Introduction to p-adic uniformization of Shimura curves

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Abstract. This is a rough note of my talks in the study group of the Shimura curves at the University of Tokyo.

1 What is p-adic uniformization?

Here we will introduce the general theory of p-adic uniformization. First we begin with Tate's uniformization of elliptic curves over local fields, which is the starting point of the theory of p-adic uniformization. Next we treat uniformizations by the Drinfeld upper half space, which are developed by Mumford, Mustafin and Kurihara.

Throughout this section, let K be a p-adic field, i.e., a finite extension of $\mathbb{Q}_p$. We denote its p-adic absolute value by $|\cdot|\colon K \longrightarrow \mathbb{R}$, its ring of integers by $\mathcal{O}_K$, and its residue field by k. For a scheme X which is (separated) of finite type over K, denote the associated rigid analytic space by X^{an}. (Nowadays there exist several formulations of rigid geometry. However, the results in this article hardly depends on the choice of the formulation. The author regards rigid spaces as adic spaces in his mind.)

1.1 Tate's uniformization

Theorem 1.1 (Tate) *For $q \in K^\times$ with $|q| < 1$, there exists an elliptic curve E_q over K such that $\mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}} \cong E_q^{\mathrm{an}}$ as rigid analytic spaces over K. The elliptic curve E_q is given by the following explicit Weierstrass equation:*

$$y^2 + xy = x^3 + a_4(q)x + a_6(q),$$

where we put

$$s_k(q) = \sum_{n=1}^{\infty} \frac{n^k q^n}{1-q^n}, \quad a_4(q) = -5s_3(q), \quad a_6(q) = -\frac{5s_3(q) + 7s_5(q)}{12}.$$

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Set theoretically, the isomorphism $\mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}} \xrightarrow{\cong} E_q^{\mathrm{an}}$ *is given by a* $G_K = \mathrm{Gal}(\overline{K}/K)$*-equivariant map*

$$\overline{K}^{\times}/q^{\mathbb{Z}} \xrightarrow{\cong} E_q(\overline{K}); \quad u \longmapsto \big(X(u,q), Y(u,q)\big),$$

where we put

$$X(u,q) = \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2s_1(q), \quad Y(u,q) = \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + s_1(q).$$

For a proof of this theorem, see [Si] for example. Note that the isomorphism $\mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}} \xrightarrow{\cong} E_q^{\mathrm{an}}$ is given by p-adically convergent power series, not polynomials. That is one of the reason why we should pass to the category of rigid analytic spaces.

Let us compare with a uniformization of an elliptic curve over $\mathbb{C}$. Recall that every elliptic curve E over $\mathbb{C}$ is the quotient of $\mathbb{C}$ by some lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ with $\mathrm{Im}\,\tau > 0$. Then E is the quotient of $\mathbb{C}/\mathbb{Z}$ by the subgroup generated by the image of τ. On the other hand, $z \longmapsto \exp(2\pi i z)$ gives an isomorphism $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^{\times}$. Hence we get an isomorphism $E \cong \mathbb{C}/\Lambda \cong \mathbb{C}^{\times}/q^{\mathbb{Z}}$ with $q = \exp 2\pi i\tau$.

By this, it is obvious that the element q in the theorem is a p-adic analogue of the q appearing in the theory of elliptic modular functions. In fact, the discriminant of E_q is given by $q\prod_{n=1}^{\infty}(1-q^n)^{24}$ and the j-invariant of E_q is given by $q^{-1} + 744 + 196884q + \cdots$, which are similar to the case over $\mathbb{C}$.

Definition 1.2 Let E be an elliptic curve over K. If there exists an element $q \in K^{\times}$ with $|q| < 1$ such that $E \cong E_q$, we say that E has a p-adic uniformization by $\mathbb{G}_m$.

Later we will give a criterion whether E has a p-adic uniformization or not.

1.2 Uniformization by the Drinfeld upper half space

Let d be an integer which is greater than 1.

Definition 1.3 Let Ω_K^d be the space obtained by deleting all the K-rational hyperplanes from $\mathbb{P}_K^{d-1}$. This has a natural structure of $d-1$-dimensional rigid analytic space over K such that the natural action of $PGL_d(K)$ on Ω_K^d is rigid-analytic.

Example 1.4 We have $\Omega_K^2(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(K)$, where $\mathbb{C}_p$ is the completion of an algebraic closure of $\mathbb{Q}_p$. Compare with $\mathfrak{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$!

Now we will give a short explanation on the rigid analytic structure of Ω_K^2. Let $\mathbb{D}^1$ be a (open) unit disk $\{[z:1] \mid |z| \le 1\}$. Since $\mathbb{D}^1$ and its translation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

covers $\mathbb{P}^1$, it suffices to give a rigid analytic structure on $\mathbb{D}^1 \cap \Omega_K^2$. For a non-negative integer n, put

$$U_n(\mathbb{C}_p) = \left\{ z \in \mathbb{D}^1(\mathbb{C}_p) = \mathcal{O}_{\mathbb{C}_p} \;\middle|\; |z-a| \geq |\pi^n| \text{ for every } a \in \mathcal{O}_K \right\},$$

where π is a uniformizer of K and q is the cardinality of k, the residue field of K. Then $\Omega_K^2(\mathbb{C}_p)$ is the increasing union of $U_n(\mathbb{C}_p)$.

Take a system of representatives $\{a_1, \ldots, a_{q^{n+1}}\}$ of $\mathcal{O}_K/\pi^{n+1}\mathcal{O}_K$. Then it is easy to see

$$U_n(\mathbb{C}_p) = \left\{ z \in \mathbb{D}^1(\mathbb{C}_p) = \mathcal{O}_{\mathbb{C}_p} \;\middle|\; |z-a_i| \geq |\pi^n| \text{ for every } i \right\}.$$

This is the underlying set of a rational subset of $\mathbb{D}^1$, which is a typical example of rigid analytic spaces. Hence U_n comes to equip a rigid analytic structure, and by glueing, so does Ω_K^2.

Theorem 1.5 (Mumford ($d = 2$) [Mum], Mustafin [Mus], Kurihara [Ku]) *Let Γ be a discrete, cocompact and torsion-free subgroup of $PGL_d(K)$. Then there exists a projective smooth scheme over K such that $\Gamma \backslash \Omega_K^d \cong X_\Gamma^{\mathrm{an}}$.*

Definition 1.6 Let X be a projective smooth scheme over K. If there exists a discrete, cocompact and torsion-free subgroup Γ of $PGL_d(K)$ such that $X \cong X_\Gamma$, then we say that X has a p-adic uniformization by Ω_K^d.

Remark 1.7 Let Γ, Γ' be discrete, cocompact and torsion-free subgroups of $PGL_d(K)$. If $X_\Gamma \cong X_{\Gamma'}$, then Γ and Γ' are conjugate. Thus a p-adic uniformization of the given scheme X is essentially unique, if it exists.

Remark 1.8 If we consider a local field with positive characteristic, the Drinfeld modular varieties are also uniformized by Ω_K^d ([Dr2]). This should be the first motivation of introducing Ω_K^d by Drinfeld. Note that the Drinfeld modular varieties are not proper. Thus this uniformization differs from that in the above theorem.

2 Applications of p-adic uniformization

What is good if we have a p-adic uniformization? In this section, we will give 2 answers for this question:

i) We may calculate the (ℓ-adic) cohomology of uniformized varieties with Galois action.
ii) We may have a "good" integral models of uniformized varieties (in fact, the integral model whose existence is ensured does not have a good reduction but has a semistable reduction).

2.1 Cohomology

Here let ℓ be a prime number which is invertible in $\mathcal{O}_K$.

2.1.1 Tate's uniformization

First we will consider the case of Tate's uniformization. Take $q \in K^\times$ with $|q| < 1$ and consider the elliptic curve E_q. Let us calculate the action of G_K on the Tate module $T_\ell E_q = H^1_{\text{ét}}((E_q)_{\overline{K}}, \mathbb{Z}_\ell(1))$. Using the isomorphism $\mathbb{G}_m^{\text{an}}/q^{\mathbb{Z}} \cong E_q^{\text{an}}$, it is easy to compute $E_q[\ell^n]$ explicitly:

$$E_q[\ell^n] = \{x \in \overline{K}^\times/q^{\mathbb{Z}} \mid x^{\ell^n} = 0\} = \bigl\{\zeta_{\ell^n}^i (q^{1/\ell^n})^j \bigm| 0 \le i, j \le \ell^n - 1\bigr\},$$

where ζ_{ℓ^n} is a primitive ℓ^n-th root of unity and q^{1/ℓ^n} is a fixed ℓ^n-th root of q. As we know, this is isomorphic to $(\mathbb{Z}/\ell^n)^2$ as modules. Moreover, the subgroup μ_{ℓ^n} generated by ζ_{ℓ^n} is stable under G_K. It is easy to see that G_K acts trivially on the quotient $E_q[\ell^n]/\mu_n$. Taking the inverse limit, we have the following:

Proposition 2.1 *As G_K-module, $T_\ell E_q$ is an extension of $\mathbb{Z}_\ell$ (with trivial G_K-action) by $\mathbb{Z}_\ell(1) = \varprojlim_n \mu_{\ell^n}$.*

Let I_K be the inertia subgroup of G_K, that is, the subgroup of G_K consisting of elements which act trivially on k, the residue field of K. As I_K-module, we can deduce more precise structure of $T_\ell E_q$. Let us fix a system $\underline{\xi} = (\xi_n)$ of elements in $\overline{K}^\times$ satisfying $\xi_0 = q$ and $\xi_{i+1}^\ell = \xi_i$. Then, for $\sigma \in I_K$, there exists a unique element $t_\ell(\sigma)$ of $\mathbb{Z}_\ell(1)$ such that $\sigma(\underline{\xi}) = t_\ell(\sigma)\underline{\xi}$. It is easy to see that $t_\ell(\sigma)$ is independent of the choice of $\underline{\xi}$ (this is the reason why we restrict ourselves to consider I_K) and that $t_\ell \colon I_K \longrightarrow \mathbb{Z}_\ell(1)$ gives a homomorphism. This t_ℓ is called the ℓ-adic tame character.

Now we can prove the following result easily:

Proposition 2.2 *Let $\underline{\varepsilon}$ be a topological generator of $\mathbb{Z}_\ell(1)$. Then $\underline{\varepsilon}$ and $\underline{\xi}$ gives a $\mathbb{Z}_\ell$-basis of $T_\ell E_q$. Under this basis, the action of $\sigma \in I_K$ is given by the matrix $\begin{pmatrix} 1 & t_\ell(\sigma) \\ 0 & 1 \end{pmatrix}$.*

Note that this proposition is an easy case of the Picard-Lefschetz theorem.

2.1.2 Uniformization by Ω_K^d

Next we pass to the case of uniformizations by Ω_K^d, which is more complicated and interesting. There is a calculation of $H^i_{\text{ét}}((\Omega_K^d)_{\overline{K}}, \mathbb{Q}_\ell)$ with G_K-action:

Theorem 2.3 (Schneider-Stuhler, dual form [Sc-St], [Da]) *We have an isomorphism*

$$H^i_{\text{ét},c}\bigl((\Omega_K^d)_{\overline{K}}, \mathbb{Q}_\ell\bigr) \cong \pi_{\{1,\ldots,i\}}(-i)$$

as $GL_d(K) \times G_K$-modules.

Here $\pi_{\{1,\ldots,i\}}$ is the unique irreducible quotient of $\operatorname{Ind}_{P_{\{1,\ldots,i\}}}^{GL_d(K)} \mathbf{1}$, where $P_{\{1,\ldots,i\}}$ is the parabolic subgroup $\{(a_{jk}) \mid a_{jk} = 0$ if $j > k$ and $j > i\}$ of $GL_d(K)$. Note that if $i = 0$ then $\pi_{\{1,\ldots,i\}} = \mathrm{St}$, the Steinberg representation of $GL_d(K)$.

Combining with the Hochschild-Serre spectral sequence and the comparison result, we have the following corollary:

Corollary 2.4 *For a discrete cocompact and torsion-free subgroup* Γ *of* $PGL_d(K)$*, we have*

$$H^i_{\text{ét}}\big((X_\Gamma)_{\overline{K}}, \mathbb{Q}_\ell\big) \cong \begin{cases} \mathbb{Q}_\ell(-\frac{i}{2}) & (i\text{: even, } 0 \le i \le 2(d-1),\ i \ne d-1), \\ 0 & (i\text{: odd, } i \ne d-1). \end{cases}$$

Moreover, we have a filtration

$$H^{d-1}_{\text{ét}}\big((X_\Gamma)_{\overline{K}}, \mathbb{Q}_\ell\big) = V = F^0V \supset F^1V \supset \cdots \supset F^{d-1}V \supset F^dV = 0$$

such that

$$F^rV/F^{r+1}V \cong \begin{cases} \mathbb{Q}_\ell(r-d+1)^{\mu(\Gamma)} & (0 \le r \le d-1, r \ne \frac{d-1}{2}), \\ \mathbb{Q}_\ell(-\frac{d-1}{2})^{\mu(\Gamma)+1} & (r = \frac{d-1}{2}). \end{cases}$$

Here $\mu(\Gamma)$ *is the multiplicity of* St *in* $\mathrm{Ind}_\Gamma^{PGL_d(K)} \mathbf{1}$.

2.2 Existence of a good integral model

2.2.1 Raynaud generic fiber

In order to explain the results on the existence of good integral models of uniformized varieties, we will recall the functor associating formal schemes with rigid analytic spaces. Basic references are [Ra] and [BL]. Let $\mathfrak{X}$ be a formal scheme (separated) locally of finite type over $\operatorname{Spf} \mathcal{O}_K$. Then we may associate $\mathfrak{X}$ with the rigid analytic space $\mathfrak{X}^{\mathrm{rig}}$ over K called the Raynaud generic fiber of $\mathfrak{X}$. We list the properties of $\mathfrak{X} \longmapsto \mathfrak{X}^{\mathrm{rig}}$:

i) For a finite extension L of K, the set $\mathfrak{X}^{\mathrm{rig}}(L)$ is equal to the set consisting of morphisms $\operatorname{Spf} \mathcal{O}_L \longrightarrow \mathfrak{X}$ of formal schemes over $\operatorname{Spf} \mathcal{O}_K$.

ii) For a scheme X over $\mathcal{O}_K$, we denote by $X^\wedge$ the completion of X along its special fiber and by X_K its generic fiber. If X is proper, then we have an isomorphism $(X^\wedge)^{\mathrm{rig}} \cong X_K^{\mathrm{an}}$.

By i), we can define the map $\mathrm{sp}_{\mathfrak{X}} \colon \mathfrak{X}^{\mathrm{rig}} \longrightarrow \mathfrak{X}$ called the specialization map. Indeed, for $x \in \mathfrak{X}^{\mathrm{rig}}$, we define $\mathrm{sp}_{\mathfrak{X}}(x)$ as the image of the unique point of $\operatorname{Spf} \mathcal{O}_L$ under the morphism $\operatorname{Spf} \mathcal{O}_L \longrightarrow \mathfrak{X}$ corresponding to x. Unless we use the formulation by Berkovich, the map $\mathrm{sp}_{\mathfrak{X}}$ is continuous.

Note that the property ii) is a consequence of i) and the valuative criterion for properness. Indeed, every K-morphism $\operatorname{Spec} L \longrightarrow X_K$ automatically extends to an $\mathcal{O}_K$-morphism $\operatorname{Spf} \mathcal{O}_L \longrightarrow X^\wedge$.

Example 2.5 Let $\mathcal{O}_K\langle T\rangle$ be the ring consisting of convergent power series with $\mathcal{O}_K$-coefficients. Then for a finite extension L of K, we have $\mathfrak{X}^{\mathrm{rig}}(L) = \mathcal{O}_L$. Thus $\mathfrak{X}^{\mathrm{rig}}$ is nothing but $\mathbb{D}^1$. Note that $\mathfrak{X}$ is the completion of $\mathbb{A}^1_{\mathcal{O}_K}$ along its special fiber. Therefore the property ii) above is not true for non-proper X.

Definition 2.6 For a rigid analytic space X over K, a formal scheme $\mathfrak{X}$ locally of finite type over $\operatorname{Spf}\mathcal{O}_K$ satisfying $\mathfrak{X}^{\mathrm{rig}} \cong X$ is called a formal model of X.

Remark 2.7 Every quasi-compact rigid analytic space has a formal model.

2.2.2 Tate's uniformization

Here we will construct a "good" formal model $\mathfrak{X}$ of $\mathbb{G}_m^{\mathrm{an}}$ and see that we may take the quotient (as a formal scheme) of $\mathfrak{X}$ by $q^{\mathbb{Z}}$. Then $\mathfrak{X}/q^{\mathbb{Z}}$ should give a formal model of E_q^{an}, and we can get an integral model of E_q by algebraizing $\mathfrak{X}/q^{\mathbb{Z}}$.

Let us begin with the blow-up Y of $\mathbb{A}^1_{\mathcal{O}_K}$ at $0 \in \mathbb{A}^1_k$. The special fiber of Y is the union of $\mathbb{A}^1$ and $\mathbb{P}^1$, which intersect transversally at one point. Put $\mathfrak{Y} = Y^\wedge$. Then, since $Y \longrightarrow \mathbb{A}^1_{\mathcal{O}_K}$ is proper, we have $\mathfrak{Y}^{\mathrm{rig}} = \mathbb{D}^1_K$. Let us consider the specialization map $\mathrm{sp}_{\mathfrak{Y}}\colon \mathbb{D}^1_K(\mathbb{C}_p) \longrightarrow Y_k$. The scheme Y_k is decomposed into three parts: the node P, $\mathbb{A}^1 \setminus \{P\}$ and $\mathbb{P}^1 \setminus \{P\}$. The inverse images of these subspaces are the following:

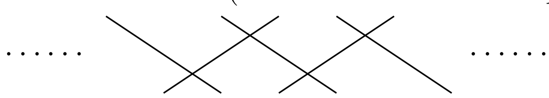
$$\mathrm{sp}_{\mathfrak{Y}}^{-1}(P) = \bigl\{z \in \mathbb{C}_p \bigm| |\pi| < |z| < 1\bigr\}, \qquad \mathrm{sp}_{\mathfrak{Y}}^{-1}\bigl(\mathbb{A}^1 \setminus \{P\}\bigr) = \bigl\{z \in \mathbb{C}_p \bigm| |z| = 1\bigr\},$$
$$\mathrm{sp}_{\mathfrak{Y}}^{-1}\bigl(\mathbb{P}^1 \setminus \{P\}\bigr) = \bigl\{z \in \mathbb{C}_p \bigm| |z| \le |\pi|\bigr\}.$$

Moreover, under a suitable coordinate of Y, the restriction of $\mathrm{sp}_{\mathfrak{Y}}$ on $\mathrm{sp}_{\mathfrak{Y}}^{-1}(\mathbb{A}^1 \setminus \{P\})$ (resp. on $\mathrm{sp}_{\mathfrak{Y}}^{-1}(\mathbb{P}^1 \setminus \{P\})$) is given by $z \longmapsto z \bmod \pi$ (resp. $z \longmapsto z/\pi \bmod \pi$). In particular, the inverse image of $Q = \mathrm{sp}_{\mathfrak{Y}}(0)$ by $\mathrm{sp}_{\mathfrak{Y}}$ is equal to $\{z \in \mathbb{C}_p \mid |z| < |\pi|\}$. Thus if we remove Q and get a formal scheme $\mathfrak{Y}'$, then $(\mathfrak{Y}')^{\mathrm{rig}}(\mathbb{C}_p)$ is equal to $\{z \in \mathbb{D}^1(\mathbb{C}_p) \mid |\pi| \le |z| \le 1\}$. In other words, $(\mathfrak{Y}')^{\mathrm{rig}}$ is an annulus over K whose inner (resp. outer) radius is $|\pi|$ (resp. 1).

On the other hand, $\mathbb{G}_m^{\mathrm{an}}$ is a union of infinitely many annuli:

$$\mathbb{G}_m^{\mathrm{an}}(\mathbb{C}_p) = \mathbb{C}_p^\times = \bigcup_{n \in \mathbb{Z}} \bigl\{z \in \mathbb{C}_p \bigm| |\pi^{n+1}| \le |z| \le |\pi^n|\bigr\}.$$

Therefore we may have a formal model $\mathfrak{X}$ of $\mathbb{G}_m^{\mathrm{an}}$ by glueing infinitely many $\mathfrak{Y}'$. The picture of its special fiber is as follows (all the irreducible components are $\mathbb{P}^1$):



Moreover, an element q of $K^\times$ acts on $\mathfrak{X}$ and maps the i-th irreducible component to $(i + v_K(q))$-th irreducible component. If $|q| < 1$, this action is obviously free and we may take the quotient $\mathfrak{X}/q^{\mathbb{Z}}$ as a formal scheme. The Raynaud generic fiber of $\mathfrak{X}/q^{\mathbb{Z}}$ is isomorphic to E_q^{an}. The formal scheme $\mathfrak{X}/q^{\mathbb{Z}}$ is semistable over $\mathcal{O}_K$, since $\mathfrak{X}$ is (strictly) semistable over $\mathcal{O}_K$. Moreover, $(\mathfrak{X}/q^{\mathbb{Z}})^{\mathrm{red}}$ is a projective curve over k. Therefore, by a usual technique, we may prove that $\mathfrak{X}/q^{\mathbb{Z}}$ is algebraizable, that is, there exists a projective scheme X over $\mathcal{O}_K$ such that $X^\wedge \cong \mathfrak{X}/q^{\mathbb{Z}}$. This X gives a semistable integral model of E_q.

Corollary 2.8 *The elliptic curve E_q has a split multiplicative reduction.*

Remark 2.9 Conversely, if an elliptic curve E over K has a split multiplicative reduction, then $E \cong E_q$ for some $q \in K^\times$ with $|q| < 1$.

2.2.3 Uniformization by Ω^d_K

This case is similar to the case of Tate's uniformization. There exists a (strictly) semistable formal model $\widehat{\Omega}^d_K$ of Ω^d_K, discovered by Deligne, such that $PGL_d(K)$ acts on $\widehat{\Omega}^d_K$ and this action induces the natural action on Ω^d_K. Here we skip the definition of $\widehat{\Omega}^d_K$ and only list basic properties of it. (Later we will see a moduli interpretation of $\widehat{\Omega}^d_K$. Thus we may consider that interpretation as a definition. All the following properties can be derived by the moduli interpretation.)

- The formal scheme $\widehat{\Omega}^d_K$ is strictly semistable over $\mathcal{O}_K$.
- The configuration of the irreducible components of $(\widehat{\Omega}^d_K)^{\mathrm{red}}$ is given by the Bruhat-Tits building of $PGL_d(K)$.
- Each irreducible component of $(\widehat{\Omega}^d_K)^{\mathrm{red}}$ is given as follows. Put $B_0 = \mathbb{P}^{d-1}_k$. Let B_1 be the blow-up along all k-rational points of $B_0 = \mathbb{P}^{d-1}_k$. Let B_2 be the blow-up along the strict transform of all k-rational lines of $\mathbb{P}^{d-1}_k$ by $B_1 \longrightarrow \mathbb{P}^{d-1}_k$. We define $B_3, \ldots, B_{d-1}$ inductively (B_{i+1} is the blow-up along the strict transform of all k-rational i-dimensional linear subspaces of $\mathbb{P}^{d-1}_k$ by $B_i \longrightarrow \mathbb{P}^{d-1}_k$). Every irreducible component is isomorphic to B_{d-1}. In particular, it is projective over k and rational.

For a discrete cocompact and torsion-free subgroup Γ of $PGL_d(K)$, we may take the quotient $\Gamma\backslash\widehat{\Omega}^d_K$. The obtained formal scheme is in fact algebraizable (we may prove that the relative dualizing sheaf is ample; thus X_Γ is a variety of general type). Therefore we have a semistable integral model of X_Γ as in the previous case.

Corollary 2.10 *If a projective smooth scheme X over K admits a p-adic uniformization by Ω^d_K, then there exist a semistable model $\mathcal{X}$ of X, a strictly semistable scheme $\mathcal{X}'$ over $\mathcal{O}_K$ such that every irreducible component of $\mathcal{X}'_k$ is isomorphic to B_{d-1}, and a finite étale $\mathcal{O}_K$-morphism $\mathcal{X}' \longrightarrow \mathcal{X}$.*

Remark 2.11 For the case $d = 2$, we have the following result of converse direction, due to Mumford. Let X is a proper smooth curve over K whose genus is greater than or equal to 2. If X has a strictly semistable model $\mathcal{X}$ over $\mathcal{O}_K$ such that every irreducible component of $\mathcal{X}_k$ is $\mathbb{P}^1$, then X has a p-adic uniformization by Ω^2_K.

3 Statement of the main theorem

3.1 Weil descent datum

Let K be a p-adic field as above, and denote by $\widehat{K}^{\mathrm{ur}}$ the completion of the maximal unramified extension of K. Let $\sigma \in \mathrm{Gal}(\widehat{K}^{\mathrm{ur}}/K)$ be the (arithmetic) Frobenius element, namely, the element that induces the q-th power map on the residue field k (here we put $q = \#k$).

Definition 3.1 Let X be a formal scheme over $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$. A Weil descent datum for X is a morphism of formal schemes $\alpha\colon X \longrightarrow X$ which makes the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ \operatorname{Spf} \mathcal{O}_{\widehat{K}^{\mathrm{ur}}} \arrow[r, "\sigma^*"] & \operatorname{Spf} \mathcal{O}_{\widehat{K}^{\mathrm{ur}}} \end{tikzcd}$$

(equivalently, a morphism $\alpha\colon X \longrightarrow \sigma_*X = X\hat{\otimes}_{\mathcal{O}_{\widehat{K}^{\mathrm{ur}}},\sigma}\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$ over $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$). There is a natural functor from the category of formal schemes over $\mathcal{O}_K$ to the category of pairs (X, α) where X is a formal scheme over $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$ and α is a Weil descent datum for X:

$$Y \longmapsto (Y \hat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}, \mathrm{id}_Y \otimes \sigma^*).$$

It is easy to see that this functor is fully faithful. If a pair (X, α) is contained in the essential image of this functor, the Weil descent datum α is said to be effective.

If α is a Weil descent datum over K, then α^r is a Weil descent datum over the degree r unramified extension of K. The following lemma is an easy consequence of étale descent.

Lemma 3.2 *If α^r is effective, then so is α.*

Definition 3.3 Put $\breve{\mathcal{M}}^{\mathrm{Dr},d} = (\widehat{\Omega}^d_K \hat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}) \times \mathbb{Z}$. This is a formal scheme over $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$. Let α be the Weil descent datum for $\breve{\mathcal{M}}^{\mathrm{Dr},d}$ defined as follows:

- On the first factor, α is the canonical one ($\widehat{\Omega}^d_K \hat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$ comes from $\mathcal{O}_K$).
- On the second factor, α is the map $+1$.

We also denote $(\breve{\mathcal{M}}^{\mathrm{Dr},d}, \alpha)$ by $\breve{\mathcal{M}}^{\mathrm{Dr},d}$. The group $GL_d(K)$ acts on $\breve{\mathcal{M}}^{\mathrm{Dr},d}$: on the first factor $g \in GL_d(K)$ acts in the same way as $\widehat{\Omega}^d_K$, and on the second factor $g \in GL_d(K)$ acts as $-v_K(\det g)$.

3.2 Statement of p-adic uniformization of Shimura curves over $\mathbb{Q}$

In this subsection, K does not denote a local field. Let B be an indefinite quaternion division algebra over $\mathbb{Q}$ which ramifies at p. We take a maximal order $\mathcal{O}_B$ of B and put $\mathcal{O}_{B_p} = \mathcal{O}_B \otimes \mathbb{Z}_p$. This is the maximal order of B_p. Let $K \subset (\mathcal{O}_B \otimes \widehat{\mathbb{Z}})^\times \subset B^\times(\mathbb{A}_f)$ be a compact open subgroup. Assume that K is decomposed into a product $K = K_p^0 \cdot K^p$, where $K_p^0 = \mathcal{O}_{B_p}^\times \subset B^\times(\mathbb{Q}_p)$ and K^p is a compact open subgroup of $B^\times(\mathbb{A}_f^p)$ ($\mathbb{A}_f^p$ denotes the finite adèle ring without p-th component).

Let us consider the Shimura curve $\mathbf{S}_K$ associated with $B^\times$ and its integral model $\mathcal{S}_K$ over $\mathbb{Z}_p$. The integral model $\mathcal{S}_K$ is given by the moduli functor associating a scheme S over $\mathbb{Z}_p$ to the set of triples $(A, \iota, \overline{\nu})$ such that:

- A is an abelian surface over S.

- $\iota\colon \mathcal{O}_B \longrightarrow \mathrm{End}(A)$ is a ring homomorphism satisfying the Kottwitz (or determinant) condition. Recall that this condition is equivalent to the following:

  Consider the action of $\mathbb{Z}_{p^2} \subset \mathcal{O}_{B_p}$ on $\mathrm{Lie}(A)$. If S is a $\mathbb{Z}_{p^2}$-scheme, then there is a canonical decomposition $\mathrm{Lie}(A) \cong \mathrm{Lie}^0(A) \oplus \mathrm{Lie}^1(A)$ such that the action of $\mathbb{Z}_{p^2}$ on $\mathrm{Lie}^0(A)$ factors through the structure map $\mathbb{Z}_{p^2} \longrightarrow \mathcal{O}_S$ and that on $\mathrm{Lie}^1(A)$ factors through $\mathbb{Z}_{p^2} \xrightarrow{\sigma} \mathbb{Z}_{p^2} \longrightarrow \mathcal{O}_S$.

  The condition is that $\mathrm{Lie}^0(A)$ and $\mathrm{Lie}^1(A)$ are projective $\mathcal{O}_S$-modules of rank one.
- $\overline{\nu}$ is a K^p-equivalence class of isomorphisms $\nu\colon T_f^pA := \prod_{\ell\neq p} T_\ell A \xrightarrow{\cong} \mathcal{O}_B \otimes \widehat{\mathbb{Z}}^p$ of $\mathcal{O}_B$-modules. If we take a positive integer N prime to p such that $K(N) := \mathrm{Ker}((\mathcal{O}_B \otimes \widehat{\mathbb{Z}})^\times \longrightarrow (\mathcal{O}_B \otimes \mathbb{Z}/N\mathbb{Z})^\times) \subset K$, then to give $\overline{\nu}$ is equivalent to give a $K/K(N)$-equivalence class of isomorphisms $A[N] \xrightarrow{\cong} \mathcal{O}_B \otimes \mathbb{Z}/N\mathbb{Z}$ of $\mathcal{O}_B$-modules.

If K^p is enough small, this functor is represented by a projective scheme $\mathcal{S}_K$ over $\mathbb{Z}_p$.

Let $\overline{B}$ be the quaternion algebra over $\mathbb{Q}$ satisfying $\mathrm{ram}\,\overline{B} = \mathrm{ram}\,B \setminus \{p\} \cup \{\infty\}$. Note that if (A,ι) is a pair as above, then $\mathrm{End}_{\mathcal{O}_B}(A) \otimes \mathbb{Q} \cong \overline{B}$. Fix an isomorphism $B^\times(\mathbb{A}_f^p) \cong \overline{B}^\times(\mathbb{A}_f^p)$ and identify them. Now we can state our main theorem.

Theorem 3.4 (p-adic uniformization of Shimura curves over $\mathbb{Q}$) *If K^p is small, then we have an isomorphism of formal schemes over $\mathcal{O}_{\widehat{K}^{\mathrm{ur}}}$ with Weil descent data:*

$$\mathcal{S}_K^\wedge \cong \overline{B}^\times(\mathbb{Q})\backslash \breve{\mathcal{M}}^{\mathrm{Dr},2} \times B^\times(\mathbb{A}_f^p)/K^p.$$

The action of $\overline{B}^\times(\mathbb{Q})$ on $\breve{\mathcal{M}}^{\mathrm{Dr},2}$ is induced from the action of $GL_2(\mathbb{Q}_p)$ on $\breve{\mathcal{M}}^{\mathrm{Dr},2}$ by the map $\overline{B}^\times(\mathbb{Q}) \hookrightarrow \overline{B}^\times(\mathbb{Q}_p) = GL_2(\mathbb{Q}_p)$. The action of $\overline{B}^\times(\mathbb{Q})$ on $B^\times(\mathbb{A}_f^p) = \overline{B}^\times(\mathbb{A}_f^p)$ is given by the multiplication from the left. The group K^p acts on $\breve{\mathcal{M}}^{\mathrm{Dr},2}$ trivially, and acts on $B^\times(\mathbb{A}_f^p)$ by the multiplication from the right.

First this theorem was proved by Čerednik ([Če]). After that, Drinfeld discovered more natural proof. In this article, we will follow the proof by Drinfeld.

Remark 3.5 Compare with the uniformization of $\mathbf{S}_K$ over $\mathbb{C}$:

$$\mathbf{S}_K(\mathbb{C}) \cong B^\times(\mathbb{Q})\backslash(\mathbb{C}\setminus\mathbb{R}) \times B^\times(\mathbb{A}_f)/K.$$

Remark 3.6 We have the following isomorphism:

$$\overline{B}^\times(\mathbb{Q})\backslash \breve{\mathcal{M}}^{\mathrm{Dr},2} \times B^\times(\mathbb{A}_f^p)/K^p \cong \overline{B}^\times(\mathbb{Q}_p)\backslash \breve{\mathcal{M}}^{\mathrm{Dr},2} \times \bigl(K^p\backslash B^\times(\mathbb{A}_f)/\overline{B}^\times(\mathbb{Q})\bigr);$$
$$(x,g) \longmapsto \bigl(x,(1,g^{-1})\bigr), \quad \bigl(g_p^{-1}x,(g^p)^{-1}\bigr) \longleftarrow \bigl(x,(g_p,g^p)\bigr).$$

Actually we will prove an isomorphism $\mathcal{S}_K^\wedge \cong GL_2(\mathbb{Q}_p)\backslash\breve{\mathcal{M}}^{\mathrm{Dr},2} \times (K^p\backslash B^\times(\mathbb{A}_f)/\overline{B}^\times(\mathbb{Q}))$.

We put $Z_K = K^p\backslash B^\times(\mathbb{A}_f)/\overline{B}^\times(\mathbb{Q})$. The orbit of the action of $GL_2(\mathbb{Q}_p)$ on Z_K is finite, and for every $x \in Z_K$, the stabilizer Γ_x of x is given by $x^{-1}K^px \cap \overline{B}^\times(\mathbb{Q})$. This is a discrete cocompact subgroup of $GL_2(\mathbb{Q}_p)$ and is torsion-free if K^p is small. Therefore we have an isomorphism

$$\mathcal{S}_K^\wedge \cong \coprod_{x\in GL_2(\mathbb{Q}_p)\backslash Z_K} \Gamma_x\backslash\breve{\mathcal{M}}^{\mathrm{Dr},2}.$$

For every x, there exists a positive integer n such that $\begin{pmatrix} p^n & 0 \\ 0 & p^n \end{pmatrix} \in \Gamma_x$. If we denote by α' the induced Weil descent datum for $\Gamma_x\backslash\breve{\mathcal{M}}^{\mathrm{Dr},2}$, then $(\alpha')^{2n}$ is clearly effective. By Lemma 3.2, α' is also effective. Therefore we may regard $\Gamma_x\backslash\breve{\mathcal{M}}^{\mathrm{Dr},2}$ as a formal scheme over $\mathbb{Z}_p$. It is easy to see that $(\Gamma_x\backslash\breve{\mathcal{M}}^{\mathrm{Dr},2})\otimes_{\mathbb{Z}_p}\mathbb{Z}_{p^{2n}}$ is a quotient of $(\widehat{\Omega}^2_{\mathbb{Q}_p}\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_{p^{2n}})\times\mathbb{Z}/2n\mathbb{Z}$ (this is a direct consequence of the proof of effectivity of α').

By the remark above, we have the following corollary:

Corollary 3.7 *For a compact open subgroup $K = K_p^0 \cdot K^p$ of $B^\times(\mathbb{A}_f)$, the scheme $\mathcal{S}_K$ is semistable over $\mathbb{Z}_p$ as long as $\mathcal{S}_K$ is representable.*

4 Moduli interpretation of $\breve{\mathcal{M}}^{\mathrm{Dr},d}$ — Drinfeld's theorem

In his paper [Dr1], Drinfeld discovered a moduli interpretation of the formal scheme $\breve{\mathcal{M}}^{\mathrm{Dr},d}$. The goal of this section is to explain his result.

In this section, again K denotes a p-adic field, not a compact open subgroup. Needless to say, the formal scheme $\breve{\mathcal{M}}^{\mathrm{Dr},d}$ is the one constructed from $\widehat{\Omega}^d_K$. We fix a uniformizer π of K. Let D be the central division algebra over K whose invariant is equal to $1/d$. Denote the maximal order of D by $\mathcal{O}_D$.

Recall an explicit construction of D and $\mathcal{O}_D$. Let K_d be the degree d unramified extension of K and $\mathcal{O}_{K_d}[\Pi]$ be the $\mathcal{O}_D$-module generated freely by $1, \Pi, \Pi^2, \ldots$. By the relation $\Pi a = \sigma(a)\Pi$ for $a \in \mathcal{O}_{K_d}$ (here $\sigma \in \mathrm{Gal}(K_d/K)$ be the Frobenius element), $\mathcal{O}_{K_d}[\Pi]$ comes to equip a structure of $\mathcal{O}_K$-algebra. Then $\mathcal{O}_D = \mathcal{O}_{K_d}[\Pi]/(\Pi^d - \pi)$. Similarly, $D = K_d[\Pi]/(\Pi^d - \pi)$.

Put $E = \widehat{K}^{\mathrm{ur}}$ and denote by $\overline{k}$ the residue field of E.

Definition 4.1 Let S be a scheme over $\mathcal{O}_E$. A special formal $\mathcal{O}_D$-module over S is a pair (X, ι) consisting of a formal group X over S and a ring homomorphism $\iota\colon \mathcal{O}_D \longrightarrow \mathrm{End}(X)$ satisfying the following conditions.

- The action of $\mathcal{O}_K$ on $\mathrm{Lie}\,X$ induced by ι coincides with that induced by $\mathcal{O}_K \hookrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_S$ (note that $\mathrm{Lie}\,X$ is an $\mathcal{O}_S$-module). Namely, the pair $(X, \iota|_{\mathcal{O}_K})$ is a formal $\mathcal{O}_K$-module over S.

– Using the action of $\mathcal{O}_{K_d} \subset \mathcal{O}_D$ on $\operatorname{Lie} X$ by ι, we can find a natural decomposition of $\operatorname{Lie} X$ as an $\mathcal{O}_S$-module: $\operatorname{Lie} X = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \operatorname{Lie}^i X$. Here $\operatorname{Lie}^i X$ is the maximal $\mathcal{O}_S$-submodule of $\operatorname{Lie} X$ where the action of $\mathcal{O}_{K_d}$ coincides with that induced by $\mathcal{O}_{K_d} \xrightarrow{\sigma^i} \mathcal{O}_{K_d} \hookrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_S$. We require that each $\operatorname{Lie}^i X$ should be a locally free $\mathcal{O}_S$-module of rank 1 (the Kottwitz condition).

Note that every formal $\mathcal{O}_D$-module is d-dimensional.

Proposition 4.2 i) *For every special formal $\mathcal{O}_S$-module over S, the $\mathcal{O}_K$-height of X is a multiple of d^2. Recall that for a formal $\mathcal{O}_K$-module X, the $\mathcal{O}_K$-height of X is the integer h such that the degree of the map $[\pi]\colon X \longrightarrow X$ is equal to q^h.*

ii) *A special formal $\mathcal{O}_D$-module of $\mathcal{O}_K$-height d^2 over $\overline{k}$ is unique up to $\mathcal{O}_D$-isogeny. We take and fix one $\mathbb{X}$ of them.*

iii) $\operatorname{End}_{\mathcal{O}_D}(\mathbb{X}) \otimes \mathbb{Q} \cong M_d(K)$.

Although this proposition is an easy consequence of the Dieudonné (or Cartier) theory, we will skip the proof.

Definition 4.3 Let $\mathrm{Nilp}_{\mathcal{O}_E}$ be the category of $\mathcal{O}_E$-schemes such that π is locally nilpotent on them. Let G be the functor from $\mathrm{Nilp}_{\mathcal{O}_E}$ to the category of sets which maps S to the set of pairs (X, ρ) such that:

– X is a special formal $\mathcal{O}_D$-module over S whose $\mathcal{O}_K$-height is d^2,

– $\rho\colon \mathbb{X} \times_{\operatorname{Spec} \overline{k}} \overline{S} \longrightarrow X \times_S \overline{S}$ is an $\mathcal{O}_D$-quasi-isogeny. Here we put $\overline{S} = S \otimes_{\mathcal{O}_E} \overline{k}$. Recall that a quasi-isogeny $X \longrightarrow X'$ between formal groups over S is a global section f of the sheaf $\underline{\operatorname{Hom}}_S(X, X') \otimes \mathbb{Q}$ such that Zariski locally there exists an integer n such that $p^n f$ is an isogeny.

Theorem 4.4 (Drinfeld, [Dr1]) *The functor G is represented by $\breve{\mathcal{M}}^{\mathrm{Dr},d}$. Furthermore, we may describe the action of $GL_d(K)$ on $\breve{\mathcal{M}}^{\mathrm{Dr},d}$, a natural map $\breve{\mathcal{M}}^{\mathrm{Dr},d} \longrightarrow \mathbb{Z}$, and the Weil descent datum of $\breve{\mathcal{M}}^{\mathrm{Dr},d}$ in terms of G as follows:*

– *for $g \in GL_d(K)$, we have $g(X, \rho) = (X, \rho \circ g^{-1})$,*

– *the morphism $\breve{\mathcal{M}}^{\mathrm{Dr},d}(S) \longrightarrow \mathbb{Z}$ maps (X, ρ) to $\deg \rho$,*

– *the Weil descent datum $\alpha\colon \breve{\mathcal{M}}^{\mathrm{Dr},d}(S) \longrightarrow \breve{\mathcal{M}}^{\mathrm{Dr},d}({}^{\sigma}S)$ is given by $(X, \rho) \longmapsto (X, \rho \circ \mathrm{Frob}^{-1})$. Here ${}^{\sigma}S$ is an $\mathcal{O}_E$-scheme such that it is equal to S as a scheme and the structure map is given by $S \longrightarrow \operatorname{Spec} \mathcal{O}_E \xrightarrow{\sigma^*} \operatorname{Spec} \mathcal{O}_E$. The quasi-isogeny $\rho \circ \mathrm{Frob}^{-1}$ is a composite of*

$$\mathbb{X} \times_{\operatorname{Spec} \overline{k} \swarrow \sigma^*} \overline{S} = (\sigma^*)^* \mathbb{X} \times_{\operatorname{Spec} \overline{k}} \overline{S} \xrightarrow{\mathrm{Frob}^{-1} \times \mathrm{id}} \mathbb{X} \times_{\operatorname{Spec} \overline{k}} \overline{S} \xrightarrow{\rho} X \times_S \overline{S}.$$

For the case $d = 2$, a detailed exposition on the proof of this theorem is given in [Bo-Ca]. See also [Ra-Zi].

Here we will not give a comment on a proof of this theorem. We only mention that it is not so difficult to prove:

i) the functor G is represented by some formal scheme $\mathcal{M}'$,

ii) the formal scheme $\mathcal{M}'$ gives a formal model of $(\Omega_K^d)_E \times \mathbb{Z}$.

The part i) is essentially given in [Ra-Zi]. The part ii) is also contained in [Ra-Zi]. I will give a rough explanation about that, by the Dieudonné theory, we may describe the map $(\mathcal{M}')^{\mathrm{rig}}(L) = \mathcal{M}'(\mathcal{O}_L) \longrightarrow \mathbb{P}^{d-1}(L)$ for a finite extension L of K.

Let us recall the Dieudonné theory briefly. Formal $\mathcal{O}_K$-modules (more generally, π-divisible $\mathcal{O}_K$-modules) over $\mathcal{O}_L$ is classified by the following two data.

- The Dieudonné module $\mathbb{D}(X)$ of X. This is a free $W_{\mathcal{O}_K}(\overline{k}) \cong \mathcal{O}_E$-module with two semi-linear operators F (called Frobenius) and V (called Verschiebung) satisfying $FV = VF = \pi$. The $\mathcal{O}_E$-rank of $\mathbb{D}(X)$ is equal to the $\mathcal{O}_K$-height of X.
- The Hodge filtration $\mathrm{Fil} \subset \mathbb{D}(X) \otimes_{\mathcal{O}_E} \mathcal{O}_L$. This is a free $\mathcal{O}_L$-submodule such that $(\mathbb{D}(X) \otimes_{\mathcal{O}_E} \mathcal{O}_L)/\mathrm{Fil} \cong \mathrm{Lie}\, X$. In particular, the corank of Fil is equal to the dimension of X.

Moreover, we have a canonical isomorphism $\mathbb{D}(X) \cong \mathbb{D}(X_{\overline{k}})$ (crystalline nature of the Dieudonné module).

Let (X, ρ) be an element of $\mathcal{M}'(\mathcal{O}_L)$. Then ρ induces an isomorphism $\rho_* \colon \mathbb{D}(\mathbb{X})_{\mathbb{Q}} \xrightarrow{\cong} \mathbb{D}(X_{\overline{k}})_{\mathbb{Q}}$ of d^2-dimensional E-vector spaces. Therefore we have an isomorphism $\mathbb{D}(\mathbb{X})_{\mathbb{Q}} \otimes_E L \xrightarrow{\cong} \mathbb{D}(X_{\overline{k}})_{\mathbb{Q}} \otimes_E L$. By the isomorphism $\mathbb{D}(X_{\overline{k}})_{\mathbb{Q}} \otimes_E L \cong \mathbb{D}(X)_{\mathbb{Q}} \otimes_E L$, the subspace $\mathrm{Fil}_{\mathbb{Q}} \subset \mathbb{D}(X)_{\mathbb{Q}} \otimes_E L$ can be regarded as an L-subspace of $\mathbb{D}(X_{\overline{k}})_{\mathbb{Q}} \otimes_E L$ whose codimension is d.

Now let us remind the action of $\mathcal{O}_D$. Then, as in the second condition of Definition 4.3, everything decomposes compatibly; we have

$$\mathbb{D}(\mathbb{X})^i_{\mathbb{Q}} \otimes_E L \underset{\cong}{\xrightarrow{\rho_*}} \mathbb{D}(X_{\overline{k}})^i_{\mathbb{Q}} \otimes_E L \supset \mathrm{Fil}^i_{\mathbb{Q}}$$

for every $i \in \mathbb{Z}/d\mathbb{Z}$. Therefore we get an L-subspace $\rho_*^{-1}(\mathrm{Fil}^0_{\mathbb{Q}})$ of $\mathbb{D}(\mathbb{X})^0_{\mathbb{Q}} \otimes_E L$ whose codimension is 1. This subspace gives an L-valued point of $\mathbb{P}(\mathbb{D}(\mathbb{X})^0_{\mathbb{Q}})$.

On the other hand, we have an isomorphism $(\mathbb{D}(\mathbb{X})^0_{\mathbb{Q}})^{V^{-1}\Pi} \otimes_K E \cong \mathbb{D}(\mathbb{X})^0_{\mathbb{Q}}$ (this is obtained in the course of the proof of Proposition 4.2). Thus $\mathbb{P}(\mathbb{D}(\mathbb{X})^0_{\mathbb{Q}})$ is the base extension of $\mathbb{P}((\mathbb{D}(\mathbb{X})^0_{\mathbb{Q}})^{V^{-1}\Pi})$ from K to E, and we get an L-valued point of $\mathbb{P}((\mathbb{D}(\mathbb{X})^0_{\mathbb{Q}})^{V^{-1}\Pi})$. Since $\mathbb{P}((\mathbb{D}(\mathbb{X})^0_{\mathbb{Q}})^{V^{-1}\Pi})$ is isomorphic to $\mathbb{P}^{d-1}_K$ over K, we have done.

Remark 4.5 i) By this description, we can explain why the obtained L-valued point lies in Ω_K^d. This has a relation to the Fontaine theory on p-adic Galois representations. Roughly speaking, Ω_K^d is the set consisting of "weakly admissible points" of $\mathbb{P}^{d-1}$. The Dieudonné module of X is the image of Fontaine's functor D_{cris}, thus (weakly) admissible. (If $K = \mathbb{Q}_p$, then the Tate module T_pX of X is mapped to $\mathbb{D}(X)_{\mathbb{Q}}$. In the general case, we require more argument.) A precise statement can be seen in [Ra-Zi].

ii) The surjectivity of the morphism $(\mathcal{M}')^{\mathrm{rig}} \longrightarrow \Omega_K^d$ can be seen as a consequence of the theorem of Colmez-Fontaine [Co-Fo] which says that weakly admissible means admissible. (However the theorem of Drinfeld is older than that of Colmez-Fontaine.)

iii) We may also determine the fiber of the map $(\mathcal{M}')^{\mathrm{rig}} \longrightarrow \Omega_K^d$ using the description above.

Remark 4.6 We may define $\widehat{\Omega}_K^d$ as the formal scheme representing the functor

$$G'\colon S \longmapsto \bigl\{(X,\rho) \in G(S) \bigm| \deg\rho = 0\bigr\}.$$

The theory of local models due to Rapoport-Zink enables us to study its local structure (for example, semistablity) from this moduli interpretation.

5 Proof of the main theorem

In this section, we use the notation introduced in 3.2. In particular, $K = K_p^0 \cdot K^p$ denotes a compact open subgroup of $B^\times(\mathbb{A}_f)$.

5.1 Interpretation of $Z_K = K^p \backslash \overline{B}^\times(\mathbb{A}_f)/\overline{B}^\times(\mathbb{Q})$

Definition 5.1 Let X be a special formal $\mathcal{O}_{B_p}$-module of height 4 over $S \in \mathrm{Nilp}_{\mathcal{O}_E}$. An algebraization of X is a pair (A, ε) consisting of:

- an abelian surface A over S with an $\mathcal{O}_B$-action,
- an $\mathcal{O}_B$-isomorphism $\varepsilon\colon \widehat{A} \xrightarrow{\cong} X$ ($\widehat{A}$ denotes the formal group associated with A).

We will denote the set of algebraizations of X by $\mathrm{Alg}(X)$.

Remark 5.2 If $(A,\varepsilon) \in \mathrm{Alg}(X)$, then the action of $\mathcal{O}_B$ on A automatically satisfies the Kottwitz condition (see 3.2).

Proposition 5.3 *We have a bijection*

$$\bigl\{(A,\varepsilon,\overline{\nu}) \bigm| (A,\varepsilon) \in \mathrm{Alg}(\mathbb{X}),\ \overline{\nu}\colon \text{a } K\text{-level structure of } A\bigr\}_{/\cong} \cong K^p \backslash \overline{B}^\times(\mathbb{A}_f)/\overline{B}^\times(\mathbb{Q}).$$

Proof. The case "$K^p = 1$", that is, the isomorphy

$$\bigl\{(A,\varepsilon,\nu) \bigm| (A,\varepsilon) \in \mathrm{Alg}(\mathbb{X}), \nu\colon T_f^p A \xrightarrow{\cong} \mathcal{O}_B \otimes \widehat{\mathbb{Z}}^p\bigr\}_{/\cong} \cong \overline{B}^\times(\mathbb{A}_f)/\overline{B}^\times(\mathbb{Q})$$

is essential. It is easy to see that the left hand side is isomorphic to the set Z' of isomorphism classes of triples (A,ε,ν) consisting of:

- an abelian surface A over $\overline{\mathbb{F}}_p$ with an $\mathcal{O}_B$-action,
- an $\mathcal{O}_B$-quasi-isogeny $\varepsilon\colon \widehat{A} \longrightarrow \mathbb{X}$,

– and an $\mathcal{O}_B$-isomorphism $\nu\colon V_f^pA := T_f^pA \otimes \mathbb{Q} \xrightarrow{\cong} B^\times(\mathbb{A}_f^p)$.

We may define the action of $g = (g_p, g^p) \in \overline{B}^\times(\mathbb{A}_f)$ on the set Z' by $g(A,\varepsilon,\nu) = (A, g_p \circ \varepsilon, R_{g^p} \circ \nu)$. Here R_{g^p} denotes the map $\overline{B}^\times(\mathbb{A}_f^p) \longrightarrow \overline{B}^\times(\mathbb{A}_f^p)$; $h \longmapsto hg^p$. Let us prove that Z' is isomorphic to $\overline{B}^\times(\mathbb{A}_f)/\overline{B}^\times(\mathbb{Q})$ as $\overline{B}^\times(\mathbb{A}_f)$-sets. First note that Z' is non-empty, since for every $(A,\iota) \in \mathcal{S}_K(\overline{\mathbb{F}}_p)$, $\widehat{A}$ is a special formal $\mathcal{O}_{B_p}$-module of height 4, and thus there exists an $\mathcal{O}_{B_p}$-quasi-isogeny $\widehat{A} \longrightarrow \mathbb{X}$ (Proposition 4.2 ii)). Furthermore, by the subsequent proposition, it is easy to see that the action of $\overline{B}^\times(\mathbb{A}_f)$ on Z' is transitive. Finally, we may prove that the stabilizer of this action is isomorphic to $\overline{B}^\times(\mathbb{Q})$. This is a consequence of the isomorphism $\mathrm{End}_{\mathcal{O}_B}(A) \otimes \mathbb{Q} \cong \overline{B}(\mathbb{Q})$. ∎

In the proof above, we used the following proposition, which is an easy consequence of the Honda-Tate theory.

Proposition 5.4 *Let A and A' be abelian schemes with $\mathcal{O}_B$-actions. Assume that each action satisfies the Kottwitz condition. Then they are $\mathcal{O}_B$-isogenous.*

Remark 5.5 In fact, such A is isogenous to the product of two supersingular elliptic curves.

5.2 Construction of the isomorphism over $\overline{\mathbb{F}}_p$

Definition 5.6 We define a map $\Theta\colon (\breve{\mathcal{M}}^{\mathrm{Dr},2})_{\overline{\mathbb{F}}_p} \times Z_K \longrightarrow (\mathcal{S}_K)_{\overline{\mathbb{F}}_p}$ as follows. Let S be an $\overline{\mathbb{F}}_p$-scheme. By Proposition 5.3, the set of S-valued points of $(\breve{\mathcal{M}}^{\mathrm{Dr},2})_{\overline{\mathbb{F}}_p} \times Z_K$ consists of 5-ples $(X,\rho,A,\varepsilon,\overline{\nu})$. From this data, we can construct canonically an algebraization (A',ε') of X and a p-quasi-isogeny $h\colon A \otimes_{\overline{\mathbb{F}}_p} S \longrightarrow A'$ which make the following diagram commutative:

$$\begin{tikzcd} \widehat{A} \otimes_{\overline{\mathbb{F}}_p} S \arrow[r, "\varepsilon", "\cong"'] \arrow[d, "\widehat{h}"] & \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} S \arrow[d, "\rho"] \\ \widehat{A}' \arrow[r, "\varepsilon'", "\cong"'] & X. \end{tikzcd}$$

If $\rho\colon \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} S \longrightarrow X$ is an isogeny, then A' is the quotient of $A \otimes_{\overline{\mathbb{F}}_p} S$ by the inverse image of $\operatorname{Ker}\rho$ under ε.

The level structure ν naturally induces a K-level structure $\overline{\nu}'$ on A'. Therefore we have a pair $(A',\overline{\nu}') \in \mathcal{S}_K(S)$. We define Θ as $(X,\rho,A,\varepsilon,\overline{\nu}) \longmapsto (A',\overline{\nu}')$.

It is easy to see that Θ induces a morphism $\overline{\Theta}\colon GL_2(\mathbb{Q}_p)\backslash(\breve{\mathcal{M}}^{\mathrm{Dr},2})_{\overline{\mathbb{F}}_p} \times Z_K \longrightarrow (\mathcal{S}_K)_{\overline{\mathbb{F}}_p}$.

Proposition 5.7 *The morphism $\overline{\Theta}$ is an isomorphism.*

Proof. First we will prove that $\overline{\Theta}$ induces a bijection on the sets of $\overline{\mathbb{F}}_p$-valued points. The injectivity is easy from the definition. The surjectivity is a consequence of Proposition 4.2 ii).

Next we will prove that $\overline{\Theta}$ is étale. We have only to prove that $\overline{\Theta}$ induces an isomorphism on deformations, which is ensured by the Serre-Tate theorem. ∎

5.3 From "over $\overline{\mathbb{F}}_p$" to "over $\mathcal{O}_E$"

Proposition 5.8 *The isomorphism $\overline{\Theta}$ extends canonically to an isomorphism*

$$\Theta\colon GL_2(\mathbb{Q}_p)\backslash\breve{\mathcal{M}}^{\mathrm{Dr},2} \times Z_K \xrightarrow{\cong} \mathcal{S}_K^{\wedge}.$$

Proof. We use the Serre-Tate theorem again. For $S \in \mathrm{Nilp}_{\mathcal{O}_E}$, let $(X, \rho, A, \varepsilon, \overline{\nu})$ be an S-valued point of $\breve{\mathcal{M}}^{\mathrm{Dr},2} \times Z_K$. By taking base change to $\overline{S}$, we get $(\overline{X}, \rho, \overline{A}, \overline{\varepsilon}, \overline{\overline{\nu}})$. By the construction in Definition 5.6, we have an algebraization $(\overline{A}', \overline{\varepsilon}')$ of $\overline{X}$ and a K-level structure $\overline{\overline{\nu}}'$ of $\overline{A}'$. Then, pulling back X by the isomorphism $\overline{\varepsilon}'$, we have a deformation of $(\overline{A}')^{\wedge}$ over S. Now by the Serre-Tate theorem, we have a deformation A' of $\overline{A}'$ over S. Since $\mathcal{O}_B$ acts on X, this abelian surface A' comes to equip the action of $\mathcal{O}_B$. Since the level structure $\overline{\overline{\nu}}'$ obviously extends to A', we get an S-valued point $(A', \overline{\nu}')$ of $\mathcal{S}_K(S)$. The map $(X, \rho, A, \varepsilon, \overline{\nu}) \longmapsto (A', \overline{\nu}')$ gives a desired extension Θ.

By the Serre-Tate theorem, it is not difficult to prove that Θ is an isomorphism. ∎

By this proposition, our proof of Theorem 3.4 is finished.

5.4 Miscellaneous remarks

Remark 5.9 It is natural to hope that we might consider more general level structures. There is the following result due to Drinfeld. Let $K = K_p^m \cdot K^p$ be a compact open subgroup of $B^\times(\mathbb{A}_f)$ where $K_p^m = 1 + \Pi^m \mathcal{O}_{B_p} \subset B^\times(\mathbb{Q}_p)$ for $m \geq 1$. Then $\mathbf{S}_K^{\mathrm{an}}$ can be uniformized by Σ_m^2, a Galois étale covering of $\Omega_{\mathbb{Q}_p}^2$. For an integer d and m, the covering Σ_m^d is obtained as $X^{\mathrm{univ}}[\Pi^m] \setminus X^{\mathrm{univ}}[\Pi^{m-1}]$, where X^{univ} is the universal special formal $\mathcal{O}_{B_p}$-module over $\Omega_{\mathbb{Q}_p}^d$. Its Galois group is isomorphic to $(\mathcal{O}_{B_p}/\Pi^m)^\times$. Note that we have neither a good formal model of Σ_m^d nor a uniformization result of $\mathcal{S}_K$ at the level of formal schemes.

The ℓ-adic cohomology of Σ_m^d is very interesting; it has a relation to the local Langlands correspondence (for GL_d) and to the local Jacquet-Langlands correspondence (for GL_d and $D^\times$). This is compatible with the fact that the ℓ-adic cohomology of $\mathbf{S}_K$ has a relation to the global Langlands correspondence.

Remark 5.10 We have a similar result for Shimura curves over a totally real field which is not equal to $\mathbb{Q}$. It was also proved by Čerednik [Če]. Boutot and Zink ([Bo-Zi]) gave another proof by generalizing Drinfeld's method explained here. First they gave a p-adic uniformization for unitary Shimura varieties using the general theory developed in [Ra-Zi] and next reduced the case of Shimura curves to the unitary case.

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