G, let $Z(G)$ denote the center of G, and $G^{\text{ad}} = G/Z(G)$ the adjoint group of G. For a field L over which G is defined, we write $\mathbf{Rep}_L(G)$ for the Tannakian category of finite-dimensional algebraic representations of G over L.

Every sheaf and cohomology are considered in the étale topology.

2   Preliminaries on Galois representations

In this section, fix a p-adic field F and its algebraic closure $\overline{F}$. Let ℓ be a prime number and G a connected reductive group over $\overline{\mathbb{Q}}_\ell$. Consider a continuous homomorphism $\phi \colon \mathrm{Gal}(\overline{F}/F) \to G(\overline{\mathbb{Q}}_\ell)$.

Definition 2.1   (i) Assume that $\ell \neq p$. We say that ϕ is potentially unramified if $\xi \circ \phi$ is potentially unramified for any $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$.

(ii) Assume that $\ell = p$. We say that ϕ is potentially crystalline if $\xi \circ \phi$ is potentially crystalline for any $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G)$.

(iii) Assume that $\ell = p$. We say that ϕ is de Rham if $\xi \circ \phi$ is de Rham (or equivalently, potentially semistable) for any $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G)$.

To measure how far ϕ is from potentially unramified or potentially crystalline, we consider the monodromy filtration on $\xi \circ \phi$ for each $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$. First, we assume that $\ell \neq p$. Then, for each $(\xi, V_\xi) \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$, we obtain the ℓ-adic representation $(\xi \circ \phi, V_\xi)$ of $\mathrm{Gal}(\overline{F}/F)$ and its monodromy filtration $M_\bullet V_\xi$.

Lemma 2.2   *Assume that $\ell \neq p$.*

(i) *The stabilizer P_ξ of the filtration $M_\bullet V_\xi \subset V_\xi$ is a parabolic subgroup of G.*

(ii) *If $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$ is faithful, then P_ξ stabilizes $M_\bullet V_{\xi'}$ for every $\xi' \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$. In particular P_ξ for faithful ξ is independent of ξ. We write P_ϕ for this P_ξ.*

(iii) *The homomorphism ϕ is potentially unramified if and only if $P_\phi = G$.*

(iv) *For the composite $\phi^{\text{ad}} \colon \mathrm{Gal}(\overline{F}/F) \xrightarrow{\phi} G(\overline{\mathbb{Q}}_\ell) \to G^{\text{ad}}(\overline{\mathbb{Q}}_\ell)$, we have $P_{\phi^{\text{ad}}} = P_\phi^{\text{ad}}$, where P_ϕ^{ad} denotes the image of P_ϕ in G^{ad}.*

(v) *For a finite extension F' of F contained in $\overline{F}$, we put $\phi' = \phi|_{\mathrm{Gal}(\overline{F}/F')}$. Then we have $P_{\phi'} = P_\phi$.*

Proof. The assertion (i) follows from [Kis10, Lemma 1.1.1, Lemma 1.1.3], since $V_\xi \mapsto M_\bullet V_\xi$ gives a filtration on the Tannakian category $\mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$.

Let us prove (ii). For integers $m, m' \geq 0$, the monodromy filtration on $V_\xi^{\otimes m} \otimes V_\xi^{\vee \otimes m'}$ can be written by using $M_\bullet V_\xi$ (see [Del80, Proposition 1.6.9]). Therefore it is stable under P_ξ. As every representation $(\xi', V_{\xi'})$ of G appears as a direct summand of $V_\xi^{\otimes m} \otimes V_\xi^{\vee \otimes m'}$ for some integers $m, m' \geq 0$, the filtration $M_\bullet V_{\xi'}$ is also preserved by P_ξ. This concludes the proof.

For (iii), note that ϕ is potentially unramified if and only if the monodromy operator N on V_ξ is zero for every $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$. If this condition is satisfied, we have $M_i V_\xi = 0$ for $i < 0$ and $M_i V_\xi = V_\xi$ for $i \geq 0$, thus $P_\phi = G$. Conversely assume that $P_\phi = G$, in other words, $M_i V_\xi$ is G-stable for every ξ and i. It suffices to show that $N = 0$ on V_ξ for each irreducible representation ξ of G. Since $M_i V_\xi$ is G-stable, there exists a unique integer i_0 such that $M_i V_\xi = 0$ $(i < i_0)$ and $M_i V_\xi = V_\xi$ $(i \geq i_0)$. Hence we have $N(V_\xi) = N(M_{i_0} V_\xi) \subset M_{i_0-2} V_\xi = 0$, as desired.

We prove (iv). Clearly we have $P_\phi^{\mathrm{ad}} \subset P_{\phi^{\mathrm{ad}}}$. For the reverse inclusion, take $g \in G(\overline{\mathbb{Q}}_\ell)$ which is mapped into $P_{\phi^{\mathrm{ad}}}(\overline{\mathbb{Q}}_\ell)$ under $G(\overline{\mathbb{Q}}_\ell) \to G^{\mathrm{ad}}(\overline{\mathbb{Q}}_\ell)$. It suffices to show that g stabilizes $M_\bullet V_\xi$ for each irreducible representation (ξ, V_ξ) of G. Put $W = V_\xi \otimes V_\xi^\vee$. Since ξ is irreducible, the center of G acts trivially on W. Therefore W can be regarded as a representation of G^{ad}. In particular, the monodromy filtration $M_\bullet W$ on W is stable under the action of g.

Let j_0 be the minimal integer such that $M_{j_0} V_\xi^\vee = V_\xi^\vee$. Fix an integer i_0. Then we have $M_{i_0+j_0} W = \sum_{i+j=i_0+j_0} M_i V_\xi \otimes M_j V_\xi^\vee$. Note that $M_{i_0} V_\xi$ can be recovered from $M_{i_0+j_0} W$ by

$$M_{i_0} V_\xi = \bigcap_{f \in \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(V_\xi^\vee, \overline{\mathbb{Q}}_\ell) \setminus \{0\}} (\mathrm{id} \otimes f)(M_{i_0+j_0} W). \qquad (*)$$

Since $M_{i_0+j_0} W = g(M_{i_0+j_0} W) = \sum_{i+j=i_0+j_0} g(M_i V_\xi) \otimes g(M_j V_\xi^\vee)$, the right hand side of $(*)$ is also equal to $g(M_{i_0} V_\xi)$. Hence we conclude that $M_\bullet V_\xi$ is stable under g.

The claim (v) is clear, since the monodromy filtration $M_\bullet V_\xi$ does not change after restricting ϕ to $\mathrm{Gal}(\overline{F}/F')$. ∎

Next we consider the case $\ell = p$. We shall introduce the notion of the monodromy filtration on a p-adic Galois representation. Let L be a finite extension of $\mathbb{Q}_p$ and V a finite-dimensional de Rham L-representation of $\mathrm{Gal}(\overline{F}/F)$. We regard V as a $\mathbb{Q}_p$-representation of $\mathrm{Gal}(\overline{F}/F)$ and consider $D_{\mathrm{pst}}(V)$, where D_{pst} is the functor introduced in [Fon94, §5.6]. If we write $\mathbb{Q}_p^{\mathrm{ur}}$ for the maximal unramified extension of $\mathbb{Q}_p$ contained in $\overline{F}$, $D_{\mathrm{pst}}(V)$ is an $L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{ur}}$-module equipped with several structures. Among them, we have the monodromy operator on $D_{\mathrm{pst}}(V)$, from which the monodromy filtration $M_\bullet D_{\mathrm{pst}}(V)$ on $D_{\mathrm{pst}}(V)$ is naturally induced.

Definition 2.3 Let V be a de Rham L-representation of $\mathrm{Gal}(\overline{F}/F)$. We say that V has the monodromy filtration if there exists a $\mathrm{Gal}(\overline{F}/F)$-stable $\mathbb{Q}_p$-subspace M_iV of V such that $D_{\mathrm{pst}}(M_iV) = M_iD_{\mathrm{pst}}(V)$ for each i. Such a subspace M_iV is unique and stable under the action of L if it exists, thanks to the fact that D_{pst} is fully faithful.

Now, let V be a finite-dimensional de Rham $\overline{\mathbb{Q}}_p$-representation of $\mathrm{Gal}(\overline{F}/F)$. We can find a subfield L of $\overline{\mathbb{Q}}_p$ which is finite over $\mathbb{Q}_p$ and a $\mathrm{Gal}(\overline{F}/F)$-stable L-subspace V_L of V such that $V_L \otimes_L \overline{\mathbb{Q}}_p = V$. The L-representation V_L is de Rham.

Definition 2.4 The condition that V_L has the monodromy filtration is independent of the choice of L and V_L. If it is the case, we say that V has the monodromy filtration, and put $M_iV = M_iV_L \otimes_L \overline{\mathbb{Q}}_p$, which is easily seen to be independent of L and V_L. We call $M_\bullet V$ the monodromy filtration of V.

Lemma 2.5 *Let V, W be finite-dimensional $\overline{\mathbb{Q}}_p$-representations of $\mathrm{Gal}(\overline{F}/F)$ which are de Rham and have the monodromy filtrations.*

(i) *Let V' be a direct summand of V as a $\mathrm{Gal}(\overline{F}/F)$-representation. Then V' is de Rham and has the monodromy filtration.*

(ii) *The representations $V \otimes W$ and $V^\vee$ are de Rham and have the monodromy filtrations. The monodromy filtration $M_\bullet(V \otimes W)$ (resp. $M_\bullet(V^\vee)$) is given by $M_n(V \otimes W) = \sum_{i+j=n} M_iV \otimes M_jW$ (resp. $M_n(V^\vee) = (V/M_{-n-1}V)^\vee$).*

Proof. Let L be a finite extension of $\mathbb{Q}_p$. We have only to consider L-representations in place of $\overline{\mathbb{Q}}_p$-representations. In the following, let V, W be finite-dimensional L-representations of $\mathrm{Gal}(\overline{F}/F)$ which are de Rham and have the monodromy filtrations.

We prove (i). We write $V = V' \oplus V''$. Then we have $D_{\mathrm{pst}}(V) = D_{\mathrm{pst}}(V') \oplus D_{\mathrm{pst}}(V'')$ and $M_iD_{\mathrm{pst}}(V) = M_iD_{\mathrm{pst}}(V') \oplus M_iD_{\mathrm{pst}}(V'')$. Since the essential image of the functor D_{pst} is stable under direct factors (see [Fon94, Théorème 5.6.7]), there exists a $\mathrm{Gal}(\overline{F}/F)$-stable subspace M_iV' of V' such that $D_{\mathrm{pst}}(M_iV') = M_iD_{\mathrm{pst}}(V')$. Hence V' has the monodromy filtration.

Next consider (ii). It is known that $V \otimes_{\mathbb{Q}_p} W$ is a de Rham representation, and $D_{\mathrm{pst}}(V \otimes_{\mathbb{Q}_p} W) = D_{\mathrm{pst}}(V) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} D_{\mathrm{pst}}(W)$. The monodromy filtration on $D_{\mathrm{pst}}(V \otimes_{\mathbb{Q}_p} W)$ is given by

$$M_nD_{\mathrm{pst}}(V \otimes_{\mathbb{Q}_p} W) = \sum_{i+j=n} M_iD_{\mathrm{pst}}(V) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} M_jD_{\mathrm{pst}}(W).$$

Therefore, if we put $M_n(V \otimes_{\mathbb{Q}_p} W) = \sum_{i+j=n} M_iV \otimes_{\mathbb{Q}_p} M_jW$, we have $D_{\mathrm{pst}}(M_n(V \otimes_{\mathbb{Q}_p} W)) = M_nD_{\mathrm{pst}}(V \otimes_{\mathbb{Q}_p} W)$ by the exactness of D_{pst}. Hence $V \otimes_{\mathbb{Q}_p} W$ has the monodromy filtration.

Let $e \in L \otimes_{\mathbb{Q}_p} L$ denote the idempotent corresponding to the diagonal component $\mathrm{Spec}\, L \hookrightarrow \mathrm{Spec}(L \otimes_{\mathbb{Q}_p} L)$. Then, we have $e(V \otimes_{\mathbb{Q}_p} W) = V \otimes_L W$. In particular

$V \otimes_L W$ is a direct summand of $V \otimes_{\mathbb{Q}_p} W$. Therefore, by (i), $V \otimes_L W$ is de Rham and has the monodromy filtration. Clearly, the monodromy filtration on $V \otimes_L W = e(V \otimes_{\mathbb{Q}_p} W)$ is given by

$$M_n(V \otimes_L W) = eM_n(V \otimes_{\mathbb{Q}_p} W) = \sum_{i+j=n} e(M_i V \otimes_{\mathbb{Q}_p} M_j W) = \sum_{i+j=n} M_i V \otimes_L M_j W.$$

The dual $V^\vee$ can be treated similarly. ∎

Lemma 2.6 *Let V be a finite-dimensional $\overline{\mathbb{Q}}_p$-representation of $\mathrm{Gal}(\overline{F}/F)$. For a finite extension F' of F contained in $\overline{F}$, we put $V' = V|_{\mathrm{Gal}(\overline{F}/F')}$. Then, V is de Rham and has the monodromy filtration if and only if so is V'. Moreover we have $M_\bullet V' = (M_\bullet V)|_{\mathrm{Gal}(\overline{F}/F')}$.*

Proof. As in the proof of Lemma 2.5, we may replace V by an L-representation of $\mathrm{Gal}(\overline{F}/F)$, where L is a finite extension of $\mathbb{Q}_p$. By definition, V is de Rham if and only if V' is de Rham. Suppose that V and V' are de Rham. We have $D_{\mathrm{pst}}(V') = D_{\mathrm{pst}}(V)|_{\mathrm{Gal}(\overline{F}/F')}$. Therefore, if V has the monodromy filtration $M_\bullet V$, then V' has the monodromy filtration $(M_\bullet V)|_{\mathrm{Gal}(\overline{F}/F')}$. Conversely, assume that V' has the monodromy filtration $M_\bullet V'$. Since $D_{\mathrm{pst}}(M_i V') = M_i D_{\mathrm{pst}}(V') = M_i D_{\mathrm{pst}}(V) \subset D_{\mathrm{pst}}(V)$ is stable under $\mathrm{Gal}(\overline{F}/F)$, so is $M_i V' \subset V' = V$. Therefore, $M_\bullet V'$ gives the monodromy filtration of V. This concludes the proof. ∎

Now, let $\phi \colon \mathrm{Gal}(\overline{F}/F) \to G(\overline{\mathbb{Q}}_p)$ be a continuous homomorphism, as in the beginning of this section.

Definition 2.7 Assume that ϕ is de Rham. We say that ϕ has the monodromy filtration if V_ξ has the monodromy filtration for every $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G)$.

Lemma 2.8 *Assume that there exists a faithful algebraic representation (ξ, V_ξ) of G such that V_ξ is de Rham and has the monodromy filtration. Then ϕ is de Rham and has the monodromy filtration.*

Proof. Thanks to Lemma 2.5, we can use the same argument as in the proof of Lemma 2.2 (ii). ∎

Lemma 2.9 *Assume that ϕ is de Rham and has the monodromy filtration.*

(i) *The stabilizer P_ξ of the filtration $M_\bullet V_\xi \subset V_\xi$ is a parabolic subgroup of G.*

(ii) *If ξ is faithful, then P_ξ stabilizes $M_\bullet V_{\xi'}$ for every representation ξ'. In particular P_ξ for faithful ξ is independent of ξ. We write P_ϕ for this P_ξ.*

(iii) *The homomorphism ϕ is potentially crystalline if and only if $P_\phi = G$.*

(iv) *The composite $\phi^{\mathrm{ad}} \colon \mathrm{Gal}(\overline{F}/F) \xrightarrow{\phi} G(\overline{\mathbb{Q}}_p) \to G^{\mathrm{ad}}(\overline{\mathbb{Q}}_p)$ is de Rham and has the monodromy filtration. Moreover we have $P_{\phi^{\mathrm{ad}}} = P_\phi^{\mathrm{ad}}$.*

Proof. This can be proved in the same way as Lemma 2.2. ∎

Lemma 2.10 *For a finite extension F' of F contained in $\overline{F}$, $\phi' = \phi|_{\mathrm{Gal}(\overline{F}/F')}$ is de Rham and has the monodromy filtration if and only if so is ϕ. Moreover, if the above conditions are satisfied, we have $P_{\phi'} = P_\phi$.*

Proof. This is an immediate consequence of Lemma 2.6. ∎

Corollary 2.11 *Let ℓ be a prime number.*

(i) *Assume that $\ell \neq p$. Then ϕ is potentially unramified if and only if ϕ^{ad} is potentially unramified.*

(ii) *Assume that $\ell = p$, ϕ is de Rham and has the monodromy filtration. Then ϕ is potentially crystalline if and only if ϕ^{ad} is potentially crystalline.*

Proof. The first assertion follows from Lemma 2.2 (iii), (iv), and the second from Lemma 2.9 (iii), (iv). ∎

Remark 2.12 In fact, Corollary 2.11 (ii) holds without assuming that ϕ has the monodromy filtration (we have only to consider the monodromy filtration on the image of D_{pst}). However we do not need this fact later.

Remark 2.13 Let ℓ be a prime number, and assume that ϕ is de Rham and has the monodromy filtration if $\ell = p$. If G is defined over $\mathbb{Q}_\ell$ and the image of ϕ is contained in $G(\mathbb{Q}_\ell)$, the parabolic subgroup P_ϕ is defined over $\mathbb{Q}_\ell$. Indeed, we can take a faithful representation ξ which is defined over $\mathbb{Q}_\ell$, and then the monodromy filtration $M_\bullet V_\xi$ is also defined over $\mathbb{Q}_\ell$.

3 Rigid geometry and semi-abelian schemes

3.1 Notation for adic spaces

Throughout this paper, we will use the framework of adic spaces introduced by Huber (*cf.* [Hub93], [Hub94], [Hub96]). Here we recall some notation briefly.

Let S be a noetherian scheme and S_0 a closed subscheme of S. We denote the formal completion of S along S_0 by $\mathcal{S}$. Put $\mathcal{S}^{\mathrm{rig}} = t(\mathcal{S})_a$, where $t(\mathcal{S})$ is the adic space associated to $\mathcal{S}$ (*cf.* [Hub94, §4]) and $t(\mathcal{S})_a$ denotes the open adic subspace of $t(\mathcal{S})$ consisting of analytic points. It is a quasi-compact analytic adic space.

Let X be a scheme of finite type over S. Put $X_0 = X \times_S S_0$ and denote the formal completion of X along X_0 by $\widehat{X}$. Then we can construct an adic space $\widehat{X}^{\mathrm{rig}}$ in the same way as $\mathcal{S}^{\mathrm{rig}}$. The induced morphism $\widehat{X}^{\mathrm{rig}} \to \mathcal{S}^{\mathrm{rig}}$ is of finite type. On the other hand, we can construct another adic space $X \times_S \mathcal{S}^{\mathrm{rig}}$. Indeed, since we have morphisms of locally ringed spaces $(\mathcal{S}^{\mathrm{rig}}, \mathcal{O}_{\mathcal{S}^{\mathrm{rig}}}) \to (t(\mathcal{S}), \mathcal{O}_{t(\mathcal{S})}) \to (S, \mathcal{O}_S)$ (for the second one, see [Hub94, Remark 4.6 (iv)]), we can make the fiber product $X \times_S \mathcal{S}^{\mathrm{rig}}$ in the sense of [Hub94, Proposition 3.8]. For simplicity, we write X^{ad}

for $X \times_S \mathcal{S}^{\mathrm{rig}}$, though it depends on (S, S_0). Since the morphism $\mathcal{S}^{\mathrm{rig}} \to S$ factors through $S^0 = S \setminus S_0$, we have $X \times_S \mathcal{S}^{\mathrm{rig}} = (X \times_S S^0) \times_{S^0} \mathcal{S}^{\mathrm{rig}}$. In particular, X^{ad} depends only on $X \times_S S^0$. The natural morphism $X^{\mathrm{ad}} \to \mathcal{S}^{\mathrm{rig}}$ is locally of finite type, but not necessarily quasi-compact; see the following example.

Example 3.1 Let R be a complete discrete valuation ring and F its fraction field. Consider the case where $S = \mathrm{Spec}\, R$ and S_0 is the closed point of S. Then, for an S-scheme X of finite type, X^{ad} can be regarded as the rigid space over F associated to a scheme $X \times_S \mathrm{Spec}\, F$ over F. For example, $(\mathbb{A}_S^1)^{\mathrm{ad}} = (\mathbb{A}_F^1)^{\mathrm{ad}}$ is the rigid-analytic affine line over F and thus is not quasi-compact. On the other hand, $(\widehat{\mathbb{A}}_S^1)^{\mathrm{rig}}$ is the unit disc "$|z| \leq 1$" in $(\mathbb{A}_F^1)^{\mathrm{ad}}$, which is quasi-compact.

Lemma 3.2 *The functors $X \mapsto \widehat{X}^{\mathrm{rig}}$ and $X \mapsto X^{\mathrm{ad}}$ commute with fiber products.*

Proof. For the functor $X \mapsto \widehat{X}^{\mathrm{rig}}$, it can be checked easily (*cf.* [Mie06, Lemma 3.4] and [Mie14, Proof of Lemma 4.4 (v)]). Consider the functor $X \mapsto X^{\mathrm{ad}}$. Let $Y \to X \leftarrow Z$ be a diagram of S-schemes of finite type. What we should prove is

$$(Y \times_X Z) \times_S \mathcal{S}^{\mathrm{rig}} \cong (Y \times_S \mathcal{S}^{\mathrm{rig}}) \times_{X \times_S \mathcal{S}^{\mathrm{rig}}} (Z \times_S \mathcal{S}^{\mathrm{rig}}).$$

It is not totally automatic, since $Y \times_X Z$ is not a fiber product in the category of locally ringed spaces. It follows from the fact that morphisms of locally ringed spaces $\mathrm{Spa}(A, A^+) \to \mathrm{Spec}\, B$ for a complete affinoid ring (A, A^+) and a ring B correspond bijectively to ring homomorphisms $B \to A$ (this fact is used implicitly in [Hub94, Remark 4.6 (iv)] to define $t(\mathcal{S}) \to S$). ∎

Let us compare $\widehat{X}^{\mathrm{rig}}$ and X^{ad}; by the commutative diagram

$$
\begin{array}{ccc}
\widehat{X}^{\mathrm{rig}} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathcal{S}^{\mathrm{rig}} & \longrightarrow & S
\end{array}
$$

and the universality of the fiber product $X \times_S \mathcal{S}^{\mathrm{rig}}$, we have a natural morphism $\widehat{X}^{\mathrm{rig}} \to X^{\mathrm{ad}}$.

Lemma 3.3 (i) *If X is separated over S, $\widehat{X}^{\mathrm{rig}} \to X^{\mathrm{ad}}$ is an open immersion.*

(ii) *If X is proper over S, $\widehat{X}^{\mathrm{rig}} \to X^{\mathrm{ad}}$ is an isomorphism.*

Proof. See [Hub94, Remark 4.6 (iv)]. ∎

Remark 3.4 Let $f \colon S' \to S$ be a morphism of finite type and $S_0' = S' \times_S S_0$. We denote by $\mathcal{S}'^{\mathrm{rig}}$ the formal completion of S' along S_0'. Then, all constructions above are compatible with the base change by f. More precisely, for a scheme X of finite type over S, we have $(X \times_S S')^{\wedge \mathrm{rig}} \cong \widehat{X}^{\mathrm{rig}} \times_{\mathcal{S}^{\mathrm{rig}}} \mathcal{S}'^{\mathrm{rig}}$ and $(X \times_S S')^{S'\text{-ad}} \cong X^{\mathrm{ad}} \times_{\mathcal{S}^{\mathrm{rig}}} \mathcal{S}'^{\mathrm{rig}}$. Here $(-)^{S'\text{-ad}}$ denotes the functor $(-)^{\mathrm{ad}}$ for the base (S', S_0'), namely, $(-)^{S'\text{-ad}} = (-) \times_{S'} \mathcal{S}'^{\mathrm{rig}}$.

In the remaining part of this subsection, assume that S is the spectrum of a complete discrete valuation ring R and S_0 is the closed point of S. For a scheme of finite type X over S, we have a natural morphism of locally and topologically ringed spaces $(t(\widehat{X}), \mathcal{O}^+_{t(\widehat{X})}) \to (\widehat{X}, \mathcal{O}_{\widehat{X}})$ (*cf.* [Hub94, Proposition 4.1]). Note that the underlying continuous map $t(\widehat{X}) \to X_0$ is different from the map $t(\widehat{X}) \to X$ considered above. We denote the composite $\widehat{X}^{\mathrm{rig}} \hookrightarrow t(\widehat{X}) \to X_0$ by $\mathrm{sp}_{\widehat{X}}$, or simply by sp.

Let Y be a closed subscheme of X_0 and $\mathcal{X}$ the formal completion of X along Y. Then we can consider the generic fiber $t(\mathcal{X})_\eta = S^0 \times_S t(\mathcal{X})$ of the adic space $t(\mathcal{X})$. This is so-called the rigid generic fiber of $\mathcal{X}$ due to Raynaud and Berthelot, in the context of adic spaces. If $Y = X_0$, then $t(\mathcal{X})_\eta = \widehat{X}^{\mathrm{rig}}$.

Lemma 3.5 *The natural morphism $t(\mathcal{X})_\eta \to \widehat{X}^{\mathrm{rig}}$ induced from $\mathcal{X} \to \widehat{X}$ is an open immersion. Its image coincides with $\mathrm{sp}^{-1}(Y)^\circ$, where $(-)^\circ$ denotes the interior in $\widehat{X}^{\mathrm{rig}}$.*

Proof. See [Hub98b, Lemma 3.13 i)]. ∎

Let X be an adic space locally of finite type over $\mathcal{S}^{\mathrm{rig}} = \mathrm{Spa}(F, R)$, where F denotes the fraction field of R. For a point x of X, we write κ_x and κ_x^+ for the residue field and the valuation ring at x, respectively. We say that $x \in X$ is classical if κ_x is a finite extension of F. We denote the set of classical points of X by $X(\mathrm{cl})$. Further, for a subset Y of X, we put $Y(\mathrm{cl}) = X(\mathrm{cl}) \cap Y$.

Lemma 3.6 *Let X be an adic space locally of finite type over $\mathrm{Spa}(F, R)$.*

(i) *For constructible subsets L_1, L_2 of X (see [Hub96, 1.1.13]), we have $L_1 \subset L_2$ if and only if $L_1(\mathrm{cl}) \subset L_2(\mathrm{cl})$. In particular, $L_1 = L_2$ if and only if $L_1(\mathrm{cl}) = L_2(\mathrm{cl})$.*

(ii) *For a constructible subset L, we write L^- (resp. L°) for the closure (resp. interior) of L in X. Then we have $L(\mathrm{cl}) = L^-(\mathrm{cl}) = L^\circ(\mathrm{cl})$.*

Proof. For (i), it suffices to show that $L_1(\mathrm{cl}) \subset L_2(\mathrm{cl})$ implies $L_1 \subset L_2$. Put $L = L_1 \setminus L_2 = L_1 \cap (X \setminus L_2)$. It is a constructible subset of X satisfying $L(\mathrm{cl}) = \varnothing$. Let U be an arbitrary affinoid open subset of X. Then, we have $(U \cap L)(\mathrm{cl}) = \varnothing$. Therefore, [Hub93, Corollary 4.3] tells us that $U \cap L = \varnothing$. Now we conclude that $L = \varnothing$, that is, $L_1 \subset L_2$.

For (ii), it suffices to prove that $L(\mathrm{cl}) = L^-(\mathrm{cl})$. Take $x \in L^- \setminus L$ and an affinoid open neighborhood U of x. Then x lies in the closure of $U \cap L$ in U. Since $U \cap L$ is a constructible subset of the spectral space U, by [Hoc69, Corollary of Theorem 1], there exists $y \in U \cap L$ such that $x \in \{y\}^-$. Therefore, by [Hub96, Lemma 1.1.10 ii)], the valuation v_x attached to x is not rank 1. In particular x is not classical. Hence we have $L(\mathrm{cl}) = L^-(\mathrm{cl})$, as desired. ∎

The following basic lemma is also used in Section 5.

Lemma 3.7 *Let $f\colon X \to Y$ be a quasi-compact quasi-separated étale morphism between adic spaces.*

(i) *For a constructible subset L of X, the image $f(L)$ is a constructible subset of Y.*

(ii) *For a locally closed subset L of X satisfying $f^{-1}(f(L)) = L$, the image $f(L)$ is a locally closed subset of Y.*

Proof. The assertion (i) can be proved in the same way as [Hub96, (1) in the proof of Lemma 2.7.4]. We recall the argument for reader's convenience. We may assume that X and Y are quasi-compact and quasi-separated. Fix $y \in f(L)$. Let Λ denote the set of constructible subsets of Y containing y. We have $\bigcap_{W \in \Lambda} W = \{y\}$, as Y is a spectral space. Since $f^{-1}(y)$ is a finite discrete subset of X, there exists a quasi-compact open subset U of X such that $U \cap f^{-1}(y) = L \cap f^{-1}(y)$. Then we have $U \cap \bigcap_{W \in \Lambda} f^{-1}(W) = L \cap \bigcap_{W \in \Lambda} f^{-1}(W)$. By the quasi-compactness of X with respect to the constructible topology, there exists $W \in \Lambda$ such that $U \cap f^{-1}(W) = L \cap f^{-1}(W)$. Put $V_y = U \cap f^{-1}(W) = L \cap f^{-1}(W)$, which is a constructible subset of X. Since f is étale, $f(U)$ is a quasi-compact open subset of Y. Therefore $f(V_y) = f(U) \cap W$ is a constructible subset of Y.

Since $L \cap f^{-1}(y) \subset V_y \subset L$, we have $L = \bigcup_{y \in f(L)} V_y$. On the other hand, L is quasi-compact under the constructible topology of X. Therefore, there exist finitely many points $y_1, \ldots, y_m \in f(L)$ such that $L = \bigcup_{i=1}^{m} V_{y_i}$. Now we conclude that $f(L) = \bigcup_{i=1}^{m} f(V_{y_i})$ is a constructible subset of Y, as desired.

Next we consider (ii). Since L is locally closed, it can be written in the form $U \cap W$, where U is an open subset of X and W is a closed subset of X. Note that $L^- \subset W$, thus $U \cap L^- = L$. For simplicity we write $L' = f(L)$. Since f is an open map, we can check that $f^{-1}(L')^- = f^{-1}(L'^-)$. Therefore we obtain

$$L = U \cap L^- = U \cap f^{-1}(L')^- = U \cap f^{-1}(L'^-)$$

and $f(L) = f(U) \cap L'^-$. As f is étale, $f(U)$ is open, hence $f(L)$ is locally closed. ∎

3.2 Etale sheaves associated to semi-abelian schemes

We continue to use the notation introduced in the beginning of the previous subsection. Let U be an open subscheme of $S^0 = S \setminus S_0$ and ℓ a prime number invertible on U. Fix an integer $m > 0$.

Let G be a semi-abelian scheme over S. Namely, G is a separated smooth commutative group scheme over S such that each fiber G_s of G at $s \in S$ is an extension of an abelian variety A_s by a torus T_s. We denote the relative dimension of G over S by d. Assume the following:

- The rank of T_s (called the toric rank of G_s) with $s \in S_0$ is a constant r.
- $G_U = G \times_S U$ is an abelian scheme.

Under the first condition, it is known that $G_0 = G \times_S S_0$ is globally an extension

$$0 \to T_0 \to G_0 \to A_0 \to 0,$$

where T_0 is a torus of rank r over S_0 and A_0 is an abelian scheme over S_0 ([FC90, Chapter I, Corollary 2.11]).

Let us consider two group spaces $\widehat{G}^{\mathrm{rig}}[\ell^m]_{U^{\mathrm{ad}}}$ and $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$ over U^{ad}, where $(-)_{U^{\mathrm{ad}}}$ denotes the restriction to U^{ad}.

Lemma 3.8 *The adic space $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$ is finite étale of degree ℓ^{2dm} over U^{ad}.*

Proof. By Lemma 3.2, we have $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}} = (G_U[\ell^m]) \times_U U^{\mathrm{ad}}$. Since $G_U[\ell^m]$ is finite étale of degree ℓ^{2dm} over U, $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$ is finite étale of degree ℓ^{2dm} over U^{ad} (see [Hub96, Corollary 1.7.3 i)]). ∎

Lemma 3.9 *The adic space $\widehat{G}^{\mathrm{rig}}[\ell^m]_{U^{\mathrm{ad}}}$ is finite étale of degree $\ell^{(2d-r)m}$ over U^{ad}.*

Proof. We may assume that $S = \mathrm{Spec}\, R$ is affine. Let $I \subset R$ be the defining ideal of S_0. By replacing R by its I-adic completion, we can reduce to the case where R is I-adically complete. Put $S_i = \mathrm{Spec}\, R/I^{i+1}$ and $G_i = G \times_S S_i$.

By [SGA3, Exposé IX, Théorème 3.6, Théorème 3.6 bis], the exact sequence

$$0 \to T_0 \to G_0 \to A_0 \to 0$$

can be lifted canonically to an exact sequence

$$0 \to T_i \to G_i \to A_i \to 0$$

over S_i, where T_i is a torus over S_i and A_i is an abelian scheme over S_i (see [Lan13, §3.3.3]). Let $\widehat{T} = \varinjlim_i T_i$ and $\widehat{A} = \varinjlim_i A_i$ be associated formal groups over $\mathcal{S}$. Then $\widehat{G}$ is an extension of $\widehat{A}$ by $\widehat{T}$.

By taking ℓ^m-torsion points, we get an exact sequence

$$0 \to \widehat{T}[\ell^m] \to \widehat{G}[\ell^m] \to \widehat{A}[\ell^m] \to 0$$

of formal groups over $\mathcal{S}$. Since $\widehat{G}^{\mathrm{rig}}[\ell^m] \cong (\widehat{G}[\ell^m])^{\mathrm{rig}}$, it suffices to see that $(\widehat{T}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$ (resp. $(\widehat{A}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$) is finite étale of degree ℓ^{rm} (resp. $\ell^{2(d-r)m}$) over U^{ad}.

First we consider $(\widehat{T}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$. Since $\widehat{T}[\ell^m] = \varinjlim_i (T_i[\ell^m])$, it is finite flat over $\mathcal{S} = \mathrm{Spf}\, R$. Therefore there exists a finite flat R-algebra R' such that $\widehat{T}[\ell^m] = \mathrm{Spf}\, R'$. Moreover, a scheme $T' = \mathrm{Spec}\, R'$ is naturally equipped with a structure of a commutative group scheme over $S = \mathrm{Spec}\, R$. Since T' is killed by ℓ^m and p is invertible on U, $T'_U = T' \times_S U$ is a finite étale group scheme over U. By Lemma 3.3 (ii), we have $(\widehat{T}[\ell^m])^{\mathrm{rig}} = (T')^{\wedge\mathrm{rig}} = T'^{\mathrm{ad}} = T' \times_S \mathcal{S}^{\mathrm{rig}}$. Therefore $(\widehat{T}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}} = T'_U \times_U U^{\mathrm{ad}}$ is finite étale over U^{ad} (see [Hub96, Corollary 1.7.3 i)]). Its degree is clearly ℓ^{rm}.

The same argument also works for $(\widehat{A}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$. ∎

By Lemma 3.8 and Lemma 3.9, we may regard $\widehat{G}^{\mathrm{rig}}[\ell^m]_{U^{\mathrm{ad}}}$ and $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$ as locally constant constructible sheaves over U^{ad}. Since we have a natural open immersion $\widehat{G}^{\mathrm{rig}} \hookrightarrow G^{\mathrm{ad}}$, $\widehat{G}^{\mathrm{rig}}[\ell^m]_{U^{\mathrm{ad}}}$ is a subsheaf of $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$.

Remark 3.10 In the setting of Remark 3.4, the construction above is clearly compatible with the base change by $f\colon S' \to S$.

In the remaining part of this subsection, we consider the case where $S = \operatorname{Spec} R$ is the spectrum of a complete discrete valuation ring R, S_0 is the closed point of S and $U = S^0 = S \setminus S_0$. Let $\bar{\eta}$ be a geometric point lying over the unique point of U^{ad}.

As in the proof of Lemma 3.9, $\widehat{G}$ is an extension

$$0 \to \widehat{T} \to \widehat{G} \to \widehat{A} \to 0$$

of a formal group $\widehat{A}$ by $\widehat{T}$. Therefore, we have $\mathbb{Z}/\ell^m\mathbb{Z}$-submodules

$$\widehat{T}^{\mathrm{rig}}[\ell^m]_{\bar{\eta}} \subset \widehat{G}^{\mathrm{rig}}[\ell^m]_{\bar{\eta}} \subset G^{\mathrm{ad}}[\ell^m]_{\bar{\eta}}.$$

By taking inverse limit and tensoring with $\mathbb{Q}_\ell$, we have

$$T_\ell \widehat{T}^{\mathrm{rig}}_{\bar{\eta}} \subset T_\ell \widehat{G}^{\mathrm{rig}}_{\bar{\eta}} \subset T_\ell G^{\mathrm{ad}}_{\bar{\eta}}, \quad V_\ell \widehat{T}^{\mathrm{rig}}_{\bar{\eta}} \subset V_\ell \widehat{G}^{\mathrm{rig}}_{\bar{\eta}} \subset V_\ell G^{\mathrm{ad}}_{\bar{\eta}},$$

where we put $V_\ell(-) = T_\ell(-) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. By Lemma 3.9 and its proof, we can deduce that $\dim_{\mathbb{Q}_\ell} V_\ell \widehat{T}^{\mathrm{rig}}_{\bar{\eta}} = r$ and $\dim_{\mathbb{Q}_\ell} V_\ell \widehat{G}^{\mathrm{rig}}_{\bar{\eta}} = 2d - r$.

Proposition 3.11 *Assume that the fraction field F of R is a finite extension of $\mathbb{Q}_p$. For a filtration*

$$0 \subset V_\ell \widehat{T}^{\mathrm{rig}}_{\bar{\eta}} \subset V_\ell \widehat{G}^{\mathrm{rig}}_{\bar{\eta}} \subset V_\ell G^{\mathrm{ad}}_{\bar{\eta}} = V_\ell G_{\bar{\eta}},$$

we have the following:

  (i) *If $\ell \neq p$, the above filtration is the weight filtration of $V_\ell G_{\bar{\eta}}$.*

 (ii) *If $\ell = p$, then the above filtration is a filtration as semistable representations of $\operatorname{Gal}(\overline{F}/F)$. Further, this filtration induces the weight filtration on $D_{\mathrm{st}}(V_p G_{\bar{\eta}})$.*

Proof. We give a proof of (ii). We can show (i) similarly. Let λ be a polarization of G_U. Then an alternating bilinear pairing

$$\langle\,,\,\rangle_\lambda \colon V_p G_{\bar{\eta}} \times V_p G_{\bar{\eta}} \to \mathbb{Q}_p(1)$$

is induced. First, we will prove that $(V_p \widehat{G}^{\mathrm{rig}}_{\bar{\eta}})^\perp = V_p \widehat{T}^{\mathrm{rig}}_{\bar{\eta}}$. Since

$$\dim_{\mathbb{Q}_p} V_p \widehat{T}^{\mathrm{rig}}_{\bar{\eta}} + \dim_{\mathbb{Q}_p} V_p \widehat{G}^{\mathrm{rig}}_{\bar{\eta}} = r + (2d - r) = 2d = \dim_{\mathbb{Q}_p} V_p G_{\bar{\eta}},$$

it is sufficient to prove that $V_p \widehat{T}^{\mathrm{rig}}_{\bar{\eta}} \subset (V_p \widehat{G}^{\mathrm{rig}}_{\bar{\eta}})^\perp$. Namely, we should prove that the homomorphism $V_p \widehat{T}^{\mathrm{rig}}_{\bar{\eta}} \otimes_{\mathbb{Q}_p} V_p \widehat{G}^{\mathrm{rig}}_{\bar{\eta}} \to \mathbb{Q}_p(1)$ induced by $\langle\,,\,\rangle_\lambda$ is zero.

Since $V_p G_{\overline{\eta}}$ is a semistable representation of $\mathrm{Gal}(\overline{F}/F)$, so are $V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}}$ and $V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}}$. We denote the residue field of F by κ_F and put $q = \#\kappa_F$. Consider the action of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}})$ and $D_{\mathrm{st}}(V_p\widehat{A}_{\overline{\eta}}^{\mathrm{rig}})$. By [SGA3, Exposé X, Théorème 3.2], $\widehat{T}$ can be algebraized into a torus T over S. Then we have $V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \cong V_p T_{\overline{\eta}}$. Therefore every eigenvalue of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}})$ is a Weil q^{-2}-number (for the definition of Weil numbers, see [TY07, p. 471]). Similarly, by [Lan13, Proposition 3.3.3.6, Remark 3.3.3.9], $\widehat{A}$ can be algebraized into an abelian scheme A over S, and we have $V_p\widehat{A}_{\overline{\eta}}^{\mathrm{rig}} \cong V_p A_{\overline{\eta}}$. By the Weil conjecture for the crystalline cohomology of abelian varieties, every eigenvalue of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p\widehat{A}_{\overline{\eta}}^{\mathrm{rig}})$ is a Weil q^{-1}-number. Therefore, every eigenvalue of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}})$ is either a Weil q^{-4}-number or a Weil q^{-3}-number. On the other hand, every eigenvalue of $\varphi^{[\kappa_F:\mathbb{F}_p]}$ on $D_{\mathrm{st}}(\mathbb{Q}_p(1))$ is equal to q^{-1}, which is a Weil q^{-2}-number. Hence any φ-homomorphism $D_{\mathrm{st}}(V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}}) \to D_{\mathrm{st}}(\mathbb{Q}_p(1))$ is zero. Since the functor D_{st} is fully faithful, any $\mathrm{Gal}(\overline{F}/F)$-equivariant homomorphism

$$V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}} \to \mathbb{Q}_p(1)$$

is zero. Hence, we have $(V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}})^{\perp} = V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}}$. Then we have a perfect pairing

$$V_p\widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \times (V_p G_{\overline{\eta}}/V_p\widehat{G}_{\overline{\eta}}^{\mathrm{rig}}) \to \mathbb{Q}_p(1).$$

The claim follows from the above arguments and this perfect pairing. ∎

Corollary 3.12 *The semistable representation $V_p G_{\overline{\eta}}$ of $\mathrm{Gal}(\overline{F}/F)$ has the monodromy filtration in the sense of Definition 2.3.*

Proof. It is well-known that in this case the weight filtration and the monodromy filtration on $D_{\mathrm{pst}}(V_p G_{\eta}) = D_{\mathrm{st}}(V_p G_{\eta})$ coincide up to shift. Therefore the claim follows from Proposition 3.11 (ii). ∎

Remark 3.13 Actually, the extension $0 \to \widehat{T} \to \widehat{G} \to \widehat{A} \to 0$ considered above can be algebraized; namely, there exists an exact sequence

$$0 \to T \to G^{\natural} \to A \to 0$$

of commutative group schemes over S, where T and A are as in the proof of Proposition 3.11, such that its formal completion along the special fiber is isomorphic to the extension above (see [Lan13, Proposition 3.3.3.6, Remark 3.3.3.9]). Such an extension is called the Raynaud extension associated to G.

Our construction above is related to the Raynaud extension in the following way. First, we have a natural isomorphism $\widehat{G}^{\mathrm{rig}}[\ell^m]_{\overline{\eta}} \xrightarrow{\cong} (G^{\natural})^{\mathrm{ad}}[\ell^m]_{\overline{\eta}}$, which is induced from an open immersion $\widehat{G}^{\mathrm{rig}} \cong (\widehat{G^{\natural}})^{\mathrm{rig}} \hookrightarrow (G^{\natural})^{\mathrm{ad}}$ (see Lemma 3.3 (i)). Moreover, the image of $\widehat{G}^{\mathrm{rig}}[\ell^m]_{\overline{\eta}} \hookrightarrow G^{\mathrm{ad}}[\ell^m]_{\overline{\eta}}$ coincides with the image of the map $G^{\natural}[\ell^m]_{\overline{\eta}} \to G[\ell^m]_{\overline{\eta}}$ in [Lan13, Corollary 4.5.3.12].

4 Shimura varieties

4.1 Notation on Shimura varieties

Let (G, X) be a Shimura datum, and $E(G, X)$ the reflex field of (G, X). We simply write E for $E(G, X)$ if there is no risk of confusion. There is the canonical model over E of the Shimura variety for (G, X), which we denote by $\{\mathrm{Sh}_K(G, X)\}_{K \subset G(\mathbb{A}^\infty)}$. Let $K \subset G(\mathbb{A}^\infty)$ be a compact open subgroup, which is always supposed to be small enough so that $\mathrm{Sh}_K(G, X)$ becomes a scheme.

4.2 Siegel modular varieties

Let $(V, \langle\ ,\ \rangle)$ be a symplectic space of dimension $2n$ over $\mathbb{Q}$, and L a self-dual $\mathbb{Z}$-lattice of V. Let $(\mathrm{GSp}_{2n}, X_{2n})$ be the Shimura datum associated to $(V, \langle\ ,\ \rangle)$. Then the Shimura variety for $(\mathrm{GSp}_{2n}, X_{2n})$ is called the Siegel modular variety. In this case the reflex field $E(\mathrm{GSp}_{2n}, X_{2n})$ equals $\mathbb{Q}$. We put

$$K(N) = \mathrm{Ker}(\mathrm{GSp}_{2n}(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_{2n}(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}))$$

for $N \geq 1$, and

$$K_{\ell, m} = \mathrm{Ker}(\mathrm{GSp}_{2n}(\mathbb{Z}_\ell) \to \mathrm{GSp}_{2n}(\mathbb{Z}/\ell^m\mathbb{Z}))$$

for a prime number ℓ and $m \geq 0$.

We recall a moduli interpretation of $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})$ using integral level structures. For simplicity, we assume that $K = K(N)$ with $N \geq 3$. We consider the functor from the category of $\mathbb{Q}$-schemes to the category of sets, that associates S to the set of isomorphism classes of triples (A, λ, η), where

- A is an abelian scheme over S,
- $\lambda \colon A \to A^\vee$ is a principal polarization, and
- $\eta \colon L/NL \xrightarrow{\cong} A[N]$ is a symplectic similitude.

This functor is represented by $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})$ (see [Del71, 4.16]).

There is another moduli interpretation using rational level structures. Let S be a connected Noetherian scheme over $\mathbb{Q}$, and fix a geometric point $\overline{s}$ of S. We put

$$T^\infty(-) = \prod_\ell T_\ell(-), \quad V^\infty(-) = T^\infty(-) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where ℓ in the product ranges over all prime numbers. Then, S-valued points of $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})$ correspond to the isogeny classes of triples $(A, \lambda, \eta K)$, where

- A is an abelian scheme over S,
- $\lambda \colon A \to A^\vee$ is a $\mathbb{Q}$-polarization, and
- ηK is a $\pi_1(S, \overline{s})$-invariant K-orbit of symplectic similitudes $V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty A$.

Using this description, the Hecke action of $g \in \mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ can be described as

$$\mathrm{Sh}_K \to \mathrm{Sh}_{g^{-1}Kg}; \quad [(A, \lambda, \eta K)] \mapsto [(A, \lambda, (\eta \circ g)g^{-1}Kg)].$$

See [Del71, 4.12] for the relation between two moduli interpretations.

Assume that $K = K_{p,0}K^p$ with a compact open subgroup K^p of $\mathrm{GSp}_{2n}(\widehat{\mathbb{Z}}^p)$. Then $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})$ has a natural integral model $\mathscr{S}_{K^p}$ over $\mathbb{Z}_p$ constructed as a moduli space of principally polarized abelian schemes with level structures (*cf.* [MFK94, Chapter 7, §3]). Let $\mathcal{A}$ denote the universal abelian scheme on $\mathscr{S}_{K^p}$.

Thanks to a work of Faltings and Chai [FC90], we have a toroidal compactification $\mathscr{S}_{K^p}^{\mathrm{tor}}$ of $\mathscr{S}_{K^p}$ over $\mathbb{Z}_p$. We have a semi-abelian scheme on $\mathscr{S}_{K^p}^{\mathrm{tor}}$ extending $\mathcal{A}$ on $\mathscr{S}_{K^p}$, for which we write the same symbol $\mathcal{A}$.

4.3 Shimura varieties of Hodge type

In this subsection, we assume that (G, X) is of Hodge type. We take an embedding $i \colon (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ of Shimura data. For a compact open subgroup $\widetilde{K}$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K, we have a natural morphism $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})$ to the Siegel modular variety, which is known to be a closed immersion if $\widetilde{K}$ is small enough. We shall recall a moduli interpretation of $\mathbb{C}$-points of $\mathrm{Sh}_K(G, X)$. Let V be the standard representation of GSp_{2n}. By [DMOS82, I, Proposition 3.1], there exists a finite collection of tensors $(s_\alpha)_{\alpha \in J'}$ with $s_\alpha \in V^{m_\alpha} \otimes V^{\vee m'_\alpha}$ such that G equals the pointwise stabilizer of $(s_\alpha)_{\alpha \in J'}$ in GSp_{2n}. We put $J = J' \amalg \{0\}$, $m_0 = m'_0 = 1$ and let s_0 be the symplectic form $\langle \ , \ \rangle \in V \otimes V^\vee$ on V.

Proposition 4.1 *A $\mathbb{C}$-valued point of $\mathrm{Sh}_K(G, X)$ corresponds to the isogeny class of triples $(A, (t_\alpha)_{\alpha \in J}, \eta K)$, where*

- *A is an abelian variety over $\mathbb{C}$,*
- *$(t_\alpha)_{\alpha \in J}$ with $t_\alpha \in H_1(A, \mathbb{Q})^{m_\alpha} \otimes H_1(A, \mathbb{Q})^{\vee m'_\alpha}$ is a finite collection of Hodge cycles on A (see [DMOS82, V, §2]) such that $\pm t_0$ is a polarization of the rational Hodge structure $H_1(A, \mathbb{Q})$,*
- *ηK is a K-orbit of $\mathbb{A}^\infty$-linear isomorphisms $V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty A$ which send s_0 to a $(\mathbb{A}^\infty)^\times$-multiple of t_0 and s_α with $\alpha \in J'$ to t_α,*

satisfying the following condition $(*)$:

$(*)$ *there exists an isomorphism $\eta_\mathbb{Q} \colon V \xrightarrow{\cong} H_1(A, \mathbb{Q})$ such that $\eta_\mathbb{Q}^{-1}$ sends t_0 to a $\mathbb{Q}^\times$-multiple of s_0, t_α with $\alpha \in J'$ to s_α, and the Hodge structure on $H_1(A, \mathbb{Q})$ to a Hodge structure on V induced by an element of X and the embedding $i \colon G \hookrightarrow \mathrm{GSp}_{2n}$.*

For a proof, see [Mil05, Theorem 7.4].

Lemma 4.2 *Let F be a p-adic field containing the reflex field E, and x an F-valued point of $\mathrm{Sh}_K(G, X)$. Choose an algebraic closure $\overline{F}$ of F and denote by $\overline{x}$ the corresponding geometric point over x.*

We take an isomorphism $\iota\colon \overline{F} \xrightarrow{\cong} \mathbb{C}$ over E, and write $\iota\overline{x}$ for the $\mathbb{C}$-valued point of $\mathrm{Sh}_K(G, X)$ determined by $\overline{x}$ and ι. Let $(A, (t_\alpha)_{\alpha\in J}, \eta K)$ be a triple in the isogeny class corresponding to $\iota\overline{x}$ such that $A = \mathcal{A}_{\overline{x}} \otimes_{\overline{F},\iota} \mathbb{C}$. Here $\mathcal{A}_{\overline{x}}$ is the abelian variety corresponding to the image of $\overline{x}$ in the Siegel modular variety. Let us choose a representative η of ηK. Under ι, it corresponds to a trivialization of the K-torsor $\pi_K^{-1}(\overline{x})$ on $\overline{x}$, where π_K denotes the natural map $\varprojlim_{K'\subset K} \mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$.

For a prime number ℓ, let $\mathcal{L}_{\mathrm{Std}\circ i,\ell}$ be the $\mathbb{Q}_\ell$-sheaf on $\mathrm{Sh}_K(G, X)$ corresponding to the representation $\mathrm{Std}\circ i$ of G on V. For the stalk $\mathcal{L}_{\mathrm{Std}\circ i,\ell,\overline{x}}$, the following hold:

(i) We have a canonical $\mathrm{Gal}(\overline{F}/F)$-equivariant isomorphism $\mathcal{L}_{\mathrm{Std}\circ i,\ell,\overline{x}} \cong V_\ell\mathcal{A}_{\overline{x}}$.

(ii) Each trivialization of the K-torsor $\pi_K^{-1}(\overline{x})$ determines an isomorphism $V_{\mathbb{Q}_\ell} \xrightarrow{\cong} \mathcal{L}_{\mathrm{Std}\circ i,\ell,\overline{x}}$. The isomorphism given by the trivialization corresponding to the chosen representative η equals the composite of $V_{\mathbb{Q}_\ell} \xrightarrow[\cong]{\eta_\ell} V_\ell A \xrightarrow[\cong]{\iota^{-1}} V_\ell\mathcal{A}_{\overline{x}} \overset{\mathrm{(i)}}{\cong} \mathcal{L}_{\mathrm{Std}\circ i,\ell,\overline{x}}$, where η_ℓ denotes the ℓ-part of η.

Proof. The first assertion is essentially a statement for the Siegel case, which is well-known. The second can be checked directly by working over $\mathbb{C}$. ∎

4.4 Shimura varieties of preabelian type

Definition 4.3 A Shimura datum (G, X) is said to be of preabelian type if there exists a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$. If a Shimura data is of preabelian type, the associated Shimura variety is said to be of preabelian type (*cf.* [Vas99, p. 402]).

Lemma 4.4 *Assume that (G, X) is of preabelian type. We take a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$. Let K'' be a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$ which contains the image of K under the map $G(\mathbb{A}^\infty) \to G^{\mathrm{ad}}(\mathbb{A}^\infty)$. We regard it as a compact open subgroup of $G'^{\mathrm{ad}}(\mathbb{A}^\infty)$ by the isomorphism $G^{\mathrm{ad}} \cong G'^{\mathrm{ad}}$. Then there exist a compact open subgroup $K' \subset G'(\mathbb{A}^\infty)$ and $g_1, \ldots, g_m \in G'^{\mathrm{ad}}(\mathbb{A}^\infty)$ such that the following hold:*

(i) *The morphism $(G', X') \to (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$ and the conjugation by g_i induces the morphism*

$$f_i\colon \mathrm{Sh}_{K'}(G', X') \to \mathrm{Sh}_{g_i^{-1}K''g_i}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}}) \to \mathrm{Sh}_{K''}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$$

for each i.

(ii) *The morphism*

$$\coprod_{1\le i\le m} f_i\colon \coprod_{1\le i\le m} \mathrm{Sh}_{K'}(G', X') \to \mathrm{Sh}_{K''}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$$

is surjective.

Proof. This follows from the definition of Shimura varieties of preabelian type and the fact that Hecke action is transitive on the connected components of a Shimura variety. ∎

5 Partition of Shimura varieties

5.1 Partition of classical points

We fix a prime number p and a finite place v of E above p. We write $\mathcal{O}_v$ for the ring of integers of E_v.

Throughout the paper, we assume that a compact open subgroup K of $G(\mathbb{A}^\infty)$ is small enough so that the following conditions are satisfied:

- The morphism $\pi_K\colon \varprojlim_{K'\subset K} \mathrm{Sh}_{K'}(G,X) \to \mathrm{Sh}_K(G,X)$ is a torsor under the quotient K_{Sh} of K by a closed subgroup of $K\cap Z(G)(\mathbb{A}^\infty)$ (*cf.* [Mil05, Theorem 5.28]).

- If the Shimura datum (G,X) satisfies the condition SV5 in [Mil05, p. 311], then K_{Sh} equals K. Note that a Shimura datum of Hodge type satisfies SV5.

Definition 5.1 Let x be a classical point of $\mathrm{Sh}_K(G,X)^{\mathrm{ad}}_{E_v}$, and $\overline{\kappa}_x$ an algebraic closure of κ_x. We write $\overline{x}$ for the geometric point corresponding to $\overline{\kappa}_x$.

By taking the pull-back of $\pi_K\colon \varprojlim_{K'\subset K} \mathrm{Sh}_{K'}(G,X) \to \mathrm{Sh}_K(G,X)$, we obtain a K_{Sh}-torsor $\pi_K^{-1}(x)$ on x. This torsor and its trivialization η over $\overline{x}$ give rise to a continuous homomorphism $\phi_{x,\eta}\colon \mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \to K_{\mathrm{Sh}}$. If we change η, the homomorphism $\phi_{x,\eta}$ changes by a K_{Sh}-conjugation.

(i) We write $\phi^{\mathrm{ad}}_{x,\eta}$ for the composite $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \xrightarrow{\phi_{x,\eta}} K_{\mathrm{Sh}} \to G^{\mathrm{ad}}(\mathbb{A}^\infty)$. If we change η, the homomorphism $\phi^{\mathrm{ad}}_{x,\eta}$ changes by a K^{ad}-conjugation, where K^{ad} denotes the image of K in $G^{\mathrm{ad}}(\mathbb{A}^\infty)$. When we are only interested in the K^{ad}-conjugacy class of $\phi^{\mathrm{ad}}_{x,\eta}$, we often drop the subscript η and simply write ϕ^{ad}_x for $\phi^{\mathrm{ad}}_{x,\eta}$.
For a prime number ℓ, we denote by $\phi^{\mathrm{ad}}_{x,\eta,\ell}$ the composite of $\phi^{\mathrm{ad}}_{x,\eta}$ and the projection $G^{\mathrm{ad}}(\mathbb{A}^\infty) \to G^{\mathrm{ad}}(\mathbb{Q}_\ell)$.

(ii) Assume that (G,X) satisfies the condition SV5. Then we write $\phi_{x,\eta}$ for the composite $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \xrightarrow{\phi_{x,\eta}} K \to G(\mathbb{A}^\infty)$. As in (i), we often write ϕ_x for $\phi_{x,\eta}$, which is well-defined up to K-conjugacy.
For a prime number ℓ, we define $\phi_{x,\eta,\ell}$ similarly.

Remark 5.2 The homomorphism $\phi^{\mathrm{ad}}_{x,\eta,\ell}$ is related to ℓ-adic automorphic étale sheaves on $\mathrm{Sh}_K(G,X)$ as follows. Let (ξ, V_ξ) be a finite-dimensional algebraic representation of G over $\overline{\mathbb{Q}}_\ell$ such that $\mathrm{Ker}\,\xi$ contains $\mathrm{Ker}(K \to K_{\mathrm{Sh}})$. Then, we have an associated smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\xi$ on $\mathrm{Sh}_K(G,X)$ (*cf.* [Mil90, Remark III.6.1]). As in Lemma 4.2 (ii), the trivialization η of $\pi^{-1}(\overline{x})$ induces an isomorphism $\mathcal{L}_{\xi,\overline{x}} \cong V_\xi$. Hence we obtain an ℓ-adic Galois representation $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \to \mathrm{GL}(\mathcal{L}_{\xi,\overline{x}}) \cong \mathrm{GL}(V_\xi)$.

(i) If ξ factors through G^{ad}, it is equal to the composite $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \xrightarrow{\phi^{\mathrm{ad}}_{x,\eta,\ell}} G^{\mathrm{ad}}(\mathbb{Q}_\ell) \xrightarrow{\xi} \mathrm{GL}(V_\xi)$.

(ii) If (G,X) satisfies SV5 (hence any ξ is allowable), it is equal to the composite $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \xrightarrow{\phi_{x,\eta,\ell}} G(\mathbb{Q}_\ell) \xrightarrow{\xi} \mathrm{GL}(V_\xi)$.

The following proposition can be checked easily.

Proposition 5.3 *Let $(G, X) \to (G', X')$ be a morphism of Shimura data such that $Z(G)$ is mapped into $Z(G')$. Let $K \subset G(\mathbb{A}^\infty)$ and $K' \subset G'(\mathbb{A}^\infty)$ be compact open subgroups such that K is mapped into K'. For $x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$, we write x' for the image of x under the induced morphism $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(G', X')$. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) & \xrightarrow{\;\phi_x^{\mathrm{ad}}\;} & G^{\mathrm{ad}}(\mathbb{A}^\infty) \\
\downarrow & & \downarrow \\
\mathrm{Gal}(\overline{\kappa}_{x'}/\kappa_{x'}) & \xrightarrow{\;\phi_{x'}^{\mathrm{ad}}\;} & G'^{\mathrm{ad}}(\mathbb{A}^\infty)
\end{array}
$$

is commutative up to K'^{ad}-conjugacy, where K'^{ad} denotes the image of K' in $G'^{\mathrm{ad}}(\mathbb{A}^\infty)$.

Proposition 5.4 *Assume that (G, X) is of preabelian type.*

(i) *For $x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$, $\phi^{\mathrm{ad}}_{x,p}$ is de Rham and has the monodromy filtration.*

(ii) *Assume that (G, X) is of Hodge type. Then, for $x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$, $\phi_{x,p}$ is de Rham and has the monodromy filtration.*

Proof. By Lemma 2.9 (iv), Lemma 2.10, Lemma 4.4 and Proposition 5.3, the assertion (i) is reduced to (ii). We prove (ii). Take an embedding $i \colon (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum and a compact open subgroup $\widetilde{K} = \widetilde{K}_p \widetilde{K}^p$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K. By shrinking K, we may assume that $\widetilde{K}_p \subset \widetilde{K}_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$ and $\widetilde{K}^p$ is small enough. Then, the morphism

$$
\mathrm{Spec}\,\kappa_x \to \mathrm{Sh}_K(G, X)_{E_v} \to \mathrm{Sh}_{\widetilde{K}_{p,0}\widetilde{K}^p}(\mathrm{GSp}_{2n}, X_{2n})_{\mathbb{Q}_p} = \mathscr{S}_{\widetilde{K}^p, \mathbb{Q}_p}
$$

uniquely extends to $\mathrm{Spec}\,\kappa_x^+ \to \mathscr{S}^{\mathrm{tor}}_{\widetilde{K}^p}$. Let $\mathcal{A}_{\kappa_x^+}$ denote the pull-back of the universal semi-abelian scheme $\mathcal{A}$ by this morphism. It extends the abelian variety $\mathcal{A}_x$ over κ_x. Therefore, the representation $V_p \mathcal{A}_{\overline{x}}$ of $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$ is semistable and has the monodromy filtration by Corollary 3.12.

Let $\mathrm{Std} \colon \mathrm{GSp}_{2n} \to \mathrm{GL}(V)$ denote the standard representation of GSp_{2n}. By Lemma 4.2 (i), we have a $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$-equivariant isomorphism $\mathcal{L}_{\mathrm{Std}\circ i, \overline{x}, p} \cong V_p \mathcal{A}_{\overline{x}}$. We fix a trivialization η of the K-torsor $\pi_K^{-1}(x)$ over $\overline{x}$. By the isomorphism $V_{\mathbb{Q}_p} \cong \mathcal{L}_{\mathrm{Std}\circ i, \overline{x}, p}$ induced from η, we regard $V_{\mathbb{Q}_p}$ as a representation of $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$. As in Remark 5.2 (ii), it is isomorphic to $\mathrm{Std} \circ i \circ \phi_{x,\eta,p}$. Summing up, we obtain a $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$-equivariant isomorphism $\mathrm{Std} \circ i \circ \phi_{x,\eta,p} \cong V_p \mathcal{A}_{\overline{x}}$. Therefore, we conclude that $\mathrm{Std} \circ i \circ \phi_{x,\eta,p}$ is semistable (hence de Rham) and has the monodromy filtration. Since $\mathrm{Std} \circ i$ is a faithful representation of G, $\phi_{x,\eta,p}$ is de Rham and has the monodromy filtration by Lemma 2.8. This completes the proof. ∎

Remark 5.5 Assume that (G, X) satisfies the condition SV6 in [Mil05, p. 312]. Recently, Liu and Zhu announced a result that the p-adic sheaf $\mathcal{L}_{\xi, x}$ is de Rham for any finite-dimensional algebraic representation ξ of G^c over $\overline{\mathbb{Q}}_p$, where G^c is the quotient of G defined in [Mil90, p. 347] (*cf.* [LZ16, Theorem 1.2]). This implies that $\phi^{\mathrm{ad}}_{x,p}$ is de Rham. We do not use this remark later.

In the sequel, we assume that (G, X) is of preabelian type. Take a finite non-empty set of primes S such that $K = K_S K^S$, where K_S is a compact open subgroup of $G(\mathbb{A}_S)$ and K^S is a hyperspecial compact open subgroup of $G(\mathbb{A}^{\infty,S})$. We write $\mathcal{P}_{G,S}(K_S)$ for the set of K_S-conjugacy classes of $\mathbb{A}_S$-parabolic subgroups of G.

Let η be a trivialization of $\pi_K^{-1}(x)$ over $\overline{x}$. By Proposition 5.4 and the results in Section 2, we can attach to $\phi_{x,\eta,\ell}^{\mathrm{ad}}$ the $\mathbb{Q}_\ell$-parabolic subgroup $P_{\phi_{x,\eta,\ell}^{\mathrm{ad}}}$ of G^{ad} for each $\ell \in S$ and $x \in \mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}(\mathrm{cl})$. By taking the product with respect to ℓ, we obtain an $\mathbb{A}_S$-parabolic subgroup of G^{ad}. It is easy to observe that the K_S^{ad}-conjugacy class $[\prod_{\ell \in S} P_{\phi_{x,\eta,\ell}^{\mathrm{ad}}}] \in \mathcal{P}_{G^{\mathrm{ad}},S}(K_S^{\mathrm{ad}})$ is independent of the choice of η. Note that the natural map $\mathcal{P}_{G,S}(K_S) \to \mathcal{P}_{G^{\mathrm{ad}},S}(K_S^{\mathrm{ad}})$; $[P] \mapsto [P^{\mathrm{ad}}]$ is bijective.

Definition 5.6 Let $[P_{x,S}] \in \mathcal{P}_{G,S}(K_S)$ be the K_S-conjugacy class that is mapped to $[\prod_{\ell \in S} P_{\phi_{x,\eta,\ell}^{\mathrm{ad}}}]$ under the bijection $\mathcal{P}_{G,S}(K_S) \to \mathcal{P}_{G^{\mathrm{ad}},S}(K_S^{\mathrm{ad}})$.

Remark 5.7 If the Shimura datum (G, X) satisfies the condition SV5, we can define $[P_{x,S}]$ directly by using $\phi_{x,\eta}$. These two ways give the same result by Lemma 2.2 (iv) and Lemma 2.9 (iv).

By the proof of Proposition 5.4, we obtain the following description of $[P_{x,S}]$ in the Hodge type case.

Corollary 5.8 *Let (G, X) be a Shimura datum of Hodge type with an embedding $(G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. Assume that $K_p \subset \mathrm{GSp}_{2n}(\mathbb{Z}_p)$ and K^p is small enough. For $x \in \mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}(\mathrm{cl})$, fix an isomorphism $\iota \colon \overline{\kappa}_x \xrightarrow{\cong} \mathbb{C}$ and let $(A, (t_\alpha), \eta K)$ be a triple in the isogeny class corresponding to the $\mathbb{C}$-point $\iota\overline{x}$ of $\mathrm{Sh}_K(G, X)$ such that $A = \mathcal{A}_{\overline{x}} \otimes_{\overline{\kappa}_x, \iota} \mathbb{C}$.*

(i) *The abelian variety $\mathcal{A}_x$ over κ_x extends to a semi-abelian scheme $\mathcal{A}_{\kappa_x^+}$ over κ_x^+. For $\ell \in S$, the monodromy filtration $M_\bullet V_\ell \mathcal{A}_{\overline{x}}$ on $V_\ell \mathcal{A}_{\overline{x}}$ is a shift of the filtration in Proposition 3.11.*

(ii) *Fix an arbitrary representative $\eta \colon V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty A$ of the K-orbit ηK. For $\ell \in S$, consider the filtration $(\eta_\ell^{-1} \circ \iota)(M_\bullet V_\ell \mathcal{A}_{\overline{x}})$ on $V_{\mathbb{Q}_\ell}$ obtained as the inverse image of $M_\bullet V_\ell \mathcal{A}_{\overline{x}}$ under $V_{\mathbb{Q}_\ell} \xrightarrow[\cong]{\eta_\ell} V_\ell A \xrightarrow[\cong]{\iota^{-1}} V_\ell \mathcal{A}_{\overline{x}}$. Then, this filtration is $G_{\mathbb{Q}_\ell}$-split in the sense of [Kis10, (1.1.2)]. Moreover, if we write $P_{x,\eta,\ell}$ for the stabilizer of this filtration, the K_S-conjugacy class $[\prod_{\ell \in S} P_{x,\eta,\ell}]$ equals $[P_{x,S}]$.*

(iii) *If $(G, X) = (\mathrm{GSp}_{2n}, X_{2n})$, then $P_{x,\eta,\ell}$ in (ii) is the stabilizer of a totally isotropic subspace of $V_{\mathbb{Q}_\ell}$ whose dimension equals the toric rank of the special fiber of $\mathcal{A}_{\kappa_x^+}$.*

Proof. The assertion (i) follows from the proofs of Proposition 5.4 and Corollary 3.12.

We prove (ii). The choice of η gives a trivialization of the K-torsor $\pi_K^{-1}(x)$ over $\overline{x}$, which is denoted by the same symbol η. By the argument in the proof of Proposition

5.4, we have $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$-equivariant isomorphisms

$$\mathrm{Std} \circ i \circ \phi_{x,\eta,\ell} = V_{\mathbb{Q}_\ell} \overset{\eta}{\underset{\cong}{=}} \mathcal{L}_{\mathrm{Std}\circ i, \overline{x}, \ell} \cong V_\ell \mathcal{A}_{\overline{x}}.$$

By Lemma 4.2 (ii), their composite is equal to

$$\mathrm{Std} \circ i \circ \phi_{x,\eta,\ell} = V_{\mathbb{Q}_\ell} \xrightarrow[\cong]{\eta_\ell} V_\ell A \xrightarrow[\cong]{\iota^{-1}} V_\ell \mathcal{A}_{\overline{x}}.$$

Hence the filtration $(\eta_\ell^{-1} \circ \iota)(M_\bullet V_\ell \mathcal{A}_{\overline{x}})$ equals the monodromy filtration $M_\bullet V_{\mathbb{Q}_\ell}$ on $V_{\mathbb{Q}_\ell}$ with respect to the action of $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$ by $\mathrm{Std} \circ i \circ \phi_{x,\eta,\ell}$. Since the monodromy filtration of $\mathrm{Std} \circ i \circ \phi_{x,\eta,\ell}$ extends to a filtration on the Tannakian category $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_{\mathbb{Q}_\ell})$, we conclude that $M_\bullet V_{\mathbb{Q}_\ell}$ is $G_{\mathbb{Q}_\ell}$-split by [Kis10, Lemma 1.1.3]. Further, by Lemma 2.2 (ii) and Lemma 2.9 (ii), we have $P_{x,\eta,\ell} = P_{\phi_{x,\eta,\ell}}$. Therefore we have $[P_{x,S}] = [\prod_{x\in S} P_{x,\eta,\ell}]$ by Remark 5.7.

The claim (iii) follows from Corollary 3.12 and the equality $(V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}})^\perp = V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}}$ (and its ℓ-adic version) in the proof of Proposition 3.11. ∎

Next we will show that the ℓ-part of $P_{x,S}$ is independent of $\ell \in S$ in some sense. To state the result, we need some preparation.

Definition 5.9 (*cf.* [Pin90, 4.5 Definition]) Let $G^{\mathrm{ad}} = G_1 \times \cdots \times G_r$ be a decomposition into $\mathbb{Q}$-simple factors. We say that a parabolic subgroup P of G is an admissible $\mathbb{Q}$-parabolic subgroup if there exists a parabolic subgroup P_i of G_i for each i such that P is the inverse image of $P_1 \times \cdots \times P_r$ and P_i is either equal to G_i or a maximal $\mathbb{Q}$-parabolic subgroup of G_i for each i. We write $\mathcal{P}_{G,\mathbb{Q}}$ for the set of $G(\mathbb{Q})$-conjugacy classes of admissible $\mathbb{Q}$-parabolic subgroups of G.

An admissible $\mathbb{A}^\infty$-parabolic subgroup means a parabolic subgroup of $G_{\mathbb{A}^\infty}$ which is $G(\mathbb{A}^\infty)$-conjugate to an admissible $\mathbb{Q}$-parabolic subgroup of G. Let $\mathcal{P}_G(K)$ denote the set of K-conjugacy classes of admissible $\mathbb{A}^\infty$-parabolic subgroups of G. Further, we write $\mathcal{P}_{G,\mathbb{A}^\infty}$ for the set of $G(\mathbb{A}^\infty)$-conjugacy classes of admissible $\mathbb{A}^\infty$-parabolic subgroups of G. We have a natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{A}^\infty}$.

Lemma 5.10 (i) *The natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G,S}(K_S)$ is injective.*

(ii) *The set $\mathcal{P}_G(K)$ is finite.*

(iii) *We take a hyperspecial compact open subgroup K''^S of $G^{\mathrm{ad}}(\mathbb{A}^{\infty,S})$ containing the image of K^S, and put $K'' = K_S^{\mathrm{ad}} K''^S$, which is a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$. Then, the natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G^{\mathrm{ad}}}(K'')$ is bijective.*

Proof. Fix a minimal parabolic subgroup P_0 of G. For an admissible $\mathbb{Q}$-parabolic subgroup P containing P_0, we write $\mathcal{P}_G(K)_P$ for the subset of $\mathcal{P}_G(K)$ consisting of K-conjugacy classes which are $G(\mathbb{A}^\infty)$-conjugate to P. Then, we have a bijection $K \backslash G(\mathbb{A}^\infty)/P(\mathbb{A}^\infty) \overset{\cong}{\to} \mathcal{P}_G(K)_P$ given by $KgP(\mathbb{A}^\infty) \mapsto gPg^{-1}$.

Let us prove (i). For an admissible $\mathbb{Q}$-parabolic subgroup P containing P_0, we have

$$K\backslash G(\mathbb{A}^\infty)/P(\mathbb{A}^\infty) = K_S\backslash G(\mathbb{A}_S)/P(\mathbb{A}_S) \times K^S\backslash G(\mathbb{A}^S)/P(\mathbb{A}^S) = K_S\backslash G(\mathbb{A}_S)/P(\mathbb{A}_S).$$

by the Iwasawa decomposition $K^S P_0(\mathbb{A}^S) = G(\mathbb{A}^S)$. This implies that the composite $\mathcal{P}_G(K)_P \hookrightarrow \mathcal{P}_G(K) \to \mathcal{P}_{G,S}(K_S)$ is injective. It suffices to show that the images of $\mathcal{P}_G(K)_P$ and $\mathcal{P}_G(K)_{P'}$ in $\mathcal{P}_{G,S}(K_S)$ are disjoint, where P and P' are distinct admissible $\mathbb{Q}$-parabolic subgroups containing P_0. If the images of $\mathcal{P}_G(K)_P$ and $\mathcal{P}_G(K)_{P'}$ intersect, then $P_{\mathbb{Q}_\ell}$ and $P'_{\mathbb{Q}_\ell}$ are $G(\mathbb{Q}_\ell)$-conjugate for each $\ell \in S$. By [BT65, Théorème 4.13], this means that P and P' are $G(\mathbb{Q})$-conjugate. Since they contain P_0, they are equal. Note that in particular we have $\mathcal{P}_G(K)_P \cap \mathcal{P}_G(K)_{P'} = \varnothing$. Hence $\mathcal{P}_G(K)$ equals $\coprod_{P \supset P_0} \mathcal{P}_G(K)_P$, where P runs through admissible $\mathbb{Q}$-parabolic subgroups of G containing P_0.

Next we prove (ii). It suffices to show that $\mathcal{P}_G(K)_P$ is a finite set for each admissible $\mathbb{Q}$-parabolic subgroup P of G containing P_0. Since

$$\mathcal{P}_G(K)_P \cong K\backslash G(\mathbb{A}^\infty)/P(\mathbb{A}^\infty) \cong K_S\backslash G(\mathbb{A}_S)/P(\mathbb{A}_S),$$

it suffices to show that $K_S\backslash G(\mathbb{A}_S)/P(\mathbb{A}_S)$ is a finite set. Let K_S^0 be the product of special compact open subgroups of $G(\mathbb{Q}_\ell)$ for $\ell \in S$. Then we have $K_S^0 P_0(\mathbb{A}_S) = G(\mathbb{A}_S)$. By shrinking K_S, we may assume that $K_S \subset K_S^0$. Then the map $K_S\backslash K_S^0 \to K_S\backslash G(\mathbb{A}_S)/P(\mathbb{A}_S)$ is surjective, hence $K_S\backslash G(\mathbb{A}_S)/P(\mathbb{A}_S)$ is finite.

Finally we prove (iii). Note that admissible $\mathbb{Q}$-parabolic subgroups of G containing P_0 are in bijection with those of G^{ad} containing P_0^{ad}. Therefore, we have only to show that $\mathcal{P}_G(K)_P \to \mathcal{P}_{G^{\mathrm{ad}}}(K'')_{P^{\mathrm{ad}}}$ is bijective for every admissible $\mathbb{Q}$-parabolic subgroup of G containing P_0. Further, it is equivalent to the bijectivity of

$$K_S\backslash G(\mathbb{A}_S)/P(\mathbb{A}_S) \xrightarrow{(*)} K_S^{\mathrm{ad}}\backslash G^{\mathrm{ad}}(\mathbb{A}_S)/P^{\mathrm{ad}}(\mathbb{A}_S).$$

By [Spr98, 15.1.4], we have

$$G(\mathbb{Q}_\ell)/P(\mathbb{Q}_\ell) \cong (G_{\mathbb{Q}_\ell}/P_{\mathbb{Q}_\ell})(\mathbb{Q}_\ell) \cong (G_{\mathbb{Q}_\ell}^{\mathrm{ad}}/P_{\mathbb{Q}_\ell}^{\mathrm{ad}})(\mathbb{Q}_\ell) \cong G^{\mathrm{ad}}(\mathbb{Q}_\ell)/P^{\mathrm{ad}}(\mathbb{Q}_\ell)$$

for each $\ell \in S$. Therefore the map $G(\mathbb{A}_S)/P(\mathbb{A}_S) \to G^{\mathrm{ad}}(\mathbb{A}_S)/P^{\mathrm{ad}}(\mathbb{A}_S)$ is bijective. The bijectivity of $(*)$ easily follows from it. ∎

By the proof above, we also obtain the following:

Corollary 5.11 *The natural map $\mathcal{P}_{G,\mathbb{Q}} \to \mathcal{P}_{G,\mathbb{A}^\infty}$ is a bijection. In particular, for a compact open subgroup K of $G(\mathbb{A}^\infty)$, we have a natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{Q}}$.*

Proof. We use the notation in the proof of Lemma 5.10. By definition, the natural map $\mathcal{P}_{G,\mathbb{Q}} \to \mathcal{P}_{G,\mathbb{A}^\infty}$ is surjective. We shall show that it is injective. Take two admissible $\mathbb{Q}$-parabolic subgroups P_1, P_2 of G containing P_0. If P_1 and P_2 are $G(\mathbb{A}^\infty)$-conjugate, then $[P_1] \in \mathcal{P}_G(K)_{P_1} \cap \mathcal{P}_G(K)_{P_2}$ for every compact open subgroup K of $G(\mathbb{A}^\infty)$. By the proof of Lemma 5.10, it implies that $P_1 = P_2$. This completes the proof. ∎

Proposition 5.12 *For $x \in \mathrm{Sh}_K(G,X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$, there uniquely exists an element $[P_x] \in \mathcal{P}_G(K)$ which is mapped to $[P_{x,S}]$ under the injection $\mathcal{P}_G(K) \hookrightarrow \mathcal{P}_{G,S}(K_S)$ in Lemma 5.10 (i). It is independent of S.*

To prove this proposition, we use the following lemma.

Lemma 5.13 *Let (G,X) be a Shimura datum of Hodge type with an embedding $i\colon (G,X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. Recall that V denotes the standard representation of GSp_{2n}.*

Let W be a totally isotropic subspace of V, and define a filtration $W_\bullet V$ on V as follows:

$$W_0 V = V, \quad W_{-1} V = W^\perp, \quad W_{-2} V = W, \quad W_{-3} V = 0.$$

We write P for the stabilizer of $W_\bullet V$ in G.

Let L be a field of characteristic 0. Assume that the filtration $W_\bullet V \otimes_{\mathbb{Q}} L$ on $V_L = V \otimes_{\mathbb{Q}} L$ is G_L-split in the sense of [Kis10, (1.1.2)]. Then, P is an admissible $\mathbb{Q}$-parabolic subgroup of G. Further, if we write P' for the stabilizer of $W_\bullet V$ in GSp_{2n}, we have $P' = i_*P$ in the notation of [MP12, 2.1.28]. Namely, the cocharacter of GSp_{2n} associated to P' as in [Pin90, 4.1] is equal to $i \circ \lambda$, where λ is the cocharacter of G associated to P.

Proof. We write U for the subgroup of P consisting of elements acting on $\mathrm{gr}^W_\bullet V$ trivially, and ν for the cocharacter $\mathbb{G}_m \to \mathrm{GL}(\mathrm{gr}^W_\bullet V)$ determined from the grading on $\mathrm{gr}^W_\bullet V$.

By [Kis10, Lemma 1.1.1], P_L is a parabolic subgroup of G_L, U_L is the unipotent radical of P_L, and the cocharacter $\nu_L \colon \mathbb{G}_m \to \mathrm{GL}(\mathrm{gr}^W_\bullet V \otimes_{\mathbb{Q}} L)$ over L factors through the closed subgroup P_L/U_L. Therefore, we conclude that P is a parabolic subgroup of G, U is the unipotent radical of P, and the cocharacter $\nu \colon \mathbb{G}_m \to \mathrm{GL}(\mathrm{gr}^W_\bullet V)$ factors through P/U. This means that the filtration $W_\bullet V$ is G-split by [Kis10, Lemma 1.1.1]. Take a cocharacter $w \colon \mathbb{G}_m \to G$ over $\mathbb{Q}$ which induces the filtration $W_\bullet V$ on V. It induces a filtration on the Tannakian category $\mathbf{Rep}_{\mathbb{Q}}(G)$. Let us prove that this filtration is Cayley in the sense of [Mil90, V, Definition 2.3]. Take an arbitrary element $h \in X^+$. By [Mil90, IV, Example 1.1 (c)] (*cf.* [Bry83, 4.2.1]), $W_\bullet V$ and h give a mixed Hodge structure on V. Therefore, [Mil90, IV, Proposition 1.3] tells us that w and h define a mixed Hodge structure on V_ξ for all objects (ξ, V_ξ) of $\mathbf{Rep}_{\mathbb{Q}}(G)$. Hence the filtration induced from w is Cayley, as desired.

Now, by applying [Mil90, V, Proposition 2.4] to each simple factor of $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = (G_1, X_1) \times \cdots \times (G_r, X_r)$, we conclude that P is admissible (we use the same argument as in the proof of Lemma 2.2 (iv) to pass to the adjoint group). The equality $P' = i_*P$ is proved in [Pin90, 4.16]. $\blacksquare$

Proof Proposition 5.12. Only the existence of $[P_x]$ requires a proof. By Lemma 2.2 (v), Lemma 2.10, Lemma 4.4, Proposition 5.3 and Lemma 5.10 (iii), we may assume that (G,X) is of Hodge type. We use the notation in Section 4.3. By shrinking K, we may assume that $K_p \subset \widetilde{K}_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$ and K^p is small enough. We use

23

the notation in Corollary 5.8. Let $M_x = (G^\natural, \underline{Y} \to G^\natural_{\kappa_x})$ be the degeneration datum corresponding to $\mathcal{A}_{\kappa_x^+}$ under the functor M in [FC90, Chapter III, Corollary 7.2]. It gives a 1-motive $M_{\overline{x}}$ over $\overline{\kappa}_x$ (*cf.* [Del74, §10.1]). For each prime $\ell \in S$, the ℓ-adic realization $H_1(M_{\overline{x}}, \mathbb{Q}_\ell)$ of $M_{\overline{x}}$ is identified with $V_\ell \mathcal{A}_{\overline{x}}$, and equipped with the weight filtration $W_{\bullet, \overline{x}, \ell}$, which coincides with the monodromy filtration $M_\bullet V_\ell \mathcal{A}_{\overline{x}}$ on $V_\ell \mathcal{A}_{\overline{x}}$ up to a shift.

The 1-motive $M_{\overline{x}}$ and the fixed isomorphism $\iota\colon \overline{\kappa}_x \xrightarrow{\cong} \mathbb{C}$ gives rise to a 1-motive $M_{\iota\overline{x}}$ over $\mathbb{C}$. Its Betti realization $H_1(M_{\iota\overline{x}}, \mathbb{Q})$ is naturally isomorphic to $H_1(A, \mathbb{Q})$ (recall that $(A, (t_\alpha)_{\alpha \in J}, \eta K)$ denotes the triple corresponding to the $\mathbb{C}$-point $\iota\overline{x}$). We denote the weight filtration on it by $W_{\bullet, \iota\overline{x}, \mathbb{Q}}$. It is known that $W_{-2, \iota\overline{x}, \mathbb{Q}}$ is totally isotropic and $W_{-1, \iota\overline{x}, \mathbb{Q}} = W_{-2, \iota\overline{x}, \mathbb{Q}}^\perp$ with respect to the polarization $\pm t_0$ on A. For $\ell \in S$, we write ε_ℓ for the comparison isomorphism $H_1(A, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{Q}_\ell \xrightarrow{\cong} V_\ell A$. The composite

$$H_1(A, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{Q}_\ell \xrightarrow[\cong]{\varepsilon_\ell} V_\ell A \xrightarrow[\cong]{\iota^{-1}} V_\ell \mathcal{A}_{\overline{x}} = H_1(M_{\overline{x}}, \mathbb{Q}_\ell)$$

carries the filtration $W_{\bullet, \iota\overline{x}, \mathbb{Q}} \otimes_\mathbb{Q} \mathbb{Q}_\ell$ onto $W_{\bullet, \overline{x}, \ell}$ defined above.

We take a representative $\eta\colon V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty A$ of the K-orbit ηK and an isomorphism $\eta_\mathbb{Q}\colon V \xrightarrow{\cong} H_1(A, \mathbb{Q})$ as in the condition $(*)$ of Proposition 4.1. For each prime ℓ, we can easily observe that $g_\ell = \eta_\ell^{-1} \circ \varepsilon_\ell \circ (\eta_\mathbb{Q} \otimes \mathbb{Q}_\ell)$ preserves the tensors $(s_\alpha)_{\alpha \in J'}$. Therefore g_ℓ lies in $G(\mathbb{Q}_\ell)$.

We put $W_\bullet V = \eta_\mathbb{Q}^{-1}(W_{\bullet, \iota\overline{x}, \mathbb{Q}})$ (it depends on the choice of $\overline{x}$, ι and $\eta_\mathbb{Q}$), and denote by P' the stabilizer of $W_\bullet V$ in G. For $\ell \in S$, we have

$$(\eta_\ell^{-1} \circ \iota)(W_{\bullet, \overline{x}, \ell}) = (\eta_\ell^{-1} \circ \varepsilon_\ell)(W_{\bullet, \iota\overline{x}, \mathbb{Q}} \otimes_\mathbb{Q} \mathbb{Q}_\ell) = g_\ell(W_\bullet V \otimes_\mathbb{Q} \mathbb{Q}_\ell).$$

By Corollary 5.8 (ii), $g_\ell(W_\bullet V \otimes_\mathbb{Q} \mathbb{Q}_\ell)$ is $G_{\mathbb{Q}_\ell}$-split. Hence $W_\bullet V \otimes_\mathbb{Q} \mathbb{Q}_\ell$ is also $G_{\mathbb{Q}_\ell}$-split. Therefore, Lemma 5.13 tells us that P' is an admissible $\mathbb{Q}$-parabolic subgroup of G. Further, by the above equality, $P_{x, \eta, \ell}$ in Corollary 5.8 (ii) is equal to $g_\ell P'_{\mathbb{Q}_\ell} g_\ell^{-1}$.

Set $g = (g_\ell)_{\ell \in S} \times 1 \in G(\mathbb{A}^\infty) = G(\mathbb{A}_S) \times G(\mathbb{A}^{\infty, S})$. Then $P_x = g P'_{\mathbb{A}^\infty} g^{-1}$ is an admissible $\mathbb{A}^\infty$-parabolic subgroup of G, and its image $[P_x]$ in $\mathcal{P}_K(G)$ is mapped to $[\prod_{\ell \in S} P_{x, \eta, \ell}] = [P_{x, S}]$ under the injection $\mathcal{P}_G(K) \hookrightarrow \mathcal{P}_{G, S}(K_S)$ (for the last equality, see Corollary 5.8 (ii)). This completes the proof. ∎

Corollary 5.14 *For $x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$ and a prime number $\ell \neq p$, the following are equivalent:*

(i) *$\phi^{\mathrm{ad}}_{x, \ell}$ is potentially unramified.*

(ii) *$\phi^{\mathrm{ad}}_{x, p}$ is potentially crystalline.*

(iii) *$[P_x] = [G]$.*

If the above conditions are satisfied, we say that x is of potentially good reduction.

Proof. By Lemma 2.2 (iii), $\phi^{\mathrm{ad}}_{x, \ell}$ is potentially unramified if and only if $P_{\phi^{\mathrm{ad}}_{x, \ell}} = G^{\mathrm{ad}}_{\mathbb{Q}_\ell}$. By definition this is clearly equivalent to $[P_x] = [G]$. Similarly we can prove the equivalence of (ii) and (iii). ∎

Definition 5.15 We denote the map $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl}) \to \mathcal{P}_G(K)$; $x \mapsto [P_x]$ by Φ_K.

Proposition 5.16 *Let (G, X) be a Shimura datum of preabelian type.*

(i) *The map Φ_K is Hecke-equivariant in the following sense. Let K, K' be compact open subgroups of $G(\mathbb{A}^\infty)$, and g an element of $G(\mathbb{A}^\infty)$ such that $g^{-1}Kg \subset K'$. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl}) & \xrightarrow{\Phi_K} & \mathcal{P}_G(K) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle [P]\mapsto[g^{-1}Pg]} \\
\mathrm{Sh}_{K'}(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl}) & \xrightarrow{\Phi_{K'}} & \mathcal{P}_G(K')
\end{array}
$$

is commutative.

(ii) *Let K be a compact open subgroup of $G(\mathbb{A}^\infty)$, and K'' a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$ containing the image of K. We write E^{ad} for the reflex field of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and v^{ad} the place of E^{ad} below v. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl}) & \xrightarrow{\Phi_K} & \mathcal{P}_G(K) \\
\downarrow & & \downarrow{\scriptstyle [P]\mapsto[P^{\mathrm{ad}}]} \\
\mathrm{Sh}_{K''}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\mathrm{ad}}_{E^{\mathrm{ad}}_{v^{\mathrm{ad}}}}(\mathrm{cl}) & \xrightarrow{\Phi_{K''}} & \mathcal{P}_{G^{\mathrm{ad}}}(K'')
\end{array}
$$

is commutative.

(iii) *Assume that (G, X) is of Hodge type, and let $i\colon (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ be an embedding into a Siegel Shimura datum. Let K be a compact open subgroup of $G(\mathbb{A}^\infty)$, and $\widetilde{K}$ a compact open subgroup of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl}) & \xrightarrow{\Phi_K} & \mathcal{P}_G(K) \\
\downarrow & & \downarrow{\scriptstyle [P]\mapsto[i_*P]} \\
\mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})^{\mathrm{ad}}_{\mathbb{Q}_p}(\mathrm{cl}) & \xrightarrow{\Phi_{\widetilde{K}}} & \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K})
\end{array}
$$

is commutative.

Proof. The assertions (i) and (ii) are immediate consequences of Proposition 5.3 and Lemma 5.10 (i). The claim (iii) follows from Lemma 5.13 and the construction of $[P_x]$ in the proof of Proposition 5.12. ∎

In the remaining part of this section, we will prove that the map Φ_K comes from a partition of $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$ into locally closed constructible subsets.

Theorem 5.17 *For each $[P] \in \mathcal{P}_G(K)$, there uniquely exists a locally closed constructible subset $C_{[P]}$ of $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$ such that*

$$
C_{[P]}(\mathrm{cl}) = \{x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl}) \mid \Phi_K(x) = [P]\}.
$$

Furthermore, the subset $C_{[G]}$ is open and quasi-compact.

Remark 5.18 The subsets $\{C_{[P]}\}_{[P] \in \mathcal{P}_G(K)}$ in Theorem 5.17 are mutually disjoint and cover $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$. Indeed, by Lemma 3.6 (i), it can be checked at the level of classical points, which is obvious.

Theorem 5.17 will be proved in Section 5.3. Admitting this theorem, we have the following definition.

Definition 5.19 We put $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}} = C_{[G]}$, and call it the potentially good reduction locus of $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$. It is a quasi-compact open subset of $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$ characterized by the following property:

- for $x \in \mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}(\mathrm{cl})$, x lies in $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}}$ if and only if x is of potentially good reduction in the sense of Corollary 5.14.

Example 5.20 When $\mathrm{Sh}_K(G, X)$ is proper over E, G has no proper parabolic subgroup defined over $\mathbb{Q}$. Therefore, we have $\mathcal{P}_G(K) = \{[G]\}$ and $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{pg}} = \mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$.

5.2 Partition in the Siegel case

In this subsection, we will give a proof of Theorem 5.17 in the Siegel case. We use the notation in Section 4.2. In particular, recall that $(V, \langle \ , \ \rangle)$ is a symplectic space of dimension $2n$ over $\mathbb{Q}$ and L is a self-dual $\mathbb{Z}$-lattice of V. For simplicity, we write $\mathcal{S}_K$ for $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})_{\mathbb{Q}_p}^{\mathrm{ad}}$.

It is well-known that conjugacy classes of maximal parabolic subgroups of GSp_{2n} are parametrized by integers $0 \le r \le n$; the class corresponding to r consists of parabolic subgroups obtained as stabilizers of r-dimensional totally isotropic subspaces of V. Namely, $\mathcal{P}_{\mathrm{GSp}_{2n}, \mathbb{Q}} \cong \{0, 1, \ldots, n\}$ under the notation in Definition 5.9. We write $\mathcal{P}_{\mathrm{GSp}_{2n}}(K)_r$ for the inverse image of r under the map

$$\mathcal{P}_{\mathrm{GSp}_{2n}}(K) \to \mathcal{P}_{\mathrm{GSp}_{2n}, \mathbb{Q}} \cong \{0, 1, \ldots, n\}$$

(see Corollary 5.11). Clearly we have $\mathcal{P}_{\mathrm{GSp}_{2n}}(K) = \coprod_{0 \le r \le n} \mathcal{P}_{\mathrm{GSp}_{2n}}(K)_r$. We put $\mathcal{P}_{\mathrm{GSp}_{2n}}(K)_{\le r} = \coprod_{r' \le r} \mathcal{P}_{\mathrm{GSp}_{2n}}(K)_{r'}$. Note that $\mathcal{P}_{\mathrm{GSp}_{2n}}(K)_0 = \mathcal{P}_{\mathrm{GSp}_{2n}}(K)_{\le 0} = \{[G]\}$.

Proposition 5.21 *There uniquely exists a constructible open subset $\mathcal{S}_{K,]\le r[}$ of $\mathcal{S}_K$ such that $x \in \mathcal{S}_K(\mathrm{cl})$ belongs to $\mathcal{S}_{K,]\le r[}$ if and only if $\Phi_K(x) \in \mathcal{P}_{\mathrm{GSp}_{2n}}(K)_{\le r}$. Moreover, $\mathcal{S}_{K,]\le 0[}$ is quasi-compact.*

Proof. By Proposition 5.16 (i), we may shrink K freely. Therefore, we may assume that K_p is contained in $K_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$.

For $r \ge 0$, let $\mathscr{S}_{K^p, \le r}^{\mathrm{tor}}$ be the subset of $\mathscr{S}_{K^p}^{\mathrm{tor}}$ consisting of $x \in \mathscr{S}_{K^p}^{\mathrm{tor}}$ such that the toric rank of the semi-abelian variety $\mathcal{A}_x$ is at most r. By [Lan13, Lemma 3.3.1.4], it is an open subset of $\mathscr{S}_{K^p}^{\mathrm{tor}}$.

Since $\mathscr{S}_{K^p}^{\mathrm{tor}}$ is proper over $\mathbb{Z}_p$, we may consider the specialization map

$$\mathrm{sp} \colon (\mathscr{S}_{K^p, \mathbb{Q}_p}^{\mathrm{tor}})^{\mathrm{ad}} = (\mathscr{S}_{K^p}^{\mathrm{tor}})^{\mathrm{ad}} = (\mathscr{S}_{K^p}^{\mathrm{tor}})^{\wedge \mathrm{rig}} \to \mathscr{S}_{K^p, \mathbb{F}_p}^{\mathrm{tor}}$$

introduced in Section 3.1 (for the second equality, see Lemma 3.3 (ii)). For an integer $r \geq 0$, we put $\mathcal{S}_{K_{p,0}K^p,]\leq r[} = \mathrm{sp}^{-1}(\mathscr{S}_{K^p,\leq r,\mathbb{F}_p}^{\mathrm{tor}}) \cap \mathscr{S}_{K^p,\mathbb{Q}_p}^{\mathrm{ad}}$. It is a constructible open subset of $\mathscr{S}_{K^p,\mathbb{Q}_p}^{\mathrm{ad}} = \mathcal{S}_{K_{p,0}K^p}$. Further, let $\mathcal{S}_{K,]\leq r[}$ be the inverse image of $\mathcal{S}_{K_{p,0}K^p,]\leq r[}$ under the natural morphism $\mathcal{S}_K \to \mathcal{S}_{K_{p,0}K^p}$. By Corollary 5.8 (iii), it satisfies the desired property. Since $\mathcal{S}_{K_{p,0}K^p,]\leq 0[} = \mathrm{sp}^{-1}(\mathscr{S}_{K^p,\mathbb{F}_p})$, it is quasi-compact. Hence $\mathcal{S}_{K,]\leq 0[}$ is also quasi-compact. ∎

For $0 \leq r \leq n$, we put $\mathcal{S}_{K,]r[} = \mathcal{S}_{K,]\leq r[} \setminus \mathcal{S}_{K,]\leq r-1[}$, where $\mathcal{S}_{K,]\leq -1[}$ means $\varnothing$. It is a locally closed constructible subset of $\mathcal{S}_K$.

Lemma 5.22 *The set $\pi_0(\mathcal{S}_{K,]r[})$ of connected components of $\mathcal{S}_{K,]r[}$ is finite, and consists of locally closed constructible subsets of $\mathcal{S}_K$. Hence $\mathcal{S}_{K,]r[}$ is topologically the disjoint union of elements of $\pi_0(\mathcal{S}_{K,]r[})$.*

Proof. By shrinking K, we may assume that K_p is contained in $K_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$. Then the claim is the case $X = \mathscr{S}_{K^p,\mathbb{Q}_p}^{\mathrm{tor}}$, $U = \mathscr{S}_{K^p,\mathbb{Q}_p}$, $U' = \mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})_{\mathbb{Q}_p}$, and $L = \mathrm{sp}^{-1}(\mathscr{S}_{K^p,\leq r,\mathbb{F}_p}^{\mathrm{tor}}) \setminus \mathrm{sp}^{-1}(\mathscr{S}_{K^p,\leq r-1,\mathbb{F}_p}^{\mathrm{tor}})$ of the subsequent general lemma. ∎

Lemma 5.23 *Let F be a p-adic field. Let X be a purely d-dimensional proper smooth scheme over F, and Y a closed subscheme of X whose dimension is less than d. We put $U = X \setminus Y$, and consider a finite étale surjection $f \colon U' \to U$.*

For a locally closed constructible subset L of X^{ad}, we put $L_{U'} = (f^{\mathrm{ad}})^{-1}(L \cap U^{\mathrm{ad}})$. Then, the set $\pi_0(L_{U'})$ is finite, and consists of locally closed constructible subsets of U'^{ad}. In particular, $L_{U'}$ is topologically the disjoint union of elements of $\pi_0(L_{U'})$.

Proof. Let X' be the normalization of X in U'. By the resolution of singularities, there exists a purely d-dimensional proper smooth scheme X'' over F and a proper birational morphism $\phi \colon X'' \to X'$ which induces an isomorphism $\phi^{-1}(U') \overset{\cong}{\to} U'$. Let us denote the composite $X'' \overset{\phi}{\to} X' \to X$ by ϕ'. By replacing X, Y, L with X'', $X'' \times_X Y$, $(\phi'^{\mathrm{ad}})^{-1}(L)$ respectively, we may assume that $U' = U$.

We denote by L° (resp. L_U°) the interior of L (resp. L_U) in X^{ad} (resp. U^{ad}). Clearly we have $L_U^\circ = L^\circ \setminus Y^{\mathrm{ad}}$. We fix a prime number $\ell \neq p$, and consider the following commutative diagram:

$$
\begin{array}{ccc}
H^0(L_{\overline{F}}, \mathbb{F}_\ell) & \longrightarrow & H^0(L_{\overline{F}}^\circ, \mathbb{F}_\ell) \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} \\
H^0(L_{U,\overline{F}}, \mathbb{F}_\ell) & \longrightarrow & H^0(L_{U,\overline{F}}^\circ, \mathbb{F}_\ell).
\end{array}
$$

Since X^{ad} is proper of finite type over $\mathrm{Spa}(F, \mathcal{O}_F)$, $H^0(L_{\overline{F}}, \mathbb{F}_\ell)$ is a finite-dimensional $\mathbb{F}_\ell$-vector space by [Hub98b, Proposition 3.16 i)]. Therefore, to show the finiteness of $\pi_0(L_U)$, it suffices to prove that the map (1) is an isomorphism. On the other hand, by [Hub98b, Theorem 3.7], the horizontal maps are isomorphisms. Hence it suffices to prove that (2) is an isomorphism. Note that, by [Hub98a, Lemma 1.3 iii)],

L° and L_U° are taut, and then $L^\circ \cap Y^{\mathrm{ad}}$ is also taut. Therefore we can consider the compactly supported cohomology of these spaces. Since $\dim(L^\circ \cap Y^{\mathrm{ad}}) < d$, we have $H_c^{2d-1}((L^\circ \cap Y^{\mathrm{ad}})_{\overline{F}}, \mathbb{F}_\ell) = H_c^{2d}((L^\circ \cap Y^{\mathrm{ad}})_{\overline{F}}, \mathbb{F}_\ell) = 0$. This implies that the natural map $H_c^{2d}(L_{U,\overline{F}}^\circ, \mathbb{F}_\ell) \to H_c^{2d}(L_{\overline{F}}^\circ, \mathbb{F}_\ell)$ is an isomorphism. By the Poincaré duality, we conclude that the map (2) is an isomorphism.

By the finiteness of $\pi_0(L_U)$, every element C of $\pi_0(L_U)$ is an open and closed subset of L_U. Since L is locally closed constructible, so is C. ∎

Lemma 5.24 *There uniquely exists a map $\Psi_K \colon \pi_0(\mathcal{S}_{K,]r[}) \to \mathcal{P}_G(K)$ satisfying the following: for $C \in \pi_0(\mathcal{S}_{K,]r[})$ and $x \in C(\mathrm{cl})$, we have $\Psi_K(C) = \Phi_K(x)$.*

Proof. Let $C \in \pi_0(\mathcal{S}_{K,]r[})$. Then, by Lemma 5.22 and Lemma 3.6 (i), we have $C(\mathrm{cl}) \neq \varnothing$. Therefore, it suffices to show that $\Phi_K(x)$ is independent of the choice of $x \in C(\mathrm{cl})$. By Proposition 5.16 (i), we may assume that K_p is contained in $K_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$. Recall that in this case $\mathcal{S}_{K,]\leq r[}$ is obtained as the inverse image under $\mathcal{S}_K \to \mathcal{S}_{K_{p,0}K^p}$ of $\mathcal{S}_{K_{p,0}K^p,]\leq r[}$, which is equal to $\mathrm{sp}^{-1}(\mathscr{S}_{K^p,\leq r,\mathbb{F}_p}^{\mathrm{tor}}) \cap \mathscr{S}_{K^p,\mathbb{Q}_p}^{\mathrm{ad}}$ (see the proof of Proposition 5.21).

Let $\mathcal{S}_{K,]r[}^\circ$ be the interior of $\mathcal{S}_{K,]r[}$ in $\mathcal{S}_K$. Then, the inverse image of $\mathcal{S}_{K_{p,0}K^p,]r[}^\circ$ under $\mathcal{S}_K \to \mathcal{S}_{K_{p,0}K^p}$ equals $\mathcal{S}_{K,]r[}^\circ$. We write C° for the interior of C in $\mathcal{S}_K$. It is connected by [Hub98b, Theorem 3.7], and included in $\mathcal{S}_{K,]r[}^\circ$. Since C is constructible in $\mathcal{S}_K$, we have $C(\mathrm{cl}) = C^\circ(\mathrm{cl})$ by Lemma 3.6 (ii). Hence it suffices to show that $\Phi_K(x)$ is independent of the choice of $x \in C^\circ(\mathrm{cl})$.

We apply the construction introduced in Section 3.2 to the case where $S = \mathscr{S}_{K^p,\leq r}^{\mathrm{tor}}$, $S_0 = \mathscr{S}_{K^p,r,\mathbb{F}_p}^{\mathrm{tor}}$, $U = \mathscr{S}_{K^p,\mathbb{Q}_p}$ and $G = \mathcal{A}|_{\mathscr{S}_{K^p,\leq r}^{\mathrm{tor}}}$. We write $\widehat{\mathcal{T}}$ for the corresponding $\widehat{T}$. By Lemma 3.5, $U^{\mathrm{ad}} = U \times_S t(\mathcal{S})_a = \mathcal{S}_{K_{p,0}K^p,]r[}^\circ$ in this case. Therefore, for each $m \geq 0$ and a prime ℓ, we have three locally constant constructible sheaves

$$\widehat{\mathcal{T}}^{\mathrm{rig}}[\ell^m]_{\mathcal{S}_{K_{p,0}K^p,]r[}^\circ} \subset \widehat{\mathcal{A}}^{\mathrm{rig}}[\ell^m]_{\mathcal{S}_{K_{p,0}K^p,]r[}^\circ} \subset \mathcal{A}^{\mathrm{ad}}[\ell^m]_{\mathcal{S}_{K_{p,0}K^p,]r[}^\circ}.$$

We put

$$\mathcal{V}_\ell = \left(\varprojlim_m \mathcal{A}^{\mathrm{ad}}[\ell^m]_{\mathcal{S}_{K_{p,0}K^p,]r[}^\circ}|_{C^\circ}\right) \otimes \mathbb{Q}_\ell, \quad \mathcal{F}_\ell = \left(\varprojlim_m \widehat{\mathcal{A}}^{\mathrm{rig}}[\ell^m]_{\mathcal{S}_{K_{p,0}K^p,]r[}^\circ}|_{C^\circ}\right) \otimes \mathbb{Q}_\ell,$$

$$\mathcal{T}_\ell = \left(\varprojlim_m \widehat{\mathcal{T}}^{\mathrm{rig}}[\ell^m]_{\mathcal{S}_{K_{p,0}K^p,]r[}^\circ}|_{C^\circ}\right) \otimes \mathbb{Q}_\ell.$$

They are smooth ℓ-adic sheaves over C°.

Now we use the moduli interpretation with rational level structures of $\mathcal{S}_K$. Fix a geometric point $\overline{x}_0$ of C°, and let ηK be the $\pi_1(C^\circ, \overline{x}_0)$-invariant K-orbit of isomorphisms $V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty \mathcal{A}_{\overline{x}_0}$ corresponding to the universal level structure on $\mathcal{A}|_{C^\circ}$. For $x \in C^\circ(\mathrm{cl})$, the rational K-level structure on $\mathcal{A}_x$ corresponding to x itself is obtained in the following manner. Fix a geometric point $\overline{x}$ lying over x. Since C° is connected, there exists an isomorphism $\pi_1(C^\circ, \overline{x}_0) \to \pi_1(C^\circ, \overline{x})$, which is canonical up to $\pi_1(C^\circ, \overline{x}_0)$-conjugacy. If we fix such an isomorphism, for a smooth sheaf $\mathcal{G}$

on C°, we have a functorial isomorphism $\mathcal{G}_{\overline{x}_0} \xrightarrow{\cong} \mathcal{G}_{\overline{x}}$ compatible with the π_1-actions. In particular, the smooth sheaf $(\varprojlim_N \mathcal{A}^{\mathrm{ad}}[N]_{C^{\circ}}) \otimes \mathbb{Q}$ determines an isomorphism $V^{\infty}\mathcal{A}_{\overline{x}_0} \xrightarrow{\cong} V^{\infty}\mathcal{A}_{\overline{x}}$. By composing it with each element of ηK, we obtain a K-orbit of isomorphisms $V_{\mathbb{A}^{\infty}} \xrightarrow{\cong} V^{\infty}\mathcal{A}_{\overline{x}}$, which turns out to be $\pi_1(C^{\circ}, \overline{x})$-invariant. Since the action of $\pi_1(x, \overline{x})$ on $V^{\infty}\mathcal{A}_{\overline{x}}$ factors through $\pi_1(x, \overline{x}) \to \pi_1(C^{\circ}, \overline{x})$, this orbit gives a rational K-level structure on $\mathcal{A}_x$.

Fix a representative η of ηK and write η_x for the composite of η and the isomorphism $V^{\infty}\mathcal{A}_{\overline{x}_0} \xrightarrow{\cong} V^{\infty}\mathcal{A}_{\overline{x}}$. We take a prime number ℓ and consider the ℓ-part

$$\eta_{x,\ell} \colon V_{\mathbb{Q}_\ell} \xrightarrow[\eta_\ell]{\cong} V_\ell \mathcal{A}_{\overline{x}_0} \xrightarrow[(*)]{\cong} V_\ell \mathcal{A}_{\overline{x}}$$

of η_x. Note that the isomorphism $(*)$ is given by the smooth ℓ-adic sheaf $\mathcal{V}_\ell$ introduced above. Moreover, by Corollary 5.8 (i), Proposition 3.11, Corollary 3.12 and its proof, the monodromy filtrations on $V_\ell \mathcal{A}_{\overline{x}_0}$ and $V_\ell \mathcal{A}_{\overline{x}}$ are given by

$$0 \subset \mathcal{T}_{\ell, \overline{x}_0} \subset \mathcal{F}_{\ell, \overline{x}_0} \subset \mathcal{V}_{\ell, \overline{x}_0}, \quad 0 \subset \mathcal{T}_{\ell, \overline{x}} \subset \mathcal{F}_{\ell, \overline{x}} \subset \mathcal{V}_{\ell, \overline{x}},$$

respectively. Since $\mathcal{F}_\ell$ and $\mathcal{T}_\ell$ are smooth sheaves, the isomorphism $(*)$ carries the first filtration to the second. Hence we have $\eta_{x,\ell}^{-1}(M_\bullet V_\ell \mathcal{A}_{\overline{x}}) = \eta_\ell^{-1}(M_\bullet V_\ell \mathcal{A}_{\overline{x}_0})$, which is independent of $x \in C^{\circ}(\mathrm{cl})$. Therefore, the parabolic subgroup $P_{x, \iota_x \circ \eta_{x,\ell}}$ in Corollary 5.8 (ii), where $\iota_x \colon \overline{\kappa}_x \xrightarrow{\cong} \mathbb{C}$ is a fixed isomorphism, is also independent of x. By Corollary 5.8 (ii) and Lemma 5.10 (i), we conclude that $\Phi_K(x) = [P_x] \in \mathcal{P}_{\mathrm{GSp}_{2n}}(K)$ is independent of x. ∎

Now we can prove Theorem 5.17 in the Siegel case.

Proof of Theorem 5.17 for the Siegel case. For $[P] \in \mathcal{P}_{\mathrm{GSp}_{2n}}(K)$, take $0 \leq r \leq n$ such that $[P]$ lies in $\mathcal{P}_{\mathrm{GSp}_{2n}}(K)_r$. We put

$$C_{[P]} = \bigcup_{\substack{C \in \pi_0(\mathcal{S}_{K,]r[}), \\ \Psi_K(C) = [P]}} C.$$

It is a constructible subset of $\mathcal{S}_K$ by Lemma 5.22. Since each $C \in \pi_0(\mathcal{S}_{K,]r[})$ is open in $\mathcal{S}_{K,]r[}$, $C_{[P]}$ is a locally closed subset of $\mathcal{S}_K$. We can also check that $x \in \mathcal{S}_K(\mathrm{cl})$ lies in $C_{[P]}$ if and only if $\Phi_K(x) = [P]$. We have already checked in Proposition 5.21 that $C_{[G]} = \mathcal{S}_{K,]\leq 0[}$ is quasi-compact open. ∎

Corollary 5.25 *For $[P] \in \mathcal{P}_G(K)_r$ and $[P'] \in \mathcal{P}_G(K)_{r'}$, assume that $r > r'$ or $r = r'$ and $[P] \neq [P']$. Then we have $C_{[P]}^- \cap C_{[P']} = \varnothing$, where $C_{[P]}^-$ denotes the closure of $C_{[P]}$ in $\mathcal{S}_K$.*

Proof. First assume that $r > r'$. Since the complement $\mathcal{S}_{K,]\leq r'[}^c$ of $\mathcal{S}_{K,]\leq r'[}$ is a closed subset of $\mathcal{S}_K$, we have

$$C_{[P]}^- \cap C_{[P']} \subset \mathcal{S}_{K,]\leq r'[}^c \cap \mathcal{S}_{K,]\leq r'[} = \varnothing.$$

Next assume that $r = r'$ and $[P] \neq [P']$. By construction, $C_{[P]}$ is closed in $\mathcal{S}_{K,]r[}$. Therefore, we have

$$C_{[P]}^- \cap C_{[P']} = (C_{[P]}^- \cap \mathcal{S}_{K,]r[}) \cap C_{[P']} = C_{[P]} \cap C_{[P']} = \varnothing.$$

For the last equality, see Remark 5.18. ∎

5.3 Existence of partition

In this subsection, we complete the proof of Theorem 5.17 by reducing to the Siegel case. First we consider the Hodge type case.

Lemma 5.26 *Let (G, X) be a Shimura datum of Hodge type with an embedding $i\colon (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. For a compact open subgroup K of $G(\mathbb{A}^\infty)$, there exists a compact open subgroup $\widetilde{K}$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K such that the map $\mathcal{P}_G(K) \to \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K}); [P] \mapsto [i_*P]$ is injective.*

Proof. It suffices to prove the following claim:

for $[P_1], [P_2] \in \mathcal{P}_G(K)$ with $[P_1] \neq [P_2]$, there exists a compact open subgroup $\widetilde{K}_{[P_1],[P_2]}$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K such that $[i_*P_1] \neq [i_*P_2]$ in $\mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K}_{[P_1],[P_2]})$.

Indeed, the intersection of $\widetilde{K}_{[P_1],[P_2]}$ for all pairs $([P_1], [P_2])$ with $[P_1] \neq [P_2]$ satisfies the desired condition.

Fix a compact open subgroup $\widetilde{K}_0$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K. Take representatives P_1, P_2 of $[P_1]$, $[P_2]$, respectively. We put $Z = \{g \in \widetilde{K}_0 \mid g(i_*P_1)g^{-1} = i_*P_2\}$, which is clearly a closed subset of $\widetilde{K}_0$. Therefore, a subset KZ of $\widetilde{K}_0$ is compact, hence closed. We prove that $1 \notin KZ$. If $1 \in KZ$, there exists $k \in K$ such that $k(i_*P_1)k^{-1} = i_*P_2$. Taking intersections with G, we obtain $kP_1k^{-1} = P_2$, which contradicts the assumption $[P_1] \neq [P_2]$. Therefore, we can find a compact open normal subgroup $\widetilde{K}_1$ of $\widetilde{K}_0$ such that $\widetilde{K}_1 \cap KZ = \varnothing$. Then, $\widetilde{K}_{[P_1],[P_2]} = K\widetilde{K}_1$ is a compact open subgroup of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ satisfying $\widetilde{K}_{[P_1],[P_2]} \cap Z = \varnothing$. This concludes the proof. ∎

Proof of Theorem 5.17 for the Hodge type case. We assume that (G, X) is of Hodge type, and take an embedding $i\colon (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. Further, we take a compact open subgroup $\widetilde{K} \subset \mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ as in Lemma 5.26.

Let $[P] \in \mathcal{P}_G(K)$. Since Theorem 5.17 is known for the Siegel case, we have a locally closed constructible subset $C_{[i_*P]}$ of $\mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})^{\mathrm{ad}}_{\mathbb{Q}_p}$. Let $C_{[P]}$ be the inverse image of $C_{[i_*P]}$ in $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$. Then, $C_{[P]}$ satisfies the desired condition by Proposition 5.16 (iii). The subset $C_{[G]}$ is open and quasi-compact, since $C_{[\mathrm{GSp}_{2n}]} = \mathcal{S}_{\widetilde{K},]\leq 0[}$ is open and quasi-compact. ∎

The following lemma is the Hodge type version of Corollary 5.25.

Lemma 5.27 *Let (G, X) be a Shimura datum of Hodge type. Take an embedding $i\colon (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. For a compact open subgroup K of $G(\mathbb{A}^\infty)$ and an integer $0 \leq r \leq n$, we write $\mathcal{P}_G(K)_r$ for the inverse image of r under the composite*

$$\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{Q}} \xrightarrow{i_*} \mathcal{P}_{\mathrm{GSp}_{2n},\mathbb{Q}} \cong \{0, 1, \ldots, n\}.$$

For $[P] \in \mathcal{P}_G(K)_r$ and $[P'] \in \mathcal{P}_G(K)_{r'}$, assume that $r > r'$ or $r = r'$ and $[P] \neq [P']$. Then we have $C^-_{[P]} \cap C_{[P']} = \varnothing$, where $C^-_{[P]}$ denotes the closure of $C_{[P]}$ in $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$.

In particular, if $[P], [P']$ are distinct elements of $\mathcal{P}_G(K)$ such that P is $G(\mathbb{A}^\infty)$-conjugate to P', then we have $C^-_{[P]} \cap C_{[P']} = \varnothing$.

Proof. We take $\widetilde{K}$ as in Lemma 5.26. By definition, we have $[i_*P] \in \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K})_r$ and $[i_*P'] \in \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K})_{r'}$. Since the map $\mathcal{P}_G(K) \to \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K})$ is injective, we have $[i_*P] \neq [i_*P']$ if $r = r'$. Hence Corollary 5.25 tells us that $C^-_{[i_*P]} \cap C_{[i_*P']} = \varnothing$.

Since the natural morphism $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v} \to \mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})^{\mathrm{ad}}_{\mathbb{Q}_p}$ maps $C_{[P]}$ and $C_{[P']}$ into $C_{[i_*P]}$ and $C_{[i_*P']}$, respectively (see the construction of $C_{[P]}$ in the proof of Theorem 5.17), the set $C^-_{[P]} \cap C_{[P']}$ is mapped into $C^-_{[i_*P]} \cap C_{[i_*P']} = \varnothing$. Therefore we conclude that $C^-_{[P]} \cap C_{[P']} = \varnothing$.

The last claim follows from the observation that if P and P' are $G(\mathbb{A}^\infty)$-conjugate, then $r = r'$. ∎

Now we can prove Theorem 5.17 for the preabelian type case.

Proof of Theorem 5.17. We choose a compact open subgroup K'' of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$ in such a way as in Proposition 5.10 (iii), and use the notation in Lemma 4.4. We write E' for the reflex field of (G', X'), and choose a place v' of E' above v^{ad} (for the definition of v^{ad}, see Proposition 5.16 (ii)).

We have natural maps

$$\mathcal{P}_{G'}(K') \to \mathcal{P}_{G'^{\mathrm{ad}}}(g_i^{-1}K''g_i) \xrightarrow{g_i} \mathcal{P}_{G'^{\mathrm{ad}}}(K'') \cong \mathcal{P}_{G^{\mathrm{ad}}}(K'') \cong \mathcal{P}_G(K)$$

for each i (see Lemma 5.10 (iii)). Let S_i be the inverse image of $[P] \in \mathcal{P}_G(K)$ under this map. We put $C^{\mathrm{ad}}_{[P]} = \bigcup_{1 \leq i \leq m} f_i(\bigcup_{[Q] \in S_i} C_{[Q]})$. Since $C_{[Q]}$ for each $[Q] \in S_i$ is constructible, Lemma 3.7 (i) tells us that $C^{\mathrm{ad}}_{[P]}$ is constructible. Let us prove that $C^{\mathrm{ad}}_{[P]}$ is locally closed. By Lemma 3.6 (i), Proposition 5.16 (i), (ii) and the constructibility of $C^{\mathrm{ad}}_{[P]}$, we can check that the inverse image of $C^{\mathrm{ad}}_{[P]}$ under $\coprod_{1 \leq i \leq m} f_i$ equals $\coprod_{1 \leq i \leq m} \bigcup_{[Q] \in S_i} C_{[Q]}$. Therefore, Lemma 3.7 (ii) tells us that it suffices to prove that $\bigcup_{[Q] \in S_i} C_{[Q]}$ is locally closed. Note that we have the commutative diagram

$$
\begin{array}{ccccccccc}
\mathcal{P}_{G'}(K') & \longrightarrow & \mathcal{P}_{G'^{\mathrm{ad}}}(g_i^{-1}K''g_i) & \xrightarrow{g_i} & \mathcal{P}_{G'^{\mathrm{ad}}}(K'') & \xleftarrow{\cong} & \mathcal{P}_{G^{\mathrm{ad}}}(K'') & \xleftarrow{\cong} & \mathcal{P}_G(K) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_{G',\mathbb{Q}} & \xrightarrow{\cong} & \mathcal{P}_{G'^{\mathrm{ad}},\mathbb{Q}} & =\!=\!=\!= & \mathcal{P}_{G'^{\mathrm{ad}},\mathbb{Q}} & \xleftarrow{\cong} & \mathcal{P}_{G^{\mathrm{ad}},\mathbb{Q}} & \xleftarrow{\cong} & \mathcal{P}_{G,\mathbb{Q}},
\end{array}
$$

where the vertical arrows are the maps in Corollary 5.11. From this diagram we can observe that all elements in S_i are $G'(\mathbb{A}^\infty)$-conjugate. Now Lemma 5.27 tells us that the closure $C_{[Q]}^-$ of $C_{[Q]}$ for $[Q] \in S_i$ does not intersect $\bigcup_{[Q'] \in S_i \setminus \{[Q]\}} C_{[Q']}$. Therefore $C_{[Q]}$ is closed (hence open) in $\bigcup_{[Q'] \in S_i} C_{[Q']}$, from which we conclude that $\bigcup_{[Q] \in S_i} C_{[Q]}$ is locally closed in $\mathrm{Sh}_{K'}(G', X')_{E'_{v'}}^{\mathrm{ad}}$, as desired.

Let $C_{[P]}$ be the inverse image of $C_{[P]}^{\mathrm{ad}}$ under the map

$$\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}} \to \mathrm{Sh}_{K''}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{E_{v^{\mathrm{ad}}}^{\mathrm{ad}}}^{\mathrm{ad}} \cong \mathrm{Sh}_{K''}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})_{E_{v^{\mathrm{ad}}}^{\mathrm{ad}}}^{\mathrm{ad}}$$

(here E^{ad} and v^{ad} are as in Proposition 5.16 (ii)). Then $C_{[P]}$ satisfies the desired condition by Proposition 5.16 (i), (ii).

If $[P] = [G]$, we have $S_i = \{[G']\}$. Hence the openness and the quasi-compactness of $C_{[G]}$ follows from those of $C_{[G']}$. ■

Lemma 5.28 *Let (G, X) be a Shimura datum of preabelian type. Take a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$, and an embedding $i \colon (G', X') \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. For a compact open subgroup K of $G(\mathbb{A}^\infty)$ and an integer $0 \le r \le n$, we write $\mathcal{P}_G(K)_r$ for the inverse image of r under the composite*

$$\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{Q}} \cong \mathcal{P}_{G^{\mathrm{ad}},\mathbb{Q}} \cong \mathcal{P}_{G'^{\mathrm{ad}},\mathbb{Q}} \cong \mathcal{P}_{G',\mathbb{Q}} \xrightarrow{i_*} \mathcal{P}_{\mathrm{GSp}_{2n},\mathbb{Q}} \cong \{0, 1, \ldots, n\}.$$

For $[P] \in \mathcal{P}_G(K)_r$ and $[P'] \in \mathcal{P}_G(K)_{r'}$, assume that $r > r'$ or $r = r'$ and $[P] \ne [P']$. Then we have $C_{[P]}^- \cap C_{[P']} = \varnothing$, where $C_{[P]}^-$ denotes the closure of $C_{[P]}$ in $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$.

In particular, if $[P]$, $[P']$ are distinct elements of $\mathcal{P}_G(K)$ such that P is $G(\mathbb{A}^\infty)$-conjugate to P', then we have $C_{[P]}^- \cap C_{[P']} = \varnothing$.

Proof. We use the same notation as in the proof of Theorem 5.17 above, and denote the morphism $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}} \to \mathrm{Sh}_{K''}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{E_{v^{\mathrm{ad}}}^{\mathrm{ad}}}^{\mathrm{ad}} \cong \mathrm{Sh}_{K''}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})_{E_{v^{\mathrm{ad}}}^{\mathrm{ad}}}^{\mathrm{ad}}$ by h. If a point x belongs to $C_{[P]}^- \cap C_{[P']}$, we can find $y \in C_{[P]}$ specializing to x (see [Hoc69, Corollary of Theorem 1]). By the construction of $C_{[P]}$, there exist $1 \le i \le m$, $[Q] \in S_i$ and $y' \in C_{[Q]}$ such that $h(y) = f_i(y')$. Since f_i is finite, there exists $x' \in \mathrm{Sh}_{K'}(G', X')_{E'_{v'}}^{\mathrm{ad}}$ which is a specialization of y' and mapped to $h(x)$ by f_i. By Remark 5.18, x' belongs to $C_{[Q'']}$ for a unique $[Q''] \in \mathcal{P}_{K'}(G')$. Take r'' such that $[Q'']$ lies in $\mathcal{P}_{K'}(G')_{r''}$. Since $x' \in C_{[Q]}^- \cap C_{[Q'']}$, Lemma 5.27 tells us that we have either $r < r''$ or $[Q] = [Q'']$.

We write $[P'']$ for the image of $[Q'']$ under the composite

$$\mathcal{P}_{G'}(K') \to \mathcal{P}_{G'^{\mathrm{ad}}}(g_i^{-1} K'' g_i) \xrightarrow{g_i} \mathcal{P}_{G'^{\mathrm{ad}}}(K'') \cong \mathcal{P}_{G^{\mathrm{ad}}}(K'') \cong \mathcal{P}_G(K).$$

It belongs to $\mathcal{P}_G(K)_{r''}$. Since $h(x) = f_i(x') \in C_{[P'']}^{\mathrm{ad}}$, the point x lies in $C_{[P'']}$. Hence Remark 5.18 tells us that $[P'] = [P'']$. In particular we have $r' = r''$, which implies $[Q] = [Q'']$. Therefore we have $[P] = [P''] = [P']$, which contradicts the assumption on $[P']$. ■

Corollary 5.29 *Let (G, X) be a Shimura datum of preabelian type. Let K and K' be compact open subgroups of $G(\mathbb{A}^\infty)$ and $g \in G(\mathbb{A}^\infty)$ with $g^{-1}Kg \subset K'$. For an element $[P']$ of $\mathcal{P}_G(K')$, the inverse image of $C_{[P']}$ under the Hecke action $g\colon \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(G, X)$ is equal to*

$$\coprod_{\substack{[P]\in\mathcal{P}_G(K),\\ [g^{-1}Pg]=[P']\ \text{in}\ \mathcal{P}_G(K')}} C_{[P]}$$

as topological spaces.

In particular, for $[P] \in \mathcal{P}_K(G)$, $C_{[P]}$ is mapped to $C_{[g^{-1}Pg]}$ under the Hecke action by g.

Proof. First note that both $g^{-1}(C_{[P']})$ and $\bigcup_{[P]\in\mathcal{P}_K(G),[g^{-1}Pg]=[P']} C_{[P]}$ are constructible subsets and have the same set of classical points by Proposition 5.16 (i). Therefore, by Lemma 3.6 (i), they are equal. The union $\bigcup_{[P]\in\mathcal{P}_K(G),[g^{-1}Pg]=[P']} C_{[P]}$ is set-theoretically disjoint by Remark 5.18. Let $[P_1]$, $[P_2]$ be distinct elements of $\mathcal{P}_G(K)$ such that $[g^{-1}P_1g] = [g^{-1}P_2g]$ in $\mathcal{P}_{K'}(G)$. Lemma 5.28 tells us that $C_{[P_1]}^- \cap C_{[P_2]} = \varnothing$. This implies that $C_{[P_1]}$ is closed (hence open) in $\bigcup_{[P]\in\mathcal{P}_K(G),[g^{-1}Pg]=[P']} C_{[P]}$. Now the proof is complete. ∎

The following lemma will be used in the next section.

Lemma 5.30 *Let the notation be as in Lemma 5.28. Put $\mathcal{P}_G(K)_{\leq r} = \bigcup_{r'\leq r} \mathcal{P}_G(K)_{r'}$. Then, $U_{r,K} = \bigcup_{[P]\in\mathcal{P}_G(K)_{\leq r}} C_{[P]}$ is a constructible open subset of $\mathrm{Sh}_K(G, \overline{X})_{E_v}^{\mathrm{ad}}$, and $U_{r,K} \setminus U_{r-1,K}$ equals $\coprod_{[P]\in\mathcal{P}_G(K)_r} C_{[P]}$ as topological spaces (here we put $U_{-1,K} = \varnothing$).*

Proof. By Lemma 5.28, the set $\bigcup_{r'>r} \bigcup_{[P]\in\mathcal{P}_G(K)_{r'}} C_{[P]}$ is closed in $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$. Therefore, by Remark 5.18, $U_{r,K}$ is open in $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$. Since $C_{[P]}$ is constructible for every $[P]$, the subset $U_{r,K}$ is also constructible.

The claim $U_{r,K} \setminus U_{r-1,K} = \coprod_{[P]\in\mathcal{P}_G(K)_r} C_{[P]}$ follows from Lemma 5.28 by the same argument as in the proof of Corollary 5.29. ∎

Remark 5.31 If (G, X) is of Hodge type (namely, $(G, X) = (G', X')$), we can also construct $U_{r,K}$ in the following way. Take a compact open subgroup $\widetilde{K} \subset \mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K. Then, $U_{r,K}$ equals the inverse image of $\mathcal{S}_{\widetilde{K},]\leq r[}$ under

$$\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}} \to \mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})_{\mathbb{Q}_p}^{\mathrm{ad}} = \mathcal{S}_{\widetilde{K}}.$$

This is an immediate consequence of Proposition 5.16 (iii) and Proposition 5.21.

6 Cohomology of Shimura varieties

6.1 Comparison of cohomology

We continue to assume that (G, X) is of preabelian type. For simplicity, we further assume that (G, X) satisfies SV6 in [Mil05, p. 311]. We simply write Sh_K for $\mathrm{Sh}_K(G, X)$, if there is no risk of confusion. Fix a prime ℓ which is different from p. Let G^c be the quotient of G defined in [Mil90, p. 347], and ξ an algebraic representation of G^c on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Then we have the associated $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\xi$ on Sh_K (see [Mil90, III, §6]). Moreover, $\mathcal{L}_\xi$ is equivariant with respect to the Hecke action.

Let p' be a prime number. Let us fix a compact open subgroup $K^{p'}$ of $G(\widehat{\mathbb{Z}}^{p'})$. We consider the compactly supported cohomology

$$H^i_c(\mathrm{Sh}_{\infty, K^{p'}, \overline{E}_v}, \mathcal{L}_\xi) = \varinjlim_{K_{p'}} H^i_c(\mathrm{Sh}_{K_{p'}K^{p'}, \overline{E}_v}, \mathcal{L}_\xi).$$

The group $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\overline{E}_v/E_v)$ naturally acts on it. By this action, $H^i_c(\mathrm{Sh}_{\infty, K^{p'}, \overline{E}_v}, \mathcal{L}_\xi)$ becomes an admissible/continuous representation of $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\overline{E}_v/E_v)$ in the sense of [HT01, §I.2].

The group $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\overline{E}_v/E_v)$ naturally acts also on

$$H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty, K^{p'}, \overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) = \varinjlim_{K_{p'}} H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{K_{p'}K^{p'}, \overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi).$$

See [Hub98c, §1] for the definition of the compactly supported ℓ-adic cohomology for adic spaces. It gives an admissible/continuous representation of $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\overline{E}_v/E_v)$ (*cf.* Lemma 6.6 in the next subsection).

Here we use the notation in [HT01, §I.2]. Let H be a locally profinite group. For an admissible/continuous representation V of $H \times \mathrm{Gal}(\overline{E}_v/E_v)$ over $\overline{\mathbb{Q}}_\ell$ and an irreducible admissible representation π of H, put $V[\pi] = \bigoplus_\sigma \sigma^{\oplus m_{\pi \boxtimes \sigma}}$, where σ runs through finite-dimensional irreducible continuous $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{Gal}(\overline{E}_v/E_v)$ and $m_{\pi \boxtimes \sigma}$ denotes the coefficient of $[\pi \boxtimes \sigma]$ in the image of V in the Grothendieck group considered in [HT01, §I.2]. It is a semisimple continuous representation of $\mathrm{Gal}(\overline{E}_v/E_v)$.

Theorem 6.1 *The kernel and the cokernel of the canonical homomorphism*

$$H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty, K^{p'}, \overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) \to H^i_c(\mathrm{Sh}_{\infty, K^{p'}, \overline{E}_v}, \mathcal{L}_\xi)$$

are non-cuspidal, namely, they have no supercuspidal subquotient of $G(\mathbb{Q}_{p'})$. In particular, for an irreducible supercuspidal representation π of $G(\mathbb{Q}_{p'})$, we have an isomorphism

$$H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty, K^{p'}, \overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi)[\pi] \cong H^i_c(\mathrm{Sh}_{\infty, K^{p'}, \overline{E}_v}, \mathcal{L}_\xi)[\pi].$$

This theorem will be proved in Section 6.2.

Remark 6.2 Let $H^i(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi) = \varinjlim_{K_{p'}} H^i(\mathrm{Sh}_{K_{p'}K^{p'},\overline{E}_v}, \mathcal{L}_\xi)$ be the ordinary cohomology of our Shimura variety. This is also an admissible/continuous representation of $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\overline{E}_v/E_v)$. By using the minimal compactification of Sh_K and its natural stratification (*cf.* [Pin92, §3.7]), it is easy to see that the kernel and the cokernel of the canonical homomorphism

$$H^i_c(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi) \to H^i(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi)$$

are non-cuspidal as $G(\mathbb{Q}_{p'})$-representations (in fact, we can use the similar argument as in the next subsection). Therefore, the kernel and the cokernel of the composite

$$H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) \to H^i_c(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi) \to H^i(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi)$$

are non-cuspidal by Theorem 6.1.

Remark 6.3 Let $IH^i(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi) = \varinjlim_{K_{p'}} H^i(\mathrm{Sh}^{\min}_{K_{p'}K^{p'},\overline{E}_v}, j_{!*}\mathcal{L}_\xi)$ be the intersection cohomology of our Shimura variety, where $j\colon \mathrm{Sh}_{K_{p'}K^{p'}} \hookrightarrow \mathrm{Sh}^{\min}_{K_{p'}K^{p'}}$ denotes the minimal compactification of $\mathrm{Sh}_{K_{p'}K^{p'}}$. Then, as in the previous remark, it is easy to see that the kernel and the cokernel of the canonical homomorphism

$$H^i_c(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi) \to IH^i(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi)$$

are non-cuspidal. Therefore, by Theorem 6.1, we have an isomorphism

$$H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi)[\pi] \cong IH^i(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi)[\pi]$$

for an irreducible supercuspidal representation π of $G(\mathbb{Q}_{p'})$.

Corollary 6.4 We put

$$H^i_c(\mathrm{Sh}_{\infty,\overline{E}_v}, \mathcal{L}_\xi) = \varinjlim_{K^{p'}} H^i_c(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi), \ H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty,\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) = \varinjlim_{K^{p'}} H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi).$$

These are admissible/continuous $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{E}_v/E_v)$-representations.

Let Π be an irreducible admissible representation of $G(\mathbb{A}^\infty)$. Assume that there exists a prime p' such that $\Pi_{p'}$ is a supercuspidal representation of $G(\mathbb{Q}_{p'})$. Then, Π does not appear as a subquotient of the kernel or the cokernel of the canonical homomorphism $H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty,\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) \to H^i_c(\mathrm{Sh}_{\infty,\overline{E}_v}, \mathcal{L}_\xi)$. In particular, we have an isomorphism of $\mathrm{Gal}(\overline{E}_v/E_v)$-representations

$$H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty,\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi)[\Pi] \cong H^i_c(\mathrm{Sh}_{\infty,\overline{E}_v}, \mathcal{L}_\xi)[\Pi].$$

Proof. We take a compact open subgroup $K^{p'} \subset G(\widehat{\mathbb{Z}}^{p'})$ such that $\Pi^{K^{p'}} \neq 0$. If Π appears as a subquotient of the kernel or the cokernel of $H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty,\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) \to H^i_c(\mathrm{Sh}_{\infty,\overline{E}_v}, \mathcal{L}_\xi)$, then $\Pi_{p'}$ appears as a subquotient of the kernel or the cokernel of $H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) \to H^i_c(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \mathcal{L}_\xi)$ (*cf.* Lemma 6.6 in the next subsection). This contradicts Theorem 6.1. ∎

6.2   Proof of Theorem 6.1

Let K be a compact open subgroup of $G(\mathbb{A}^\infty)$. We regard $C_{[P]}$ for $[P] \in \mathcal{P}_G(K)$ as a pseudo-adic space (*cf.* [Hub96, §1.10]). See [Hub98c, Proposition 2.6 (i)] for the definition of the compactly supported ℓ-adic cohomology of pseudo-adic spaces.

Proposition 6.5 *Let $[P]$ be an element of $\mathcal{P}_G(K)$.*

(i) *For a constructible ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell$ on $\mathrm{Sh}_{K,\overline{E}_v}$, $H^i_c(C_{[P],\overline{E}_v}, \mathcal{F}^{\mathrm{ad}})$ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space.*

(ii) *For a constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf $\mathcal{F}$ on $\mathrm{Sh}_{K,\overline{E}_v}$, $H^i_c(C_{[P],\overline{E}_v}, \mathcal{F}^{\mathrm{ad}})$ is a finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-module.*

Before proving this proposition, we note the following general lemmas.

Lemma 6.6 *Let k be an algebraically closed non-archimedean field, and X an adic space locally of finite type, separated and taut over k. Let L be a locally closed constructible subset of X, which is regarded as a pseudo-adic space. Let $\pi \colon X' \to X$ be a finite étale Galois covering with Galois group H. We put $L' = \pi^{-1}(L)$. For an ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell$ on X, the natural map*

$$H^i_c(L, \mathcal{F}) \to H^i_c(L', \pi^*\mathcal{F})^H$$

is an isomorphism.

Proof. This lemma might be well-known, but we include its proof for reader's convenience. We put $\mathcal{F}'_n = \pi_*\pi^*\mathcal{F}_n$ and $\mathcal{F}' = (\mathcal{F}'_n) \otimes \overline{\mathbb{Q}}_\ell$. The group H acts on $\mathcal{F}'_n$, and we have $(\mathcal{F}'_n)^H = \mathcal{F}_n$. Consider the map $\psi = \sum_{h \in H} h \colon \mathcal{F}'_n \to \mathcal{F}_n$. The composite $\mathcal{F}_n \hookrightarrow \mathcal{F}'_n \xrightarrow{\psi} \mathcal{F}_n$ equals the multiplication by $\#H$. By taking the cohomology, we have a commutative diagram

$$\begin{array}{ccc}
H^i_c(L, (\mathcal{F}_n)_n) \otimes \overline{\mathbb{Q}}_\ell \longrightarrow H^i_c(L, (\mathcal{F}'_n)_n) \otimes \overline{\mathbb{Q}}_\ell \xrightarrow{H^i_c(\psi)} H^i_c(L, (\mathcal{F}_n)_n) \otimes \overline{\mathbb{Q}}_\ell \\
\quad\quad\searrow{\pi^*} \quad\quad\quad \| \quad\quad\quad\quad\quad\quad\quad \downarrow{\pi^*} \\
H^i_c(L', (\pi^*\mathcal{F}_n)_n) \otimes \overline{\mathbb{Q}}_\ell \xrightarrow{\sum_{h \in H} h} H^i_c(L', (\pi^*\mathcal{F}_n)_n) \otimes \overline{\mathbb{Q}}_\ell.
\end{array}$$

The composite of two upper horizontal arrows is the multiplication by $\#H$, which is an isomorphism. Therefore π^* is injective and $H^i_c(\psi)$ is surjective. The surjectivity of $H^i_c(\psi)$ implies that the image of π^* is equal to that of $\sum_{h \in H} h$, that is, the H-invariant part of $H^i_c(L', (\pi^*\mathcal{F}_n)_n) \otimes \overline{\mathbb{Q}}_\ell$. ∎

Remark 6.7 By the same method, we can also prove that the natural map

$$\varprojlim_n H^i_c(L, \mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell \to \left(\varprojlim_n H^i_c(L', \pi^*\mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell \right)^H$$

is an isomorphism.

Lemma 6.8 *Let k be an algebraically closed non-archimedean field. Let X be an adic space locally of finite type, separated and taut over k, and U a constructible open subset of X. Set $Z = X \setminus U$, which is regarded as a pseudo-adic space. For an ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell$ over X, we have a long exact sequence*

$$\cdots \to H_c^i(U, \mathcal{F}) \to H_c^i(X, \mathcal{F}) \to H_c^i(Z, \mathcal{F}) \to H_c^{i+1}(U, \mathcal{F}) \to \cdots.$$

Proof. Since the open immersion $j \colon U \hookrightarrow X$ is quasi-compact, we have $H_c^i(U, \mathcal{F}) = H_c^i(X, j_! j^* \mathcal{F})$ by the definition of the compactly supported cohomology. Therefore, the claim follows from [Hub98c, Proposition 2.6 (i)]. ∎

Proof of Proposition 6.5. We consider (i). Note that Corollary 5.29 and Lemma 6.6 enable us to shrink K arbitrarily.

First we consider the Hodge type case. Take an embedding $(G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. We have a constructible open subset $U_{r,K}$ for each $0 \le r \le n$ by Lemma 5.30. By the long exact sequence

$$\cdots \to H_c^i(U_{r-1,K,\overline{E}_v}, \mathcal{F}) \to H_c^i(U_{r,K,\overline{E}_v}, \mathcal{F}) \to H_c^i(U_{r,K,\overline{E}_v} \setminus U_{r-1,K,\overline{E}_v}, \mathcal{F})$$
$$\to H_c^{i+1}(U_{r-1,K,\overline{E}_v}, \mathcal{F}) \to \cdots$$

(see Lemma 6.8) and Lemma 5.30, it suffices to show that $H_c^i(U_{r,K,\overline{E}_v}, \mathcal{F})$ is finite-dimensional for each $0 \le r \le n$.

Take a compact open subgroup $\widetilde{K}$ of $G(\mathbb{A}^\infty)$ so that we have a natural embedding $\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})_E$. By shrinking K, we may assume that $\widetilde{K} = \widetilde{K}_p \widetilde{K}^p$ with $\widetilde{K}_p \subset \mathrm{GSp}_{2n}(\mathbb{Z}_p)$. Let $\mathscr{S}_{\widetilde{K}}^{\mathrm{nor}}$ be the normalization of $\mathscr{S}_{\widetilde{K}^p}^{\mathrm{tor}}$ in $\mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})_{\mathbb{Q}_p}$, and $\mathscr{S}_{\widetilde{K}, \le r, \mathbb{F}_p}^{\mathrm{nor}}$ the inverse image of $\mathscr{S}_{\widetilde{K}^p, \le r, \mathbb{F}_p}^{\mathrm{tor}}$ in $\mathscr{S}_{\widetilde{K}}^{\mathrm{nor}}$. We write Y for the closure of $\mathrm{Sh}_K(G, X)_{E_v}$ in $\mathscr{S}_{\widetilde{K}, E_v}^{\mathrm{nor}}$ and put $Z = Y \setminus \mathrm{Sh}_K(G, X)_{E_v}$. Let V denote the inverse image of $\mathscr{S}_{\widetilde{K}, \le r, \mathbb{F}_p}^{\mathrm{nor}}$ under the composite

$$Y^{\mathrm{ad}} \hookrightarrow (\mathscr{S}_{\widetilde{K}}^{\mathrm{nor}})_{E_v}^{\mathrm{ad}} \to (\mathscr{S}_{\widetilde{K}}^{\mathrm{nor}})_{\mathbb{Q}_p}^{\mathrm{ad}} = (\mathscr{S}_{\widetilde{K}}^{\mathrm{nor}})^{\wedge \mathrm{rig}} \xrightarrow{\mathrm{sp}} \mathscr{S}_{\widetilde{K}, \mathbb{F}_p}^{\mathrm{nor}},$$

which is a quasi-compact open subset of Y^{ad}. Note that $V \setminus (V \cap Z^{\mathrm{ad}}) = U_{r,K}$ by Remark 5.31. Therefore, [Hub98c, Theorem 3.3 (i)] tells us that $H_c^i(U_{r,K,\overline{E}_v}, \mathcal{F})$ is finite-dimensional. This completes the proof in the Hodge type case.

Next we consider the preabelian type case. We choose a compact open subgroup of K'' of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$ in such a way as in Proposition 5.10 (iii), and use the notation in Lemma 4.4. Then, for $[P] \in \mathcal{P}_K(G)$, the inverse image of $C_{[P^{\mathrm{ad}}]}$ under $\pi \colon \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K''}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is equal to $C_{[P]}$. Therefore, by pushing forward sheaves by π, we may assume that $G = G'^{\mathrm{ad}}$ (note that $(\pi_* \mathcal{F}_n)^{\mathrm{ad}} = \pi_*^{\mathrm{ad}} \mathcal{F}_n^{\mathrm{ad}}$ by [Hub96, Theorem 3.7.2]). Since the Hecke action is transitive on the connected components of $\mathrm{Sh}_K(G, X)_{\overline{E}_v}$, we may work on a connected component $\mathrm{Sh}_K(G, X)_{\overline{E}_v}^0$ of $\mathrm{Sh}_K(G, X)_{\overline{E}_v}$ which is a quotient of a connected component $\mathrm{Sh}_{K'}(G', X')_{\overline{E}_v}^0$ of

37

$\mathrm{Sh}_{K'}(G', X')_{\overline{E}_v}$ by a free action of a finite group H for some K'. By Corollary 5.29, the inverse image of $C_{[P],\overline{E}_v} \cap \mathrm{Sh}_K(G, X)^{0,\mathrm{ad}}_{\overline{E}_v}$ for $[P] \in \mathcal{P}_K(G)$ under

$$f \colon \mathrm{Sh}_{K'}(G', X')^{0,\mathrm{ad}}_{\overline{E}_v} \to \mathrm{Sh}_K(G, X)^{0,\mathrm{ad}}_{\overline{E}_v}$$

equals

$$\coprod_{[P'] \in \mathcal{P}_{K'}(G'), [P'] \mapsto [P]} C_{[P'],\overline{E}_v} \cap \mathrm{Sh}_{K'}(G', X')^{0,\mathrm{ad}}_{\overline{E}_v}.$$

Since $H^i_c(C_{[P'],\overline{E}_v}, f^*\mathcal{F})$ is finite-dimensional, so is

$$H^i_c\big(f^{-1}(C_{[P],\overline{E}_v} \cap \mathrm{Sh}_K(G, X)^{0,\mathrm{ad}}_{\overline{E}_v}), f^*\mathcal{F}\big).$$

By Lemma 6.6, $H^i_c(C_{[P],\overline{E}_v} \cap \mathrm{Sh}_K(G, X)^{0,\mathrm{ad}}_{\overline{E}_v}, \mathcal{F})$ is equal to the H-invariant part of the above, hence finite-dimensional. This concludes the proof of (i).

The assertion (ii) can be proved in the same way, by using the Hochschild-Serre spectral sequence in place of Lemma 6.6 when taking a quotient. ∎

Remark 6.9 By the same method and Remark 6.7, we can also prove that the natural map $H^i_c(C_{[P],\overline{E}_v}, \mathcal{F}^{\mathrm{ad}}) \to \varprojlim_n H^i_c(C_{[P],\overline{E}_v}, \mathcal{F}^{\mathrm{ad}}_n) \otimes \overline{\mathbb{Q}}_\ell$ is an isomorphism. However, we do not need this fact.

Now let $K^{p'}$ be as in Section 6.1, $K_{p'}$ a compact open subgroup of $G(\mathbb{Q}_{p'})$, and $K = K_{p'}K^{p'}$. Take a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$, and an embedding $i \colon (G', X') \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. Then, as in Lemma 5.28 and Lemma 5.30, we obtain an increasing sequence

$$\{[G]\} = \mathcal{P}_G(K)_{\leq 0} \subset \mathcal{P}_G(K)_{\leq 1} \subset \cdots \subset \mathcal{P}_G(K)_{\leq n} = \mathcal{P}_G(K)$$

of subsets of $\mathcal{P}_G(K)$. Note that $\mathcal{P}_G(K)_r = \mathcal{P}_G(K)_{\leq r} \setminus \mathcal{P}_G(K)_{\leq r-1}$ is a union of fibers of the natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{A}^\infty}$. Therefore, by refining the sequence above, we can find an increasing sequence

$$\{[G]\} = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_m = \mathcal{P}_G(K)$$

of subsets of $\mathcal{P}_G(K)$ satisfying the following conditions:

- For every $0 \leq r \leq n$, there exists $0 \leq j \leq m$ such that $S_j = \mathcal{P}_G(K)_{\leq r}$.
- For $[P_1], [P_2] \in \mathcal{P}_G(K)$, P_1 and P_2 are conjugate by $G(\mathbb{Q}_{p'})$ if and only if $[P_1], [P_2] \in S_j \setminus S_{j-1}$ for some $0 \leq j \leq m$ (here we put $S_{-1} = \varnothing$).

For $0 \leq j \leq m$, we put $T_{j,K} = \bigcup_{[P] \in S_j} C_{[P]}$. By Lemma 5.30, it is a constructible open subset of $\mathrm{Sh}^{\mathrm{ad}}_{K,E_v}$. Further, we put $Z_{j,K} = T_{j,K} \setminus T_{j-1,K}$ ($T_{-1,K}$ is regarded as $\varnothing$). Lemma 5.30 tells us that $Z_{j,K} = \coprod_{[P] \in S_j \setminus S_{j-1}} C_{[P]}$ as topological spaces. For a compact open subgroup $K'_{p'} \subset K_{p'}$, we put $K' = K'_{p'}K^{p'}$, and let S'_j be the inverse image of S_j under the natural map $\mathcal{P}_G(K') \to \mathcal{P}_G(K)$. The sequence $\{S'_j\}_{0 \leq j \leq m}$

satisfies the same conditions as $\{S_j\}_{0 \leq j \leq m}$ does. Therefore, we can define $T_{j,K'}$ and $Z_{j,K'}$ in the same way as $T_{j,K}$ and $Z_{j,K}$. Note that $T_{j,K'}$ (resp. $Z_{j,K'}$) is the inverse image of $T_{j,K}$ (resp. $Z_{j,K}$) under $\mathrm{Sh}^{\mathrm{ad}}_{K',E_v} \to \mathrm{Sh}^{\mathrm{ad}}_{K,E_v}$. We put

$$V^i_{\leq j} = \varinjlim_{K'_{p'}} H^i_c(T_{j,K',\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi), \quad V^i_j = \varinjlim_{K'_{p'}} H^i_c(Z_{j,K',\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi),$$

where $K'_{p'}$ runs through compact open subgroups of $K_{p'}$.

Lemma 6.10 (i) *The group $G(\mathbb{Q}_{p'})$ naturally acts on $V^i_{\leq j}$ and V^i_j, and these are admissible $G(\mathbb{Q}_{p'})$-representations.*

(ii) *We have the following long exact sequence of $G(\mathbb{Q}_{p'})$-modules:*

$$\cdots \to V^i_{\leq j-1} \to V^i_{\leq j} \to V^i_j \to V^{i+1}_{\leq j-1} \to \cdots .$$

Proof. First, by Corollary 5.29, the group $G(\mathbb{Q}_{p'})$ acts on $V^i_{\leq j}$ and V^i_j. By Lemma 6.8, we have a long exact sequence as in (ii), which is obviously $G(\mathbb{Q}_{p'})$-equivariant.

Let us prove (i). Clearly $V^i_{\leq j}$ and V^i_j are smooth $G(\mathbb{Q}_{p'})$-representations. We will show the admissibility of them. Take a compact open subgroup $K_{p'}$ of $G(\mathbb{Q}_{p'})$ and its open normal subgroup $K'_{p'}$, and put $K = K_{p'}K^{p'}$, $K' = K'_{p'}K^{p'}$. Then, we have

$$H^i_c(Z_{j,K',\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi)^{K_{p'}} = H^i_c(Z_{j,K,\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi)$$

by Lemma 6.6. Taking the inductive limit with respect to $K'_{p'}$, we have

$$(V^i_j)^{K_{p'}} = H^i_c(Z_{j,K,\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) = \bigoplus_{[P] \in S_j \setminus S_{j-1}} H^i_c(C_{[P],\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi).$$

By Proposition 6.5 (i), it is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Therefore we conclude that V^i_j is an admissible representation of $G(\mathbb{Q}_{p'})$. By the long exact sequence in (ii) and the obvious identity $V^i_0 = V^i_{\leq 0}$, we can see the admissibility of $V^i_{\leq j}$ inductively. ∎

Proposition 6.11 *We take a representative P_j of an element of $S_j \setminus S_{j-1}$. For a compact open subgroup $K'_{p'} \subset K_{p'}$, let $[P_j]_{K'}$ denote the class of P_j in $\mathcal{P}_G(K')$. We put $W^i_j = \varinjlim_{K'_{p'}} H^i_c(C_{[P_j]_{K'},\overline{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi)$, where $K'_{p'}$ runs through compact open subgroups of $K_{p'}$.*

(i) *We have a natural smooth action of $P_j(\mathbb{Q}_{p'})$ on W^i_j.*

(ii) *We have a natural $G(\mathbb{Q}_{p'})$-equivariant isomorphism $V^i_j \cong \mathrm{Ind}^{G(\mathbb{Q}_{p'})}_{P_j(\mathbb{Q}_{p'})} W^i_j$.*

(iii) *The $G(\mathbb{Q}_{p'})$-representation V^i_j is non-cuspidal.*

Proof. The claim (i) is clear from Corollary 5.29. Let us prove (ii). We follow the proof of [IM10, Proposition 5.20]. By the Frobenius reciprocity, we have a homomorphism of $G(\mathbb{Q}_{p'})$-modules $\mathrm{Ind}^{G(\mathbb{Q}_{p'})}_{P_j(\mathbb{Q}_{p'})} W^i_j \to V^i_j$. We shall observe that this

is bijective. We take a special maximal compact subgroup $K_{p'}^0$ of $G(\mathbb{Q}_{p'})$. For a compact open subgroup $K_{p'}' \subset K_{p'}$ which is normal in $K_{p'}^0$, we have

$$
\begin{aligned}
H_c^i(Z_{j,K'}, \overline{E}_v, \mathcal{L}_\xi^{\mathrm{ad}}) &\overset{(1)}{=} \bigoplus_{[P] \in S_j' \setminus S_{j-1}'} H_c^i(C_{[P], \overline{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) \\
&\overset{(2)}{=} \bigoplus_{g \in K_{p'}' \backslash G(\mathbb{Q}_{p'})/P_j(\mathbb{Q}_{p'})} H_c^i(C_{[gP_jg^{-1}]_{K'}, \overline{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) \\
&\overset{(3)}{=} \bigoplus_{g \in K_{p'}' \backslash K_{p'}^0/P_j(\mathbb{Q}_{p'}) \cap K_{p'}^0} H_c^i(C_{[gP_jg^{-1}]_{K'}, \overline{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) \\
&\overset{(4)}{\cong} \mathrm{Ind}_{(P_j(\mathbb{Q}_{p'}) \cap K_{p'}^0)/(P_j(\mathbb{Q}_{p'}) \cap K_{p'}')}^{K_{p'}^0/K_{p'}'} H_c^i(C_{[P_j]_{K'}, \overline{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}).
\end{aligned}
$$

Here (1) follows from $Z_{j,K'} = \coprod_{[P] \in S_j' \setminus S_{j-1}'} C_{[P]}$ mentioned before, (2) from the definitions of S_j' and P_j, and (3) from the Iwasawa decomposition $G(\mathbb{Q}_{p'}) = P_j(\mathbb{Q}_{p'})K_{p'}^0$. The isomorphism (4) is a consequence of Corollary 5.29 and [Boy99, Lemme 13.2]. By taking the inductive limit, we obtain $K_{p'}^0$-isomorphisms

$$
V_j^i \cong \mathrm{Ind}_{P_j(\mathbb{Q}_{p'}) \cap K_{p'}^0}^{K_{p'}^0} W_j^i \overset{\cong}{\Longleftarrow} \mathrm{Ind}_{P_j(\mathbb{Q}_{p'})}^{G(\mathbb{Q}_{p'})} W_j^i
$$

(the second map is an isomorphism by the Iwasawa decomposition). By the proof of [Boy99, Lemme 13.2], it is easy to see that the first isomorphism above is nothing but the $K_{p'}^0$-homomorphism obtained by the Frobenius reciprocity for $P_j(\mathbb{Q}_{p'}) \cap K_{p'}^0 \subset K_{p'}^0$. Therefore the composite of the two isomorphisms above coincides with the $G(\mathbb{Q}_{p'})$-homomorphism introduced at the beginning of our proof of (ii). Thus we conclude the proof of (ii).

Finally consider (iii). By (ii), we have only to prove that the unipotent radical of $P_j(\mathbb{Q}_{p'})$ acts trivially on W_j^i. By [Boy99, Lemme 13.2.3], it suffices to prove that W_j^i is an admissible $P_j(\mathbb{Q}_{p'})$-representation. For any compact open subgroup $K_{p'}'$ of $K_{p'}^0$, the vector space $(W_j^i)^{P_j(\mathbb{Q}_{p'}) \cap K_{p'}'}$ is a subspace of $(\mathrm{Ind}_{P_j(\mathbb{Q}_{p'})}^{G(\mathbb{Q}_{p'})} W_j^i)^{K_{p'}'}$. By (ii) and Lemma 6.10 (i), $(\mathrm{Ind}_{P_j(\mathbb{Q}_{p'})}^{G(\mathbb{Q}_{p'})} W_j^i)^{K_{p'}'} \cong (V_j^i)^{K_{p'}'}$ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Hence W_j^i is an admissible $P_j(\mathbb{Q}_{p'})$-representation, as desired. ∎

Proof of Theorem 6.1. The claim follows from Lemma 6.10 (ii) and Proposition 6.11 (iii), because $V_{\leq 0}^i = H_c^i(\mathrm{Sh}_{\infty,K^{p'}, \overline{E}_v}^{\mathrm{pg}}, \mathcal{L}_\xi^{\mathrm{ad}})$ and $V_{\leq m}^i = H_c^i(\mathrm{Sh}_{\infty,K^{p'}, \overline{E}_v}, \mathcal{L}_\xi)$. ∎

6.3 Torsion coefficients

Since our method of proving Theorem 6.1 is totally geometric, we may also obtain an analogous result for ℓ-torsion coefficients. For simplicity, we will only consider a

constant coefficient $\overline{\mathbb{F}}_\ell$. We assume that $p' \neq \ell$, and put

$$H_c^i(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell) = \varinjlim_{K_{p'}} H_c^i(\mathrm{Sh}_{K_{p'}K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell),$$

$$H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell) = \varinjlim_{K_{p'}} H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{K_{p'}K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell).$$

They are naturally endowed with actions of $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\overline{E}_v/E_v)$. They are admissible/continuous $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\overline{E}_v/E_v)$-representations; note that we have

$$H_c^i(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell)^{K_{p'}} = H_c^i(\mathrm{Sh}_{K_{p'}K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell),$$

$$H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell)^{K_{p'}} = H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{K_{p'}K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell),$$

if $K_{p'}$ is a pro-p' group (*cf.* [Mie10, Proposition 2.5]).

The following theorem can be proved in exactly the same way as Theorem 6.1 (we use Proposition 6.5 (ii) in place of Proposition 6.5 (i)).

Theorem 6.12 *We assume that $p' \neq \ell$. The kernel and the cokernel of the canonical homomorphism*

$$H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell) \to H_c^i(\mathrm{Sh}_{\infty,K^{p'},\overline{E}_v}, \overline{\mathbb{F}}_\ell)$$

have no supercuspidal subquotient of $G(\mathbb{Q}_{p'})$. For the definition of supercuspidal representations over $\overline{\mathbb{F}}_\ell$, see [Vig96, II.2.5].

7 PEL type case

7.1 Notation for Shimura varieties of PEL type

In this section, we are interested in Shimura varieties of PEL type considered in [Kot92b, §5] (see also [Lan13, §1.4]). We recall it briefly. Fix a prime p. Consider a 6-tuple $(B, \mathcal{O}_B, *, V, L, \langle\ ,\ \rangle)$, where

- B is a finite-dimensional simple $\mathbb{Q}$-algebra such that $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$,
- $\mathcal{O}_B$ is an order of B whose p-adic completion is a maximal order of $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$,
- $*$ is a positive involution of B (namely, an involution such that $\mathrm{Tr}(bb^*) > 0$ for every non-zero $b \in B$) which preserves $\mathcal{O}_B$,
- V is a non-zero finite B-module,
- L is a $\mathbb{Z}$-lattice of V preserved by $\mathcal{O}_B$, and
- $\langle\ ,\ \rangle \colon V \times V \to \mathbb{Q}$ is a non-degenerate alternating $*$-Hermitian pairing with respect to the B-action such that $\langle x, y \rangle \in \mathbb{Z}$ for every $x, y \in L$, and that $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a self-dual lattice of $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

From $(B, V, \langle \, , \, \rangle)$, we define a simple $\mathbb{Q}$-algebra $C = \mathrm{End}_B(V)$ with a unique involution $\#$ satisfying $\langle cv, w \rangle = \langle v, c^\# w \rangle$ for every $c \in C$ and $v, w \in V$. Moreover we define an algebraic group G over $\mathbb{Q}$ by

$$G(R) = \{ g \in (C \otimes_{\mathbb{Q}} R)^\times \mid gg^\# \in R^\times \}$$

for every $\mathbb{Q}$-algebra R. The condition $gg^\# \in R^\times$ is equivalent to the existence of $c(g) \in R^\times$ such that $\langle gv, gw \rangle = c(g) \langle v, w \rangle$ for every $v, w \in V \otimes_{\mathbb{Q}} R$. By the presence of the lattice L, G can be naturally extended to a group scheme over $\mathbb{Z}$, which is also denoted by the same symbol G.

Consider an $\mathbb{R}$-algebra homomorphism $h \colon \mathbb{C} \to C \otimes_{\mathbb{Q}} \mathbb{R}$ preserving involutions (on $\mathbb{C}$, we consider the complex conjugation) such that the symmetric real-valued bilinear form $(v, w) \mapsto \langle v, h(i)w \rangle$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ is positive definite. Such a 7-tuple $(B, \mathcal{O}_B, *, V, L, \langle \, , \, \rangle, h)$ is said to be an unramified integral PEL datum. Note that the map h induces a homomorphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$, which is also denoted by h.

Let F be the center of B and F^+ the subfield of F consisting of elements fixed by $*$. The existence of h tells us that $N = [F : F^+](\dim_F C)^{1/2}/2$ is an integer. An unramified integral PEL datum falls into the following three types:

type (A) $[F : F^+] = 2$.

type (C) $[F : F^+] = 1$ and $C \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a product of $M_{2N}(\mathbb{R})$.

type (D) $[F : F^+] = 1$ and $C \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a product of $M_N(\mathbb{H})$.

For simplicity, we will exclude the type (D) case.

Using $h \colon \mathbb{C} \to C \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow C \otimes_{\mathbb{Q}} \mathbb{C}$, we can decompose the $B \otimes_{\mathbb{Q}} \mathbb{C}$-module $V \otimes_{\mathbb{Q}} \mathbb{C}$ as $V \otimes_{\mathbb{Q}} \mathbb{C} = V_1 \oplus V_2$, where V_1 (resp. V_2) is the subspace of $V \otimes_{\mathbb{Q}} \mathbb{C}$ on which $h(z)$ acts by z (resp. $\bar{z}$) for every $z \in \mathbb{C}$. We denote by E the field of definition of the isomorphism class of the $B \otimes_{\mathbb{Q}} \mathbb{C}$-module V_1, and call it the reflex field. It is a subfield of $\mathbb{C}$ which is finite over $\mathbb{Q}$.

In the sequel, we fix an unramified integral PEL datum $(B, \mathcal{O}_B, *, V, L, \langle \, , \, \rangle, h)$. For a compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, consider the functor $\mathscr{S}_{K^p}$ from the category of $\mathcal{O}_{E,(p)}$-schemes to the category of sets, that associates S to the set of isomorphism classes of quadruples (A, i, λ, η^p), where

- A is an abelian scheme over S,
- $\lambda \colon A \to A^\vee$ is a prime-to-p polarization,
- $i \colon \mathcal{O}_B \to \mathrm{End}_S(A)$ is an algebra homomorphism such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$ for every $b \in \mathcal{O}_B$,
- η^p is a level-K^p structure of (A, i, λ) of type $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p, \langle \, , \, \rangle)$ in the sense of [Lan13, Definition 1.3.7.6],

satisfying the equality of polynomials $\det_{\mathcal{O}_S}(b; \mathrm{Lie}\, A) = \det_E(b; V_1)$ in the sense of [Kot92b, §5]. Recall that two quadruples (A, i, λ, η^p) and $(A', i', \lambda', \eta'^p)$ are said to be isomorphic if there exists an isomorphism $f \colon A \to A'$ of abelian schemes such that

– $\lambda = f^\vee \circ \lambda' \circ f$,

– $f \circ i(b) = i'(b) \circ f$ for every $b \in \mathcal{O}_B$,

– and $f \circ \eta^p = \eta'^p$ in the sense of [Lan13, Definition 1.4.1.4].

If K^p is neat (*cf.* [Lan13, Definition 1.4.1.8]), the functor $\mathscr{S}_{K^p}$ is represented by a quasi-projective smooth $\mathcal{O}_{E,(p)}$-scheme (see [Lan13, Corollary 7.2.3.9]), which is also denoted by $\mathscr{S}_{K^p}$. Here we will call it a Shimura variety of PEL type. The group $G(\mathbb{A}^{\infty,p})$ naturally acts on the tower of schemes $(\mathscr{S}_{K^p})_{K^p \subset G(\widehat{\mathbb{Z}}^p)}$ as Hecke correspondences.

Let ℓ be a prime number different from p. For an algebraic representation ξ of G on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space, we can define a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\xi$ on Sh_K (see [Mil90, III, §6]). It is equivariant with respect to the Hecke action.

Remark 7.1 (i) Our definition of $\mathscr{S}_{K^p}$, due to [Lan13], is slightly different from that in [Kot92b], but they give the same moduli space. See [Lan13, Proposition 1.4.3.4].

(ii) Let us recall the relation between $\mathscr{S}_{K^p}$ and Shimura varieties in Section 4. See [Kot92b, §8] for detail. Let X denote the $G(\mathbb{R})$-orbit of the homomorphism $h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m \to G_\mathbb{R}$. Then, the pair (G, X) forms a Shimura datum, and $\mathscr{S}_{K^p,E}$ is isomorphic to a disjoint union of $\#\ker^1(\mathbb{Q}, G)$ copies of $\mathrm{Sh}_{G(\mathbb{Z}_p)K^p}(G, X)$. In the cases of type (C) or type (A) with N even, it is known that $\ker^1(\mathbb{Q}, G) = 1$.

So far in this section, we have only considered level structures which are prime to p. Now we add p^m-level structures on the universal abelian scheme of the generic fiber $\mathscr{S}_{K^p,E}$. Let $\mathscr{S}_{m,K^p,E}$ be the scheme over $\mathscr{S}_{K^p,E}$ classifying principal level-m structures (*cf.* [Lan13, Definition 1.3.6.2]) of the universal object $(\mathcal{A}, i^{\mathrm{univ}}, \lambda^{\mathrm{univ}})$ over $\mathscr{S}_{K^p,E}$. We denote the structure morphism $\mathscr{S}_{m,K^p,E} \to \mathscr{S}_{K^p,E}$ by pr_m, which is finite and étale. We write $\mathcal{L}_{\xi,m}$ or $\mathcal{L}_\xi$ for the inverse image of $\mathcal{L}_\xi$ by pr_m.

Let $K_{p,m}$ be the compact open subgroup of $G(\mathbb{Q}_p)$ defined as the kernel of $G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^m\mathbb{Z})$. Then $\mathscr{S}_{m,K^p,E}$ coincides with a disjoint union of the Shimura variety $\mathrm{Sh}_{K_{p,m}K^p}(G, X)$, where we use the notation in Remark 7.1 (ii).

7.2 Compactly supported cohomology and nearby cycle cohomology

Fix a place v of E over p. We write E_v for the completion of E at v, $\mathcal{O}_v$ for the ring of integers of E_v, and κ_v the residue field of $\mathcal{O}_v$. We put $\mathscr{S}_{K^p,\eta} = \mathscr{S}_{K^p,E_v}$, $\mathscr{S}_{K^p,\overline{\eta}} = \mathscr{S}_{K^p,\overline{E}_v}$, $\mathscr{S}_{K^p,v} = \mathscr{S}_{K^p,\kappa_v}$ and $\mathscr{S}_{K^p,\overline{v}} = \mathscr{S}_{K^p,\overline{\kappa}_v}$. Further, for $m \geq 0$ we set $\mathscr{S}_{m,K^p,\eta} = \mathscr{S}_{m,K^p,E} \otimes_E E_v$ and $\mathscr{S}_{m,K^p,\overline{\eta}} = \mathscr{S}_{m,K^p,E} \otimes_E \overline{E}_v$.

Let p' be a prime number, and $K^{p'} \subset G(\widehat{\mathbb{Z}}^{p'})$ a compact open subgroup. If $p' \neq p$, we assume that $K^{p'} = K_{p,m_0}K^{p',p}$ for some $m_0 \geq 0$ and compact open subgroup $K^{p',p}$

of $G(\widehat{\mathbb{Z}}^{p',p})$. We put

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{\eta}},\mathcal{L}_\xi) = \begin{cases} \varinjlim_m H_c^i(\mathscr{S}_{m,K^p,\overline{\eta}},\mathcal{L}_\xi) & \text{if } p'=p, \\ \varinjlim_{K_{p'}} H_c^i(\mathscr{S}_{m0,K_{p'}K^{p',p},\overline{\eta}},\mathcal{L}_\xi) & \text{if } p'\neq p, \end{cases}$$

which is an admissible/continuous $G(\mathbb{Q}_{p'})\times\mathrm{Gal}(\overline{E}_v/E_v)$-representation. We are also interested in the nearby cycle cohomology defined as follows:

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{v}},R\psi\mathcal{L}_\xi) = \begin{cases} \varinjlim_m H_c^i\big(\mathscr{S}_{K^p,\overline{v}},R\psi(\mathrm{pr}_{m*}\mathcal{L}_\xi)\big) & \text{if } p'=p, \\ \varinjlim_{K_{p'}} H_c^i\big(\mathscr{S}_{K_{p'}K^{p',p},\overline{v}},R\psi(\mathrm{pr}_{m0*}\mathcal{L}_\xi)\big) & \text{if } p'\neq p. \end{cases}$$

Obviously the group $\mathrm{Gal}(\overline{E}_v/E_v)$ acts on it. The following lemma gives an action of $G(\mathbb{Q}_{p'})$ on $H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{v}},R\psi\mathcal{L}_\xi)$.

Lemma 7.2 *We have a natural action of $G(\mathbb{Q}_{p'})$ on $H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{v}},R\psi\mathcal{L}_\xi)$. By this action, $H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{v}},R\psi\mathcal{L}_\xi)$ becomes an admissible/continuous $G(\mathbb{Q}_{p'})\times\mathrm{Gal}(\overline{E}_v/E_v)$-representation.*

Proof. We show the claim in the case $p'=p$. The other cases are easier. To ease notation, we omit the subscript K^p.

As in [Man05, §6], we can construct a tower $(\mathscr{S}_m)_{m\geq 0}$ of schemes over $\mathcal{O}_v$ with finite transition maps such that $\mathscr{S}_m$ gives an integral model of $\mathscr{S}_{m,\eta}$ and $\mathscr{S}_0=\mathscr{S}$. In this situation, we have

$$H_c^i\big(\mathscr{S}_{\overline{v}},R\psi(\mathrm{pr}_{m*}\mathcal{L}_\xi)\big) = H_c^i(\mathscr{S}_{m,\overline{v}},R\psi\mathcal{L}_\xi),$$

where $\mathscr{S}_{m,\overline{v}} = \mathscr{S}_m\otimes_{\mathcal{O}_v}\overline{\kappa}_v$.

We put $G^+(\mathbb{Q}_p) = \{g\in G(\mathbb{Q}_p)\mid g^{-1}L_p\subset L_p\}$. For $g\in G^+(\mathbb{Q}_p)$, let $e(g)$ be the minimal non-negative integer such that $gL_p\subset p^{-e(g)}L_p$. Then we can construct a tower $(\mathscr{S}_{m,g})_{m\geq e(g)}$ of schemes over $\mathcal{O}_v$ and two morphisms

$$\mathrm{pr}\colon \mathscr{S}_{m,g}\to\mathscr{S}_m, \quad [g]\colon \mathscr{S}_{m,g}\to\mathscr{S}_{m-e(g)}$$

which are compatible with the transition maps. It is known that these are proper morphisms, pr induces an isomorphism on the generic fibers, and $[g]$ induces the Hecke action of g on the generic fibers (see [Man05, Proposition 16, Proposition 17]). In particular, we have a canonical cohomological correspondence (*cf.* [SGA5, Exposé III], [Fuj97])

$$c_g\colon [g]_\eta^*\mathcal{L}_{\xi,m-e(g)} \xrightarrow{\cong} \mathrm{pr}_\eta^*\mathcal{L}_{\xi,m} = R\,\mathrm{pr}_\eta^!\mathcal{L}_{\xi,m}.$$

Let

$$R\psi(c_g)\colon [g]_{\overline{v}}^*R\psi\mathcal{L}_{\xi,m-e(g)} \to R\,\mathrm{pr}_{\overline{v}}^!R\psi\mathcal{L}_{\xi,m}$$

be the specialization of c_g (*cf.* [Fuj97, §1.5], [IM10, §6]). Since $[g]_{\overline{v}}$ is proper, this induces a homomorphism

$$H_c^i(\mathscr{S}_{m-e(g),\overline{v}}, R\psi\mathcal{L}_\xi) \xrightarrow{H_c^i(R\psi(c_g))} H_c^i(\mathscr{S}_{m,\overline{v}}, R\psi\mathcal{L}_\xi).$$

Taking the inductive limit, we get

$$\gamma_g \colon H_c^i(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi) \to H_c^i(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi).$$

From an obvious relation $c_{gg'} = c_g \circ g^* c_{g'}$ for $g, g' \in G^+(\mathbb{Q}_p)$, we deduce $\gamma_{gg'} = \gamma_g \circ \gamma_{g'}$ (*cf.* [IM10, Corollary 6.3]). On the other hand, by [Man05, Proposition 16 (3), Proposition 17 (3)], $\gamma_{p^{-1}}$ is the identity. Since $G(\mathbb{Q}_p)$ is generated by $G^+(\mathbb{Q}_p)$ and p as a monoid, we can extend γ_g to the whole $G(\mathbb{Q}_p)$. By [Man05, Proposition 16 (4)], the restriction of this action to $K_{p,0} = G(\mathbb{Z}_p)$ coincides with the inductive limit of the natural action of $K_{p,0}$ on $H_c^i(\mathscr{S}_{m,\overline{v}}, R\psi\mathcal{L}_\xi)$. In particular, it is a smooth action. Furthermore, for integers $m' \geq m \geq 1$, we have

$$H_c^i(\mathscr{S}_{m',\overline{v}}, R\psi\mathcal{L}_\xi)^{K_{p,m}/K_{p,m'}} = H_c^i(\mathscr{S}_{m,\overline{v}}, R\psi\mathcal{L}_\xi)$$

(see [Mie10, Proposition 2.5]). Taking inductive limit, we obtain

$$H_c^i(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi)^{K_{p,m}} = H_c^i(\mathscr{S}_{m,\overline{v}}, R\psi\mathcal{L}_\xi).$$

This implies that $H_c^i(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\mathcal{L}_\xi)$ is an admissible/continuous representation of $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{E}_v/E_v)$. ∎

Corollary 7.3 *The kernel and the cokernel of the canonical homomorphism*

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{v}}, R\psi\mathcal{L}_\xi) \to H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{\eta}}, \mathcal{L}_\xi)$$

(cf. [SGA7, Exposé XIII, (2.1.7.3)]) are non-cuspidal. In particular, for an irreducible supercuspidal representation π of $G(\mathbb{Q}_{p'})$, we have an isomorphism

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{\eta}}, \mathcal{L}_\xi)[\pi] \cong H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{v}}, R\psi\mathcal{L}_\xi)[\pi].$$

Similar results hold for the coefficient $\overline{\mathbb{F}}_\ell$ for a prime number $\ell \neq p, p'$.

Proof. Analogues of Theorem 5.17 and Theorem 6.1 are also valid in the PEL type case in this section by Remark 7.1 (ii). Let $\mathscr{S}_{K^p,E_v}^{\mathrm{pg}}$ be the potentially good reduction locus of $\mathscr{S}_{K^p,E_v}^{\mathrm{ad}}$. As in the Siegel case, $\mathscr{S}_{K^p,E_v}^{\mathrm{pg}}$ coincides with the rigid generic fiber of the completion of $\mathscr{S}_{K^p,\mathcal{O}_v}$ along the special fiber. Hence, we have an isomorphism

$$H_c^i(\mathscr{S}_{K^p,\overline{E}_v}^{\mathrm{pg}}, \mathrm{pr}_{m*}\mathcal{L}_\xi^{\mathrm{ad}}) \cong H_c^i(\mathscr{S}_{K^p,\overline{v}}, R\psi(\mathrm{pr}_{m*}\mathcal{L}_\xi))$$

for any non-negative integer m by [Hub96, Theorem 3.7.2, Theorem 5.7.6] and [Hub98c, Theorem 3.1]. Taking inductive limits, we have an isomorphism

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{E}_v}^{\mathrm{pg}}, \mathcal{L}_\xi^{\mathrm{ad}}) \cong H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{v}}, R\psi\mathcal{L}_\xi).$$

Hence the claim follows from the analogue of Theorem 6.1. ∎

Remark 7.4 (i) The case where $p' \neq p$ in Corollary 7.3 was previously obtained by Tetsushi Ito and the second author. In that case, we can use minimal compactifications over $\mathcal{O}_v$ to show the claim.

(ii) In [LS15], Lan and Stroh obtained a stronger result that the canonical homomorphism in Corollary 7.3 is in fact an isomorphism. Their method is totally different from ours.

7.3 Example

In this subsection, we give a very simple application of Corollary 7.3. Proofs in this subsection are rather sketchy, since the technique is more or less well-known.

Here we consider the Shimura variety for $GU(1, n-1)$ over $\mathbb{Q}$. Let F be an imaginary quadratic extension of $\mathbb{Q}$ and $\mathrm{Spl}_{F/\mathbb{Q}}$ the set of rational primes over which $F/\mathbb{Q}$ splits. We fix a field embedding $F \hookrightarrow \mathbb{C}$ and regard F as a subfield of $\mathbb{C}$. For an integer $n \geq 2$, consider the integral PEL datum $(B, \mathcal{O}_B, *, V, L, \langle \ , \ \rangle, h)$ as follows:

- $B = F$, $\mathcal{O}_B = \mathcal{O}_F$ and $*$ is the unique non-trivial element of $\mathrm{Gal}(F/\mathbb{Q})$.
- $V = F^n$ and $L = \mathcal{O}_F^n$.
- $\langle \ , \ \rangle \colon V \times V \to \mathbb{Q}$ is an alternating pairing satisfying the following conditions:
    - $\langle x, y \rangle \in \mathbb{Z}$ for every $x, y \in L$,
    - $\langle bx, y \rangle = \langle x, b^*y \rangle$ for every $x, y \in V$ and $b \in F$, and
    - $G_{\mathbb{R}} \cong GU(1, n-1)$ (for the definition of G, see Section 7.1).
- $h \colon \mathbb{C} \to \mathrm{End}_F(V) \otimes_{\mathbb{Q}} \mathbb{R} \cong M_n(\mathbb{C})$ is given by $z \mapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z}I_{n-1} \end{pmatrix}$, where the last isomorphism is induced by the fixed embedding $F \hookrightarrow \mathbb{C}$.

In this case, the reflex field E is equal to F. To a neat compact open subgroup K of $G(\widehat{\mathbb{Z}})$, we can attach the Shimura variety Sh_K of PEL type, which is not proper over $\mathrm{Spec}\, F$.

Put $\Sigma = \{p \in \mathrm{Spl}_{F/\mathbb{Q}} \mid L_p = L_p^{\perp}\}$. Then our integral Shimura datum is unramified at every $p \in \Sigma$. Moreover, for such p, $G_{\mathbb{Q}_p}$ is isomorphic to $\mathrm{GL}_n(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$ (*cf.* [Far04, §1.2.3]). If $K = K_{p,0}K^p$ for some compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, we have $\mathrm{Sh}_K = \mathscr{S}_{K^p} \otimes_{\mathcal{O}_{F,(p)}} F$, where $\mathscr{S}_{K^p}$ is the moduli space introduced in Section 7.1.

Let us fix a prime number ℓ. We put

$$H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell) = \varinjlim_K H_c^i(\mathrm{Sh}_K \otimes_F \overline{F}, \overline{\mathbb{Q}}_\ell).$$

It is an admissible/continuous $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{F}/F)$-representation over $\overline{\mathbb{Q}}_\ell$.

Theorem 7.5 *Let Π be an irreducible admissible representation of $G(\mathbb{A}^\infty)$ over $\overline{\mathbb{Q}}_\ell$. Assume that there exists a prime $p \in \Sigma$ such that Π_p is a supercuspidal representation of $G(\mathbb{Q}_p)$. Then $H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)[\Pi] = 0$ unless $i = n - 1$.*

Remark 7.6 For proper Shimura varieties, an analogous result is known ([Clo91], [HT01, Corollary IV.2.7]). It would be possible to give an "automorphic" proof of Theorem 7.5 by using results in [Mor10]. However, the authors think that our proof, consisting of purely local arguments, is simpler and has some importance.

Proof. Let ℓ' be another prime number and fix an isomorphism of fields $\iota \colon \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_{\ell'}$. Then ι induces an isomorphism $H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)[\Pi] \cong H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_{\ell'})[\iota\Pi]$, where $\iota\Pi$ is the representation of $G(\mathbb{A}^\infty)$ over $\overline{\mathbb{Q}}_{\ell'}$ induced by Π and ι. It is easy to observe that Π_p is supercuspidal if and only if $(\iota\Pi)_p$ is supercuspidal. Therefore, we can change our ℓ freely, and thus we can assume that there exists a prime $p \in \Sigma \setminus \{\ell\}$ such that Π_p is supercuspidal. Fix such p and take a place v of F lying over p. Then, for an integer $m \geq 0$ and a neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, $\mathrm{Sh}_{K_{p,m}K^p} \otimes_F F_v$ is isomorphic to $\mathscr{S}_{m,K^p,\eta}$ introduced in Section 7.2. Therefore we have an isomorphism

$$H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell) \cong \varinjlim_{m,K^p} H_c^i(\mathscr{S}_{m,K^p,\overline{\eta}}, \overline{\mathbb{Q}}_\ell) = \varinjlim_{K^p} H_c^i(\mathscr{S}_{\infty,K^p,\overline{\eta}}, \overline{\mathbb{Q}}_\ell).$$

Thus it suffices to show that $H_c^i(\mathscr{S}_{\infty,K^p,\overline{\eta}}, \overline{\mathbb{Q}}_\ell)[\pi] = 0$ for a supercuspidal representation π of $G(\mathbb{Q}_p)$, a neat compact open subgroup K^p, and an integer $i \neq n - 1$. By Corollary 7.3, it is equivalent to showing that $H_c^i(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell)[\pi] = 0$.

For an integer $h \geq 0$, let $\mathscr{S}_{K^p,v}^{[h]}$ be the reduced closed subscheme of $\mathscr{S}_{K^p,v}$ consisting of points x such that the étale rank of $\mathcal{A}_x[v^\infty]$ is less than or equal to h (*cf.* [HT01, p. 111]), where $\mathcal{A}$ denotes the universal abelian scheme over $\mathscr{S}_{K^p}$. Put $\mathscr{S}_{K^p,v}^{(h)} = \mathscr{S}_{K^p,v}^{[h]} \setminus \mathscr{S}_{K^p,v}^{[h-1]}$. Our proof of the theorem is divided into the subsequent two lemmas. ∎

Lemma 7.7 *For every supercuspidal representation π of $G(\mathbb{Q}_p)$, we have*

$$H_c^i(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell)[\pi] = \left(\varinjlim_m H_c^i\big(\mathscr{S}_{K^p,\overline{v}}^{[0]}, (R\psi\,\mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p,\overline{v}}^{[0]}}\big) \right)[\pi].$$

Proof. First recall that $\mathscr{S}_{m,K^p,\eta}$ has a good integral model over $\mathcal{O}_v$. For an integer $m \geq 0$, consider the functor from the category of $\mathcal{O}_v$-schemes to the category of sets, that associates S to the set of isomorphism classes of 6-tuples $(A, i, \lambda, \eta^p, \eta_v, \eta_{p,0})$, where

- $[(A, i, \lambda, \eta^p)] \in \mathscr{S}_{K^p}(S)$,
- $\eta_v \colon L \otimes_{\mathbb{Z}} (v^{-m}\mathcal{O}_v/\mathcal{O}_v) \to A[v^m]$ is a Drinfeld v^m-level structure (*cf.* [HT01, II.2]), and
- $\eta_{p,0} \colon p^{-m}\mathbb{Z}/\mathbb{Z} \to \mu_{p^m,S}$ is a Drinfeld p^m-level structure.

Then it is easy to see that this functor is represented by a scheme $\mathscr{S}_{m,K^p}$ which is finite over $\mathscr{S}_{K^p}$. Moreover the generic fiber of $\mathscr{S}_{m,K^p}$ can be naturally identified with $\mathscr{S}_{m,K^p,\eta}$ (*cf.* the moduli problem $\mathfrak{X}'_U$ introduced in [HT01, p. 92]). As in [HT01, III.4], we can extend the Hecke action of $G(\mathbb{Q}_p)$ on $(\mathscr{S}_{m,K^p,\eta})_{m\geq 0}$ to the tower $(\mathscr{S}_{m,K^p})_{m\geq 0}$. We have a $G(\mathbb{Q}_p)$-equivariant isomorphism

$$H_c^i(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell) \cong \varinjlim_m H_c^i(\mathscr{S}_{m,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell).$$

Let us denote by $\mathscr{S}_{m,K^p,v}^{[h]}$ (resp. $\mathscr{S}_{m,K^p,v}^{(h)}$) the inverse image of $\mathscr{S}_{K^p,v}^{[h]}$ (resp. $\mathscr{S}_{K^p,v}^{(h)}$) under $\mathscr{S}_{m,K^p} \to \mathscr{S}_{K^p}$. For an integer $h \geq 0$, it is easy to observe that

$$\varinjlim_m H_c^i\big(\mathscr{S}_{K^p,\overline{v}}^{[h]}, (R\psi\,\mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p,\overline{v}}^{[h]}}\big) \cong \varinjlim_m H_c^i\big(\mathscr{S}_{m,K^p,\overline{v}}^{[h]}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{m,K^p,\overline{v}}^{[h]}}\big),$$

$$\varinjlim_m H_c^i\big(\mathscr{S}_{K^p,\overline{v}}^{(h)}, (R\psi\,\mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p,\overline{v}}^{(h)}}\big) \cong \varinjlim_m H_c^i\big(\mathscr{S}_{m,K^p,\overline{v}}^{(h)}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{m,K^p,\overline{v}}^{(h)}}\big)$$

and that they are admissible $G(\mathbb{Q}_p)$-representations. Moreover, by considering the kernel of the universal Drinfeld v^m-level structure η_v^{univ}, we can decompose $\mathscr{S}_{m,K^p,v}^{(h)}$ into finitely many open and closed subsets indexed by the set consisting of direct summands of $L \otimes_{\mathbb{Z}} (v^{-m}\mathcal{O}_v/\mathcal{O}_v)$ with rank $n-h$ (*cf.* [Boy99, Définition 10.4.1, Proposition 10.4.2] and [IM10, Definition 5.1, Lemma 5.3]). Using this partition, when $h > 0$, we can prove that the $G(\mathbb{Q}_p)$-representation

$$\varinjlim_m H_c^i\big(\mathscr{S}_{m,K^p,\overline{v}}^{(h)}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{m,K^p,\overline{v}}^{(h)}}\big)$$

is parabolically induced from a proper parabolic subgroup of $G(\mathbb{Q}_p)$. Therefore, by the same argument as in the proof of Theorem 6.1, we can conclude that the kernel and the cokernel of

$$\varinjlim_m H_c^i(\mathscr{S}_{m,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell) \to \varinjlim_m H_c^i\big(\mathscr{S}_{m,K^p,\overline{v}}^{[0]}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{m,K^p,\overline{v}}^{[0]}}\big)$$

are non-cuspidal. This completes the proof of the lemma. ∎

Lemma 7.8 *Let π be a supercuspidal representation of $G(\mathbb{Q}_p)$. If $i \neq n-1$, we have*

$$\Big(\varinjlim_m H_c^i\big(\mathscr{S}_{K^p,\overline{v}}^{[0]}, (R\psi\,\mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p,\overline{v}}^{[0]}}\big)\Big)[\pi] = 0.$$

Proof. Let $\mu_h \colon \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ be the homomorphism of algebraic groups over $\mathbb{C}$ defined as the composite of

$$\mathbb{G}_{m,\mathbb{C}} \xrightarrow{z \mapsto (z,1)} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \stackrel{(*)}{\cong} (\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{m,\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{h_{\mathbb{C}}} G_{\mathbb{C}},$$

where $(*)$ is given by $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\cong} \mathbb{C} \times \mathbb{C}$; $a \otimes b \mapsto (ab, a\overline{b})$. Fix an isomorphism of fields $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ and denote by $\mu \colon \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ the induced cocharacter of $G_{\overline{\mathbb{Q}}_p}$. Let b be a unique basic element of $B(G_{\mathbb{Q}_p}, \mu)$ (for the definition of $B(G, \mu)$, we refer to [Far04, §2.1.1]), and denote by $\mathcal{M}$ the Rapoport-Zink space associated to the local unramified PEL datum $(F \otimes_{\mathbb{Q}} \mathbb{Q}_p, \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p, *, V_p, L_p, \langle\ ,\ \rangle, b, \mu)$ (*cf.* [Far04, §2.3.5]). The Rapoport-Zink space $\mathcal{M}$ is equipped with an action of the group $J(\mathbb{Q}_p)$, where J denotes the algebraic group over $\mathbb{Q}_p$ associated to b (*cf.* [RZ96, Proposition 1.12]). By [Far04, §2.3.7.1], $\mathcal{M}$ is isomorphic to $\mathcal{M}_{\mathrm{LT}} \times \mathbb{Q}_p^\times/\mathbb{Z}_p^\times$, where $\mathcal{M}_{\mathrm{LT}}$ is the Lubin-Tate space for $\mathrm{GL}_n(\mathbb{Q}_p)$. Furthermore, $J(\mathbb{Q}_p)$ is isomorphic to $D^\times \times \mathbb{Q}_p^\times$, where D

denotes the central division algebra over $\mathbb{Q}_p$ with invariant $1/n$. The action of $J(\mathbb{Q}_p)$ on $\mathcal{M}$ is identified with the well-known action of $D^\times \times \mathbb{Q}_p^\times$ on $\mathcal{M}_{\mathrm{LT}} \times \mathbb{Q}_p^\times/\mathbb{Z}_p^\times$.

By the p-adic uniformization theorem of Rapoport-Zink ([RZ96, Theorem 6.30], [Far04, Corollaire 3.1.9]), we have an isomorphism

$$\coprod_{\ker^1(\mathbb{Q},G)} I(\mathbb{Q})\backslash \mathcal{M} \times G(\mathbb{A}^{\infty,p})/K^p \cong \mathscr{S}_{K^p}^\wedge,$$

where I is an algebraic group over $\mathbb{Q}$ satisfing $I(\mathbb{A}^\infty) \cong J(\mathbb{Q}_p) \times G(\mathbb{A}^{\infty,p})$ and $\mathscr{S}_{K^p}^\wedge$ denotes the formal completion of $\mathscr{S}_{K^p} \otimes_{\mathcal{O}_v} W(\overline{\mathbb{F}}_p)$ along $\mathscr{S}_{K^p,\overline{v}}^{[0]}$, the basic locus of $\mathscr{S}_{K^p,\overline{v}}$. By this isomorphism, we know that $\mathscr{S}_{K^p,\overline{v}}^{[0]}$, which coincides with $\mathscr{S}_{K^p}^\wedge$ as topological spaces, consists of finitely many closed points; indeed, the left hand side of the isomorphism above is a finite disjoint union of formal schemes of the form $\Gamma\backslash\mathcal{M}$, where $\Gamma \subset J(\mathbb{Q}_p)$ is a discrete cocompact subgroup (*cf.* [Far04, Lemme 3.1.7]). Therefore, by [Ber96, Theorem 3.1], we have an isomorphism

$$H_c^i\big(\mathscr{S}_{K^p,\overline{v}}^{[0]}, (R\psi\,\mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p,\overline{v}}^{[0]}}\big) = H^i\big(\mathscr{S}_{K^p,\overline{v}}^{[0]}, (R\psi\,\mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p,\overline{v}}^{[0]}}\big)$$
$$\cong H^i\big(\mathscr{S}_{m,K^p,\overline{\eta}}(b), \overline{\mathbb{Q}}_\ell\big),$$

where $\mathscr{S}_{m,K^p,\overline{\eta}}(b) = \mathrm{pr}_m^{-1}(\mathrm{sp}^{-1}(\mathscr{S}_{K^p,v}^{[0]})^\circ)_{\overline{F}_v}$.

Now we use the Hochschild-Serre spectral sequence (see [Far04, Théorème 4.5.12])

$$E_2^{r,s} = \varinjlim_m \mathrm{Ext}_{J(\mathbb{Q}_p)\text{-smooth}}^r \big(H_c^{2(n-1)-s}(\mathcal{M}_{K_{p,m}}, \overline{\mathbb{Q}}_\ell)(n-1), \mathcal{A}(I)_{\mathbf{1}}^{K^p}\big)$$
$$\Rightarrow \varinjlim_m H^{r+s}\big(\mathscr{S}_{m,K^p,\overline{\eta}}(b), \overline{\mathbb{Q}}_\ell\big),$$

where $\mathcal{M}_{K_{p,m}}$ is the Rapoport-Zink space of level $K_{p,m}$, and $\mathcal{A}(I)_{\mathbf{1}}$ is the space of automorphic forms on $I(\mathbb{A}^\infty)$ (see [Far04, Définition 4.5.8] for detail). Since $J(\mathbb{Q}_p) = D^\times \times \mathbb{Q}_p^\times$ is anisotropic modulo center, it is easy to see that $E_2^{r,s} = 0$ unless $r = 0$. If $r = 0$, we have

$$E_2^{0,s} = \varinjlim_m \mathrm{Hom}_{J(\mathbb{Q}_p)}\big(H_c^{2(n-1)-s}(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)(n-1), \mathcal{A}(I)_{\mathbf{1}}^{K^p}\big)^{K_{p,m}},$$

where we put $H_c^i(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell) = \varinjlim_m H_c^i(\mathcal{M}_{K_{p,m}}, \overline{\mathbb{Q}}_\ell)$.

By [Mie10], the $G(\mathbb{Q}_p)$-representation $H_c^{2(n-1)-s}(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)(n-1)$ has non-zero supercuspidal part only if $s = n - 1$. Indeed, for an irreducible supercuspidal representation $\pi = \pi_1 \otimes \chi$ of $G(\mathbb{Q}_p) = \mathrm{GL}_n(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$, where π_1 is an irreducible supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ and χ is a character of $\mathrm{GL}_1(\mathbb{Q}_p)$, we have

$$H_c^i(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)[\pi] = H_c^i(\mathcal{M}_{\mathrm{LT},\infty}, \overline{\mathbb{Q}}_\ell)[\pi_1] \otimes \chi,$$

as we see in [Far04, p. 168]. Therefore $E_2^{0,s}$ has a supercuspidal subquotient only if $s = n - 1$.

Hence we conclude that

$$\varinjlim_m H^i_c\big(\mathscr{S}^{[0]}_{K^p,\overline{v}},(R\psi\,\mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}^{[0]}_{K^p,\overline{v}}}\big) \cong \varinjlim_m H^i\big(\mathscr{S}_{m,K^p,\overline{\eta}}(b),\overline{\mathbb{Q}}_\ell\big)$$

has non-zero supercuspidal part only if $i = n-1$. ∎

We also have a similar result for the torsion coefficient case. For a neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, we put

$$H^i_c(\mathrm{Sh}_{K^p},\overline{\mathbb{F}}_\ell) = \varinjlim_m H^i_c(\mathrm{Sh}_{K_{p,m}K^p}\otimes_F \overline{F},\overline{\mathbb{F}}_\ell).$$

It is an admissible/continuous $G(\mathbb{Q}_p)\times\mathrm{Gal}(\overline{F}/F)$-representation over $\overline{\mathbb{F}}_\ell$.

Theorem 7.9 *Let p be a prime in $\Sigma\setminus\{\ell\}$ and π an irreducible supercuspidal $\overline{\mathbb{F}}_\ell$-representation of $G(\mathbb{Q}_p)$. Then, for every neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, we have $H^i_c(\mathrm{Sh}_{K^p},\overline{\mathbb{F}}_\ell)[\pi]=0$ unless $i=n-1$.*

Remark 7.10 (i) Theorem 7.9 for proper Shimura varieties is due to Shin [Shi15]. His method, using Mantovan's formula, is slightly different from ours. The non-proper cases are also covered in his paper using results in our paper.

(ii) Using the result in [Dat12], it is possible to describe the action of $W_{\mathbb{Q}_p}$ on $H^{n-1}_c(\mathrm{Sh}_{K^p},\overline{\mathbb{F}}_\ell)[\pi]$ by means of the mod-ℓ local Langlands correspondence. Such study has also been carried out by Shin when the Shimura variety is proper.

Proof. Almost all arguments in the proof of Theorem 7.5 work well. The only one point which should be modified is about the vanishing of the supercuspidal part of $E_2^{r,s}$ for $(r,s)\neq(0,n-1)$ in the proof of Lemma 7.8; note that an irreducible $\overline{\mathbb{F}}_\ell$-representation of $J(\mathbb{Q}_p)$, being supercuspidal, is not necessarily injective in the category of smooth $\overline{\mathbb{F}}_\ell$-representations of $J(\mathbb{Q}_p)$ with the fixed central character. For this point, we can use the same argument as that by Shin (see [Shi15, §3.2]), in which he uses the vanishing of the supercuspidal part $H^i_c(\mathcal{M}_{\mathrm{LT},\infty},\overline{\mathbb{F}}_\ell)_{\mathrm{sc}}$ for $i\neq n-1$ (*cf.* [Dat12, proof of Proposition 3.1.1, Remarque 3.1.5]) and the projectivity of the $D^\times$-representation $H^{n-1}_c(\mathcal{M}_{\mathrm{LT},\infty},\overline{\mathbb{F}}_\ell)_{\mathrm{sc}}$ (*cf.* [Dat12, §3.2.2, Remarque iii)]). ∎

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