Integral log crystalline cohomology and algebraic correspondences

Yoichi Mieda

1 Introduction

In this article, we will consider the action of an algebraic correspondence on the compactly supported cohomology of a smooth scheme over a field. Let k be a field and X a purely d-dimensional scheme which is separated and smooth over k (note that we do not assume that X is proper over k). Let Γ be a closed subscheme of $X \times X$ with pure codimension d. Such a closed subscheme is called an algebraic correspondence. We denote the composite $\Gamma \hookrightarrow X \times X \xrightarrow{\mathrm{pr}_i} X$ by γ_i. Throughout this article, we assume that γ_1 is proper.

Let us consider some compactly supported cohomology theory H_c^*. Then we can often define the homomorphism $\Gamma^*\colon H_c^i(X) \longrightarrow H_c^i(X)$, which is called the action of Γ on $H_c^i(X)$. For example, if H_c^i is étale cohomology or Betti cohomology (and $k = \mathbb{C}$), then Γ^* is obtained as the composite $H_c^i(X) \xrightarrow{\gamma_1^*} H_c^i(\Gamma) \xrightarrow{\gamma_{2*}} H_c^i(X)$. Here γ_1^* can be defined since γ_1 is proper, and γ_{2*} can be defined since X is smooth over k.

In this article, we will consider the case where k is perfect with characteristic $p > 0$ and H_c^* is a p-adic cohomology. We are interested in the trace $\mathrm{Tr}(\Gamma^*; H_c^i(X))$.

Which p-adic cohomology do we consider? Since the characteristic of k is p, the p-adic étale cohomology theory does not work at all. Crystalline cohomology is known to be a good p-adic cohomology theory for a scheme which is proper and smooth over k, but it does not work well for a non-proper scheme. Here we take H_c^i as (compactly supported) rigid cohomology introduced by Berthelot ([Be1]).

Let us recall it briefly. Let V be a complete discrete valuation ring with mixed characteristic $(0, p)$, whose residue field is k. Denote the fraction field of V by K. Then, for a scheme Y which is separated of finite type over k, we can define a finite-dimensional K-vector space $H^i_{\mathrm{rig}}(Y/K)$ (resp. $H^i_{\mathrm{rig},c}(Y/K)$) which is called the rigid cohomology (resp. the compactly supported rigid cohomology) of Y.

To construct them, we need rigid geometry. For simplicity, we will assume that Y is quasi-projective. Then Y has a projective compactification $Y \hookrightarrow \overline{Y}$. Moreover, $\overline{Y}$ can be embedded into a formal scheme $\mathfrak{Y}$ which is proper smooth over $\mathrm{Spf}\, V$ (we

Faculty of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka, 819–0395 Japan
E-mail address: mieda@math.kyushu-u.ac.jp

may take $\mathfrak{Y} = \widehat{\mathbb{P}}^n_V$ for some n). We denote the Raynaud generic fiber of $\mathfrak{Y}$ by $\mathfrak{Y}^{\mathrm{rig}}$. This is a rigid space (in the sense of Tate) over K. We denote the inverse image of Y (resp. $\overline{Y}$) under the specialization map $\mathrm{sp}_{\mathfrak{Y}}\colon \mathfrak{Y}^{\mathrm{rig}} \longrightarrow \mathfrak{Y}_k$ by $]Y[$ (resp. $]\overline{Y}[$). First assume that Y is projective over k, namely, $Y = \overline{Y}$. In this case we define $H^i_{\mathrm{rig}}(Y/K) = H^i(]Y[, \Omega^\bullet_{]Y[})$. We can prove that this is independent of the choice of $\mathfrak{Y}$. In particular, if Y is projective smooth over k and there exists a projective smooth V-scheme $\mathcal{Y}$ satisfying $\mathcal{Y}_k \cong Y$, then we have $H^i_{\mathrm{rig}}(Y/K) = H^i(\widehat{\mathcal{Y}}^{\mathrm{rig}}, \Omega^\bullet_{\widehat{\mathcal{Y}}^{\mathrm{rig}}}) \cong H^i(\mathcal{Y}_K, \Omega^\bullet_{\mathcal{Y}_K}) = H^i_{\mathrm{dR}}(\mathcal{Y}_K/K)$, where $\widehat{\mathcal{Y}}$ is the p-adic completion of $\mathcal{Y}$. In the case where $Y \subsetneq \overline{Y}$, we may define the complex $j^\dagger\Omega^\bullet_{]\overline{Y}[}$ on $]\overline{Y}[$ consisting of differential forms which are "overconvergent" around $]Y[$. We put $H^i_{\mathrm{rig}}(Y/K) = H^i(]\overline{Y}[, j^\dagger\Omega^\bullet_{]\overline{Y}[})$. It is known to be independent of the choice of $\overline{Y}$ and $\mathfrak{Y}$. The definition of $H^i_{\mathrm{rig},c}(Y/K)$ is a little more involving and we will omit it.

If Y is smooth over k and equidimensional, then $H^i_{\mathrm{rig}}(Y/K)$ and $H^{2\dim Y - i}_{\mathrm{rig},c}(Y/K)$ is dual to each other (the Poincaré duality theorem, [Be2]).

On $H^i_{\mathrm{rig},c}(X/K)$, we can define the action of Γ as the composite

$$H^i_{\mathrm{rig},c}(X/K) \xrightarrow{\gamma_1^*} H^i_{\mathrm{rig},c}(\Gamma/K) \xrightarrow{\cup\,\mathrm{cl}_{X\times X}(\Gamma)} H^i_{\mathrm{rig},c}(X \times X/K) \xrightarrow{\mathrm{pr}_{2*}} H^i_{\mathrm{rig},c}(X/K),$$

where $\mathrm{cl}_{X\times X}(\Gamma) \in H^{2d}_{\mathrm{rig},\Gamma}(X \times X/K)$ is the refined cycle class of Γ due to Petrequin ([Pe]). Note that this construction is compatible with composition of algebraic correspondences. For an integral Γ, we may also give an equivalent definition of Γ^* by using the alteration theorem to Γ (if Γ is smooth over k, then we can define the push-forward map γ_{2*} by using the Poincaré duality). I think this alternative definition is simpler, but under this definition it is more difficult to observe the compatibility with composition.

Now we can state our main theorem in this article.

Theorem 1.1 $\mathrm{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K)) \in V$.

We will give some remarks on this theorem. If $k = \mathbb{C}$ and H^*_c is Betti cohomology $H^*_c(-, \mathbb{Q})$, then $H^i_c(X, \mathbb{Q})$ has the natural $\mathbb{Z}$-lattice $\mathrm{Im}(H^i_c(X, \mathbb{Z}) \longrightarrow H^i_c(X, \mathbb{Q}))$ that is preserved by Γ^*. Therefore $\mathrm{Tr}(\Gamma^*; H^i_c(X, \mathbb{Q}))$ is an integer. Similarly, if H^*_c is ℓ-adic étale cohomology where ℓ is a prime number distinct from p, then the trace $\mathrm{Tr}(\Gamma^*; H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell))$ lies in $\mathbb{Z}_\ell$. However, we cannot obtain the theorem above directly by the similar method, since we have no integral structure of rigid cohomology.

It is strongly believed that $\mathrm{Tr}(\Gamma^*; H^i_c(X, \mathbb{Q}_\ell))$ ($\ell \neq p$) and $\mathrm{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K))$ are integers and all of them are equal. This is actually proved in the case where X is proper over k ([KM]). Nevertheless, for a non-proper X, we have no way to prove the integrality; it is known as a very difficult open problem in this area.

2 Log crystalline cohomology

In this section, we will sketch the proof of Theorem 1.1. First notice the following lemma.

Lemma 2.1 *In order to show Theorem 1.1, we may assume that X has a compactification $X \hookrightarrow \overline{X}$ such that $\overline{X}$ is smooth (and proper) over k and $D := \overline{X} \setminus X$ is a simple normal crossing divisor of X (for simplicity, we will call such a compactification a good compactification).*

Proof. We use the alteration theorem due to de Jong; for a scheme X which is separated and smooth over k, there exist a proper surjective generically finite morphism $\pi \colon Y \longrightarrow X$ and a good compactification $Y \hookrightarrow \overline{Y}$. Assume that Theorem 1.1 holds for Y. Then we have $\mathrm{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K)) \in (\deg \pi)^{-1} V$. Since $\deg \pi$ is independent of Γ, we have $\mathrm{Tr}((\Gamma^*)^m; H^i_{\mathrm{rig},c}(X/K)) \in (\deg \pi)^{-1} V$ for every positive integer m. By [Kl, Lemma 2.8], we may conclude that $\mathrm{Tr}((\Gamma^*)^m; H^i_{\mathrm{rig},c}(X/K)) \in V$. ∎

In the remaining part of this section, we will assume that X has a good compactification $X \hookrightarrow \overline{X}$ and put $D = \overline{X} \setminus X$. For such X, log crystalline cohomology gives an integral structure on $H^i_{\mathrm{rig},c}(X/K)$. Let us recall log crystalline cohomology. Denote by W the ring of Witt vectors of k. We can define the log crystalline cohomology $H^i_{\mathrm{crys}}((\overline{X}, D)/W)$ and the "compactly supported" (or "with minus log pole") log crystalline cohomology $H^i_{\mathrm{crys}}((\overline{X}, -D)/W)$. These are W-modules. Roughly speaking, $H^i_{\mathrm{crys}}((\overline{X}, \pm D)/W)$ is obtained as the Zariski cohomology $H^i(\overline{X}, W\Omega^\bullet_{\overline{X}}(\pm \log D))$ of the de Rham-Witt complex $W\Omega^\bullet_{\overline{X}}(\pm \log D)$, which is the de Rham-Witt analogue of the de Rham complex with log poles $\Omega^\bullet_{\overline{X}}(\pm \log D)$. Recall that $\Omega^\bullet_{\overline{X}}(-\log D) = \mathcal{I} \otimes \Omega^\bullet_{\overline{X}}(\log D)$, where $\mathcal{I} \subset \mathcal{O}_{\overline{X}}$ is the defining ideal of $D \subset \overline{X}$. We may also define these by using log crystalline site (for example, see [Ts]). Our starting point is the following theorem.

Theorem 2.2 (Shiho [Sh]) *We have the natural and functorial isomorphisms*

$$H^i_{\mathrm{rig}}(X/K) \cong H^i_{\mathrm{crys}}\big((\overline{X}, D)/W\big) \otimes_W K, \quad H^i_{\mathrm{rig},c}(X/K) \cong H^i_{\mathrm{crys}}\big((\overline{X}, -D)/W\big) \otimes_W K.$$

The former isomorphism is proved in [Sh]. The latter follows from the former and the Poincaré duality for rigid cohomology ([Be2]) and for log crystalline cohomology ([Ts]).

By this theorem, our main theorem is reduced to the following:

Theorem 2.3 *We can define Γ^* on $H^i_{\mathrm{crys}}((\overline{X}, -D)/W)$ so that the following diagram commutes:*

$$\begin{tikzcd}
H^i_{\mathrm{rig},c}(X/K) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\Gamma^*"] & H^i_{\mathrm{crys}}\big((\overline{X}, -D)/W\big) \otimes_W K \arrow[d, "\Gamma^* \otimes \mathrm{id}"] \\
H^i_{\mathrm{rig},c}(X/K) \arrow[r, leftrightarrow, "\cong"] & H^i_{\mathrm{crys}}\big((\overline{X}, -D)/W\big) \otimes_W K.
\end{tikzcd}$$

Since log crystalline cohomology does not work well for non-smooth schemes, we cannot define Γ^* in the same manner as for étale or rigid cohomology. The key of the proof is to construct the cycle class $\mathrm{cl}(\Gamma)$ in the partially supported log crystalline cohomology $H^{2d}_{\mathrm{crys}}((\overline{X}\times\overline{X}, D_1 - D_2)/W)$, where $D_1 = D\times\overline{X}$ and $D_2 = \overline{X}\times D$. The cohomology above is the Zariski cohomology $H^i(\overline{X}\times\overline{X}, W\Omega^{\bullet}_{\overline{X}\times\overline{X}}(D_1, D_2))$ of the de Rham-Witt complex $W\Omega^{\bullet}_{\overline{X}\times\overline{X}}(D_1, D_2)$, which is the de Rham-Witt analogue of "$\Omega^{\bullet}_{(\overline{X}\times\overline{X})/k}(D_1, D_2)$" appearing in [DI, (4.2.1.2)]. It is a crystalline analogue of $H^i((\overline{X}\times\overline{X})\setminus D_1, j_!\mathbb{Z}_\ell)$, where j denotes the open immersion $(\overline{X}\times\overline{X})\setminus(D_1\cup D_2)\hookrightarrow(\overline{X}\times\overline{X})\setminus D_1$. Moreover we should prove various functorialities of $\mathrm{cl}(\Gamma)$; for example, the image of $\mathrm{cl}(\Gamma)$ in $H^{2d}_{\mathrm{crys}}(((\overline{X}\times\overline{X})\setminus(D_1\cup D_2))/W)$ coincides with the usual crystalline cycle class $\mathrm{cl}(\Gamma)$ due to Gros ([Gr]).

Let me explain the construction of $\mathrm{cl}(\Gamma)$. Denote the closure of Γ in $\overline{X}\times\overline{X}$ by $\overline{\Gamma}$. Since we are assuming that γ_1 is proper, we have $\overline{\Gamma}\cap D_2\subset\overline{\Gamma}\cap D_1$. We will use Gros' method ([Gr]); namely, we will construct $\mathrm{cl}(\Gamma)$ in the local Hodge-Witt cohomology $H^d_{\overline{\Gamma}}(\overline{X}\times\overline{X}, W\Omega^d_{\overline{X}\times\overline{X}}(D_1, D_2))$. If $\overline{\Gamma}$ does not intersect $D_1\cap D_2$, then this local cohomology is isomorphic to $H^d_{\overline{\Gamma}}(\overline{X}\times\overline{X}, W\Omega^d_{\overline{X}\times\overline{X}}(\log D_1))$ and we may define $\mathrm{cl}(\Gamma)$ as the image of Gros' cycle class $\mathrm{cl}(\overline{\Gamma})\in H^d_{\overline{\Gamma}}(\overline{X}\times\overline{X}, W\Omega^d_{\overline{X}\times\overline{X}})$ under the natural homomorphism $H^d_{\overline{\Gamma}}(\overline{X}\times\overline{X}, W\Omega^d_{\overline{X}\times\overline{X}})\longrightarrow H^d_{\overline{\Gamma}}(\overline{X}\times\overline{X}, W\Omega^d_{\overline{X}\times\overline{X}}(\log D_1))$. Thus, in order to define $\mathrm{cl}(\Gamma)$ in the general case, we want to "remove" $D_1\cap D_2$. In other words, we need some vanishing results on the local cohomology of $W_m\Omega^d_{\overline{X}\times\overline{X}}(D_1, D_2)$. This is the most difficult part of this work, which involves direct calculations. The functorialities of $\mathrm{cl}(\Gamma)$ follows directly from the construction and the functorialities of Gros' cycle class.

Once we get the class $\mathrm{cl}(\Gamma)$, then we can define Γ^* as the composite

$$\begin{aligned}&H^i_{\mathrm{crys}}\big((\overline{X}, -D)/W\big)\xrightarrow{\mathrm{pr}_1^*}H^i_{\mathrm{crys}}\big((\overline{X}\times\overline{X}, -D_1)/W\big)\\&\qquad\xrightarrow{\cup\,\mathrm{cl}(\Gamma)}H^{i+2d}_{\mathrm{crys}}\big((\overline{X}\times\overline{X}, -D_1-D_2)/W\big)\xrightarrow{\mathrm{pr}_{2*}}H^i_{\mathrm{crys}}\big((\overline{X}, -D)/W\big).\end{aligned}$$

We should compare it with Γ^* on the rigid cohomology. It is a slightly complicated task requiring careful blow-ups of $\overline{X}\times\overline{X}$ and an alteration of (the strict transform of) $\overline{\Gamma}$, but is eventually deduced from the functorialities of our cycle class and those of Shiho's comparison map.

3 Consequences of the main theorem

In this section, we will give some easy consequences of Theorem 1.1. Here we use the notation introduced in Section 1 (we do not assume that X has a good compactification).

Corollary 3.1 *Assume that k is a finite field $\mathbb{F}_q$. Then the alternating sum of the traces $\sum_{i=0}^{2d}(-1)^i\,\mathrm{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K))$ is an integer.*

If X is proper over k, then $\sum_{i=0}^{2d}(-1)^i\,\mathrm{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K))$ is equal to the intersection number $(\Gamma, \Delta_X)_{X\times X}$ ($\Delta_X\subset X\times X$ denotes the diagonal) by the Lefschetz trace formula. In particular, it is an integer.

To prove this corollary, we use a p-adic analogue of Fujiwara's trace formula.

Proposition 3.2 *There exists an integer N such that for every $n \geq N$ we have*

$$\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^* \circ (\mathrm{Fr}_X^*)^n; H^i_{\mathrm{rig},c}(X/K)\bigr) = (\Gamma^{(n)}, \Delta_X)_{X\times X},$$

where Fr_X is the qth power Frobenius morphism and $\Gamma^{(n)} = (\mathrm{Fr}_X^n \times \mathrm{id})_*\Gamma$.

This proposition can be proved by the similar method as in [KS].

Proof of Corollary 3.1. By the rationality and the functional equation of the congruence zeta function and the Weil conjecture for rigid cohomology (these follows from [KM] since rigid cohomology is a Weil cohomology), every eigenvalue of Fr_X^* on $H^i_{\mathrm{rig},c}(X/K)$ lies in $\overline{\mathbb{Z}}[1/p]$, where $\overline{\mathbb{Z}}$ denotes the ring of algebraic integers. Therefore, by Proposition 3.2 and an easy linear algebra (see [Mi1, Lemma 2.1.3]), we have $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K)) \in \mathbb{Z}[1/p]$. On the other hand, Theorem 1.1 says that $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K)) \in V$. Therefore $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H^i_{\mathrm{rig},c}(X/K))$ lies in $\mathbb{Z}[1/p] \cap V = \mathbb{Z}$. ∎

By using Proposition 3.2, we can derive an analogous result for ℓ-adic cohomology from Corollary 3.1.

Corollary 3.3 *Here let k be an arbitrary field and ℓ be a prime number which is invertible in k. Then the alternating sum of the traces $\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}(\Gamma^*; H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell))$ is an integer.*

Proof. By the standard specialization argument, we may reduce to the case where k is a finite field. By (original) Fujiwara's trace formula and Proposition 3.2, there exists an integer N such that for every $n \geq N$ we have

$$\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^* \circ (\mathrm{Fr}_X^*)^n; H^i_{\mathrm{rig},c}(X/K)\bigr) = \sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^* \circ (\mathrm{Fr}_X^*)^n; H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell)\bigr).$$

By the van der Mond argument, we have the equality

$$\sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^*; H^i_{\mathrm{rig},c}(X/K)\bigr) = \sum_{i=0}^{2d}(-1)^i \operatorname{Tr}\bigl(\Gamma^*; H^i_c(X_{\overline{k}}, \mathbb{Q}_\ell)\bigr).$$

Since the left hand side is an integer by Corollary 3.1, so is the right hand side. ∎

Actually, this corollary has been already proved in [BE] by using relative motivic cohomology defined by Levine. However I think that our p-adic proof is also interesting.

For more detailed survey on this work, please see [Mi2].

References

[Be1] P. Berthelot, *Finitude et pureté cohomologique en cohomologie rigide*, Invent. Math. **128** (1997), no. 2, 329–377.

[Be2] P. Berthelot, *Dualité de Poincaré et formule de Künneth en cohomologie rigide*, C. R. Acad. Sci. Paris Ser. I Math. **325** (1997), no. 5, 493–498.

[BE] S. Bloch, H. Esnault, *Künneth projectors for open varieties*, Algebraic cycles and motives. Vol. 2, 54–72, London Math. Soc. Lecture Note Ser., 344, Cambridge Univ. Press, Cambridge, 2007.

[DI] P. Deligne, L. Illusie, *Relèvements modulo* p^2 *et décomposition du complexe de de Rham*, Invent. Math. **89** (1987), no. 2, 247–270.

[Fu] K. Fujiwara, *Rigid geometry, Lefschetz-Verdier trace formula and Deligne's conjecture*, Invent. Math. **127** (1997), no. 3, 489–533.

[Gr] M. Gros, *Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique*, Mem. Soc. Math. France, No. 21 (1985), 87 pp.

[KS] K. Kato, T. Saito, *Ramification theory for varieties over a perfect field*, Ann. of Math. (2) **168** (2008), no. 1, 33–96.

[KM] N. Katz, W. Messing, *Some consequences of the Riemann hypothesis for varieties over finite fields*, Invent. Math. **23** (1974), 73–77.

[Kl] S. L. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix esposés sur la cohomologie des schémas, pp. 359–386. North-Holland, Amsterdam; Masson, Paris, 1968.

[Mi1] Y. Mieda, *On* ℓ*-independence for the étale cohomology of rigid spaces over local fields*, Compos. Math. **143** (2007), no. 2, 393–422.

[Mi2] Y. Mieda, *Cycle classes, Lefschetz trace formula and integrality for* p*-adic cohomology*, RIMS Kôkyûroku Bessatsu **B12** (2009), 57–66.

[Pe] D. Petrequin, *Classes de Chern et classes de cycles en cohomologie rigide*, Bull. Soc. Math. France **131** (2003), no. 1, 59–121.

[Sh] A. Shiho, *Crystalline fundamental groups II. Log convergent cohomology and rigid cohomology*, J. Math. Sci. Univ. Tokyo **9** (2002), no. 1, 1–163.

[Ts] T. Tsuji, *Poincaré duality for logarithmic crystalline cohomology*, Compositio Math. **118** (1999), no. 1, 11–41.