Geometric approach to the explicit local Langlands correspondence

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Abstract. We propose a geometric strategy of giving explicit description of the Langlands parameter of an irreducible supercuspidal representation of $\mathrm{GL}(n)$ over a non-archimedean local field. The key is to compare the cohomology of an affinoid in the Lubin-Tate space at infinite level and that of the reduction of its formal model. As examples, we treat the cases of depth 0 supercuspidal representations and simple supercuspidal representations.

1 Introduction

Let F be a non-archimedean local field. By the local Langlands correspondence for $\mathrm{GL}_n(F)$ ([HT01], [LRS93]), irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ are parameterized by n-dimensional irreducible smooth representations of the Weil group W_F of F. On the other hand, irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ are completely classified by the theory of types [BK93]. However, in general it is very difficult to determine the parameter corresponding to each supercuspidal representation explicitly. This problem, called "the explicit local Langlands correspondence", is extensively studied by Bushnell and Henniart; see [BH05a], [BH05b], [BH10], [Hen06] and references therein. Recently, Imai and Tsushima [IT15b] also gave a result in this direction, which is not covered by the works cited above. The methods in all of these works are purely algebraic, and sometimes involve very detailed and complicated computations.

The aim of this paper is to propose a new geometric method for the explicit local Langlands correspondence. It is well-known that the local Langlands correspondence for $\mathrm{GL}_n(F)$ has a nice geometric realization; it appears in the middle degree ℓ-adic cohomology of the Lubin-Tate tower. This fact is called the non-abelian Lubin-Tate theory after Carayol [Car90], and has been proved by Harris-Taylor [HT01] and Boyer [Boy99]. Therefore, it is quite natural to expect that we can understand the local Langlands correspondence by studying the geometry of the Lubin-Tate tower.

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Apart from many results in the case $n = 2$, the first result in this direction was given by Yoshida [Yos10]. He constructed a semistable model of the first layer of the Lubin-Tate tower, and discovered the Deligne-Lusztig variety for GL_n in the reduction of it. By using this result, he concluded that a part of the non-abelian Lubin-Tate theory boils down to the Deligne-Lusztig theory [DL76]. More recently, the Lubin-Tate space at infinite level, namely the projective limit of the Lubin-Tate tower, turns out to be much simpler than individual layers. This idea goes back to Faltings [Fal02] and Fargues [Far08], and now is formulated by the theory of perfectoid spaces [Sch12]; see [SW13] and [Wei] for detail. Based on this theory, Weinstein and Boyarchenko [BW16] constructed a family of affinoids in the Lubin-Tate space at infinite level. They also constructed formal models of the affinoids, and observed that a part of the local Langlands correspondence is encoded in the ℓ-adic cohomology of their reduction. Following this work, some other authors also obtain similar results; see [IT15a], [IT16] and [Tok16]. The common strategy in these works is to combine an explicit computation of the reduction of the formal models and a previously known explicit description of the local Langlands correspondence. The idea of this article is to reverse this process; we combine a computation of the reduction and the non-abelian Lubin-Tate theory to obtain a result in the explicit local Langlands correspondence. To carry it out, we need to relate the cohomology of the reduction with that of the Lubin-Tate tower itself, which is not included in the previous works. The main ingredient of this article is to give a useful condition which ensures the existence of such a relation.

Here is a key technical result for us:

Theorem 1.1 (Definition 3.17, Proposition 3.21, Theorem 3.22) *Let k be a complete algebraically closed non-archimedean field, k° its valuation ring, and κ the residue field of k°. Assume that the characteristic of κ is positive. Fix a non-zero element ϖ in the maximal ideal of k°. Let A be a ϖ-adically complete flat k°-algebra endowed with an action of a profinite group K_0. We write $\widetilde{A}$ for the integral closure of A in $A[1/\varpi]$. We assume the following two conditions (see Section 3 for detail):*

- *The space $\mathrm{Spa}(A[1/\varpi], \widetilde{A})$ is the projective limit of a tower $\{X_K\}_K$ of smooth affinoids over k indexed by open normal subgroups K of K_0.*
- *The reduction $\mathrm{Spec}(A \otimes_{k^\circ} \kappa)$ of $\mathrm{Spf}\, A$ is the perfection of an affine scheme Y of finite type over κ.*

(i) *For a prime number ℓ invertible in κ, we have a map*

$$\mathrm{sp}^* \colon H^i_c(Y, \overline{\mathbb{Q}}_\ell) \to \varinjlim_K H^i_c(X_K, \overline{\mathbb{Q}}_\ell),$$

which we call the specialization map. It is functorial with respect to automorphisms of A of finite level (i.e., automorphisms coming from those of the tower $\{X_K\}_K$).

(ii) *Assume moreover that Y is pure-dimensional and smooth over κ. Let $\{Z_K\}_K$ be a tower of rigid spaces which contains $\{X_K\}_K$ as a tower of open rigid*

subspaces. For a subspace $V \subset H^i_c(Y, \overline{\mathbb{Q}}_\ell)$ such that the composite

$$V \hookrightarrow H^i_c(Y, \overline{\mathbb{Q}}_\ell) \to H^i(Y, \overline{\mathbb{Q}}_\ell)$$

is injective, the composite

$$V \hookrightarrow H^i_c(Y, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} \varinjlim_K H^i_c(X_K, \overline{\mathbb{Q}}_\ell) \to \varinjlim_K H^i(Z_K, \overline{\mathbb{Q}}_\ell)$$

is also injective.

In the application, $\{Z_K\}$ becomes the Lubin-Tate tower, $\mathrm{Spa}(A[1/\varpi], \widetilde{A})$ an affinoid in the Lubin-Tate space at infinite level, and $\mathrm{Spf}\, A$ a formal model of it. We remark that, unlike the case of finite type, the specialization map sp^* is not necessarily surjective, even if Y is smooth (see Example 3.23).

To demonstrate how Theorem 1.1 is used to give an explicit description of the local Langlands correspondence, we choose two classes of irreducible supercuspidal representations as examples; one is depth 0 supercuspidal representations and the other is simple supercuspidal representations. Recall that the latter are supercuspidal representations with minimal positive depth; see [GR10], [RY14], [KL15]. By geometric method, we will obtain the following result:

Theorem 1.2 *Let π be either a depth 0 supercuspidal representation or a simple supercuspidal representation. Then, we can describe the Langlands parameter of π explicitly.*

For a more precise statement, see Theorems 5.3 and 6.4. We can also determine the image of π under the local Jacquet-Langlands correspondence between $\mathrm{GL}_n(F)$ and $D^\times$, where D is the central division algebra over F with invariant $1/n$. Note that the resulting description in Theorem 1.2 is not new, as in Remarks 5.4, 6.5. However our proof is totally different from the previous ones. We do not need to investigate involved representations closely. In fact, we do not have to compute neither characters nor epsilon factors. The author expects that the same method can be applied to supercuspidal representations with larger depth. This will be considered in our future works.

The idea to see the map $H^i_c(Y, \overline{\mathbb{Q}}_\ell) \to H^i(Y, \overline{\mathbb{Q}}_\ell)$ in Theorem 1.1 (ii) emerged from discussion with Takahiro Tsushima. After we wrote this paper, we are informed that Tsushima is also working on a similar problem in the case where $n = 2$ and F is an odd equal characteristic local field. He also pays attention to the map $H^i_c(Y, \overline{\mathbb{Q}}_\ell) \to H^i(Y, \overline{\mathbb{Q}}_\ell)$, but his basic strategy is to find a family of affinoids with good reduction in the Lubin-Tate spaces at suitable finite levels, which is different from ours.

The outline of this paper is as follows. In Section 2, we give an analogous result as Theorem 1.1 in the finite level setting by using the theory of formal nearby cycles. In Section 3, we deduce Theorem 1.1 from the finite level case. To enable this

step, we should allow bad reduction in Section 2, even if the reduction appearing in Theorem 1.1 is the perfection of a smooth affine scheme over κ. In Section 4, we recall the definition of the Lubin-Tate tower and the statement of the non-abelian Lubin-Tate theory. We also verify that Theorem 1.1 is applicable to an affinoid in the Lubin-Tate space at infinite level. In Section 5, we explicitly describe the Langlands parameter of a depth 0 supercuspidal representation. We use a formal model of an affinoid whose reduction becomes the perfection of the Deligne-Lusztig variety for GL_n. This formal model can be seen as the infinite level version of a piece of the model constructed by Yoshida [Yos10]. In Section 6, we give a description of the Langlands parameter of a simple supercuspidal representation. Here we use results in [IT15a] and [IT16]. Appendix A contains some results for algebras over the valuation ring k° which do not satisfy any finiteness condition. The results on the cohomology of the Artin-Schreier sheaves in Appendix B are used in Section 6.

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2 Specialization map: the case of finite level

Let k be a complete algebraically closed non-archimedean field. We write k° for the valuation ring of k and $\mathfrak{m}$ the maximal ideal of k°. The residue field $k^\circ/\mathfrak{m}$ of k° is denoted by κ. We fix a prime number ℓ invertible in κ and an integer $m \geq 1$, and put $\Lambda = \mathbb{Z}/\ell^m\mathbb{Z}$.

Let $\mathcal{X}$ be a quasi-compact admissible formal scheme over $\operatorname{Spf} k^\circ$ in the sense of [BL93]; namely, $\mathcal{X}$ is of topologically finite type and flat over $\operatorname{Spf} k^\circ$. We put $\mathcal{X}_s = \mathcal{X} \otimes_{k^\circ} \kappa$, which is a scheme of finite type over κ. The closed immersion $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}_s$ induces an isomorphism between étale sites. In the following we identify the étale sites $(\mathcal{X}_{\mathrm{red}})_{\text{ét}}$ and $\mathcal{X}_{s,\text{ét}}$ by this isomorphism.

As in [Hub96, §1.9] and [Hub98a, §2], we can attach to $\mathcal{X}$ an adic space $d(\mathcal{X})$ of finite type over $\operatorname{Spa}(k, k^\circ)$. Moreover, we have a morphism of sites $\lambda_{\mathcal{X}} \colon d(\mathcal{X})_{\text{ét}} \to (\mathcal{X}_{\mathrm{red}})_{\text{ét}} = \mathcal{X}_{s,\text{ét}}$. We denote the right derived functor of $\lambda_{\mathcal{X}}$ by $R\Psi_{\mathcal{X}}$ (or simply by $R\Psi$), and call it the formal nearby cycle functor. By adjointness, we have a natural map $\mathrm{sp}^* \colon \Lambda \to R\Psi_{\mathcal{X}}\Lambda$.

Let us assume that $\mathcal{X}$ is pseudo-compactifiable in the sense of [Mie14b, Definition 4.24]. Then, by [Mie14b, Corollary 4.29], there exists a functorial isomorphism $R\Gamma_c(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda) \xrightarrow{\cong} R\Gamma_c(d(\mathcal{X}), \Lambda)$. Hence the map $\mathrm{sp}^* \colon \Lambda \to R\Psi_{\mathcal{X}}\Lambda$ induces a morphism

$$R\Gamma_c(\mathcal{X}_s, \Lambda) \to R\Gamma_c(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda) \xrightarrow{\cong} R\Gamma_c(d(\mathcal{X}), \Lambda),$$

which is also denoted by sp^*.

In the following, assume that $d(\mathcal{X})$ is smooth over $\mathrm{Spa}(k, k^\circ)$. Since $\mathcal{X}$ is admissible, the dimensions of $\mathcal{X}_s$ and $d(\mathcal{X})$ are equal, for which we write d. Then, we have the trace map $\mathrm{Tr}_{d(\mathcal{X})}\colon R\Gamma_c(d(\mathcal{X}), \Lambda) \to \Lambda(-d)[-2d]$. By composing the isomorphism above, we obtain a map $R\Gamma_c(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda) \to \Lambda(-d)[-2d]$. If we denote the structure morphism $\mathcal{X}_s \to \mathrm{Spec}\,\kappa$ by f_s, $R\Gamma_c(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda)$ is identical to $Rf_{s!}R\Psi_{\mathcal{X}}\Lambda$. Hence, by adjointness we have the map $\mathrm{cosp}^*\colon R\Psi_{\mathcal{X}}\Lambda \to Rf_s^!\Lambda(-d)[-2d]$.

The main result in this section is as follows:

Theorem 2.1 *The composite of* $\Lambda \xrightarrow{\mathrm{sp}^*} R\Psi_{\mathcal{X}}\Lambda \xrightarrow{\mathrm{cosp}^*} Rf_s^!\Lambda(-d)[-2d]$ *is equal to the Gysin map* $\mathrm{Gys}_{f_s}\colon \Lambda \to Rf_s^!\Lambda(-d)[-2d]$ *with respect to* f_s*, that is, the adjoint of the trace map* $\mathrm{Tr}_{f_s}\colon Rf_{s!}\Lambda \to \Lambda(-d)[-2d]$.

Lemma 2.2 *To prove Theorem 2.1, we may assume that* $\mathcal{X}$ *is affine.*

Proof. Since $\mathcal{X}_s$ is d-dimensional, we have $H_c^i(\mathcal{X}_s, \Lambda(d)) = 0$ for $i > 2d$. Therefore, by adjointness, it suffices to show that the composite of

$$H_c^{2d}(\mathcal{X}_s, \Lambda(d)) \xrightarrow{\mathrm{sp}^*} H_c^{2d}(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda(d)) \xrightarrow{\cong} H_c^{2d}(d(\mathcal{X}), \Lambda(d)) \xrightarrow{\mathrm{Tr}_{d(\mathcal{X})}} \Lambda$$

is equal to $\mathrm{Tr}_{\mathcal{X}_s}\colon H_c^{2d}(\mathcal{X}_s, \Lambda(d)) \to \Lambda$.

Let $\mathcal{U}$ be an open formal subscheme of $\mathcal{X}$ such that $\dim(\mathcal{X}_s \setminus \mathcal{U}_s) < d$. We prove that Theorem 2.1 for $\mathcal{U}$ implies that for $\mathcal{X}$. Consider the following diagram:

$$\begin{tikzcd}
H_c^{2d}(\mathcal{U}_s, \Lambda(d)) \arrow[r, "\mathrm{sp}^*"] \arrow[d] & H_c^{2d}(\mathcal{U}_s, R\Psi_{\mathcal{U}}\Lambda(d)) \arrow[r, "\cong"] \arrow[d] & H_c^{2d}(d(\mathcal{U}), \Lambda(d)) \arrow[r, "\mathrm{Tr}_{d(\mathcal{U})}"] \arrow[d] & \Lambda \arrow[d, equal] \\
H_c^{2d}(\mathcal{X}_s, \Lambda(d)) \arrow[r, "\mathrm{sp}^*"] & H_c^{2d}(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda(d)) \arrow[r, "\cong"] & H_c^{2d}(d(\mathcal{X}), \Lambda(d)) \arrow[r, "\mathrm{Tr}_{d(\mathcal{X})}"] & \Lambda.
\end{tikzcd}$$

Since the restriction of $\mathrm{sp}^*\colon \Lambda \to R\Psi_{\mathcal{X}}\Lambda$ equals $\mathrm{sp}^*\colon \Lambda \to R\Psi_{\mathcal{U}}\Lambda$, the left rectangle is commutative. By [Mie14b, Lemma 4.28 (iii)], the middle rectangle commutes. As $d(\mathcal{U}) \hookrightarrow d(\mathcal{X})$ is an open immersion, the right rectangle is also commutative. Therefore, if Theorem 2.1 holds for $\mathcal{U}$, then the composite of

$$H_c^{2d}(\mathcal{U}_s, \Lambda(d)) \to H_c^{2d}(\mathcal{X}_s, \Lambda(d)) \xrightarrow{\mathrm{sp}^*} H_c^{2d}(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda(d)) \xrightarrow{\cong} H_c^{2d}(d(\mathcal{X}), \Lambda(d)) \xrightarrow{\mathrm{Tr}_{d(\mathcal{X})}} \Lambda$$

coincides with $\mathrm{Tr}_{\mathcal{U}_s}\colon H_c^{2d}(\mathcal{U}_s, \Lambda(d)) \to \Lambda$, which is equal to the composite of

$$H_c^{2d}(\mathcal{U}_s, \Lambda(d)) \to H_c^{2d}(\mathcal{X}_s, \Lambda(d)) \xrightarrow{\mathrm{Tr}_{\mathcal{X}_s}} \Lambda.$$

On the other hand, by the exact sequence

$$H_c^{2d}(\mathcal{U}_s, \Lambda(d)) \to H_c^{2d}(\mathcal{X}_s, \Lambda(d)) \to H_c^{2d}(\mathcal{X}_s \setminus \mathcal{U}_s, \Lambda(d))$$

and the assumption $\dim(\mathcal{X}_s \setminus \mathcal{U}_s) < d$, the map $H_c^{2d}(\mathcal{U}_s, \Lambda(d)) \to H_c^{2d}(\mathcal{X}_s, \Lambda(d))$ is surjective. Hence we conclude that the composite of

$$H_c^{2d}(\mathcal{X}_s, \Lambda(d)) \xrightarrow{\mathrm{sp}^*} H_c^{2d}(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda(d)) \xrightarrow{\cong} H_c^{2d}(d(\mathcal{X}), \Lambda(d)) \xrightarrow{\mathrm{Tr}_{d(\mathcal{X})}} \Lambda$$

is equal to $\mathrm{Tr}_{\mathcal{X}_s}\colon H_c^{2d}(\mathcal{X}_s,\Lambda(d))\to\Lambda$.

By this observation, we may replace $\mathcal{X}$ by its open formal subscheme $\mathcal{U}$ satisfying $\dim(\mathcal{X}_s\backslash\mathcal{U}_s)<d$. First we remove from $\mathcal{X}$ the intersections of irreducible components of $\mathcal{X}_s$. Then, by considering each d-dimensional connected component, we may assume that $\mathcal{X}_s$ is irreducible. Finally, by taking an arbitrary non-empty affine open formal subscheme of $\mathcal{X}$, we may reduce to the case where $\mathcal{X}$ is affine. ■

Later we assume that $\mathcal{X}$ is affine. Fix a non-zero element $\varpi\in\mathfrak{m}$.

Lemma 2.3 *Let $\mathcal{X}$ be an affine formal scheme of topologically finite type and flat over* $\mathrm{Spf}\,k^\circ$. *Assume that $d(\mathcal{X})$ is d-dimensional and smooth over* $\mathrm{Spa}(k,k^\circ)$. *Then there exists an affine scheme X over* $\mathrm{Spec}\,k^\circ$ *satisfying the following conditions:*

- *X is of finite presentation and flat over k°,*
- *the generic fiber of X is d-dimensional and smooth over k, and*
- *the ϖ-adic formal completion of X is isomorphic to $\mathcal{X}$ over* $\mathrm{Spf}\,k^\circ$.

Proof. Write $\mathcal{X}=\mathrm{Spf}\,A$. Since $d(\mathcal{X})$ is smooth over $\mathrm{Spa}(k,k^\circ)$, A is formally smooth outside $V(\varpi)$ in the sense of [Elk73, p. 581]; see [Tem08, Proposition 3.3.2] (in [Tem08, §3.3], k is assumed to be a discrete valuation field of characteristic 0, but it does not play a role in the proof of [Tem08, Proposition 3.3.2]). Further, by [BL93, Propositions 1.1 (c), 1.3], A can be written in the form $k^\circ\langle T_1,\dots,T_n\rangle/I$, where I is a finitely presented ideal of $k^\circ\langle T_1,\dots,T_n\rangle$. Therefore, by [Elk73, Théorème 7 and Remarque 2 (c)], there exists a finitely generated k°-algebra B such that $B\otimes_{k^\circ}k$ is smooth over k and the ϖ-adic completion of B is isomorphic to A. Put $B_{\varpi\text{-tors}}=\{b\in B\mid \varpi^m b=0 \text{ for some } m\geq 0\}$. Since A is ϖ-torsion free, [FK13, Chapter 0, Proposition 7.4.5] tells us that the ϖ-adic completion of $B/B_{\varpi\text{-tors}}$ is again isomorphic to A. Hence, replacing B by $B/B_{\varpi\text{-tors}}$, we may suppose that B is flat over k°. Then, by [RG71, Corollaire 3.4.7], B is a finitely presented k°-algebra.

We put $X'=\mathrm{Spec}\,B$, and denote the structure morphism $X'\to\mathrm{Spec}\,k^\circ$ by g'. We write X'_s (resp. X'_η) for the special (resp. generic) fiber of g'. By Chevalley's semi-continuity theorem [EGA, IV, Théorème 13.1.3], the subset $Y=\{x\in X'\mid \dim_x g'^{-1}(g'(x))\geq d+1\}$ is closed. On the other hand, since $X'_s=\mathcal{X}_s$ is d-dimensional, Y is contained in X'_η. Therefore Y is equal to the disjoint union of all connected components of X'_η whose dimensions are greater than d. In particular, Y is open in X'. Put $X=X'\setminus Y$, which is an open and closed subscheme of X' containing X'_s. We will see that X satisfies the conditions in the lemma. Clearly, X is an affine scheme of finite presentation and flat over k°, and has a smooth generic fiber X_η whose dimension is at most d. Since X is an open subscheme of X' containing X'_s, the ϖ-adic completion of X coincides with that of X'. Therefore the ϖ-adic completion of X is isomorphic to $\mathcal{X}$. Hence we have an open immersion $d(\mathcal{X})\hookrightarrow X_\eta^{\mathrm{ad}}$, where X_η^{ad} denotes the adic space associated to X_η. As $d(\mathcal{X})$ is d-dimensional, we conclude that X_η is d-dimensional. This completes the proof. ■

Fix an affine scheme X as in Lemma 2.3 and denote the structure morphism $X \to \operatorname{Spec} k^\circ$ by g. We put $S = \operatorname{Spec} k^\circ$ and write s (resp. η) for the closed (resp. generic) point of S. Consider the following diagram whose rectangles are cartesian:

$$\begin{tikzcd}
\mathcal{X}_s \arrow[r, equal] \arrow[d, "f_s"] & X_s \arrow[r, "i"] \arrow[d, "g_s"] & X \arrow[d, "g"] & X_\eta \arrow[l, "j"'] \arrow[d, "g_\eta"] \\
s \arrow[r, equal] & s \arrow[r, "i"] & S & \eta. \arrow[l, "j"']
\end{tikzcd}$$

As usual, we put $R\psi_X\Lambda = i^*Rj_*\Lambda$. By adjointness, we have a natural map $\mathrm{sp}^*\colon \Lambda \to R\psi_X\Lambda$. On the other hand, since X_η is d-dimensional, we have the trace map

$$Rg_{s!}R\psi_X\Lambda \to R\psi_S Rg_{\eta!}\Lambda \xrightarrow{\mathrm{Tr}_{X_\eta}} R\psi_S\Lambda(-d)[-2d] = \Lambda(-d)[-2d].$$

The map $R\psi_X\Lambda \to Rg_s^!\Lambda(-d)[-2d]$ obtained by adjointness is denoted by cosp^*.

Lemma 2.4 *There exists a natural isomorphism $R\psi_X\Lambda \to R\Psi_{\mathcal{X}}\Lambda$, which makes the following diagram commute:*

$$\begin{tikzcd}
\Lambda \arrow[r, "\mathrm{sp}^*"] \arrow[d, equal] & R\psi_X\Lambda \arrow[r, "\mathrm{cosp}^*"] \arrow[d, "\cong"] & Rg_s^!\Lambda(-d)[-2d] \arrow[d, equal] \\
\Lambda \arrow[r, "\mathrm{sp}^*"] & R\Psi_{\mathcal{X}}\Lambda \arrow[r, "\mathrm{cosp}^*"] & Rf_s^!\Lambda(-d)[-2d].
\end{tikzcd}$$

Proof. The construction of an isomorphism $R\psi_X\Lambda \to R\Psi_{\mathcal{X}}\Lambda$ is due to [Hub96, Theorem 3.5.13]. We shall recall it briefly. Consider the following diagram of sites (see [Hub96, 3.5.12]):

$$\begin{tikzcd}
d(\mathcal{X})_{\text{ét}} \arrow[r, "e"] \arrow[d, "\lambda"] & X_{\eta,\text{ét}} \arrow[d, "j"] \\
X_{s,\text{ét}} \arrow[r, "i"] & X_{\text{ét}}.
\end{tikzcd}$$

As in [Hub96, Theorem 3.5.13], one can construct a natural morphism of functors $\phi\colon \lambda^{-1} \circ i^{-1} \to e^{-1} \circ j^{-1}$. For every sheaf L on $X_{\text{ét}}$, a map

$$\begin{aligned} R\psi_X j^*L = i^*Rj_*j^*L &\xrightarrow{\mathrm{adj}} i^*Rj_*Re_*e^*j^*L \xrightarrow{\phi} i^*Ri_*R\lambda_*e^*j^*L \\ &\xrightarrow{\mathrm{adj}} R\lambda_*e^*j^*L = R\Psi_{\mathcal{X}}e^*j^*L \end{aligned}$$

is induced. If we put $L = \Lambda$, we get the map $R\psi_X\Lambda \to R\Psi_{\mathcal{X}}\Lambda$, which is in fact an isomorphism.

Let us prove the commutativity of the left rectangle. For a sheaf L on $X_{\text{ét}}$, we have the following commutative diagram (see the subsequent Lemma 2.5 (iii)):

$$\begin{tikzcd}
L \arrow[r, "\mathrm{adj}"] \arrow[d, equal] & R(j \circ e)_*(j \circ e)^*L \arrow[r, "\phi"] & R(i \circ \lambda)_*(j \circ e)^*L \arrow[d, equal] \\
L \arrow[r, "\mathrm{adj}"] & R(i \circ \lambda)_*(i \circ \lambda)^*L \arrow[r, "\phi"] & R(i \circ \lambda)_*(j \circ e)^*L.
\end{tikzcd}$$

By the adjointness of i^* and Ri_*, we obtain the commutative diagram

$$\begin{tikzcd}
i^*L \arrow[r, "\mathrm{adj}"] \arrow[d, equal] & i^*Rj_*Re_*e^*j^*L \arrow[r, "\phi"] & i^*Ri_*R\lambda_*e^*j^*L \arrow[r, "\mathrm{adj}"] & R\lambda_*e^*j^*L \arrow[d, equal] \\
i^*L \arrow[r, "\mathrm{adj}"] & R\lambda_*\lambda^*i^*L \arrow[rr, "\phi"] & & R\lambda_*e^*j^*L.
\end{tikzcd}$$

Now put $L = \Lambda$. By definition, the composite of the top row is equal to that of $\Lambda \xrightarrow{\mathrm{sp}^*} R\psi_X\Lambda \to R\Psi_{\mathcal{X}}\Lambda$. On the other hand, by Lemma 2.5 (ii), the map $\lambda^*i^*\Lambda \xrightarrow{\phi} e^*j^*\Lambda$ is the identity on the constant sheaf Λ. Hence the composite of the bottom row equals $\mathrm{sp}^*\colon \Lambda \to R\Psi_{\mathcal{X}}\Lambda$. This concludes the commutativity of the left rectangle.

Next we consider the right rectangle. By exactly the same method as in the proof of [Mie14b, Proposition 4.42], we can check that the rectangle in the following diagram is commutative (recall that X_η^{ad} denotes the adic space over $\mathrm{Spa}(k, k^\circ)$ associated to X_η):

$$\begin{tikzcd}
H_c^{2d}(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda(d)) \arrow[d, "\cong"] & & H_c^{2d}(X_s, R\psi_X\Lambda(d)) \arrow[ll, "\cong"'] \arrow[d] \\
H_c^{2d}(d(\mathcal{X}), \Lambda(d)) \arrow[r] \arrow[dr, "\mathrm{Tr}_{d(\mathcal{X})}"'] & H_c^{2d}(X_\eta^{\mathrm{ad}}, \Lambda(d)) \arrow[d, "\mathrm{Tr}_{X_\eta^{\mathrm{ad}}}"] & H_c^{2d}(X_\eta, \Lambda(d)) \arrow[l, "\cong"'] \arrow[dl, "\mathrm{Tr}_{X_\eta}"] \\
& \Lambda. &
\end{tikzcd}$$

The lower left triangle clearly commutes. The lower right triangle is commutative by [Mie14a, Proposition 2.2]. Hence we obtain the following commutative diagram:

$$\begin{tikzcd}
R\Gamma_c(X_s, R\psi_X\Lambda(d)[2d]) \arrow[r] \arrow[d, "\cong"] & R\Gamma_c(X_\eta, \Lambda(d)[2d]) \arrow[r, "\mathrm{Tr}_{X_\eta}"] & \Lambda \arrow[d, equal] \\
R\Gamma_c(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda(d)[2d]) \arrow[r, "\cong"] & R\Gamma_c(d(\mathcal{X}), \Lambda(d)[2d]) \arrow[r, "\mathrm{Tr}_{d(\mathcal{X})}"] & \Lambda.
\end{tikzcd}$$

By adjointness, we conclude that the diagram

$$\begin{tikzcd}
R\psi_X\Lambda \arrow[r, "\mathrm{cosp}^*"] \arrow[d, "\cong"] & Rg_s^*\Lambda(-d)[-2d] \arrow[d, equal] \\
R\Psi_{\mathcal{X}}\Lambda \arrow[r, "\mathrm{cosp}^*"] & Rf_s^*\Lambda(-d)[-2d]
\end{tikzcd}$$

commutes, as desired. ■

In the proof of the lemma above, we have used the following general fact, whose proof is immediate:

Lemma 2.5 *Let $\mathcal{C}$, $\mathcal{C}'$ be sites, $f, g\colon \mathcal{C} \to \mathcal{C}'$ morphisms of sites, and $\phi\colon f^{-1} \to g^{-1}$ a morphism of functors.*

(i) *The morphism ϕ induces morphisms of functors $f^* \to g^*$ and $g_* \to f_*$, where (f^*, f_*) (resp. (g^*, g_*)) is the morphism of toposes induced by f (resp. g).*

(ii) *For a constant sheaf L on $\mathcal{C}'$, the morphism $\phi\colon f^*L \to g^*L$ is the identity.*

(iii) *The composite of $\mathrm{id} \xrightarrow{\mathrm{adj}} Rf_*f^* \xrightarrow{\phi} Rf_*g^*$ equals that of $\mathrm{id} \xrightarrow{\mathrm{adj}} Rg_*g^* \xrightarrow{\phi} Rf_*g^*$.*

By Lemma 2.4, Theorem 2.1 is reduced to the following lemma:

Lemma 2.6 *Let $g\colon X \to S = \operatorname{Spec} k^\circ$ be as in Lemma 2.3. Then, the composite of $\Lambda \xrightarrow{\mathrm{sp}^*} R\psi_X\Lambda \xrightarrow{\mathrm{cosp}^*} Rg_s^!\Lambda(-d)[-2d]$ is equal to the Gysin map $\mathrm{Gys}_{g_s}\colon \Lambda \to Rg_s^!\Lambda(-d)[-2d]$ with respect to g_s.*

Proof. By adjointness, it suffices to show that the composite of

$$Rg_{s!}\Lambda(d)[2d] \xrightarrow{\mathrm{sp}^*} Rg_{s!}R\psi_X\Lambda(d)[2d] \to R\psi_S Rg_{\eta!}\Lambda(d)[2d] \xrightarrow{\mathrm{Tr}_{X_\eta}=\mathrm{Tr}_{g_\eta}} R\psi_S\Lambda \qquad (*)$$

is equal to that of $Rg_{s!}\Lambda(d)[2d] \xrightarrow{\mathrm{Tr}_{g_s}} \Lambda \xrightarrow{\cong} R\psi_S\Lambda$. As $g\colon X \to S$ is of finite presentation and flat, we have the trace map $\mathrm{Tr}_g\colon Rg_!\Lambda(d)[2d] \to \Lambda$ (see [SGA4, Exposé XVIII, Théorème 2.9]). Since the trace map is compatible with base change, it induces the following commutative diagram:

$$\begin{tikzcd}
Rg_!\Lambda(d)[2d] \arrow[r, "\mathrm{adj}"] \arrow[d, "\mathrm{Tr}_g"] & Rg_!Rj_*j^*\Lambda(d)[2d] \arrow[r] & Rj_*Rg_{\eta!}\Lambda(d)[2d] \arrow[d, "Rj_*\,\mathrm{Tr}_{g_\eta}"] \\
\Lambda \arrow[rr, "\mathrm{adj}"] & & Rj_*j^*\Lambda \arrow[r, equal] & Rj_*\Lambda.
\end{tikzcd}$$

By taking i^*, we obtain the commutativity of the lower rectangle of the following diagram:

$$\begin{tikzcd}
Rg_{s!}\Lambda(d)[2d] \arrow[r, "\mathrm{sp}^*"] \arrow[d, "\cong"] & Rg_{s!}R\psi_X\Lambda(d)[2d] \arrow[d, "\cong"] \\
i^*Rg_!\Lambda(d)[2d] \arrow[r, "\mathrm{adj}"] \arrow[dd, "i^*\,\mathrm{Tr}_g"] & i^*Rg_!Rj_*j^*\Lambda(d)[2d] \arrow[d] \\
& R\psi_S Rg_{\eta!}\Lambda(d)[2d] \arrow[d, "R\psi_S\,\mathrm{Tr}_{g_\eta}"] \\
\Lambda \arrow[r] & R\psi_S\Lambda.
\end{tikzcd}$$

The upper rectangle is clearly commutative. Hence the composite of $(*)$ is equal to that of

$$Rg_{s!}\Lambda(d)[2d] \xrightarrow{\cong} i^*Rg_!\Lambda(d)[2d] \xrightarrow{i^*\,\mathrm{Tr}_g} \Lambda \xrightarrow{\cong} R\psi_S\Lambda.$$

Since the trace map is compatible with base change, the composite of the first two maps equals Tr_{g_s}. This concludes the proof. ■

Corollary 2.7 *Let Y be a purely d-dimensional separated smooth scheme of finite type over κ. Assume that a finite surjective morphism $\pi\colon \mathcal{X}_s \to Y$ over κ is given. Take a decomposition $Y = \coprod_j Y_j$ into connected components, and let $\delta_j \geq 1$ be the generic degree of $\pi|_{Y_j}\colon \pi^{-1}(Y_j) \to Y_j$. We denote by $\deg \pi$ the locally constant function on Y that equals δ_j on Y_j. It induces an endomorphism $\Lambda \xrightarrow{\times \deg \pi} \Lambda$ of a sheaf over Y, and thus that of $H^i_c(Y,\Lambda)$. Concretely, it is described as follows:*

$$\times \deg \pi\colon H^i_c(Y,\Lambda) = \bigoplus_j H^i_c(Y_j,\Lambda) \xrightarrow{\bigoplus_j(\times\delta_j)} \bigoplus_j H^i_c(Y_j,\Lambda) = H^i_c(Y,\Lambda).$$

Then, we have the following commutative diagram:

$$\begin{tikzcd}
H^i_c(Y,\Lambda) \arrow[r, "\pi^*"] \arrow[d, "\times\deg\pi"'] & H^i_c(\mathcal{X}_s,\Lambda) \arrow[r, "\mathrm{sp}^*"] \arrow[d, "\mathrm{Gys}_{f_s}"] & H^i_c(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda) \arrow[r, "\cong"] \arrow[d, equal] & H^i_c(d(\mathcal{X}),\Lambda) \arrow[d, equal] \\
H^i_c(Y,\Lambda) \arrow[d] & H^i_c(\mathcal{X}_s, Rf_s^!\Lambda(-d)[-2d]) \arrow[l, "\pi_*"'] \arrow[d] & H^i_c(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda) \arrow[l, "\mathrm{cosp}^*"'] \arrow[r, "\cong"] \arrow[d] & H^i_c(d(\mathcal{X}),\Lambda) \arrow[d] \\
H^i(Y,\Lambda) & H^i(\mathcal{X}_s, Rf_s^!\Lambda(-d)[-2d]) \arrow[l, "\pi_*"'] & H^i(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda) \arrow[l, "\mathrm{cosp}^*"'] \arrow[r, equal] & H^i(d(\mathcal{X}),\Lambda).
\end{tikzcd}$$

Here π_* are the maps induced from

$$\pi_* Rf_s^!\Lambda(-d)[-2d] = R\pi_! R\pi^! Rh^!\Lambda(-d)[-2d] \underset{\cong}{\xleftarrow{\mathrm{Gys}_h}} R\pi_! R\pi^!\Lambda \xrightarrow{\mathrm{adj}} \Lambda,$$

where $h\colon Y \to \operatorname{Spec}\kappa$ denotes the structure map of Y.

Proof. By Theorem 2.1, the upper middle rectangle commutes. To see the commutativity of the upper left rectangle, it suffices to prove that the composite of

$$\begin{aligned}
\Lambda \xrightarrow{\mathrm{adj}} \pi_*\pi^*\Lambda = R\pi_!\Lambda &\xrightarrow{\mathrm{Gys}_{f_s}} R\pi_! Rf_s^!\Lambda(-d)[-2d] \\
&= R\pi_! R\pi^! Rh^!\Lambda(-d)[-2d] \underset{\cong}{\xleftarrow{\mathrm{Gys}_h}} R\pi_! R\pi^!\Lambda \xrightarrow{\mathrm{adj}} \Lambda \qquad (*)
\end{aligned}$$

is equal to the multiplication by $\deg \pi$. Since this problem is local on Y, we may assume that Y is connected. Further, by shrinking Y, we may assume that $\pi\colon \mathcal{X}_s \to Y$ is flat ([EGA, IV, Théorème 6.9.1]). Then, as the trace map is compatible with composition, we can easily observe that the composite of $(*)$ is equal to that of

$$\Lambda \xrightarrow{\mathrm{adj}} \pi_*\pi^*\Lambda = \pi_!\pi^*\Lambda \xrightarrow{\mathrm{Tr}_\pi} \Lambda.$$

Hence the claim follows from [SGA4, Exposé XVIII, Théorème 2.9 (Var 4)].

The commutativity of the lower right rectangle follows from the construction of the isomorphism $H^i_c(\mathcal{X}_s, R\Psi_{\mathcal{X}}\Lambda) \to H^i_c(d(\mathcal{X}),\Lambda)$ (see [Mie14b, Definition 4.27]) and the proof of [Mie14b, Lemma 4.8]. The other rectangles are obviously commutative.■

Note that $H^i_c(d(\mathcal{X}),\mathbb{Z}_\ell) = \varprojlim_m H^i_c(d(\mathcal{X}),\mathbb{Z}/\ell^m\mathbb{Z})$ by [Hub98b, Theorem 3.1]; recall that we are assuming that $d(\mathcal{X})$ is smooth, thus locally algebraic by [Hub96, (1.7.7)]. Hence, by taking projective limit, we obtain the specialization maps

$$\mathrm{sp}^*\colon H^i_c(\mathcal{X}_s,\mathbb{Z}_\ell) \to H^i_c(d(\mathcal{X}),\mathbb{Z}_\ell), \quad \mathrm{sp}^*\colon H^i_c(\mathcal{X}_s,\overline{\mathbb{Q}}_\ell) \to H^i_c(d(\mathcal{X}),\overline{\mathbb{Q}}_\ell)$$

between ℓ-adic cohomology.

Theorem 2.8 *Let Y and $\pi\colon \mathcal{X}_s \to Y$ be as in Corollary 2.7. Let V be a subspace of $H^i_c(Y,\overline{\mathbb{Q}}_\ell)$ such that the composite $V \hookrightarrow H^i_c(Y,\overline{\mathbb{Q}}_\ell) \to H^i(Y,\overline{\mathbb{Q}}_\ell)$ is an injection. Then, for any open immersion $d(\mathcal{X}) \hookrightarrow Z$ into an adic space Z which is locally of finite type, separated and taut over $\mathrm{Spa}(k,k^\circ)$, the composite*

$$V \hookrightarrow H^i_c(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\pi^*} H^i_c(\mathcal{X}_s,\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} H^i_c(d(\mathcal{X}),\overline{\mathbb{Q}}_\ell) \to H^i_c(Z,\overline{\mathbb{Q}}_\ell)$$

is also an injection.

Proof. By Corollary 2.7, we have the following commutative diagram for each integer $m \geq 1$:

$$\begin{tikzcd}
H^i_c(Y,\mathbb{Z}/\ell^m\mathbb{Z}) \arrow[r, "\mathrm{sp}^*\circ\pi^*"] \arrow[d, "\times\deg\pi"'] & H^i_c(d(\mathcal{X}),\mathbb{Z}/\ell^m\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^i_c(Z,\mathbb{Z}/\ell^m\mathbb{Z}) \arrow[d, equal] \\
H^i_c(Y,\mathbb{Z}/\ell^m\mathbb{Z}) \arrow[d] & H^i_c(d(\mathcal{X}),\mathbb{Z}/\ell^m\mathbb{Z}) \arrow[l, "\pi_*\circ\mathrm{cosp}^*"'] \arrow[r] \arrow[d] & H^i_c(Z,\mathbb{Z}/\ell^m\mathbb{Z}) \arrow[d] \\
H^i(Y,\mathbb{Z}/\ell^m\mathbb{Z}) & H^i(d(\mathcal{X}),\mathbb{Z}/\ell^m\mathbb{Z}) \arrow[l, "\pi_*\circ\mathrm{cosp}^*"'] & H^i(Z,\mathbb{Z}/\ell^m\mathbb{Z}). \arrow[l]
\end{tikzcd}$$

Therefore, the composite $\phi\colon H^i_c(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\times\deg\pi} H^i_c(Y,\overline{\mathbb{Q}}_\ell) \to H^i(Y,\overline{\mathbb{Q}}_\ell)$ can be decomposed into

$$H^i_c(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*\circ\pi^*} H^i_c(d(\mathcal{X}),\overline{\mathbb{Q}}_\ell) \xrightarrow{(*)} \left(\varprojlim_m H^i_c(Z,\mathbb{Z}/\ell^m\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell \to H^i(Y,\overline{\mathbb{Q}}_\ell).$$

Clearly $(*)$ is the composite of

$$H^i_c(d(\mathcal{X}),\overline{\mathbb{Q}}_\ell) \to H^i_c(Z,\overline{\mathbb{Q}}_\ell) \to \left(\varprojlim_m H^i_c(Z,\mathbb{Z}/\ell^m\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell.$$

Hence ϕ has a decomposition

$$H^i_c(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*\circ\pi^*} H^i_c(d(\mathcal{X}),\overline{\mathbb{Q}}_\ell) \to H^i_c(Z,\overline{\mathbb{Q}}_\ell) \to H^i(Y,\overline{\mathbb{Q}}_\ell).$$

Since $\phi|_V\colon V \hookrightarrow H^i_c(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\phi} H^i(Y,\overline{\mathbb{Q}}_\ell)$ is the composite of

$$V \hookrightarrow H^i_c(Y,\overline{\mathbb{Q}}_\ell) \to H^i(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\times\deg\pi} H^i(Y,\overline{\mathbb{Q}}_\ell)$$

and $\times\deg\pi$ is an isomorphism, $\phi|_V$ is an injection by the assumption on V. Therefore, the composite

$$V \hookrightarrow H^i_c(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*\circ\pi^*} H^i_c(d(\mathcal{X}),\overline{\mathbb{Q}}_\ell) \to H^i_c(Z,\overline{\mathbb{Q}}_\ell)$$

is also an injection, as desired. ■

3 Specialization map: the case of infinite level

Let k, k°, $\mathfrak{m}$ and κ be as in the previous section. Assume that the characteristic of κ is $p > 0$. Fix a non-zero element $\varpi \in \mathfrak{m}$.

In this section, we consider a flat k°-algebra A which is ϖ-adically complete. We denote by $\widetilde{A}$ the integral closure of A in $A[1/\varpi]$. Then, $(A[1/\varpi], \widetilde{A})$ is an affinoid (k, k°)-algebra (for the topology on $A[1/\varpi]$, see [Hub93, Examples 1.1 (iv)]). Suppose that A is equipped with an action over k° of a profinite group K_0. Then, $\widetilde{A}$ is a K_0-stable subring of $A[1/\varpi]$.

We will assume two conditions on A and the action of K_0. The first is that the affinoid (k, k°)-algebra $(A[1/\varpi], \widetilde{A})$ is obtained as a limit of a tower of smooth affinoid (k, k°)-algebras. The precise statement is as follows:

Assumption 3.1 For each open normal subgroup K of K_0, we are given a complete f-adic k-algebra B_K of topologically finite type endowed with an action of K_0/K. For open normal subgroups K, K' with $K' \subset K$, we are given a continuous K_0-equivariant k-homomorphism $B_K \to B_{K'}$ so that $\{B_K\}_{K \lhd K_0}$ becomes an inductive system. Further we impose the following conditions:

(a) For an open normal subgroup K of K_0, the adic space $X_K = \mathrm{Spa}(B_K, B_K^\circ)$ is smooth over $\mathrm{Spa}(k, k^\circ)$.

(b) For open normal subgroups K, K' of K_0 with $K' \subset K$, the transition map $X_{K'} \to X_K$ is a finite étale Galois covering with Galois group K/K'. Note that this implies that $X_{K'} \to X_K$ is surjective and $B_K \to (B_{K'})^{K/K'}$ is an isomorphism.

(c) There exists a K_0-equivariant homomorphism $\{(B_K, B_K^\circ)\}_{K \lhd K_0} \to (A[1/\varpi], \widetilde{A})$ of inductive systems of affinoid (k, k°)-algebras (here $(A[1/\varpi], \widetilde{A})$ means the constant inductive system), such that
   - the induced continuous map $\mathrm{Spa}(A[1/\varpi], \widetilde{A}) \to \varprojlim_{K \lhd K_0} \mathrm{Spa}(B_K, B_K^\circ)$ is a homeomorphism, and
   - the induced homomorphism $\varinjlim_K B_K \to A[1/\varpi]$ has dense image.

The second assumption is that the reduction of A is the perfection of a finitely generated κ-algebra. More precisely,

Assumption 3.2 There exist a finitely generated κ-algebra R and an isomorphism

$$\varinjlim_{x \mapsto x^p} R \xrightarrow{\cong} A/\mathfrak{m}A$$

of κ-algebras. We fix such an R and an isomorphism, and put $Y = \mathrm{Spec}\, R$.

Under these assumptions, we will compare two cohomology groups $H^i_c(Y, \overline{\mathbb{Q}}_\ell)$ and $\varinjlim_K H^i_c(X_K, \overline{\mathbb{Q}}_\ell)$. We begin with basic properties of the inductive system $\{B_K\}_{K \lhd K_0}$.

Lemma 3.3 *For each open normal subgroup K of K_0, the map* $\mathrm{Spa}(A[1/\varpi], \widetilde{A}) \to \mathrm{Spa}(B_K, B_K^\circ) = X_K$ *is surjective.*

Proof. By Assumption 3.1, it suffices to show the surjectivity of the map

$$\varprojlim_{K' \lhd K_0} X_{K'} = \varprojlim_{K' \lhd K_0, K' \subset K} X_{K'} \to X_K.$$

This follows from the surjectivity and the finiteness of $X_{K'} \to X_K$; recall that the filtered projective limit of non-empty finite sets is non-empty. ■

Lemma 3.4 *The homomorphism $B_K \to A[1/\varpi]$ is injective.*

Proof. First note that B_K is reduced, as $\mathrm{Spa}(B_K, B_K^\circ)$ is assumed to be smooth over $\mathrm{Spa}(k, k^\circ)$. Therefore we have $0 = \sqrt{0} = \bigcap_{\mathfrak{n} \subset B_K} \mathfrak{n}$, where $\mathfrak{n}$ runs through maximal ideals of B_K (recall that B_K is a Jacobson ring; see [FK13, Chapter 0, Proposition 9.3.10]).

We denote the homomorphism $B_K \to A[1/\varpi]$ by ϕ. Take an arbitrary maximal ideal $\mathfrak{n}$ of B_K, and let $v_\mathfrak{n} \in \mathrm{Spa}(B_K, B_K^\circ)$ be the classical point corresponding to $\mathfrak{n}$. By Lemma 3.3, we can find $\widetilde{v}_\mathfrak{n} \in \mathrm{Spa}(A[1/\varpi], \widetilde{A})$ satisfying $v_\mathfrak{n} = \widetilde{v}_\mathfrak{n} \circ \phi$. For $a \in \mathrm{Ker}\,\phi$, we have $v_\mathfrak{n}(a) = \widetilde{v}_\mathfrak{n}(\phi(a)) = \widetilde{v}_\mathfrak{n}(0) = 0$. Hence $\mathrm{Ker}\,\phi \subset \mathrm{supp}\, v_\mathfrak{n} = \mathfrak{n}$. Therefore we conclude that $\mathrm{Ker}\,\phi \subset \bigcap_{\mathfrak{n} \subset B_K} \mathfrak{n} = 0$, namely, ϕ is an injection. ■

By this lemma, we regard B_K as a k-subalgebra of $A[1/\varpi]$.

Definition 3.5 We put $B' = \bigcup_{K \lhd K_0} B_K \subset A[1/\varpi]$, $A' = A \cap B'$ and $\widetilde{A}' = \widetilde{A} \cap B'$. Note that B' is a dense K_0-stable k-subalgebra of $A[1/\varpi]$, while A' and $\widetilde{A}'$ are K_0-stable k°-subalgebras of B'. For an open normal subgroup K of K_0, we put $A_K = A \cap B_K$ and $\widetilde{A}_K = \widetilde{A} \cap B_K$. These are open K_0-stable k°-subalgebras of B_K.

Lemma 3.6 *The inclusion $A' \hookrightarrow A$ induces isomorphisms $A'/\varpi^m A' \xrightarrow{\cong} A/\varpi^m A$ for every integer $m \geq 0$ and $A'/\mathfrak{m}A' \xrightarrow{\cong} A/\mathfrak{m}A$.*

Proof. Let $m \geq 0$ be an integer. Since A' is dense in A, we have $A = A' + \varpi^m A$. Thus the map $A'/\varpi^m A' \to A/\varpi^m A$ is surjective. Let us show the injectivity. Take an element $x \in A' \cap \varpi^m A$ and write $x = \varpi^m y$ with $y \in A$. Since B' is a k-subalgebra of $A[1/\varpi]$, we have $y = \varpi^{-m} x \in \varpi^{-m} A' \subset B'$. Hence y lies in $A \cap B' = A'$ and x lies in $\varpi^m A'$. This means that $A'/\varpi^m A' \to A/\varpi^m A$ is injective.

In particular, the map $A'/\varpi A' \to A/\varpi A$ is an isomorphism. By taking the base change $(-) \otimes_{k^\circ/\varpi k^\circ} k^\circ/\mathfrak{m}$, we conclude that the homomorphism $A'/\mathfrak{m}A' \to A/\mathfrak{m}A$ is also an isomorphism. ■

Lemma 3.7 *For an open normal subgroup K of K_0, we have $(B')^K = B_K$, $(A')^K = A_K$ and $(\widetilde{A}')^K = \widetilde{A}_K$.*

Proof. For an open normal subgroup K' of K_0 contained in K, we have $(B_{K'})^K = B_K$ by assumption. By taking the union with respect to such K', we conclude that $(B')^K = B_K$. From this equality, we can deduce $(A')^K = A \cap (B')^K = A \cap B_K = A_K$ and $(\widetilde{A}')^K = \widetilde{A} \cap (B')^K = \widetilde{A} \cap B_K = \widetilde{A}_K$. ■

Lemma 3.8 (i) *The ring $\widetilde{A}'$ is integral over its subring A'.*

(ii) *For an open normal subgroup K of K_0, $\widetilde{A}_K$ is integral over its subring A_K.*

(iii) *For an open normal subgroup K of K_0, $\widetilde{A}_K$ coincides with B_K°.*

Proof. We prove (i). Take $x \in \widetilde{A}' \subset \widetilde{A}$. Since $\widetilde{A}$ is integral over A, there exist $n \geq 1$ and $a_1, \dots, a_n \in A$ such that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$. As $A \subset \widetilde{A} \subset A[1/\varpi]$, we can find an integer $m \geq 0$ such that $\varpi^m x^i \in A$ for every $0 \leq i \leq n-1$. Since B' is dense in $A[1/\varpi]$, we have $(a_i + \varpi^m A) \cap B' \neq \varnothing$; in other words, for each i there exists $b_i \in B'$ such that $b_i - a_i \in \varpi^m A$. Note that $b_i \in A \cap B' = A'$.

For such b_i's, we have

$$x^n + b_1 x^{n-1} + \cdots + b_n = (b_1 - a_1)x^{n-1} + \cdots + (b_n - a_n).$$

The left hand side lies in B', while the right hand side lies in A. If we put $c = (b_1 - a_1)x^{n-1} + \cdots + (b_n - a_n)$, it is an element of $A \cap B' = A'$ and x is a root of the monic polynomial $T^n + b_1 T^{n-1} + \cdots + b_n - c \in A'[T]$. Hence x is integral over A'.

Next we prove (ii). By (i), it suffices to show that A' is integral over A_K. Take an arbitrary element $a \in A'$ and an open normal subgroup K' of K_0 contained in K such that $a \in B_{K'}$. Then $a \in A' \cap B_{K'} = A_{K'}$. The element a is a root of the monic polynomial $\prod_{g \in K/K'}(T - g(a))$, whose coefficients belong to $(A_{K'})^K = A_K$. Therefore a is integral over A_K.

Finally we prove (iii). Since we have a homomorphism $(B_K, B_K^\circ) \to (A[1/\varpi], \widetilde{A})$ of affinoid (k, k°)-algebras, we have $B_K^\circ \subset \widetilde{A} \cap B_K = \widetilde{A}_K$. Let us prove the reverse inclusion $\widetilde{A}_K \subset B_K^\circ$. Take $a \in \widetilde{A}_K$. By [Hub93, Lemma 3.3 (i)], we have

$$B_K^\circ = \{x \in B_K \mid v(x) \leq 1 \text{ for every } v \in \mathrm{Spa}(B_K, B_K^\circ)\}.$$

Therefore, it suffices to observe that $v(a) \leq 1$ for every $v \in \mathrm{Spa}(B_K, B_K^\circ)$. By Lemma 3.3, there exists $\widetilde{v} \in \mathrm{Spa}(A[1/\varpi], \widetilde{A})$ such that $\widetilde{v}|_{B_K} = v$. As $a \in \widetilde{A}_K \subset \widetilde{A}$, we have $v(a) = \widetilde{v}(a) \leq 1$, as desired. ■

Corollary 3.9 *For an open normal subgroup K of K_0, B_K° contains A_K over which B_K° is integral.*

Proof. Clear from Lemma 3.8 (ii), (iii). ■

Recall that a k°-algebra C is said to be topologically finitely generated if it is isomorphic to a quotient of $k^\circ\langle T_1, \dots, T_n\rangle$ for some $n \geq 0$ (see [FK13, Definition 8.4.1 and Proposition 8.4.4]). Such C is ϖ-adically complete. We say that $c_1, \dots, c_{n'} \in C$

are topological generators of C if the k°-homomorphism $k^\circ\langle T_1, \dots, T_{n'}\rangle \to C$ given by $T_i \mapsto c_i$ is surjective.

We would like to write the k°-algebra A' as an inductive limit of topologically finitely generated k°-algebras. First we focus on each B_K. In the following proposition, we slightly change the notation for simplicity.

Proposition 3.10 *Let B be a complete f-adic k-algebra of topologically finite type. Let A be an open subring of B° such that B° is integral over A. We write $\mathcal{D}_A$ for the set consisting of topologically finitely generated k°-subalgebras of A which are rings of definition of A (or equivalently, B). Then, the following hold:*

(i) *The set $\mathcal{D}_A$ is a filtered ordered set with respect to inclusions.*

(ii) *We have $A = \bigcup_{A_0 \in \mathcal{D}_A} A_0$.*

(iii) *For $A_0 \in \mathcal{D}_A$, A and B° are integral over A_0.*

Proof. First we prove that $\mathcal{D}_A$ is non-empty. Since B is of topologically finite type, we can take a surjective continuous k-homomorphism $\phi\colon k\langle T_1, \dots, T_n\rangle \to B$. By the open mapping theorem, ϕ is an open map. Moreover, by [Hub93, Proposition 1.10], ϕ is an adic map. Hence the image B_0 of $k^\circ\langle T_1, \dots, T_n\rangle$ under ϕ is a topologically finitely generated k°-subalgebra of B which is a ring of definition of B. Therefore ϖB_0 is an ideal of definition of B_0 (see [Hub93, Proposition 1.5 (ii)]), and B_0 is ϖ-adically complete. Further, by [Hub94, proof of Lemma 4.4], B° is the integral closure of B_0 inside B.

Put $b_i = \phi(T_i) \in B_0 \subset B^\circ$. Since B° is integral over A, for each $1 \le i \le n$ there exists a monic polynomial $f_i \in A[T]$ such that $f_i(b_i) = 0$. Since A is open, there exists an integer $m \ge 1$ such that $\varpi^m B_0 \subset A$. Let A_0 be the k°-subalgebra of A generated by the coefficients of $f_1, \dots, f_n$ and the elements of $\varpi^m B_0$. Since A_0 contains $\varpi^m B_0$, it is an open subring of B. As the coefficients of $f_1, \dots, f_n$ are power-bounded in B, A_0 is bounded in B. Hence A_0 is a ring of definition of B. In particular, A_0 is ϖ-adically complete. We will show that $A_0 \in \mathcal{D}_A$.

Let us prove that $B_0 \subset A_0[b_1, \dots, b_n] \subset B^\circ$. For every $x \in B_0$, there exists $f \in k^\circ[T_1, \dots, T_n]$ such that $x - f(b_1, \dots, b_n) \in \varpi^m B_0$. Thus $x \in f(b_1, \dots, b_n) + \varpi^m B_0 \subset A_0[b_1, \dots, b_n]$. The inclusion $A_0[b_1, \dots, b_n] \subset B^\circ$ is clear. We put $B_0' = A_0[b_1, \dots, b_n]$. As $b_1, \dots, b_n$ are power-bounded in B, B_0' is also a ring of definition of B. Since b_i is integral over A_0 by construction, B_0' is a finite A_0-algebra. Note also that $\varpi^m B_0 \subset A_0$ implies that $\varpi^m B_0' \subset A_0$.

Now we prove $A_0 \in \mathcal{D}_A$. The k°-algebra $A_0/\varpi^m B_0$ is generated by the coefficients of $f_1, \dots, f_n$. Therefore, its quotient $A_0/\varpi^m B_0'$ is also finitely generated over k°. Let us observe that $A_0/\varpi^{m+1} B_0'$ is finitely generated over k°. For simplicity, we set $R = A_0/\varpi^{m+1} B_0'$ and $I = \varpi^m B_0'/\varpi^{m+1} B_0'$. Since B_0' is finitely generated as an A_0-module, I is a finitely generated ideal of R. As $m \ge 1$, we have $I^2 = 0$. Take $a_1, \dots, a_r \in R$ so that their images generate R/I as a k°-algebra, and generators $x_1, \dots, x_s \in I$ as an R-module. Then it is immediate to see that R is generated by $a_1, \dots, a_r, x_1, \dots, x_s$ as a k°-algebra. Since $\varpi^{m+1} B_0' \subset \varpi A_0$, we conclude that

$A_0/\varpi A_0$ is a finitely generated k°-algebra. By [FK13, Chapter 0, Proposition 8.4.2], this means that A_0 is topologically finitely generated over k°, as desired.

Next we prove (iii). For $A_0 \in \mathcal{D}_A$, we can take a surjection $\phi\colon k\langle T_1, \dots, T_n\rangle \to B$ above so that $\phi(k^\circ\langle T_1, \dots, T_n\rangle) = A_0$. As mentioned above, B° is integral over A_0. Hence A is also integral over A_0.

We prove (i). Take $A_0, A_0' \in \mathcal{D}_A$ and topological generators $a_1', \dots, a_n'$ of A_0' over k°. Then, by the same argument as in the third paragraph of this proof, we can show that $A_0' \subset A_0[a_1', \dots, a_n']$. Clearly $A_0[a_1', \dots, a_n']$ is open and bounded in B, hence a ring of definition of B. In particular, $A_0[a_1', \dots, a_n']$ is ϖ-adically complete. On the other hand, by (iii), $a_i' \in A$ is integral over A_0. Therefore $A_0[a_1', \dots, a_n']$ is a finite A_0-algebra, hence topologically finitely generated k°-algebra. Thus $A_0[a_1', \dots, a_n']$ is an element of $\mathcal{D}_A$ containing A_0 and A_0'.

Finally we prove (ii). Fix $A_0 \in \mathcal{D}_A$. For $a \in A$, consider the subring $A_0[a]$ of A. Since $a \in A \subset B^\circ$, $A_0[a]$ is a ring of definition of B. In particular, $A_0[a]$ is ϖ-adically complete. By (iii), a is integral over A_0. Therefore $A_0[a]$ is finite over A_0, and thus topologically finitely generated over k°. Hence $A_0[a]$ is an element of $\mathcal{D}_A$ containing a. This completes the proof. ■

We return to the original setting. For an open normal subgroup K of K_0, we simply write $\mathcal{D}_K$ for $\mathcal{D}_{A_K}$.

Lemma 3.11 *For open normal subgroups K, K' of K_0 with $K' \subset K$ and $A_0 \in \mathcal{D}_K$, there exists an element of $\mathcal{D}_{K'}$ containing A_0.*

Proof. Take an arbitrary element A_0' of $\mathcal{D}_{K'}$ and topological generators $a_1, \dots, a_n$ of A_0 over k°. Clearly $A_0'[a_1, \dots, a_n]$ is a subring of $A_{K'}$. Since $a_1, \dots, a_n$ are power-bounded in B_K and $B_K \to B_{K'}$ is adic, they are power-bounded in $B_{K'}$. Therefore $A_0'[a_1, \dots, a_n]$ is a ring of definition of $B_{K'}$. By Proposition 3.10 (iii), $a_i \in A_K \subset A_{K'}$ is integral over A_0'. Therefore $A_0'[a_1, \dots, a_n]$ is a finite A_0'-algebra. In particular, it is topologically finitely generated over k°. Thus $A_0'[a_1, \dots, a_n]$ belongs to $\mathcal{D}_{K'}$. On the other hand, since $A_0'[a_1, \dots, a_n] \cap B_K$ is open in B_K, there exists $m \geq 1$ such that $\varpi^m A_0 \subset A_0'[a_1, \dots, a_n]$. Therefore, by the same way as in the third paragraph of the proof of Proposition 3.10, we can check that $A_0 \subset A_0'[a_1, \dots, a_n]$. This concludes the proof. ■

Put $\mathcal{D} = \coprod_{K \lhd K_0} \mathcal{D}_K$. We define a partial order on $\mathcal{D}$ as follows: for $A_0 \in \mathcal{D}_K$ and $A_0' \in \mathcal{D}_{K'}$, $A_0 \leq A_0'$ if $K' \subset K$ and $A_0 \subset A_0'$.

Corollary 3.12 *This makes $\mathcal{D}$ a filtered ordered set.*

Proof. Clear from Proposition 3.10 (i) and Lemma 3.11. ■

Corollary 3.13 *We have*

$$A' = \bigcup_{A_0 \in \mathcal{D}} A_0 \cong \varinjlim_{A_0 \in \mathcal{D}} A_0, \quad A/\mathfrak{m}A \cong \varinjlim_{A_0 \in \mathcal{D}} A_0/\mathfrak{m}A_0.$$

Proof. The first claim is clear from Proposition 3.10 (ii) and Corollary 3.12. As tensor product commutes with inductive limit, we have $A'/\mathfrak{m}A' \cong \varinjlim_{A_0 \in \mathcal{D}} A_0/\mathfrak{m}A_0$. Therefore the second follows from Lemma 3.6. ■

Lemma 3.14 *Let A_0, A_0' be elements of $\mathcal{D}$ such that $A_0 \leq A_0'$. Then, the inclusion $A_0 \hookrightarrow A_0'$ is finite and continuous.*

Proof. Let K and K' be open normal subgroups of K_0 such that $A_0 \in \mathcal{D}_K$ and $A_0' \in \mathcal{D}_{K'}$. Since $(A_{K'})^{K/K'} = A_K$, $A_{K'}$ is integral over A_K. By Proposition 3.10 (iii), A_K is integral over A_0. As $A_0 \subset A_0' \subset A_{K'}$, we conclude that A_0' is integral over A_0. In particular, $A_0'/\varpi A_0'$ is integral over $A_0/\varpi A_0$. On the other hand, as A_0' is a topologically finitely generated k°-algebra, $A_0'/\varpi A_0'$ is finitely generated over k°. In particular, $A_0'/\varpi A_0'$ is finitely generated over $A_0/\varpi A_0$. Thus $A_0'/\varpi A_0'$ is finite over $A_0/\varpi A_0$. By [FK13, Chapter 0, Proposition 7.2.4], we conclude that A_0' is finite over A_0.

Since the topology of A_0 and A_0' are ϖ-adic ([Hub93, Proposition 1.5 (ii)]), the k°-homomorphism $A_0 \to A_0'$ is clearly continuous. ■

Recall that we have fixed a κ-isomorphism $\varinjlim_{x \mapsto x^p} R \cong A/\mathfrak{m}A$ in Assumption 3.2. In particular, we are given a κ-homomorphism $R \to A/\mathfrak{m}A$.

Definition 3.15 Let $\mathcal{C}$ be the set consisting of pairs (A_0, π), where

- A_0 is an element of $\mathcal{D}$, and
- $\pi\colon R \to A_0/\mathfrak{m}A_0$ is a finite κ-homomorphism such that the composite of π and the homomorphism $A_0/\mathfrak{m}A_0 \to A/\mathfrak{m}A$ induced from the inclusion $A_0 \hookrightarrow A' \hookrightarrow A$ is equal to the fixed homomorphism $R \to A/\mathfrak{m}A$.

It is equipped with the partial order induced from that of $\mathcal{D}$. Namely, $(A_0, \pi) \leq (A_0', \pi')$ if and only if $A_0 \leq A_0'$ and the composite $R \xrightarrow{\pi} A_0/\mathfrak{m}A_0 \to A_0'/\mathfrak{m}A_0'$ equals π'. By Corollaries 3.12, 3.13 and Lemma 3.14, this order is filtered (recall that R is finitely generated over κ).

Lemma 3.16 (i) *The set $\mathcal{C}$ is non-empty.*

(ii) *For $(A_0, \pi) \in \mathcal{C}$, the morphism $\operatorname{Spec} A_0/\mathfrak{m}A_0 \to \operatorname{Spec} R$ induced by π is a homeomorphism on the underlying topological spaces.*

Proof. We prove (i). Since $A/\mathfrak{m}A = \varinjlim_{A_0 \in \mathcal{D}} A_0/\mathfrak{m}A_0$ and R is finitely presented over κ, we can find $A_1 \in \mathcal{D}$ and a κ-homomorphism $R \to A_1/\mathfrak{m}A_1$ such that the composite $R \to A_1/\mathfrak{m}A_1 \to A/\mathfrak{m}A$ is the fixed one. On the other hand, Since $A/\mathfrak{m}A = \varinjlim_{x \to x^p} R$, there exist a factorization $A_1/\mathfrak{m}A_1 \to R \to A/\mathfrak{m}A$ of the homomorphism $A_1/\mathfrak{m}A_1 \to A/\mathfrak{m}A$ and an integer $m \geq 1$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[rr, "x \mapsto x^{p^m}"] \arrow[dr] & & R \arrow[r, "(*)"] & A/\mathfrak{m}A. \\ & A_1/\mathfrak{m}A_1 \arrow[ur] & & \end{tikzcd}$$

Apply the same argument to the homomorphism $(*)$, we can find $A_0 \in \mathcal{D}$ with $A_1 \leq A_0$ and a homomorphism $R \to A_0/\mathfrak{m}A_0$ such that the right triangle of the following diagram commutes:

$$\begin{tikzcd}
R \arrow[rr, "x\mapsto x^{p^m}"] \arrow[dr] & & R \arrow[rr] \arrow[dr, "(1)"] & & A/\mathfrak{m}A. \\
& A_1/\mathfrak{m}A_1 \arrow[ur] \arrow[rr, "(2)"'] & & A_0/\mathfrak{m}A_0 \arrow[ur] &
\end{tikzcd}$$

Replacing A_0 by larger one, we can also make the lower triangle commute.

By Lemma 3.14, the homomorphism (2) is finite. Therefore, the homomorphism (1) is also finite. If we write π for the composite $R \xrightarrow{x\mapsto x^{p^m}} R \xrightarrow{(1)} A_0/\mathfrak{m}A_0$, it is a finite κ-homomorphism, and (A_0, π) belongs to $\mathcal{C}$.

Next we prove (ii). Since $\operatorname{Spec} A_0/\mathfrak{m}A_0 \to \operatorname{Spec} R$ is finite and therefore closed, it suffices to show that it is a bijection. By the same argument as above, we can construct the commutative diagram

$$\begin{tikzcd}
R \arrow[rr, "x\mapsto x^{p^{m'}}"] \arrow[dr] & & R \arrow[rr] \arrow[dr] & & A/\mathfrak{m}A, \\
& A_0/\mathfrak{m}A_0 \arrow[ur] \arrow[rr] & & A_2/\mathfrak{m}A_2 \arrow[ur] &
\end{tikzcd}$$

where $m' \geq 1$ is an integer and $A_2 \in \mathcal{D}$ with $A_0 \leq A_2$. We will observe that $\operatorname{Spec} A_2/\mathfrak{m}A_2 \to \operatorname{Spec} A_0/\mathfrak{m}A_0$ is surjective. Take open normal subgroups K, K' of K_0 such that $A_0 \in \mathcal{D}_K$ and $A_2 \in \mathcal{D}_{K'}$. Put $\mathcal{X} = \operatorname{Spf} A_0$ and $\mathcal{X}' = \operatorname{Spf} A_2$. By Lemma 3.14, a morphism $\mathcal{X}' \to \mathcal{X}$ is induced. By Proposition 3.10 (iii), we have $d(\mathcal{X}) = \operatorname{Spa}(B_K, B_K^\circ) = X_K$ and $d(\mathcal{X}') = \operatorname{Spa}(B_{K'}, B_{K'}^\circ) = X_{K'}$. Further, we have the following commutative diagram of topological spaces (see [Hub96, Proposition 1.9.1]):

$$\begin{tikzcd}
\mathcal{X}'_{\mathrm{red}} \arrow[d] & d(\mathcal{X}') \arrow[l, "\lambda_{\mathcal{X}'}"'] \arrow[d] \arrow[r, equal] & X_{K'} \arrow[d] \\
\mathcal{X}_{\mathrm{red}} & d(\mathcal{X}) \arrow[l, "\lambda_{\mathcal{X}}"'] \arrow[r, equal] & X_K.
\end{tikzcd}$$

By assumption the map $X_{K'} \to X_K$ is surjective. On the other hand, as $\mathcal{X}$ is flat over k°, the map $\lambda_{\mathcal{X}}$ is surjective (see [FK13, Chapter 2, Proposition 3.1.5, Theorem A.4.7]). Hence the map $\mathcal{X}'_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ is also surjective. As topological spaces, $\mathcal{X}_{\mathrm{red}}$ (resp. $\mathcal{X}'_{\mathrm{red}}$) is identified with $\operatorname{Spec} A_0/\mathfrak{m}A_0$ (resp. $\operatorname{Spec} A_2/\mathfrak{m}A_2$). Therefore the map $\operatorname{Spec} A_2/\mathfrak{m}A_2 \to \operatorname{Spec} A_0/\mathfrak{m}A_0$ is surjective, as desired.

By this surjectivity, the map $\operatorname{Spec} R \to \operatorname{Spec} A_0/\mathfrak{m}A_0$ is also surjective. On the other hand, as the composite $\operatorname{Spec} R \to \operatorname{Spec} A_0/\mathfrak{m}A_0 \to \operatorname{Spec} R$ is the identity on the underlying topological space, $\operatorname{Spec} R \to \operatorname{Spec} A_0/\mathfrak{m}A_0$ is injective. Hence $\operatorname{Spec} R \to \operatorname{Spec} A_0/\mathfrak{m}A_0$ is bijective, and so is $\operatorname{Spec} A_0/\mathfrak{m}A_0 \to \operatorname{Spec} R$, as desired.■

Now we can construct the specialization map $\mathrm{sp}^*\colon H^i_c(Y,\overline{\mathbb{Q}}_\ell) \to \varinjlim_K H^i_c(X_K,\overline{\mathbb{Q}}_\ell)$.

Definition 3.17 Take $(A_0,\pi)\in\mathcal{C}$ and put $\mathcal{X}=\mathrm{Spf}\,A_0$. Then, by Proposition 3.10 (iii), we have $d(\mathcal{X})=\mathrm{Spa}(B_K,B_K^\circ)=X_K$, where K is the open normal subgroup of K_0 such that $A_0\in\mathcal{D}_K$. We define $\mathrm{sp}^*\colon H^i_c(Y,\overline{\mathbb{Q}}_\ell)\to\varinjlim_K H^i_c(X_K,\overline{\mathbb{Q}}_\ell)$ by the composite of

$$H^i_c(Y,\overline{\mathbb{Q}}_\ell)\xrightarrow{\pi^*}H^i_c(\mathcal{X}_s,\overline{\mathbb{Q}}_\ell)\xrightarrow{\mathrm{sp}^*}H^i_c(d(\mathcal{X}),\overline{\mathbb{Q}}_\ell)=H^i_c(X_K,\overline{\mathbb{Q}}_\ell)\to\varinjlim_K H^i_c(X_K,\overline{\mathbb{Q}}_\ell),$$

where we also write π for the morphism $\mathcal{X}_s\to Y$ induced by $\pi\colon R\to A_0/\mathfrak{m}A_0$.

Lemma 3.18 *The map* sp^* *in Definition 3.17 is independent of the choice of* $(A_0,\pi)\in\mathcal{C}$.

Proof. Since $\mathcal{C}$ is filtered, it suffices to compare sp^* for (A_0,π) and that for (A_0',π') under the condition $(A_0,\pi)\le(A_0',\pi')$. Put $\mathcal{X}'=\mathrm{Spf}\,A_0'$. By Lemma 3.14, we have a finite morphism $\mathcal{X}'\to\mathcal{X}$, which we denote by f. We have only to prove that the following diagram is commutative for $\Lambda=\mathbb{Z}/\ell^m\mathbb{Z}$:

$$\begin{tikzcd}
H^i_c(\mathcal{X}_s,\Lambda) \arrow[r, "\mathrm{sp}^*"] \arrow[d, "f_s^*"] & H^i_c(d(\mathcal{X}),\Lambda) \arrow[d, "d(f)^*"] \\
H^i_c(\mathcal{X}'_s,\Lambda) \arrow[r, "\mathrm{sp}^*"] & H^i_c(d(\mathcal{X}'),\Lambda).
\end{tikzcd}$$

Note that we have the following commutative diagram of sites ([Hub96, (3.5.4)]):

$$\begin{tikzcd}
d(\mathcal{X}')_{\text{ét}} \arrow[r, "\lambda_{\mathcal{X}'}"] \arrow[d, "d(f)"] & \mathcal{X}'_{s,\text{ét}} \arrow[d, "f_s"] \\
d(\mathcal{X})_{\text{ét}} \arrow[r, "\lambda_{\mathcal{X}}"] & \mathcal{X}_{s,\text{ét}}.
\end{tikzcd}$$

This gives rise to the following commutative diagram in $D^+(\mathcal{X}_{s,\text{ét}},\Lambda)$:

$$\begin{tikzcd}
\Lambda \arrow[r, "\mathrm{adj}"] \arrow[dd, "\mathrm{adj}"] & R\lambda_{\mathcal{X}*}\Lambda \arrow[d, "\mathrm{adj}"] \\
 & R\lambda_{\mathcal{X}*}Rd(f)_*\Lambda \arrow[d, equal] \\
Rf_{s*}\Lambda \arrow[r, "\mathrm{adj}"] & Rf_{s*}R\lambda_{\mathcal{X}'*}\Lambda.
\end{tikzcd}$$

By taking H^i_c of this diagram, we get the desired commutativity. ■

Next we discuss the functoriality of sp^*. Here we only consider automorphisms of A which come from those on the tower $\{X_K\}$.

Definition 3.19 Let σ be an automorphism of A which preserves k° (i.e., $\sigma(k^\circ) = k^\circ$). We say that σ is of finite level if it satisfies $\sigma(A') = A'$.

Lemma 3.20 *Let σ be an automorphism of finite level of A. Then the following hold.*

(i) *For each open normal subgroup K of K_0, there exists an open normal subgroup K' of K_0 such that $\sigma(B_K) \subset B_{K'}$.*

(ii) *The automorphism σ naturally induces an automorphism of $\varinjlim_K H^i_c(X_K, \overline{\mathbb{Q}}_\ell)$.*

(iii) *For any $A_0 \in \mathcal{D}$, there exists $A_1 \in \mathcal{D}$ such that $\sigma(A_0) \subset A_1$. For such A_1, the homomorphism $\sigma\colon A_0 \to A_1$ is finite and continuous.*

Proof. First note that $\sigma(\varpi)$ belongs to $\mathfrak{m}$, hence there exists an integer $r \geq 1$ such that $\sigma(\varpi)^r \in \varpi k^\circ$. In particular $\sigma\colon k^\circ \to k^\circ$ is continuous. The same holds for σ^{-1}.

We prove (i). For an open normal subgroup K of K_0, take $A_0 \in \mathcal{D}_K$ and its topological generators $a_1, \dots, a_n \in A_0$ over k°. Since σ preserves A', we have $\sigma(a_1), \dots, \sigma(a_n) \in A'$. Therefore, we can find an open normal subgroup K' of K_0 such that $\sigma(a_1), \dots, \sigma(a_n)$ are fixed by K'. We will prove that $\sigma(A_0)$ is fixed by K'. Take $g \in K'$ and $x \in A_0$. Then, for every integer $m \geq 1$, there exists $f \in k^\circ[T_1, \dots, T_n]$ such that $x - f(a_1, \dots, a_n) \in \varpi^{rm} A_0 \subset \varpi^{rm} A$. Then we have $\sigma(x) - (\sigma f)(\sigma(a_1), \dots, \sigma(a_n)) \in \sigma(\varpi)^{rm} A \subset \varpi^m A$. Since the action of $g \in K'$ is an automorphism over k° and $\sigma(a_1), \dots, \sigma(a_n)$ are fixed by K', we obtain $g(\sigma(x)) - (\sigma f)(\sigma(a_1), \dots, \sigma(a_n)) \in \varpi^m A$, and thus $\sigma(x) - g(\sigma(x)) \in \varpi^m A$ for every $m \geq 1$. As A is ϖ-adically complete, this means that $\sigma(x)$ is fixed by $g \in K'$. Hence we have $\sigma(A_0) \subset (A')^{K'} = A_{K'}$. Now we conclude that

$$\sigma(B_K) = \sigma(A_0[1/\varpi]) = \sigma(A_0)[1/\varpi] \subset A_{K'}[1/\varpi] = B_{K'},$$

hence (i).

If we twist the k-algebra structure of $B_{K'}$ by $\sigma\colon k \to k$, $B_{K'}$ is again a complete f-adic k-algebra of topologically finite type, and the map $\sigma\colon B_K \to B_{K'}$ becomes a homomorphism over k. By [BGR84, Theorem 6.1.3/1], σ is automatically continuous. The composite of $B_K \xrightarrow{\sigma} B_{K'} \hookrightarrow B'$ is equal to that of $B_K \hookrightarrow B' \xrightarrow{\sigma} B'$, which is integral (see the proof of Lemma 3.8 (ii)). Hence $\sigma\colon B_K \to B_{K'}$ is also integral. By [BGR84, Theorem 6.3.5/1], this means that $\sigma\colon B_K \to B_{K'}$ is finite. Therefore a finite morphism $X_{K'} = \mathrm{Spa}(B_{K'}, B^\circ_{K'}) \to \mathrm{Spa}(B_K, B^\circ_K) = X_K$ and a map $H^i_c(X_K, \overline{\mathbb{Q}}_\ell) \to H^i_c(X_{K'}, \overline{\mathbb{Q}}_\ell)$ are induced. Passing to the inductive limit, we obtain $\varinjlim_K H^i_c(X_K, \overline{\mathbb{Q}}_\ell) \to \varinjlim_K H^i_c(X_K, \overline{\mathbb{Q}}_\ell)$, which we denote by σ_*. Applying the same argument to σ^{-1}, we obtain $(\sigma^{-1})_*\colon \varinjlim_K H^i_c(X_K, \overline{\mathbb{Q}}_\ell) \to \varinjlim_K H^i_c(X_K, \overline{\mathbb{Q}}_\ell)$. It is immediate to show that σ_* and $(\sigma^{-1})_*$ are inverse to each other. Therefore σ_* is an isomorphism. This completes the proof of (ii).

Let us prove (iii). By the argument above, for $A_0 \in \mathcal{D}_K$, we can find an open normal subgroup K' of K_0 such that $\sigma(A_0) \subset A_{K'}$, and the map $\sigma\colon A_0 \to A_{K'}$ is continuous. Take a system of topological generators $a_1, \dots, a_n$ of A_0 over k° and an element $A'_0 \in \mathcal{D}_{K'}$. Put $A_1 = A'_0[\sigma(a_1), \dots, \sigma(a_n)]$. Since $\sigma(a_1), \dots, \sigma(a_n) \in A_{K'}$,

we can observe that $A_1 \in \mathcal{D}_{K'}$ in the same way as in the proof of Lemma 3.11. On the other hand, by the continuity of σ, there exists an integer $m \geq 1$ such that $\sigma(\varpi^m A_0) \subset A'_0$. Therefore, by the same way as in the third paragraph of the proof of Proposition 3.10, we can check that $\sigma(A_0) \subset A_1$. This concludes the existence of A_1.

Next, we take an arbitrary $A_1 \in \mathcal{D}$ such that $\sigma(A_0) \subset A_1$. The composite of $A_0 \xrightarrow{\sigma} A_1 \hookrightarrow A'$ is equal to that of $A_0 \hookrightarrow A' \xrightarrow{\sigma} A'$, which is integral by Proposition 3.10 (iii) and the proof of Lemma 3.8 (ii). Hence $\sigma\colon A_0 \to A_1$ is also integral. This implies that $\sigma\colon A_0/\varpi A_0 \to A_1/\sigma(\varpi)A_1$ is integral. On the other hand, as $A_1/\sigma(\varpi)A_1$ is finitely generated over k° and σ preserves k°, the map $\sigma\colon A_0/\varpi A_0 \to A_1/\sigma(\varpi)A_1$ is of finite type. Hence $\sigma\colon A_0/\varpi A_0 \to A_1/\sigma(\varpi)A_1$ is finite, and so is $\sigma\colon A_0 \to A_1$ by [FK13, Chapter 0, Proposition 7.2.4]. By [Hub93, Proposition 1.5 (ii)], the topology of A_0 and A_1 are ϖ-adic. Since $\sigma(\varpi^{rm}A_0) \subset \varpi^m A_1$, $\sigma\colon A_0 \to A_1$ is continuous. This completes the proof of (iii). ■

Proposition 3.21 *Let σ be an automorphism of finite level of A and $\overline{\sigma}$ an automorphism of R. Assume that the fixed isomorphism $A/\mathfrak{m}A \cong \varinjlim_{x \mapsto x^p} R$ is compatible with σ and $\overline{\sigma}$. Then, the specialization map $\mathrm{sp}^*\colon H^i_c(Y, \overline{\mathbb{Q}}_\ell) \to \varinjlim_K H^i_c(X_K, \overline{\mathbb{Q}}_\ell)$ is compatible with the actions of $\overline{\sigma}$ and σ.*

Proof. We may assume that $R \neq 0$. Then $A/\mathfrak{m}A \neq 0$. Since $\kappa \subset A/\mathfrak{m}A$ is preserved by σ, we conclude that $\kappa \subset R$ is preserved by $\overline{\sigma}$ and $\overline{\sigma}|_\kappa = \sigma|_{k^\circ} \bmod \mathfrak{m}$.

Take $(A_0, \pi) \in \mathcal{C}$. By Lemma 3.20 (iii), we can find $A_1 \in \mathcal{D}$ containing $\sigma(A_0)$. We write π' for the composite $R \xrightarrow{\overline{\sigma}^{-1}} R \xrightarrow{\pi} A_0/\mathfrak{m}A_0 \xrightarrow{\sigma} A_1/\mathfrak{m}A_1$, which is a κ-homomorphism. By the commutative diagram

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, "\overline{\sigma}"] & A_0/\mathfrak{m}A_0 \arrow[r] \arrow[d, "\sigma"] & A/\mathfrak{m}A \arrow[d, "\sigma"] \\ R \arrow[r, "\pi'"] & A_1/\mathfrak{m}A_1 \arrow[r] & A/\mathfrak{m}A \end{tikzcd}$$

and Lemma 3.20 (iii), we can observe that $(A_1, \pi') \in \mathcal{C}$. Put $\mathcal{X} = \operatorname{Spf} A_0$ and $\mathcal{X}' = \operatorname{Spf} A_1$. By Lemma 3.20 (iii), the homomorphism $\sigma\colon A_0 \to A_1$ induces a finite morphism $\sigma\colon \mathcal{X}' \to \mathcal{X}$ of formal schemes. As in the proof of Lemma 3.18, the commutative diagram of sites

$$\begin{tikzcd} d(\mathcal{X}')_{\text{ét}} \arrow[r, "\lambda_{\mathcal{X}'}"] \arrow[d, "\sigma"] & \mathcal{X}'_{s,\text{ét}} \arrow[r, "\pi'"] \arrow[d, "\sigma"] & Y_{\text{ét}} \arrow[d, "\overline{\sigma}"] \\ d(\mathcal{X})_{\text{ét}} \arrow[r, "\lambda_{\mathcal{X}}"] & \mathcal{X}_{s,\text{ét}} \arrow[r, "\pi"] & Y_{\text{ét}} \end{tikzcd}$$

gives rise to the following commutative diagram for $\Lambda = \mathbb{Z}/\ell^m\mathbb{Z}$:

$$\begin{tikzcd} H^i_c(Y, \Lambda) \arrow[r, "\pi"] \arrow[d, "\overline{\sigma}"] & H^i_c(\mathcal{X}_s, \Lambda) \arrow[r, "\mathrm{sp}^*"] \arrow[d, "\sigma"] & H^i_c(d(\mathcal{X}), \Lambda) \arrow[r, equal] \arrow[d, "\sigma"] & H^i_c(X_K, \Lambda) \arrow[d, "\sigma"] \\ H^i_c(Y, \Lambda) \arrow[r, "\pi'"] & H^i_c(\mathcal{X}'_s, \Lambda) \arrow[r, "\mathrm{sp}^*"] & H^i_c(d(\mathcal{X}'), \Lambda) \arrow[r, equal] & H^i_c(X_{K'}, \Lambda). \end{tikzcd}$$

Here K (resp. K') is the open normal subgroup of K_0 such that $A_0 \in \mathcal{D}_K$ (resp. $A_1 \in \mathcal{D}_{K'}$). By the construction of the specialization map, this implies the commutativity of the diagram

$$\begin{array}{ccc} H^i_c(Y,\overline{\mathbb{Q}}_\ell) & \xrightarrow{\mathrm{sp}^*} & \varinjlim_K H^i_c(X_K,\overline{\mathbb{Q}}_\ell) \\ \downarrow{\scriptstyle\overline{\sigma}} & & \downarrow{\scriptstyle\sigma} \\ H^i_c(Y,\overline{\mathbb{Q}}_\ell) & \xrightarrow{\mathrm{sp}^*} & \varinjlim_K H^i_c(X_K,\overline{\mathbb{Q}}_\ell), \end{array}$$

which concludes the proof. ■

Here is the infinite level version of Theorem 2.8.

Theorem 3.22 *Let $\{Z_K\}_{K\lhd K_0}$ be a projective system of adic spaces locally of finite type, separated and taut over $\mathrm{Spa}(k,k^\circ)$ with proper transition maps. Suppose that we are given a system of open immersions $\{X_K \hookrightarrow Z_K\}_{K\lhd K_0}$ over $\mathrm{Spa}(k,k^\circ)$ which makes the cartesian diagram*

$$\begin{array}{ccc} X_{K'} & \longrightarrow & Z_{K'} \\ \downarrow & & \downarrow \\ X_K & \longrightarrow & Z_K \end{array}$$

for every open normal subgroups K, K' of K_0 with $K' \subset K$.

Assume that Y is pure-dimensional and smooth over κ. Let V be a subspace of $H^i_c(Y,\overline{\mathbb{Q}}_\ell)$ such that the composite $V \hookrightarrow H^i_c(Y,\overline{\mathbb{Q}}_\ell) \to H^i(Y,\overline{\mathbb{Q}}_\ell)$ is an injection. Then, the composite

$$V \hookrightarrow H^i_c(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} \varinjlim_K H^i_c(X_K,\overline{\mathbb{Q}}_\ell) \to \varinjlim_K H^i_c(Z_K,\overline{\mathbb{Q}}_\ell)$$

is also an injection.

Proof. Fix $(A_0,\pi) \in \mathcal{C}$ with $A_0 \in \mathcal{D}_K$. Let K' be an open normal subgroup of K_0 contained in K. Then, by Lemmas 3.11 and 3.14, we can find $(A'_0,\pi') \in \mathcal{C}$ such that $(A_0,\pi) \le (A'_0,\pi')$ and $A'_0 \in \mathcal{D}_{K'}$. Put $\mathcal{X}' = \mathrm{Spf}\, A'_0$. It is of topologically finite type and flat over $\mathrm{Spf}\, k^\circ$, and pseudo-compactifiable (see [Mie14b, Example 4.25 (i)]). By Proposition 3.10 (iii), we have $d(\mathcal{X}') = X_{K'}$, which is smooth over $\mathrm{Spa}(k,k^\circ)$. By Lemma 3.16 (ii), the morphism $\pi'\colon \mathcal{X}'_s \to \mathrm{Spec}\, R = Y$ is finite and surjective, and the dimension of $\mathcal{X}'_s$ is equal to $\dim Y$. Therefore, Theorem 2.8 tells us that the composite

$$V \hookrightarrow H^i_c(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*\circ\pi'} H^i_c(d(\mathcal{X}'),\overline{\mathbb{Q}}_\ell) = H^i_c(X_{K'},\overline{\mathbb{Q}}_\ell) \to H^i_c(Z_{K'},\overline{\mathbb{Q}}_\ell)$$

is injective. Obviously, the inductive limit of this map with respect to K' coincides with

$$V \hookrightarrow H^i_c(Y,\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} \varinjlim_{K'} H^i_c(X_{K'},\overline{\mathbb{Q}}_\ell) \to \varinjlim_{K'} H^i_c(Z_{K'},\overline{\mathbb{Q}}_\ell)$$

considered in the theorem. Thus we obtain the desired injectivity. ■

We will end this section by giving a simple but helpful example.

Example 3.23 Here we assume that $p \neq 2$. Take $\varpi' \in \mathfrak{m}$ such that $\varpi'^2 = \varpi$, and fix a sequence $(\varpi'^{1/p^m})_{m\geq 0}$ of p^mth roots of ϖ'. Put $\varpi^{1/p^m} = (\varpi'^{1/p^m})^2$. Let I be the ideal of $k^\circ\langle T^{p^{-\infty}}\rangle$ (see Example A.5) generated by $T^{2/p^m} - \varpi^{1/p^m}$ for every $m \geq 0$, and $\overline{I}$ the closure of I in $k^\circ\langle T^{p^{-\infty}}\rangle$. Let us consider $A = k^\circ\langle T^{p^{-\infty}}\rangle/\overline{I}$.

(i) The k°-algebra A is ϖ-adically complete and flat.

(ii) Define continuous k-homomorphisms

$$\phi\colon k[S]/(S^2-1) \to A[1/\varpi], \quad \psi\colon A[1/\varpi] \to k \times k$$

by $\phi(S) = \varpi'^{-1}T$ and $\psi(T^{1/p^m}) = (\varpi'^{1/p^m}, -\varpi'^{1/p^m})$. Then, ϕ and ψ are isomorphisms. In particular, $A[1/\varpi]$ is a complete f-adic k-algebra of topologically finite type.

(iii) We have $\psi(A) = \{(x,y) \in k^\circ \times k^\circ \mid x - y \in \mathfrak{m}\}$. In particular, A is a ring of definition of $A[1/\varpi]$ which is not topologically finitely generated over k°.

(iv) We have $A/\mathfrak{m}A \cong \kappa$.

(v) Let $\widetilde{A}$ be the integral closure of A in $A[1/\varpi]$. We have $\psi(\widetilde{A}) = k^\circ \times k^\circ$. Namely, $\widetilde{A}$ coincides with $A[1/\varpi]^\circ$.

(vi) Put $A_m = k^\circ\langle T_m\rangle/(T_m^2 - \varpi^{1/p^m}) = k^\circ[T_m]/(T_m^2 - \varpi^{1/p^m})$. We have a k°-homomorphism $A_m \to A$; $T_m \mapsto T^{1/p^m}$, which turns out to be injective. We write ψ_m for the composite $A_m \to A \xrightarrow{\psi} k \times k$. Then, we have $\psi_m(A_m) = \{(x,y) \in k^\circ \times k^\circ \mid x - y \in \varpi'^{1/p^m}k^\circ\}$. In particular, $A \cong \varinjlim_m A_m$.

By (i), (ii), (iv), (v), this A with the obvious action of $K_0 = 1$ satisfy Assumptions 3.1, 3.2. By (iv), $Y = \operatorname{Spec}\kappa$ is a point, while $\operatorname{Spa}(A[1/\varpi], A[1/\varpi]^\circ)$ consists of two points by (ii). In particular, the specialization map sp^* on H^0_c in Definition 3.17 is not surjective.

Proof. The assertion (i) follows from Corollary A.4, as the image of $T^2 - \varpi$ in $\kappa[T^{p^{-\infty}}]$ is non-zero.

We prove (ii). It is easy to see that ϕ and ψ are well-defined and $\psi \circ \phi$ is an isomorphism. Therefore ϕ is an injection. Put $C = k^\circ[S]/(S^2-1)$. Since $\phi(\varpi' C) \subset A$, ϕ is continuous. We shall prove that ϕ is surjective. First we prove that $A \subset \phi(C) + \varpi A$. It suffices to show that $T^{1/p^m} \in \phi(C)$. Note that $(T^{1/p^m}/\varpi'^{1/p^m})^2 = T^{2/p^m}/\varpi^{1/p^m} = 1$ in $A[1/\varpi]$. Since we are assuming $p \neq 2$, we have

$$\phi(S) = \frac{T}{\varpi'} = \left(\frac{T^{1/p^m}}{\varpi'^{1/p^m}}\right)^{p^m} = \frac{T^{1/p^m}}{\varpi'^{1/p^m}}.$$

Hence $T^{1/p^m} = \phi(\varpi'^{1/p^m}S) \in \phi(C)$, as desired. Now we prove the surjectivity of ϕ. Since $k[S]/(S^2-1)$ is ϖ-divisible, it suffices to see that every $a_0 \in A$ lies in $\phi(C)$. By $A \subset \phi(C) + \varpi A$, we can find $x_0 \in C$ and $a_1 \in A$ such that $a_0 = \phi(x_0) + \varpi a_1$. Continuing this process, we get $(x_m)_{m\geq 0}$ and $(a_m)_{m\geq 1}$ such that $x_m \in C$, $a_m \in A$

and $a_m = \phi(x_m) + \varpi a_{m+1}$. Put $x = \sum_{m=0}^{\infty} \varpi^m x_m \in C$. Since ϕ is continuous and A is ϖ-adically separated, we have

$$\phi(x) = \lim_{n\to\infty} \phi\Big(\sum_{m=0}^{n} \varpi^m x_m\Big) = \lim_{n\to\infty} (a_0 - \varpi^{n+1} a_{n+1}) = a_0.$$

Hence a_0 lies in $\phi(C)$. Therefore ϕ is a bijection. As $A \subset \phi(C)$, ϕ^{-1} is continuous. Thus ϕ is an isomorphism. Since $\psi \circ \phi$ is an isomorphism, so is ψ.

Let us prove (iii). We can easily check that $\psi(A) \subset \{(x,y) \in k^\circ \times k^\circ \mid x-y \in \mathfrak{m}\}$. We prove the reverse inclusion. Let $x, y \in k^\circ$ be elements such that $x-y \in \mathfrak{m}$. Since $(x,y) = (\psi \circ \phi)(\frac{x+y}{2} + \frac{x-y}{2}S)$, we have

$$\psi^{-1}(x,y) = \phi\Big(\frac{x+y}{2} + \frac{x-y}{2}S\Big) = \frac{x+y}{2} + \frac{x-y}{2} \cdot \frac{T}{\varpi'} = \frac{x+y}{2} + \frac{x-y}{2} \cdot \frac{T^{1/p^m}}{\varpi'^{1/p^m}}.$$

Since $x-y \in \mathfrak{m}$, for a sufficiently large m, $(x-y)/\varpi'^{1/p^m}$ belongs to k°. Hence we have $\psi^{-1}(x,y) \in A$, that is, $(x,y) \in \psi(A)$. Suppose that $\psi(A)$ is topologically finitely generated over k°, and take topological generators $(x_1,y_1), \dots, (x_n,y_n) \in \psi(A)$. Then, there exists an integer $m \geq 0$ such that $x_i - y_i \in \varpi^{1/p^m} k^\circ$ for every $1 \leq i \leq n$. Since $\psi(A)/\varpi^{1/p^m}(k^\circ \times k^\circ)$ is generated by the images of $(x_1,y_1), \dots, (x_n,y_n)$, it is included in $\{(x,x) \mid x \in k^\circ/\varpi^{1/p^m} k^\circ\}$. It is absurd, as $\psi(A)/\varpi^{1/p^m}(k^\circ \times k^\circ)$ contains an element $(\varpi^{1/p^{m+1}}, 0)$. Thus $\psi(A)$ is not topologically finitely generated over k°.

For (iv), it suffices to prove that the natural map $\kappa = k^\circ/\mathfrak{m} \to A/\mathfrak{m}A$ is surjective, or equivalently, $A = k^\circ + \mathfrak{m}A$. By (ii), we may replace A by $\psi(A)$. Take $(x,y) \in \psi(A)$. By (iii), we have $x, y \in k^\circ$ and $x - y \in \mathfrak{m}$. Therefore we have $(x,y) = (y,y) + (x-y, 0) \in k^\circ + \mathfrak{m}\psi(A)$. Now we conclude that $\psi(A) = k^\circ + \mathfrak{m}\psi(A)$.

We prove (v). Since $k^\circ \times k^\circ$ is integrally closed in $k \times k$, it suffices to show that $k^\circ \times k^\circ$ is integral over $\psi(A)$. Take any element $x = (a,b) \in k^\circ \times k^\circ$. Then (a,a) and (b,b) belong to $\psi(A)$, and x is a root of the polynomial $(T-(a,a))(T-(b,b))$ in $\psi(A)[T]$. This means that x is integral over $\psi(A)$.

Finally consider (vi). It is immediate to see that the homomorphism $A_m[1/\varpi] \to k \times k$ induced from ψ_m is an isomorphism. Therefore $A_m[1/\varpi] \to A[1/\varpi]$ is also an isomorphism. Since A_m is ϖ-torsion free, we conclude that $A_m \to A$ is an injection. Let us prove that $\psi_m(A_m) = \{(x,y) \in k^\circ \times k^\circ \mid x - y \in \varpi'^{1/p^m} k^\circ\}$. As $\psi_m(T_m) = (\varpi'^{1/p^m}, -\varpi'^{1/p^m})$, $\psi_m(A_m)$ is contained in the right hand side. Since the k°-algebra $\{(x,y) \in k^\circ \times k^\circ \mid x - y \in \varpi'^{1/p^m} k^\circ\}$ is generated by $(\varpi'^{1/p^m}, -\varpi'^{1/p^m})$, the desired equality holds. ■

4 The Lubin-Tate tower

Let F be a non-archimedean local field. We denote by $\mathcal{O}_F$ the ring of integers of F and $\mathfrak{p}_F$ the maximal ideal of $\mathcal{O}_F$. Write q for the characteristic of the residue field $\mathcal{O}_F/\mathfrak{p}_F$, and p for the characteristic of $\mathbb{F}_q = \mathcal{O}_F/\mathfrak{p}_F$. The normalized valuation of F is denoted by v_F. Fix a uniformizer $\varpi \in \mathcal{O}_F$.

We fix an algebraic closure $\overline{F}$ of F. Let F^{ur} (resp. F^{ab}) be the maximal unramified (resp. abelian) extension of F inside $\overline{F}$. Denote by $\breve{F}$, $\widehat{F}^{\mathrm{ab}}$ and C for the completion of F^{ur}, F^{ab} and $\overline{F}$, respectively. The residue field of $\breve{F}$ is denoted by $\overline{\mathbb{F}}_q$, which is an algebraic closure of $\mathbb{F}_q$. We write W_F for the Weil group of F, and $\mathrm{Art}_F\colon F^\times \xrightarrow{\cong} W_F^{\mathrm{ab}}$ for the isomorphism of the local class field theory, which is normalized so that $\mathrm{Art}_F(\varpi)$ is a lift of the geometric Frobenius automorphism on $\overline{\mathbb{F}}_q$.

We denote by **Nilp** the category of schemes over $\mathcal{O}_{\breve{F}}$ on which ϖ is locally nilpotent, and by **Set** the category of sets. For $S \in \mathbf{Nilp}$, we set $\overline{S} = S \otimes_{\mathcal{O}_{\breve{F}}} \overline{\mathbb{F}}_q$.

Let $n \geq 1$ be an integer. We fix a one-dimensional formal $\mathcal{O}_F$-module $\mathbb{X}$ of height n over $\overline{\mathbb{F}}_q$, which is unique up to isomorphism. Put $D = \mathrm{End}_{\mathcal{O}_F}(\mathbb{X}) \otimes_{\mathcal{O}_F} F$. It is known to be a central division algebra over F with invariant $1/n$.

For an integer $m \geq 0$, let $\mathcal{M}_m\colon \mathbf{Nilp} \to \mathbf{Set}$ be the functor that sends S to the set of isomorphism classes of triples (X, ρ, η), where

- X is a formal $\mathcal{O}_F$-module on S,
- $\rho\colon \mathbb{X} \otimes_{\overline{\mathbb{F}}_q} \overline{S} \to X \times_S \overline{S}$ is an $\mathcal{O}_F$-quasi-isogeny, and
- η is a Drinfeld m-level structure on X.

For an integer δ, let $\mathcal{M}_m^{(\delta)}$ be the subfunctor of $\mathcal{M}_m$ corresponding to the triple (X, ρ, η) with $\mathrm{height}_{\mathcal{O}_F} \rho = \delta$. Then, $\mathcal{M}_m^{(0)}$ can be identified with the deformation functor of $\mathbb{X}$ with Drinfeld m-level structures. In [Dri74, §4], it is proved that $\mathcal{M}_m^{(0)}$ is represented by $\mathrm{Spf}\, A_m$, where A_m is an n-dimensional Noetherian regular complete local $\mathcal{O}_{\breve{F}}$-algebra. Since $\mathcal{M}_m^{(\delta)}$ is (non-canonically) isomorphic to $\mathcal{M}_m^{(0)}$, we conclude that $\mathcal{M}_m = \coprod_{\delta \in \mathbb{Z}} \mathcal{M}_m^{(\delta)}$ is represented by a locally Noetherian formal scheme over $\mathcal{O}_{\breve{F}}$. Moreover, $\{\mathcal{M}_m\}_{m \geq 0}$ form a projective system of formal schemes with finite flat transition morphisms. We call it the Lubin-Tate tower.

The Lubin-Tate tower is equipped with several actions. First, the group $D^\times = \mathbf{QIsog}_{\mathcal{O}_F}(\mathbb{X})$ of self-quasi-isogenies of $\mathbb{X}$ acts on each $\mathcal{M}_m$ on the right. The formal scheme $\mathcal{M}_m$ also has a natural Weil descent datum in the sense of [RZ96, Definition 3.45]. These are compatible with the transition maps of $\{\mathcal{M}_m\}_{m \geq 0}$. Further, the group $\mathrm{GL}_n(F)$ acts on the tower $\{\mathcal{M}_m\}_{m \geq 0}$ on the right as a pro-object. This action may change the level m. However, the subgroup $\mathrm{GL}_n(\mathcal{O}_F)$ preserves each $\mathcal{M}_m$, and the mth principal congruence subgroup $K_m = \mathrm{Ker}(\mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p}_F^m))$ acts trivially on $\mathcal{M}_m$.

We denote by M_m the rigid generic fiber of $\mathcal{M}_m$, which is known to be an $n-1$-dimensional smooth rigid space over $\breve{F}$. From $\{\mathcal{M}_m\}_{m \geq 0}$ we obtain a projective system $\{M_m\}_{m \geq 0}$ of rigid spaces over $\breve{F}$, whose transition maps are finite, étale and Galois. For $m' \geq m \geq 0$, the action of K_m on $M_{m'}$ gives identification between $K_m/K_{m'}$ and the Galois group of $M_{m'} \to M_m$. For each compact open subgroup K of $\mathrm{GL}_n(\mathcal{O}_F)$, we can define the Lubin-Tate space M_K of level K as follows: take an integer $m \geq 0$ such that $K_m \subset K$ and put $M_K = M_m/(K/K_m)$, which is in fact independent of the choice of m. Now we obtain a projective system $\{M_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_F)}$, whose transition maps are finite and étale. It is also equipped with actions of $D^\times$

and $\mathrm{GL}_n(F)$, and a Weil descent datum.

For our purpose, it is convenient to consider the quotient of M_K by the discrete subgroup $\varpi^{\mathbb{Z}} \subset F^\times \subset D^\times$, which is denoted by $M_{K,\varpi^{\mathbb{Z}}}$. Note that the decomposition $\mathcal{M}_m = \coprod_{\delta\in\mathbb{Z}} \mathcal{M}_m^{(\delta)}$ induces $M_K = \coprod_{\delta\in\mathbb{Z}} M_K^{(\delta)}$, and $\varpi \in D^\times$ maps $M_K^{(\delta)}$ to $M_K^{(\delta+n)}$. Therefore, the quotient $M_{K,\varpi^{\mathbb{Z}}}$ is isomorphic to $\coprod_{0\le\delta<n} M_K^{(\delta)}$.

Take a prime number $\ell \neq p$. Fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and identify them. We consider the compactly supported ℓ-adic cohomology of the tower $\{M_{K,\varpi^{\mathbb{Z}}}\}_{K\subset \mathrm{GL}_n(\mathcal{O}_F)}$.

Definition 4.1 We put $H_{\mathrm{LT}} = \varinjlim_K H_c^{n-1}(M_{K,\varpi^{\mathbb{Z}}} \otimes_{\breve{F}} C, \overline{\mathbb{Q}}_\ell)$.

The group $\mathrm{GL}_n(F) \times D^\times$ naturally acts on H_{LT}. This action is known to be smooth (see [Str08, Lemma 2.5.1]). By [RZ96, Lemma 5.36], the subgroup $\varpi^{\mathbb{Z}} \subset F^\times \subset \mathrm{GL}_n(F)$ acts trivially on H_{LT}. On the other hand, by using the Weil descent datum on $M_{K,\varpi^{\mathbb{Z}}}$, one can define an action of W_F on H_{LT}. As a result, H_{LT} becomes a representation of $(\mathrm{GL}_n(F)/\varpi^{\mathbb{Z}}) \times (D^\times/\varpi^{\mathbb{Z}}) \times W_F$. The $\mathrm{GL}_n(F)$-supercuspidal part of this representation can be described by using the local Langlands correspondence and the local Jacquet-Langlands correspondence:

Theorem 4.2 (Non-abelian Lubin-Tate theory, [HT01], [Boy99]) *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ whose central character is trivial on $\varpi^{\mathbb{Z}}$. We write $\mathrm{rec}_F(\pi)$ (resp. $\mathrm{JL}(\pi)$) for the irreducible smooth representation of W_F (resp. $D^\times$) corresponding to π under the local Langlands correspondence (resp. the local Jacquet-Langlands correspondence). Then we have*

$$H_{\mathrm{LT},\pi^\vee} = \pi^\vee \boxtimes \mathrm{JL}(\pi) \boxtimes \mathrm{rec}_F(\pi)(\tfrac{1-n}{2})$$

as representations of $\mathrm{GL}_n(F) \times D^\times \times W_F$, where $H_{\mathrm{LT},\pi^\vee}$ denotes the $\pi^\vee$-isotypic part of H_{LT}.

To simplify our argument, we also use the following result, which is in some sense stronger than the theorem above:

Theorem 4.3 *The action of W_F on H_{LT} is smooth.*

Proof. Since $D^\times/\varpi^{\mathbb{Z}}$ is compact, we have the isotypic decomposition

$$H_{\mathrm{LT}} = \bigoplus_\rho H_{\mathrm{LT},\rho},$$

where ρ runs through irreducible smooth representations of $D^\times$ whose central characters are trivial on $\varpi^{\mathbb{Z}}$, and $H_{\mathrm{LT},\rho}$ denotes the ρ-isotypic part of H_{LT}. We fix ρ and denote by π the discrete series representation of $\mathrm{GL}_n(F)$ satisfying $\mathrm{JL}(\pi) = \rho$. Such π can be written as $\mathrm{St}_s(\pi')$, where s is an integer dividing n and π' is an irreducible supercuspidal representation of $\mathrm{GL}_{n/s}(F)$. By [Boy09b] and [Boy09a], we have

$$H_{\mathrm{LT},\rho} = \pi^\vee \boxtimes \rho \boxtimes \mathrm{rec}_F(\pi')(\tfrac{m}{2})$$

for some $m \in \mathbb{Z}$. Since $\mathrm{rec}_F(\pi')$ is a smooth representation of W_F, we conclude that the action of W_F on $H_{\mathrm{LT},\rho}$ is smooth. ■

Let us recall the determinant map. We write $\{\mathcal{N}_m\}_m$ for the Lubin-Tate tower with respect to $\wedge^n\mathbb{X}$, where $\wedge^n$ is the exterior product defined in [Hed14]. Then, as in [Wei, §2.5], we can construct the determinant map $\mathcal{M}_m^{(0)} \to \mathcal{N}_m^{(0)}$. On the other hand, by the classical Lubin-Tate theory, we have $\mathcal{N}_m^{(0)} = \operatorname{Spf} \mathcal{O}_{\breve{F}_m}$ where $\breve{F}_m$ denotes the completion of the mth Lubin-Tate extension of F^{ur}. Choose an $\breve{F}$-embedding $\varinjlim_m \breve{F}_m \hookrightarrow \widehat{F}^{\mathrm{ab}}$ and consider the base change $\mathcal{M}_{m,\mathcal{O}_C}^{(0)} = \mathcal{M}_m^{(0)} \widehat{\otimes}_{\mathcal{O}_{\breve{F}_m}} \mathcal{O}_C$ of $\mathcal{M}_m^{(0)}$ under $\mathcal{O}_{\breve{F}_m} \hookrightarrow \mathcal{O}_{\widehat{F}^{\mathrm{ab}}} \hookrightarrow \mathcal{O}_C$. We put

$$G^1 = \{(g, d, \sigma) \in \mathrm{GL}_n(F) \times D^\times \times W_F \mid \det(g)^{-1} \operatorname{Nrd}(d) \operatorname{Art}_F^{-1}(\sigma) = 1\},$$

where $\operatorname{Nrd}\colon D^\times \to F^\times$ denotes the reduced norm. By using the action of $\mathrm{GL}_n(F) \times D^\times$ and the Weil descent datum on $\{\mathcal{M}_m\}_m$, we can define an action of G^1 on the tower $\{\mathcal{M}_{m,\mathcal{O}_C}^{(0)}\}_m$. The action of $(g, d, \sigma) \in G^1$ makes the following diagram commute:

$$\begin{tikzcd}
\{\mathcal{M}_{m,\mathcal{O}_C}^{(0)}\}_m \arrow[r, "(g,d,\sigma)"] \arrow[d] & \{\mathcal{M}_{m,\mathcal{O}_C}^{(0)}\}_m \arrow[d] \\
\operatorname{Spf} \mathcal{O}_C \arrow[r, "\sigma^*"] & \operatorname{Spf} \mathcal{O}_C.
\end{tikzcd}$$

Passing to the rigid generic fiber, we obtain the rigid space $M_{m,C}^{(0)} = M_m^{(0)} \otimes_{\breve{F}_m} C$ over C. For an integer $m \geq 0$, put $K'_m = \mathrm{SL}_n(\mathcal{O}_F) \cap K_m$. It is easy to observe that the transition map $M_{m',C}^{(0)} \to M_{m,C}^{(0)}$ for $m' \geq m \geq 0$ is finite étale Galois with Galois group $K'_m/K'_{m'}$. Therefore, by the same method as in the case of $\{M_K\}_{K \subset \mathrm{GL}_n(\mathcal{O}_F)}$, we can extend $\{M_{m,C}^{(0)}\}_m$ to the projective system $\{M_{K,C}^{(0)}\}_{K \subset \mathrm{SL}_n(\mathcal{O}_F)}$, where K runs through open subgroups of $\mathrm{SL}_n(\mathcal{O}_F)$. The group G^1 acts on this tower.

We put $H'_{\mathrm{LT}} = \varinjlim_K H_c^{n-1}(M_{K,C}^{(0)}, \overline{\mathbb{Q}}_\ell)$. The following proposition gives a connection between H_{LT} and H'_{LT}.

Proposition 4.4 *Put $G = \mathrm{GL}_n(F) \times (D^\times/\varpi^{\mathbb{Z}}) \times W_F$. The group G^1 can be regarded as a cocompact closed normal subgroup of G. Moreover, the action of G^1 on H'_{LT} is smooth, and we have a G-equivariant isomorphism*

$$\operatorname{Ind}_{G^1}^G H'_{\mathrm{LT}} \cong H_{\mathrm{LT}}.$$

Proof. First we will prove that G^1 is a cocompact closed normal subgroup of G. The natural homomorphism $G^1 \to G$ is clearly injective. Further, its image is equal to the kernel of the homomorphism $\nu\colon G \to F^\times/\varpi^{n\mathbb{Z}}$; $(g, \overline{d}, \sigma) \mapsto \overline{\det(g)^{-1} \operatorname{Nrd}(d) \operatorname{Art}_F^{-1}(\sigma)}$. Hence the image of G^1 is closed normal and cocompact in G.

Next we construct a G^1-equivariant surjection $H_{\mathrm{LT}} \to H'_{\mathrm{LT}}$. Take an integer $m \geq 0$. We have $M_m^{(0)} \otimes_{\breve{F}} C = \coprod_{\sigma \in \mathrm{Gal}(\breve{F}_m/\breve{F})} M_m^{(0)} \otimes_{\breve{F}_m, \sigma} C$. Therefore, the morphism $M_{m,C}^{(0)} = M_m^{(0)} \otimes_{\breve{F}_m} C \hookrightarrow M_m^{(0)} \otimes_{\breve{F}} C \hookrightarrow M_{m,\varpi^{\mathbb{Z}}} \otimes_{\breve{F}} C$ induces a surjective map

$$H_c^{n-1}(M_{m,\varpi^{\mathbb{Z}}} \otimes_{\breve{F}} C, \overline{\mathbb{Q}}_\ell) \to H_c^{n-1}(M_m^{(0)} \otimes_{\breve{F}_m} C, \overline{\mathbb{Q}}_\ell).$$

By taking inductive limit, we obtain a surjective linear map $\phi\colon H_{\mathrm{LT}} \to H'_{\mathrm{LT}}$. It is easy to see that this is G^1-equivariant. By Theorem 4.3, the action of G^1 on H_{LT} is smooth. Therefore H'_{LT} is also a smooth representation of G^1.

By the Frobenius reciprocity, we obtain a desired G-equivariant homomorphism $\widetilde{\phi}\colon H_{\mathrm{LT}} \to \mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}}$. We shall prove that this map is an isomorphism. It suffices to show that $\widetilde{\phi}\colon H_{\mathrm{LT}}^{K_m} \to (\mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}})^{K_m}$ is an isomorphism for each $m \geq 1$. We have

$$H_{\mathrm{LT}}^{K_m} = H_c^{n-1}(M_{m,\varpi^{\mathbb{Z}}} \otimes_{\breve{F}} C, \overline{\mathbb{Q}}_\ell) = \bigoplus_{0\leq\delta<n} H_c^{n-1}(M_m^{(\delta)} \otimes_{\breve{F}} C, \overline{\mathbb{Q}}_\ell).$$

On the other hand, note that $\nu\colon G \to F^\times/\varpi^{n\mathbb{Z}}$ induces a bijection $G^1\backslash G/K_m \xrightarrow{\cong} F^\times/\varpi^{n\mathbb{Z}}(1+\mathfrak{p}_F^m)$. In particular, it is a finite set with cardinality $r = n(\mathcal{O}_F^\times : 1+\mathfrak{p}_F^m)$. Since $\mathrm{Nrd}\colon D^\times/\varpi^{\mathbb{Z}} \to F^\times/\varpi^{n\mathbb{Z}}$ is surjective, we can take a system of representatives $\{(1,d_i,1)\}_{1\leq i\leq r}$ of $G^1\backslash G/K_m$ such that $d_1 = 1$. We have an injection

$$(\mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}})^{K_m} \hookrightarrow \bigoplus_{1\leq i\leq r} (H'_{\mathrm{LT}})^{K'_m} = \bigoplus_{1\leq i\leq r} H_c^{n-1}(M_{m,C}^{(0)}, \overline{\mathbb{Q}}_\ell); \quad f \mapsto (f(1,d_i,1))_{1\leq i\leq r}.$$

Take an element $x \in H_c^{n-1}(M_{m,C}^{(0)}, \overline{\mathbb{Q}}_\ell) \subset H_c^{n-1}(M_m^{(0)} \otimes_{\breve{F}} C, \overline{\mathbb{Q}}_\ell) \subset H_{\mathrm{LT}}^{K_m}$ and consider $\widetilde{\phi}(x) \in (\mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}})^{K_m}$. Let d be an element of $D^\times$ and write $\overline{d}$ for the image of d in $D^\times/\varpi^{\mathbb{Z}}$. If $\mathrm{Nrd}(d) \notin \varpi^{n\mathbb{Z}}\mathcal{O}_F^\times$, $(1,\overline{d},1)$ maps $M_{m,C}^{(0)}$ into $M_m^{(\delta)} \otimes_{\breve{F}} C$ with $\delta \neq 0$. If $\mathrm{Nrd}(d) \in \varpi^{n\mathbb{Z}}\mathcal{O}_F^\times \setminus \varpi^{n\mathbb{Z}}(1+\mathfrak{p}_F^m)$, $(1,\overline{d},1)$ maps $M_{m,C}^{(0)}$ to $M_m^{(0)} \otimes_{\breve{F}_{m,\sigma}} C$ with $\sigma \neq 1$. Therefore, unless $d \in \varpi^{n\mathbb{Z}}(1+\mathfrak{p}_F^m)$, we have $\phi((1,\overline{d},1)x) = 0$. Hence we conclude that

$$(\widetilde{\phi}(x))(1,d_i,1) = \phi((1,d_i,1)x) = \begin{cases} x & i = 1, \\ 0 & i \neq 1. \end{cases}$$

If we put $x_j = (1,d_j,1)^{-1}x \in H_{\mathrm{LT}}^{K_m}$, we have

$$(\widetilde{\phi}(x_j))(1,d_i,1) = \phi((1,d_id_j^{-1},1)x) = \begin{cases} x & i = j, \\ 0 & i \neq j. \end{cases}$$

This means that the composite $H_{\mathrm{LT}}^{K_m} \xrightarrow{\widetilde{\phi}} (\mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}})^{K_m} \hookrightarrow \bigoplus_{1\leq i\leq r} H_c^{n-1}(M_{m,C}^{(0)}, \overline{\mathbb{Q}}_\ell)$ is surjective. Therefore we conclude that $(\mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}})^{K_m} \cong \bigoplus_{1\leq i\leq r} H_c^{n-1}(M_{m,C}^{(0)}, \overline{\mathbb{Q}}_\ell)$, and $\widetilde{\phi}$ is surjective.

Now we have only to compare the dimensions of $H_{\mathrm{LT}}^{K_m}$ and $(\mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}})^{K_m}$. As $M_m^{(\delta)} \cong M_m^{(0)}$ (an isomorphism is given by an element $d \in D^\times$ with $v_F(\mathrm{Nrd}(d)) = -\delta$) and $M_m^{(0)} \otimes_{\breve{F}_{m,\sigma}} C \cong M_{m,C}^{(0)}$ for $\sigma \in \mathrm{Gal}(\breve{F}_m/\breve{F})$, we obtain

$$\dim H_{\mathrm{LT}}^{K_m} = n \dim H_c^{n-1}(M_m^{(0)} \otimes_{\breve{F}} C, \overline{\mathbb{Q}}_\ell) = n[\breve{F}_m : \breve{F}] \dim H_c^{n-1}(M_{m,C}^{(0)}, \overline{\mathbb{Q}}_\ell).$$

On the other hand, we have $\dim(\mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}})^{K_m} = r \dim H_c^{n-1}(M_{m,C}^{(0)}, \overline{\mathbb{Q}}_\ell)$. Since $r = n(\mathcal{O}_F^\times : 1+\mathfrak{p}_F^m) = n[\breve{F}_m : \breve{F}]$, we conclude that $\dim H_{\mathrm{LT}}^{K_m} = \dim(\mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}})^{K_m}$. ■

Next we introduce the infinite level setting. Let A_∞ be the $\mathfrak{m}_0$-adic completion of $\varinjlim_m A_m$, where $\mathfrak{m}_0$ denotes the maximal ideal of A_0. Put $A_{m,\mathcal{O}_C} = A_m \widehat{\otimes}_{\mathcal{O}_{\breve{F}_m}} \mathcal{O}_C$ and $A_{\infty,\mathcal{O}_C} = A_\infty \widehat{\otimes}_{\mathcal{O}_{\widehat{F}^{\mathrm{ab}}}} \mathcal{O}_C$. We can observe that $A_{\infty,\mathcal{O}_C}$ is the $\mathfrak{m}_0$-adic completion of $\varinjlim_m A_{m,\mathcal{O}_C}$. The group G^1 acts on $A_{\infty,\mathcal{O}_C}$. The action of $(g,d,\sigma) \in G^1$ preserves $\mathcal{O}_C \subset A_{\infty,\mathcal{O}_C}$ and induces σ on $\mathcal{O}_C$. It is known that the ring $A_{\infty,\mathcal{O}_C}$ with the G^1-action has very simple description; see [Wei, §2] and [IT15a, §1].

We put $\mathcal{M}^{(0)}_{\infty,\mathcal{O}_C} = \operatorname{Spf} A_{\infty,\mathcal{O}_C}$, $t(\mathcal{M}^{(0)}_{\infty,\mathcal{O}_C}) = \operatorname{Spa}(A_{\infty,\mathcal{O}_C}, A_{\infty,\mathcal{O}_C})$, and write $M^{(0)}_{\infty,C}$ for the generic fiber (i.e., the open subspace given by the condition $|\varpi(x)| \neq 0$) of $t(\mathcal{M}^{(0)}_{\infty,\mathcal{O}_C})$. By [Hub93] and [Hub94], $t(\mathcal{M}^{(0)}_{\infty,\mathcal{O}_C})$ and $M^{(0)}_{\infty,C}$ are topological spaces equipped with structure presheaves. In fact $M^{(0)}_{\infty,C}$ is known to be a perfectoid space ([Wei, Lemma 2.10.1]) and its structure presheaf is a sheaf, but we do not need this result. Since $\mathcal{M}^{(0)}_{m,\mathcal{O}_C} = \operatorname{Spf} A_{m,\mathcal{O}_C}$ and $M^{(0)}_{m,C}$ is the generic fiber of $t(\mathcal{M}^{(0)}_{m,\mathcal{O}_C}) = \operatorname{Spa}(A_{m,\mathcal{O}_C}, A_{m,\mathcal{O}_C})$, we have a morphism $M^{(0)}_{\infty,C} \to M^{(0)}_{m,C}$ compatible with the transition maps of $\{M^{(0)}_{m,C}\}_m$. Hence we obtain a morphism $M^{(0)}_{\infty,C} \to \{M^{(0)}_{K,C}\}_{K \subset \mathrm{SL}_n(\mathcal{O}_F)}$, which is clearly G^1-equivariant.

The following proposition enables us to apply the results in Section 3 to an affinoid of $M^{(0)}_{\infty,C}$.

Proposition 4.5 *Let U be a rational subset of $t(\mathcal{M}^{(0)}_{\infty,\mathcal{O}_C})$ contained in $M^{(0)}_{\infty,C}$, and A a ϖ-adically complete flat $\mathcal{O}_C$-algebra such that $\mathcal{X} = \operatorname{Spf} A$ is a formal model of U; in other words, $(A[1/\varpi], \widetilde{A})$ is isomorphic to $(\mathcal{O}(U), \mathcal{O}^+(U))$ over $(C, \mathcal{O}_C)$ (recall that $\widetilde{A}$ denotes the integral closure of A in $A[1/\varpi]$). Assume that there exists an open subgroup K_U of $\mathrm{SL}_n(\mathcal{O}_F)$ which stabilizes U and whose induced action on U extends to $\mathcal{X}$; namely, the induced action of K_U on $\mathcal{O}(U) = A[1/\varpi]$ stabilizes A.*

(i) *There exist an integer $m \geq 0$ such that $K'_m \subset K_U$ and an open affinoid U_m in $M^{(0)}_{m,C}$ whose inverse image under $M^{(0)}_{\infty,C} \to M^{(0)}_{m,C}$ is equal to U. We put $K_0 = K'_m$ and $U_{K_0} = U_m$.*

(ii) *For an open normal subgroup K of K_0, let $U_K = \operatorname{Spa}(B_K, B_K^\circ)$ denote the affinoid obtained as the inverse image of U_{K_0} under $M^{(0)}_{K,C} \to M^{(0)}_{K_0,C}$. Then, $(A, \{B_K\}_{K \lhd K_0})$ satisfies Assumption 3.1.*

(iii) *Assume that $g \in G^1$ satisfies $Ug = U$ and extends to an automorphism of $\mathcal{X}$. Then, the induced automorphism of A is of finite level in the sense of Definition 3.19.*

Proof. We write $U = R(\frac{t_1,\dots,t_r}{s})$, where $t_1,\dots,t_r,s$ are elements of $A_{\infty,\mathcal{O}_C}$ such that $t_1,\dots,t_r$ generate an open ideal of $A_{\infty,\mathcal{O}_C}$. As $A_{\infty,\mathcal{O}_C}$ is the completion of $\varinjlim_m A_{m,\mathcal{O}_C}$, [Hub93, Lemma 3.10] tells us that we may assume that $t_1,\dots,t_r,s \in A_{m,\mathcal{O}_C}$ for some $m \geq 0$. We take such m so that $K'_m \subset K_U$. Since $|\varpi(x)| \neq 0$ for $x \in U$, by [Hub93, Lemma 3.11] there exists an integer $j \geq 0$ such that $|a(x)| < |\varpi(x)|$ for every $a \in \mathfrak{m}_0^j A_{\infty,\mathcal{O}_C}$ and $x \in U$. Take generators $b_1,\dots,b_l \in \mathfrak{m}_0^j$ and consider the rational subset $U_m = R(\frac{t_1,\dots,t_r}{s}, \frac{b_1,\dots,b_l}{\varpi})$ of $t(\mathcal{M}^{(0)}_{m,\mathcal{O}_C})$. Then, it is

contained in the generic fiber $M^{(0)}_{m,C}$ and its inverse image under $M^{(0)}_{\infty,C} \to M^{(0)}_{m,C}$ is equal to U. This proves (i).

Next we prove (ii). Since $K_0 = K'_m \subset K_U$, A is equipped with an action of K_0. The conditions (a) and (b) in Assumption 3.1 follow from the corresponding properties for $\{M^{(0)}_{K,C}\}_{K \lhd K_0}$. Consider the condition (c). For $m' \geq m$, we simply write $U_{m'}$ and $B_{m'}$ for $U_{K'_{m'}}$ and $B_{K'_{m'}}$, respectively. As the map $t(\mathcal{M}^{(0)}_{\infty,\mathcal{O}_C}) \to t(\mathcal{M}^{(0)}_{m',\mathcal{O}_C})$ carries U to $U_{m'}$, we have a continuous homomorphism of affinoid rings $(B_{m'}, B^\circ_{m'}) \to (\mathcal{O}(U), \mathcal{O}^+(U)) \cong (A[1/\varpi], \widetilde{A})$ (see [Hub94, Proposition 1.3 (i)]). Therefore we obtain a homomorphism $\{(B_{m'}, B^\circ_{m'})\}_{m' \geq m} \to (A[1/\varpi], \widetilde{A})$, which is clearly K_0-equivariant. This extends to a K_0-equivariant homomorphism $\{(B_K, B^\circ_K)\}_{K \lhd K_0} \to (A[1/\varpi], \widetilde{A})$. By [Hub94, Lemma 1.5 (ii)], the induced map

$$\mathrm{Spa}(A[1/\varpi], \widetilde{A}) \to \varprojlim_{K \lhd K_0} \mathrm{Spa}(B_K, B^\circ_K) = \varprojlim_{m' \geq m} \mathrm{Spa}(B_{m'}, B^\circ_{m'})$$

is identified with $U \to \varprojlim_{K \lhd K_0} U_K = \varprojlim_{m' \geq m} U_{m'}$. On the other hand, by [SW13, Proposition 2.4.2], the continuous map between underlying topological spaces

$$t(\mathcal{M}^{(0)}_{\infty,\mathcal{O}_C}) \to \varprojlim_{m'} t(\mathcal{M}^{(0)}_{m',\mathcal{O}_C})$$

is a homeomorphism. Therefore $U \to \varprojlim_{m' \geq m} U_{m'}$ is also a homeomorphism. The density of the image of $\varinjlim_K B_K = \varinjlim_m B_m \to A[1/\varpi]$ follows from the density of the image of $\varinjlim_m A_{m,\mathcal{O}_C} \to A_{\infty,\mathcal{O}_C}$; see the proof of [Hub96, Remark 2.4.3 ii)]. This completes the proof of (ii).

We prove (iii). Clearly g preserves $\mathcal{O}_C \subset A$. By the definition of the action of G^1 on $\{M^{(0)}_{K,C}\}_{K \subset \mathrm{SL}_n(\mathcal{O}_F)}$, for an open normal subgroup K of K_0 and $g \in G^1$, there exists an open normal subgroup K_g of K_0 such that g gives a morphism $M^{(0)}_{K_g,C} \to M^{(0)}_{K,C}$. As the maps $U \to U_K$ and $U \to U_{K_g}$ are surjective (see Lemma 3.3), U_{K_g} is mapped to U_K by g. Therefore a continuous homomorphism $B_K \to B_{K_g}$ is induced. Passing to the inductive limit, we conclude that the action of g on $A[1/\varpi]$ preserves the image of $\varinjlim_{K \lhd K_0} B_K \to A[1/\varpi]$. Since we are assuming that g preserves $A \subset A[1/\varpi]$, this means that the action of g is of finite level. ■

Corollary 4.6 *Let U and $\mathcal{X} = \mathrm{Spf}\, A$ be as in Proposition 4.5. Let J be a subgroup of G^1 whose action on $M^{(0)}_{\infty,C}$ stabilizes U and extends to an action on $\mathcal{X}$. Assume that there exists an affine scheme Y of finite type over $\overline{\mathbb{F}}_q$ endowed with a J-action such that $\mathcal{X}_s = \mathrm{Spec}(A \otimes_{\mathcal{O}_C} \overline{\mathbb{F}}_q)$ is isomorphic to the perfection of Y as schemes over $\overline{\mathbb{F}}_q$ with J-actions.*

(i) *We have a J-equivariant homomorphism*

$$\mathrm{sp}^* \colon H^{n-1}_c(Y, \overline{\mathbb{Q}}_\ell) \to \varinjlim_K H^{n-1}_c(M^{(0)}_{K,C}, \overline{\mathbb{Q}}_\ell) = H'_{\mathrm{LT}}.$$

(ii) *Assume moreover that Y is pure-dimensional and smooth over $\overline{\mathbb{F}}_q$. Let V be a subspace of $H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell)$ such that the composite $V \hookrightarrow H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell) \to H^{n-1}(Y, \overline{\mathbb{Q}}_\ell)$ is injective. Then, the composite $V \hookrightarrow H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} H'_{\mathrm{LT}}$ is also injective.*

(iii) *In the situation of (ii), suppose that V is stable under J. Let J' be a subgroup of G^1 which contains J as a finite index subgroup. Assume that U and Ug are disjoint for every $g \in J' \setminus J$. Then, the composite*

$$\mathrm{Ind}_J^{J'} V \hookrightarrow \mathrm{Ind}_J^{J'} H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell) \to H'_{\mathrm{LT}}$$

is injective, where the second arrow denotes the J'-homomorphism induced from sp^* by the Frobenius reciprocity.

Proof. Let K_0, $\{U_K\}_{K_0 \lhd K}$ and $\{B_K\}_{K_0 \lhd K}$ be as in Proposition 4.5. Write $Y = \operatorname{Spec} R$. By Proposition 4.5 (ii), $(A, \{B_K\}_{K \lhd K_0}, R)$ satisfies Assumptions 3.1 and 3.2. Therefore, by Definition 3.17, we have a specialization map

$$\mathrm{sp}^* \colon H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell) \to \varinjlim_K H_c^{n-1}(U_K, \overline{\mathbb{Q}}_\ell).$$

By Propositions 3.21 and 4.5 (iii), this map is J-equivariant. Composing with the natural map $\varinjlim_K H_c^{n-1}(U_K, \overline{\mathbb{Q}}_\ell) \to \varinjlim_K H_c^{n-1}(M_{K,C}^{(0)}, \overline{\mathbb{Q}}_\ell)$, we obtain a desired J-equivariant map. The second assertion is a direct consequence of Theorem 3.22.

Let us prove (iii). Take a system of representatives $g_1, \dots, g_r$ of $J \backslash J'$ such that $g_1 = 1$. Put $U_i = Ug_i$. Then, $\mathcal{X} = \operatorname{Spf} A$ gives a formal model of U_i by the isomorphism $(A[1/\varpi], \widetilde{A}) \cong (\mathcal{O}(U), \mathcal{O}^+(U)) \xrightarrow[\cong]{g_i^{-1}} (\mathcal{O}(U_i), \mathcal{O}^+(U_i))$. We write $\mathcal{X}_i$ for this formal model. Then, the pair $(U_i, \mathcal{X}_i)$ satisfies the condition in Proposition 4.5; indeed, we can take K_{U_i} as $g_i^{-1} K_U g_i$. Let $m \geq 0$ be an integer which satisfies the condition in Proposition 4.5 (i) for every $1 \leq i \leq r$. Then, by Proposition 4.5 (ii), we obtain towers $\{U_{i,K}\}_{K \lhd K_0}$. Put $\mathcal{X}' = \coprod_{1 \leq i \leq r} \mathcal{X}_i$ and $U'_K = \coprod_{1 \leq i \leq r} U'_{i,K}$. The reduction $\mathcal{X}'_s$ is identified with the perfection of $Y' = \coprod_{1 \leq i \leq r} Y_i$, where Y_i is a copy of Y. By assumption, the natural morphism $U'_K \to M_{K,C}^{(0)}$ is an open immersion. Therefore, by Definition 3.17 we have homomorphisms

$$\begin{aligned} \bigoplus_{1 \leq i \leq r} H_c^{n-1}(Y_i, \overline{\mathbb{Q}}_\ell) = H_c^{n-1}(Y', \overline{\mathbb{Q}}_\ell) &\xrightarrow{\mathrm{sp}^*} \varinjlim_K H_c^{n-1}(U'_K, \overline{\mathbb{Q}}_\ell) \\ &\to \varinjlim_K H_c^{n-1}(M_{K,C}^{(0)}, \overline{\mathbb{Q}}_\ell) = H'_{\mathrm{LT}}, \end{aligned}$$

whose composite is denoted by sp'^*. The restriction of sp'^* to $H_c^{n-1}(Y_1, \overline{\mathbb{Q}}_\ell)$ coincides with sp^* in (i). Furthermore, by Theorem 3.22, the composite

$$\bigoplus_{1 \leq i \leq r} V \hookrightarrow \bigoplus_{1 \leq i \leq r} H_c^{n-1}(Y_i, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}'^*} H'_{\mathrm{LT}}$$

is injective.

On the other hand, we can define an action of J' on $\mathcal{X}'$ as follows. Take $h \in J'$. For $1 \leq i \leq r$, let j_i be the unique integer such that $g_i h \in J g_{j_i}$. The map $i \mapsto j_i$ is a permutation of $\{1, \dots, r\}$. Define $h\colon \mathcal{X}_i \to \mathcal{X}_{j_i}$ as

$$\mathcal{X}_i = \mathcal{X} \xrightarrow{g_i h g_{j_i}^{-1}} \mathcal{X} = \mathcal{X}_{j_i},$$

and $h\colon \mathcal{X}' \to \mathcal{X}'$ as the disjoint union of them. Similarly, we can define an action of J' on Y'. Since $\bigcup_{1 \leq i \leq r} U_i$ is stable under J', the action of J' on $\{M_{K,C}^{(0)}\}_K$ induces that on $\{U'_K\}_K$, and it is compatible with the action of J' on $\mathcal{X}'$ (see the proof of Proposition 4.5 (iii)). Therefore, the action of J' on $\mathcal{X}'$ is of finite level and the homomorphism $\mathrm{sp}'^*\colon H_c^{n-1}(Y', \overline{\mathbb{Q}}_\ell) \to H'_{\mathrm{LT}}$ is J'-equivariant.

It is immediate to see that the J'-equivariant map

$$\mathrm{Ind}_J^{J'} H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell) \to H_c^{n-1}(Y', \overline{\mathbb{Q}}_\ell)$$

induced from $H_c^{n-1}(Y_1, \overline{\mathbb{Q}}_\ell) \hookrightarrow H_c^{n-1}(Y', \overline{\mathbb{Q}}_\ell)$ is an isomorphism. Under this isomorphism, $\mathrm{Ind}_J^{J'} V \subset \mathrm{Ind}_J^{J'} H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell)$ is mapped to the subspace $\bigoplus_{1 \leq i \leq r} V \subset \bigoplus_{1 \leq i \leq r} H_c^{n-1}(Y_i, \overline{\mathbb{Q}}_\ell) = H_c^{n-1}(Y', \overline{\mathbb{Q}}_\ell)$. The composite

$$\mathrm{Ind}_J^{J'} H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell) \cong H_c^{n-1}(Y', \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}'^*} H'_{\mathrm{LT}}$$

is clearly the J'-homomorphism induced from sp^* by the Frobenius reciprocity. Therefore, we conclude that the composite

$$\mathrm{Ind}_J^{J'} V \hookrightarrow \mathrm{Ind}_J^{J'} H_c^{n-1}(Y, \overline{\mathbb{Q}}_\ell) \to H'_{\mathrm{LT}}$$

is injective. ■

We end this section by giving representation-theoretic lemmas used in the next two sections.

Lemma 4.7 *Let G be a totally disconnected locally compact group and J an open subgroup of G.*

(i) *For a smooth representation π of J, there exists a J-equivariant injection $\pi \hookrightarrow \mathrm{Ind}_J^G \pi$.*

(ii) *Let N be a closed subgroup of G satisfying $NJ = JN$, and ρ a smooth representation of N. Then, we have a J-equivariant isomorphism $(\mathrm{Ind}_N^{NJ} \rho)|_J \cong \mathrm{Ind}_{N \cap J}^J (\rho|_{N \cap J})$ (note that NJ is an open subgroup of G).*

Proof. We write V for the representation space of π. For $v \in V$, define the map $\phi_v\colon G \to V$ as follows:

$$\phi_v(g) = \begin{cases} gv & g \in J, \\ 0 & g \notin J. \end{cases}$$

It is immediate to see that $\phi_v \in \mathrm{Ind}_J^G V$ and $v \mapsto \phi_v$ gives a J-equivariant injection $\pi \hookrightarrow \mathrm{Ind}_J^G \pi$. This concludes the proof of (i).

Next consider (ii). We denote by algInd the algebraic induction functor. Then, we can easily check that

$$(\mathrm{algInd}_N^{NJ} \rho)|_J \cong \mathrm{algInd}_{N\cap J}^J(\rho|_{N\cap J}).$$

Indeed, the map $\mathrm{algInd}_N^{NJ} \rho \to \mathrm{algInd}_{N\cap J}^J \rho$ is given by $\phi \mapsto \phi|_J$, and the map $\mathrm{algInd}_{N\cap J}^J \rho \to \mathrm{algInd}_N^{NJ} \rho$ by $\psi \mapsto \psi'$, where $\psi'(ng) = n\psi(g)$ for $n \in N$ and $g \in J$. These maps are inverse to each other. By taking the J-smooth part $(-)^{J\text{-sm}}$, we have

$$\big((\mathrm{algInd}_N^{NJ} \rho)|_J\big)^{J\text{-sm}} \cong \mathrm{Ind}_{N\cap J}^J(\rho|_{N\cap J}).$$

Since J is an open subgroup of NJ, for any representation W of NJ, we have $W^{J\text{-sm}} = W^{NJ\text{-sm}}$. Therefore, we have $(\mathrm{algInd}_N^{NJ} \rho)^{J\text{-sm}} = \mathrm{Ind}_N^{NJ} \rho$. This concludes the proof of (ii). ■

Lemma 4.8 *Let G and H be totally disconnected locally compact groups and Δ a closed subgroup of $G \times H$. Assume that the composite $\Delta \hookrightarrow G \times H \twoheadrightarrow G$ is surjective. Let (π, V) (resp. (ρ, W)) be a smooth representation of Δ (resp. H).*

(i) *The group G acts on the space $\mathrm{Hom}_{\Delta\cap H}(\rho, \pi)$ as follows: for $g \in G$ and $\phi \in \mathrm{Hom}_{\Delta\cap H}(\rho, \pi)$,*

$$(g\phi)(w) = (g, h_g)\phi(h_g^{-1}w) \quad (w \in W).$$

Here h_g denotes an element of H such that $(g, h_g) \in \Delta$, whose existence follows from the assumption on Δ. Note that the right hand side is independent of the choice of h_g.

(ii) *If ρ is finitely generated as a representation of H, we have a natural isomorphism $\mathrm{Hom}_{\Delta\cap H}(\rho, \pi)^{G\text{-sm}} \cong \mathrm{Hom}_H(\rho, \mathrm{Ind}_\Delta^{G\times H} \pi)$ as representations of G, where $(-)^{G\text{-sm}}$ denotes the G-smooth part.*

(iii) *Assume that ρ is finitely generated as a representation of H. For a closed subgroup Δ' of $G \times H$ containing Δ, we have*

$$\mathrm{Hom}_{\Delta\cap H}(\rho, \pi)^{G\text{-sm}} \cong \mathrm{Hom}_{\Delta'\cap H}(\rho, \mathrm{Ind}_\Delta^{\Delta'} \pi)^{G\text{-sm}}$$

as representations of G.

Proof. As in the proof of Lemma 4.7 (ii), we denote by algInd the algebraic induction functor. First we prove the following:

$$\mathrm{Hom}_{\Delta\cap H}(\rho, \pi) \cong \mathrm{Hom}_H(\rho, \mathrm{algInd}_\Delta^{G\times H} \pi). \tag{$*$}$$

By the surjectivity of $\Delta \hookrightarrow G \times H \twoheadrightarrow G$, we have $\Delta H = H\Delta = G \times H$. Therefore, as in the proof of Lemma 4.7 (ii), we can prove that

$$(\mathrm{algInd}_\Delta^{G\times H} \pi)|_H \cong \mathrm{algInd}_{\Delta\cap H}^H(\pi|_{\Delta\cap H}).$$

Therefore, by the Frobenius reciprocity we have

$$\mathrm{Hom}_{\Delta\cap H}(\rho,\pi)\cong\mathrm{Hom}_H(\rho,\mathrm{algInd}_{\Delta\cap H}^{H}\,\pi)\cong\mathrm{Hom}_H(\rho,\mathrm{algInd}_{\Delta}^{G\times H}\,\pi).$$

This concludes the proof of $(*)$. Under the isomorphism $(*)$, $\phi\in\mathrm{Hom}_{\Delta\cap H}(\rho,\pi)$ is mapped to $\Phi\colon W\to\mathrm{algInd}_{\Delta}^{G\times H}\,V$ as follows:

$$\Phi(w)(g,h)=(g,h_g)\phi(h_g^{-1}hw)\quad(w\in W).$$

The inverse map $\Phi\mapsto\phi$ is given by $\phi(w)=\Phi(w)(1,1)$ $(w\in W)$.

Note that G acts on $\mathrm{Hom}_H(\rho,\mathrm{algInd}_{\Delta}^{G\times H}\,\pi)$. Therefore, by using the isomorphism $(*)$, we can define an action of G on $\mathrm{Hom}_{\Delta\cap H}(\rho,\pi)$. Let us describe it explicitly. For $g\in G$, $w\in W$ and $\phi\in\mathrm{Hom}_{\Delta\cap H}(\rho,\pi)$ corresponding to $\Phi\in\mathrm{Hom}_H(\rho,\mathrm{algInd}_{\Delta}^{G\times H}\,\pi)$, we have

$$\begin{aligned}(g\phi)(w)&=\big((g\Phi)(w)\big)(1,1)=\big((g,1)\Phi(w)\big)(1,1)=\big((g,h_g)(1,h_g^{-1})\Phi(w)\big)(1,1)\\&=\big((g,h_g)\Phi(h_g^{-1}w)\big)(1,1)=\Phi(h_g^{-1}w)(g,h_g)=(g,h_g)\big(\Phi(h_g^{-1}w)(1,1)\big)\\&=(g,h_g)\phi(h_g^{-1}w).\end{aligned}$$

This action coincides with that in (i). This concludes the proof of (i).

Next we prove (ii). By $(*)$ and the smoothness of ρ, we have an isomorphism

$$\mathrm{Hom}_{\Delta\cap H}(\rho,\pi)\cong\mathrm{Hom}_H\big(\rho,(\mathrm{algInd}_{\Delta}^{G\times H}\,\pi)^{H\text{-sm}}\big),$$

which is G-equivariant by the proof of (i). By taking the G-smooth parts, we have

$$\mathrm{Hom}_{\Delta\cap H}(\rho,\pi)^{G\text{-sm}}\cong\mathrm{Hom}_H\big(\rho,(\mathrm{algInd}_{\Delta}^{G\times H}\,\pi)^{H\text{-sm}}\big)^{G\text{-sm}}.$$

Since $\mathrm{Ind}_{\Delta}^{G\times H}\,\pi=(\mathrm{algInd}_{\Delta}^{G\times H}\,\pi)^{(G\times H)\text{-sm}}=((\mathrm{algInd}_{\Delta}^{G\times H}\,\pi)^{H\text{-sm}})^{G\text{-sm}}$, it suffices to show the following:

for a $G\times H$-representation U, we have $\mathrm{Hom}_H(\rho,U)^{G\text{-sm}}=\mathrm{Hom}_H(\rho,U^{G\text{-sm}})$.

Clearly we have $\mathrm{Hom}_H(\rho,U)^{G\text{-sm}}\subset\mathrm{Hom}_H(\rho,U^{G\text{-sm}})$. Take $\phi\in\mathrm{Hom}_H(\rho,U^{G\text{-sm}})$. Since ρ is finitely generated, we can find a finite system of generators $x_1,\dots,x_n\in W$. There exists an open subgroup K of G which stabilizes $\phi(x_1),\dots,\phi(x_n)\in U^{G\text{-sm}}$. Then, for every $x\in W$, $\phi(x)$ is fixed by K. In other words, ϕ lies in $\mathrm{Hom}_H(\rho,U)^K$. This proves that $\mathrm{Hom}_H(\rho,U^{G\text{-sm}})\subset\mathrm{Hom}_H(\rho,U)^{G\text{-sm}}$. Now the proof of (ii) is complete.

For (iii), just note that

$$\begin{aligned}\mathrm{Hom}_{\Delta\cap H}(\rho,\pi)^{G\text{-sm}}&\cong\mathrm{Hom}_H(\rho,\mathrm{Ind}_{\Delta}^{G\times H}\,\pi)\cong\mathrm{Hom}_H(\rho,\mathrm{Ind}_{\Delta'}^{G\times H}\,\mathrm{Ind}_{\Delta}^{\Delta'}\,\pi)\\&\cong\mathrm{Hom}_{\Delta'\cap H}(\rho,\mathrm{Ind}_{\Delta}^{\Delta'}\,\pi)^{G\text{-sm}}.\end{aligned}$$

■

5 Example I: depth 0 supercuspidal representations

In this section, we determine $\mathrm{rec}_F(\pi)$ and $\mathrm{JL}(\pi)$ for a depth 0 supercuspidal representation π of $\mathrm{GL}_n(F)$. First, we will introduce some notation. Let $\theta\colon \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$ be a character. We assume that θ is regular, namely, it does not factor through the norm map $\mathrm{Nr}_{\mathbb{F}_{q^n}/\mathbb{F}_{q^m}}\colon \mathbb{F}_{q^n}^\times \to \mathbb{F}_{q^m}^\times$ to any subfield $\mathbb{F}_{q^m} \subsetneq \mathbb{F}_{q^n}$. It is well-known that we can attach to θ an irreducible cuspidal representation R_θ of $\mathrm{GL}_n(\mathbb{F}_q)$. Here we recall a construction of R_θ by the Deligne-Lusztig theory [DL76]. Let DL_n be the affine algebraic variety over $\overline{\mathbb{F}}_q$ defined by the equation

$$\left(\det(x_i^{q^{j-1}})_{1\le i,j\le n}\right)^{q-1} = (-1)^{n-1}.$$

The group $\mathrm{GL}_n(\mathbb{F}_q)$ naturally acts on DL_n on the right. On the other hand, the group $\mathbb{F}_{q^n}^\times$ acts on DL_n (on the right) by $x_i \mapsto \zeta x_i$ ($\zeta \in \mathbb{F}_{q^n}^\times$). The representation R_θ is given as the θ-isotypic part $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$ of the $(n-1)$th compactly supported cohomology $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)$ of DL_n. The central character of R_θ equals $\theta|_{\mathbb{F}_q^\times}$ ([DL76, Corollary 1.22]).

We denote by F_n the degree n unramified extension of F. Let $\chi\colon F_n^\times \to \mathbb{C}^\times$ be a tame character, namely, a character which is trivial on $1+\varpi\mathcal{O}_{F_n}$. Its restriction to $\mathcal{O}_{F_n}^\times$ factors as $\mathcal{O}_{F_n}^\times \twoheadrightarrow \mathbb{F}_{q^n}^\times \xrightarrow{\overline{\chi}} \mathbb{C}^\times$. We say that χ is regular if $\overline{\chi}$ is regular.

Definition 5.1 For a regular tame character $\chi\colon F_n^\times \to \mathbb{C}^\times$, let $\overline{\pi}_\chi$ be the representation of $F^\times \mathrm{GL}_n(\mathcal{O}_F)$ such that

- $\overline{\pi}_\chi|_{\mathrm{GL}_n(\mathcal{O}_F)}$ is the inflation of $R_{\overline{\chi}}$ by $\mathrm{GL}_n(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F}_q)$, and
- $F^\times$ acts on $\overline{\pi}_\chi$ by $\chi|_{F^\times}$.

We put $\pi_\chi = \text{c-Ind}_{F^\times \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)} \overline{\pi}_\chi$, which is known to be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$.

From a regular tame character χ, we can also construct a representation of $D^\times$. Recall that the quotient of $\mathcal{O}_D$ by its Jacobson radical is equal to $\mathbb{F}_{q^n}$. Therefore we have a surjection $\mathcal{O}_D^\times \twoheadrightarrow \mathbb{F}_{q^n}^\times$.

Definition 5.2 For a regular tame character $\chi\colon F_n^\times \to \mathbb{C}^\times$, let $\overline{\rho}_\chi$ be the character of $F^\times\mathcal{O}_D^\times$ such that

- $\overline{\rho}_\chi|_{\mathcal{O}_D^\times}$ is the composite $\mathcal{O}_D^\times \twoheadrightarrow \mathbb{F}_{q^n}^\times \xrightarrow{\overline{\chi}} \mathbb{C}^\times$, and
- $\overline{\rho}_\chi|_{F^\times} = \chi|_{F^\times}$.

We put $\rho_\chi = \text{c-Ind}_{F^\times\mathcal{O}_D^\times}^{D^\times} \overline{\rho}_\chi$, which is known to be an irreducible smooth representation of $D^\times$.

The goal of this section is the following:

Theorem 5.3 *For a regular tame character $\chi\colon F_n^\times \to \mathbb{C}^\times$, we have*

$$\mathrm{rec}_F(\pi_\chi) = \mathrm{Ind}_{W_{F_n}}^{W_F} \chi\delta^{n-1}, \quad \mathrm{JL}(\pi_\chi) = \rho_\chi,$$

where $\delta\colon F_n^\times \to \mathbb{C}^\times$; $a \mapsto (-1)^{v_{F_n}(a)}$ is the unramified quadratic character. As usual, we regard characters of $F_n^\times$ as those of W_{F_n} by the isomorphism $\mathrm{Art}_{F_n}\colon F_n^\times \xrightarrow{\cong} W_{F_n}^{\mathrm{ab}}$.

Remark 5.4 (i) The identity $\mathrm{rec}_F(\pi_\chi) = \mathrm{Ind}_{W_{F_n}}^{W_F} \chi\delta^{n-1}$ was proved in [Hen92]. The proof in [Hen92] requires case-by-case argument when (n,q) is either of $(2,2)$, $(2,3)$, $(4,2)$, $(6,2)$. Our proof works for any case uniformly.

(ii) The identity $\mathrm{rec}_F(\pi_\chi) = \rho_\chi$ was obtained in [SZ05, Theorem 3]; see also [BH11, Theorem 1].

Remark 5.5 Let $\xi\colon F^\times \to \mathbb{C}^\times$ be an unramified character. For a tame character $\chi\colon F_n^\times \to \mathbb{C}^\times$, put $\chi_\xi = \chi(\xi \circ \mathrm{Nr}_{F_n/F})$. Then, χ_ξ is also tame and $\overline{\chi_\xi} = \overline{\chi}$. Therefore, χ is regular if and only if χ_ξ is regular. Moreover, we have $\overline{\pi}_{\chi_\xi} = \overline{\pi}_\chi \otimes (\xi \circ \det)$ and

$$\pi_\chi \otimes (\xi \circ \det) = \text{c-Ind}_{F^\times \mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)}(\overline{\pi}_\chi \otimes (\xi \circ \det)) = \pi_{\chi_\xi}.$$

Similarly we have $\rho_\chi \otimes (\xi \circ \mathrm{Nrd}) = \rho_{\chi_\xi}$ and $(\mathrm{Ind}_{W_{F_n}}^{W_F} \chi) \otimes \xi = \mathrm{Ind}_{W_{F_n}}^{W_F} \chi_\xi$. As rec_F and JL are compatible with character twist, Theorem 5.3 for χ is equivalent to that for χ_ξ.

If we take ξ such that $\xi(\varpi)^n = \chi(\varpi)^{-1}$, we have $\chi_\xi(\varpi) = 1$. Therefore, to prove Theorem 5.3 we may assume that $\chi(\varpi) = 1$. In this case the central characters of π_χ and ρ_χ are trivial on $\varpi^{\mathbb{Z}} \subset F^\times$.

Later, for a regular character $\theta\colon \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$, we also write θ for the regular tame character $F_n^\times = \varpi^{\mathbb{Z}} \times \mathcal{O}_{F_n}^\times \twoheadrightarrow \mathcal{O}_{F_n}^\times \twoheadrightarrow \mathbb{F}_{q^n}^\times \xrightarrow{\theta} \mathbb{C}^\times$. It suffices to show that $\mathrm{rec}_F(\pi_\theta) = \mathrm{Ind}_{W_{F_n}}^{W_F} \theta\delta^{n-1}$ and $\mathrm{JL}(\pi_\theta) = \rho_\theta$.

For $b \in \overline{\mathbb{F}}_q$ with $b^{q-1} = (-1)^{n-1}$, let Y_b be the affine algebraic variety over $\overline{\mathbb{F}}_q$ defined by the equation $\det(x_i^{q^{j-1}})_{1\le i,j\le n} = b$. Clearly we have $\mathrm{DL}_n = \coprod_{b^{q-1}=(-1)^{n-1}} Y_b$. In the following, we construct an affinoid in $M_{\infty,C}^{(0)}$ and its formal model whose reduction is isomorphic to Y_b for some $b \in \overline{\mathbb{F}}_q$ with $b^{q-1} = (-1)^{n-1}$. Since it is well-known to specialists, we omit the detail.

Let $\widetilde{\mathbb{X}}$ be the formal $\mathcal{O}_F$-module over $\mathcal{O}_F$ whose logarithm is $\sum_{i=0}^\infty \frac{T^{q^{in}}}{\varpi^i}$ (it is called the standard formal $\mathcal{O}_F$-module in [BW16, 2.3]). In the following, take $\mathbb{X}$ as $\widetilde{\mathbb{X}} \otimes_{\mathcal{O}_F} \overline{\mathbb{F}}_q$. Let $y = (\xi^{q^{-m}})_{m\ge 0} \in T\widetilde{\mathbb{X}}(\mathcal{O}_C)$, $\alpha_1, \dots, \alpha_n \in \mathcal{O}_{F_n}$ and $t = (\tau^{q^{-m}})_{m\ge 0}$ be as in [BW16, 3.7]. Since $\widetilde{\mathbb{X}} \otimes_{\mathcal{O}_F} \mathcal{O}_C$ has CM by $\mathcal{O}_{F_n}$, we have embeddings $F_n \hookrightarrow M_n(F)$ and $F_n \hookrightarrow D$, by which we regard F_n as subfields of $M_n(F)$ and D (see [BW16, 3.1]).

By [Wei, 2.9.2], we have

$$A_{\infty,\mathcal{O}_C} \cong \mathcal{O}_C[[X_1^{q^{-\infty}}, \dots, X_n^{q^{-\infty}}]]/(\delta(X_1, \dots, X_n)^{q^{-m}} - \tau^{q^{-m}} \mid m \ge 0)^-,$$

where $(-)^-$ denotes the closure. For the definition of $\delta^{q^{-m}}$, see [IT15a, §1.1].

Let U be the subset of $M^{(0)}_{\infty,C}$ defined by $|X_i| \leq |\varpi|^{\frac{1}{q^n-1}}$ for every $1 \leq i \leq n$. It is clearly a rational subset of $t(\mathcal{M}^{(0)}_{\infty,\mathcal{O}_C})$. Recall that $|\xi| = |\varpi|^{\frac{1}{q^n-1}}$ ([BW16, 3.7]), and put $x_i^{q^{-m}} = X_i^{q^{-m}}/\xi^{q^{-m}}$. Then,

$$\delta'(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)^{q^{-m}} = (\xi^{q^{-m}})^{-(1+q+\cdots+q^{n-1})}\delta(\boldsymbol{\xi}\boldsymbol{x}_1, \dots, \boldsymbol{\xi}\boldsymbol{x}_n)^{q^{-m}}$$

lies in $\mathcal{O}_C\langle x_1^{q^{-\infty}}, \dots, x_n^{q^{-\infty}}\rangle$, where $\boldsymbol{x}_i$ (resp. $\boldsymbol{\xi}$) denotes the system $(x_i^{q^{-m}})$ (resp. $(\xi^{q^{-m}})$). On the other hand, since $|\tau| = |\varpi|^{\frac{1}{q-1}} = |\xi|^{1+q+\cdots+q^{n-1}}$,

$$\tau'^{q^{-m}} = (\xi^{q^{-m}})^{-(1+q+\cdots+q^{n-1})}\tau^{q^{-m}}$$

lies in $\mathcal{O}_C^\times$. Put

$$\mathcal{X} = \operatorname{Spf} \mathcal{O}_C\langle x_1^{q^{-\infty}}, \dots, x_n^{q^{-\infty}}\rangle/(\delta'(x_1, \dots, x_n)^{q^{-m}} - \tau'^{q^{-m}} \mid m \geq 0)^-.$$

As in [IT15a, Theorem 2.5], we can check that $\mathcal{X}$ gives a formal model of U.

Proposition 5.6 *The formal scheme $\mathcal{X}$ is flat over $\mathcal{O}_C$ and its reduction $\mathcal{X}_s$ is isomorphic to the perfection of Y_b for some $b \in \overline{\mathbb{F}}_q$ with $b^{q-1} = (-1)^{n-1}$.*

Proof. First we compute the reduction $\mathcal{X}_s$. By [BW16, Lemma 2.10.4], the image of $\delta'^{q^{-m}}$ in $\overline{\mathbb{F}}_q[x_1^{q^{-\infty}}, \dots, x_n^{q^{-\infty}}]$ is equal to $(\det(x_i^{q^{j-1}})_{1\leq i,j\leq n})^{q^{-m}}$. On the other hand, [BW16, Lemma 3.7.1] tells us that the image of $\tau'^{q^{-m}}$ in $\overline{\mathbb{F}}_q$ is equal to $(\det(\overline{\alpha}_i^{q^{j-1}}))^{q^{-m}}$. Put $b = \det(\overline{\alpha}_i^{q^{j-1}})$. Since $\alpha_i \in \mathcal{O}_{F_n}$, we have $\overline{\alpha}_i^{q^n} = \overline{\alpha}_i$. Therefore, we have

$$b^q = \det(\overline{\alpha}_i^{q^j}) = (-1)^{n-1}b.$$

By definition, $\mathcal{X}_s$ is identified with

$$\operatorname{Spec} \overline{\mathbb{F}}_q[x_1^{q^{-\infty}}, \dots, x_n^{q^{-\infty}}]/(\det(x_i^{q^{j-1}})^{q^{-m}} - b^{q^{-m}} \mid m \geq 0),$$

which is the perfection of Y_b.

In particular, we have $\dim \mathcal{X}_s = \dim Y_b = n-1$. Hence Corollary A.6 tells us that $\mathcal{X}$ is flat over $\mathcal{O}_C$. This completes the proof. ■

Next we consider the group action on U and $\mathcal{X}$.

Definition 5.7 We put $J = F^\times \operatorname{GL}_n(\mathcal{O}_F) \times (\mathcal{O}_D^\times/\varpi^{\mathbb{Z}}) \times W_{F_n} \subset G$ and $J^1 = G^1 \cap J$. For $\sigma \in W_{F_n}$, we put $n_\sigma = v_{F_n}(\operatorname{Art}_{F_n}^{-1}(\sigma)) \in \mathbb{Z}$ and $u_\sigma = \varpi^{-n_\sigma}\operatorname{Art}_{F_n}^{-1}(\sigma) \in \mathcal{O}_{F_n}^\times$. We define a homomorphism $\Theta\colon J \to \operatorname{GL}_n(\mathbb{F}_q) \times \mathbb{F}_{q^n}^\times \times \mathbb{Z}$ by

$$(\varpi^m g, d, \sigma) \mapsto (\overline{g}, \overline{d^{-1}u_\sigma^{-1}}, n_\sigma),$$

where $m \in \mathbb{Z}$, $g \in \operatorname{GL}_n(\mathcal{O}_F)$, $d \in \mathcal{O}_D^\times$, and $\sigma \in W_{F_n}$.

As explained before, the group $\mathrm{GL}_n(\mathbb{F}_q)\times\mathbb{F}_{q^n}^\times$ acts on DL_n. We can also define an action of $\mathbb{Z}$ on DL_n over $\mathbb{F}_{q^n}$ (not $\overline{\mathbb{F}}_q$) by $m\colon x_i\mapsto x_i$, $a\mapsto a^{q^{-mn}}$ $(a\in\overline{\mathbb{F}}_q)$, which commutes with the action of $\mathrm{GL}_n(\mathbb{F}_q)\times\mathbb{F}_{q^n}^\times$. Therefore, J acts on DL_n through the homomorphism Θ.

Recall that we have a decomposition $\mathrm{DL}_n=\coprod_{b^{q-1}=(-1)^{n-1}}Y_b$. It is easy to see that an element $(g,a)\in\mathrm{GL}_n(\mathbb{F}_q)\times\mathbb{F}_{q^n}^\times$ maps Y_b to $Y_{b'}$, where $b'=\det(g)\,\mathrm{Nr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(a)b$. In particular, the group

$$(\mathrm{GL}_n(\mathbb{F}_q)\times\mathbb{F}_{q^n}^\times)^1=\{(g,a)\in\mathrm{GL}_n(\mathbb{F}_q)\times\mathbb{F}_{q^n}^\times\mid\det(g)\,\mathrm{Nr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(a)=1\}$$

acts on Y_b. On the other hand, if $b^q=(-1)^{n-1}b$, we have

$$b^{q^n}=(-1)^{(n-1)(1+q+\cdots+q^{n-1})}b=b,$$

hence $b\in\mathbb{F}_{q^n}$. Therefore the action of $\mathbb{Z}$ on DL_n preserves Y_b.

Since the image of J^1 under Θ is contained in $(\mathrm{GL}_n(\mathbb{F}_q)\times\mathbb{F}_{q^n})^1\times\mathbb{Z}$, the action of J^1 on DL_n preserves Y_b.

Lemma 5.8 *The group J^1 is generated by the following elements:*

- $(g,1,1)$ *for* $g\in\mathrm{GL}_n(\mathcal{O}_F)$ *with* $\det g=1$,
- $(1,d,1)$ *for* $d\in\mathcal{O}_D^\times$ *with* $\mathrm{Nrd}\,d=1$,
- $(a,a,1)$ *for* $a\in F_n^\times$,
- $(1,\mathrm{Art}_{F_n}^{-1}(\sigma)^{-1},\sigma)$ *for* $\sigma\in I_{F_n}$, *and*
- $(1,\varpi^{-1},\sigma)$ *for* $\sigma\in W_{F_n}$ *with* $\mathrm{Art}_{F_n}^{-1}(\sigma)=\varpi$.

Here I_{F_n} denotes the inertia group of F_n.

Proof. This is immediate from the surjectivity of $\mathrm{Nr}_{F_n/F}\colon\mathcal{O}_{F_n}^\times\to\mathcal{O}_F^\times$. ■

Proposition 5.9 (i) *The action of $J^1\subset G^1$ on $M_{\infty,C}^{(0)}$ stabilizes U and extends to $\mathcal{X}$.*

(ii) *The induced action of J^1 on $\mathcal{X}_s$ is compatible with the action of J^1 on Y_b under the isomorphism in Proposition 5.6.*

Proof. This follows easily from Lemma 5.8 and the description of the group action in [IT15a, §1.2]. For the element $(1,\mathrm{Art}_{F_n}^{-1}(\sigma)^{-1},\sigma)$ with $\sigma\in I_{F_n}$, note that $|\xi|=|\varpi|^{\frac{1}{q^n-1}}$ implies that $\xi\in\varpi^{\frac{1}{q^n-1}}\mathcal{O}_C^\times$, hence $\sigma(\xi)/\xi\equiv\sigma(\varpi^{\frac{1}{q^n-1}})/\varpi^{\frac{1}{q^n-1}}\equiv\mathrm{Art}_{F_n}^{-1}(\sigma)$ modulo the maximal ideal of $\mathcal{O}_C$. For the element $(1,\varpi^{-1},\sigma)$ with $\mathrm{Art}_{F_n}^{-1}(\sigma)=\varpi$, we use the fact $\sigma(\xi)=\xi$, which is a consequence of the classical Lubin-Tate theory. ■

Lemma 5.10 *Let $\theta\colon\mathbb{F}_{q^n}^\times\to\mathbb{C}^\times$ be a regular character. Then, $H_c^{n-1}(\mathrm{DL}_n,\overline{\mathbb{Q}}_\ell)_\theta$ is a J^1-stable subspace of $H_c^{n-1}(\mathrm{DL}_n,\overline{\mathbb{Q}}_\ell)$. Moreover, we have an isomorphism*

$$\mathrm{Hom}_{J^1\cap(\mathcal{O}_D^\times/\varpi^{\mathbb{Z}})}\bigl(\overline{\rho}_{\theta^{-1}},H_c^{n-1}(\mathrm{DL}_n,\overline{\mathbb{Q}}_\ell)_\theta\bigr)\cong\overline{\pi}_\theta\boxtimes(\theta\delta^{n-1})^{-1}(\tfrac{1-n}{2})$$

of representations of $F^\times\,\mathrm{GL}_n(\mathcal{O}_F)\times W_{F_n}$. Note that the action of $F^\times\,\mathrm{GL}_n(\mathcal{O}_F)\times W_{F_n}$ on the left hand side is defined as in Lemma 4.8 (i), since the composite $J^1\hookrightarrow J\twoheadrightarrow F^\times\,\mathrm{GL}_n(\mathcal{O}_F)\times W_{F_n}$ is surjective.

Proof. Since J^1 obviously normalizes $(\mathcal{O}_D^\times/\varpi^{\mathbb{Z}}, \theta)$, $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$ is a J^1-stable subspace of $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)$. As $J^1 \cap (\mathcal{O}_D^\times/\varpi^{\mathbb{Z}})$ acts on $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$ by $\overline{\rho}_{\theta^{-1}} = \theta^{-1}$, as a vector space we have

$$\mathrm{Hom}_{J^1 \cap (\mathcal{O}_D^\times/\varpi^{\mathbb{Z}})}\big(\overline{\rho}_{\theta^{-1}}, H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta\big) = H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta.$$

Under this identification, the action of $F^\times \mathrm{GL}_n(\mathcal{O}_F) \times W_{F_n}$ on the left hand side is described as follows:

- $(g, 1)$ for $g \in \mathrm{GL}_n(\mathcal{O}_F)$ with $\det g = 1$ acts as $\overline{g} \in \mathrm{GL}_n(\mathbb{F}_q)$ on $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$,
- $(\varpi, 1)$ acts trivially,
- $(a, 1)$ for $a \in \mathcal{O}_{F_n}^\times$ acts as $\overline{a} \in \mathrm{GL}_n(\mathbb{F}_q)$ on $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$,
- $(1, \sigma)$ for $\sigma \in I_{F_n}$ acts by the scalar $\theta(\mathrm{Art}_{F_n}^{-1}(\sigma))^{-1}$,
- $(1, \sigma)$ for $\sigma \in W_{F_n}$ with $\mathrm{Art}_{F_n}(\sigma) = \varpi$ acts as Frob_q^n on $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$.

By [DM85], Frob_q^n acts on $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta$ by $(-1)^{n-1} q^{\frac{n(n-1)}{2}} = (\delta^{1-n}(\frac{1-n}{2}))(\varpi)$ (see also the proof of [Wan14, Théorème 3.1.12]). Therefore we conclude that

$$\mathrm{Hom}_{J \cap (\mathcal{O}_D^\times/\varpi^{\mathbb{Z}})}\big(\overline{\rho}_{\theta^{-1}}, H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta\big) \cong \overline{\pi}_\theta \boxtimes (\theta\delta^{n-1})^{-1}(\tfrac{1-n}{2}). \qquad \blacksquare$$

Proposition 5.11 *There exists a J^1-equivariant injection $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta \to H'_{\mathrm{LT}}$.*

Proof. Let $b \in \overline{\mathbb{F}}_q$ be as in Proposition 5.6. We put

$$(\mathbb{F}_{q^n}^\times)^1 = \{a \in \mathbb{F}_{q^n}^\times \mid \mathrm{Nr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(a) = 1\},$$

and write $H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta$ for the $\theta|_{(\mathbb{F}_{q^n}^\times)^1}$-isotypic part of $H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)$. By Propositions 5.6 and 5.9, the pair $(U, \mathcal{X})$ satisfies the condition in Proposition 4.5 (we may take $K_U = J^1 \cap \mathrm{SL}_n(\mathcal{O}_F)$). By Proposition 5.9 and Corollary 4.6 (i), we have a J^1-equivariant map

$$\mathrm{sp}^* \colon H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell) \to H'_{\mathrm{LT}}.$$

Now note that Y_b is purely $n-1$-dimensional and smooth over $\overline{\mathbb{F}}_q$. Since Y_b is $(\mathbb{F}_{q^n}^\times)^1$-equivariantly isomorphic to the Deligne-Lusztig variety for $\mathrm{SL}_n(\mathbb{F}_q)$, [DL76, Theorem 9.8] tells us that the composite

$$H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta \hookrightarrow H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell) \to H^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)$$

is an injection. Therefore, Corollary 4.6 (ii) tells us that the composite

$$H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta \hookrightarrow H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{sp}^*} H'_{\mathrm{LT}}$$

is an injection.

Thus we have only to prove that the pull-back map $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell) \to H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)$, which is J^1-equivariant, induces an isomorphism $H_c^{n-1}(\mathrm{DL}_n, \overline{\mathbb{Q}}_\ell)_\theta \xrightarrow{\cong} H_c^{n-1}(Y_b, \overline{\mathbb{Q}}_\ell)_\theta$.

It is easy to see that $H_c^{n-1}(\mathrm{DL}_n,\overline{\mathbb{Q}}_\ell)\cong \mathrm{Ind}_{(\mathbb{F}_{q^n}^\times)^1}^{\mathbb{F}_{q^n}^\times} H_c^{n-1}(Y_b,\overline{\mathbb{Q}}_\ell)$ as representations of $\mathbb{F}_{q^n}^\times$. By the Frobenius reciprocity, we have

$$\begin{aligned}H_c^{n-1}(\mathrm{DL}_n,\overline{\mathbb{Q}}_\ell)_\theta &= \mathrm{Hom}_{\mathbb{F}_{q^n}^\times}(\theta, H_c^{n-1}(\mathrm{DL}_n,\overline{\mathbb{Q}}_\ell)) \cong \mathrm{Hom}_{\mathbb{F}_{q^n}^\times}\big(\theta, \mathrm{Ind}_{(\mathbb{F}_{q^n}^\times)^1}^{\mathbb{F}_{q^n}^\times} H_c^{n-1}(Y_b,\overline{\mathbb{Q}}_\ell)\big)\\ &= \mathrm{Hom}_{(\mathbb{F}_{q^n}^\times)^1}\big(\theta|_{(\mathbb{F}_{q^n}^\times)^1}, H_c^{n-1}(Y_b,\overline{\mathbb{Q}}_\ell)\big) = H_c^{n-1}(Y_b,\overline{\mathbb{Q}}_\ell)_\theta.\end{aligned}$$

It is easy to see that the composite of the above identifications is equal to the pull-back map $H_c^{n-1}(\mathrm{DL}_n,\overline{\mathbb{Q}}_\ell)_\theta \to H_c^{n-1}(Y_b,\overline{\mathbb{Q}}_\ell)_\theta$. This concludes the proof. ■

Proof of Theorem 5.3. By Proposition 5.11, we have an injection

$$\mathrm{Hom}_{J^1\cap(\mathcal{O}_D^\times/\varpi^{\mathbb{Z}})}\big(\overline{\rho}_{\theta^{-1}}, H_c^{n-1}(\mathrm{DL}_n,\overline{\mathbb{Q}}_\ell)_\theta\big) \hookrightarrow \mathrm{Hom}_{J^1\cap(\mathcal{O}_D^\times/\varpi^{\mathbb{Z}})}(\overline{\rho}_{\theta^{-1}}, H'_{\mathrm{LT}}),$$

which is $F^\times\,\mathrm{GL}_n(\mathcal{O}_F)\times W_{F_n}$-equivariant. Lemma 5.10 says that

$$\mathrm{Hom}_{J^1\cap(\mathcal{O}_D^\times/\varpi^{\mathbb{Z}})}\big(\overline{\rho}_{\theta^{-1}}, H_c^{n-1}(\mathrm{DL}_n,\overline{\mathbb{Q}}_\ell)_\theta\big) \cong \overline{\pi}_\theta \boxtimes (\theta\delta^{n-1})^{-1}(\tfrac{1-n}{2}),$$

which is a smooth representation of $F^\times\,\mathrm{GL}_n(\mathcal{O}_F)\times W_{F_n}$. Therefore, we obtain an $F^\times\,\mathrm{GL}_n(\mathcal{O}_F)\times W_{F_n}$-equivariant injection

$$\overline{\pi}_\theta \boxtimes (\theta\delta^{n-1})^{-1}(\tfrac{1-n}{2}) \hookrightarrow \mathrm{Hom}_{J^1\cap(\mathcal{O}_D^\times/\varpi^{\mathbb{Z}})}(\overline{\rho}_{\theta^{-1}}, H'_{\mathrm{LT}})^{\mathrm{sm}},$$

where $(-)^{\mathrm{sm}}$ denotes the $F^\times\,\mathrm{GL}_n(\mathcal{O}_F)\times W_{F_n}$-smooth part. On the other hand, Lemma 4.8 (ii) tells us that

$$\mathrm{Hom}_{J^1\cap(\mathcal{O}_D^\times/\varpi^{\mathbb{Z}})}(\overline{\rho}_{\theta^{-1}}, H'_{\mathrm{LT}})^{\mathrm{sm}} \cong \mathrm{Hom}_{\mathcal{O}_D^\times/\varpi^{\mathbb{Z}}}(\overline{\rho}_{\theta^{-1}}, \mathrm{Ind}_{J^1}^J H'_{\mathrm{LT}}).$$

Since $J\subset G$ is open, $G^1\subset G$ is closed normal and $J^1=G^1\cap J$, by Lemma 4.7 (ii) we have a J-equivariant isomorphism $\mathrm{Ind}_{J^1}^J H'_{\mathrm{LT}}\cong \mathrm{Ind}_{G^1}^{G^1J} H'_{\mathrm{LT}}$. Further, as G^1J is an open subgroup of G, we have a G^1J-equivariant injection $\mathrm{Ind}_{G^1}^{G^1J} H'_{\mathrm{LT}}\hookrightarrow \mathrm{Ind}_{G^1J}^G(\mathrm{Ind}_{G^1}^{G^1J} H'_{\mathrm{LT}}) = \mathrm{Ind}_{G^1}^G H'_{\mathrm{LT}}$ by Lemma 4.7 (i). Together with Proposition 4.4, we obtain an $F^\times\,\mathrm{GL}_n(\mathcal{O}_F)\times W_{F_n}$-equivariant injection

$$\overline{\pi}_\theta \boxtimes (\theta\delta^{n-1})^{-1}(\tfrac{1-n}{2}) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_D^\times/\varpi^{\mathbb{Z}}}(\overline{\rho}_{\theta^{-1}}, H_{\mathrm{LT}}),$$

which induces a non-zero J-equivariant map $\overline{\pi}_\theta\boxtimes\overline{\rho}_{\theta^{-1}}\boxtimes(\theta\delta^{n-1})^{-1}(\frac{1-n}{2})\to H_{\mathrm{LT}}$. By the Frobenius reciprocity, this corresponds to a non-zero G-equivariant map

$$\pi_\theta\boxtimes\rho_{\theta^{-1}}\boxtimes\mathrm{Ind}_{W_{F_n}}^{W_F}(\theta\delta^{n-1})^{-1}(\tfrac{1-n}{2})\to H_{\mathrm{LT}}.$$

Since π_θ is supercuspidal and its central character is trivial on $\varpi^{\mathbb{Z}}$, Theorem 4.2 tells us that there exists a non-zero map $\rho_{\theta^{-1}}\boxtimes\mathrm{Ind}_{W_{F_n}}^{W_F}(\theta\delta^{n-1})^{-1}\to \mathrm{JL}(\pi_\theta)^\vee\boxtimes\mathrm{rec}_F(\pi_\theta)^\vee$. As $\rho_{\theta^{-1}}$ and $\mathrm{JL}(\pi_\theta)^\vee$ are irreducible, we have $\rho_{\theta^{-1}}=\mathrm{JL}(\pi_\theta)^\vee$ and $\mathrm{JL}(\pi_\theta)=\rho_{\theta^{-1}}^\vee=\rho_\theta$. As $\mathrm{rec}_F(\pi_\theta)^\vee$ is irreducible and $\dim\mathrm{Ind}_{W_{F_n}}^{W_F}(\theta\delta^{n-1})^{-1}=n=\dim\mathrm{rec}_F(\pi_\theta)^\vee$, we conclude that $\mathrm{rec}_F(\pi_\theta)^\vee=\mathrm{Ind}_{W_{F_n}}^{W_F}(\theta\delta^{n-1})^{-1}$, hence $\mathrm{rec}_F(\pi_\theta)=\mathrm{Ind}_{W_{F_n}}^{W_F}(\theta\delta^{n-1})$. ■

6 Example II: simple supercuspidal representations

In this section, we determine $\mathrm{rec}_F(\pi)$ and $\mathrm{JL}(\pi)$ for a simple supercuspidal representation π of $\mathrm{GL}_n(F)$. In the following we fix a uniformizer ϖ of F and a non-trivial additive character $\psi_0\colon \mathbb{F}_p \to \mathbb{C}^\times$. We write ψ for the composite $\mathbb{F}_q \xrightarrow{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}} \mathbb{F}_p \xrightarrow{\psi_0} \mathbb{C}^\times$.

We briefly recall the notion of simple supercuspidal representations of $\mathrm{GL}_n(F)$ and $D^\times$. See [Mie16, §4.1] for more detail. Let Iw be the standard Iwahori subgroup of $\mathrm{GL}_n(F)$ and Iw_+ the pro-p radical of Iw. The character ψ determines a character of Iw_+:

$$\psi\colon \mathrm{Iw}_+ \to \mathbb{C}^\times; (a_{ij}) \mapsto \psi(\overline{a_{12}} + \overline{a_{23}} + \cdots + \overline{a_{n-1,n}} + \overline{\varpi^{-1}a_{n1}}).$$

Here the image of $a \in \mathcal{O}_F$ under the map $\mathcal{O}_F \twoheadrightarrow \mathbb{F}_q$ is denoted by $\overline{a}$. Put

$$\varphi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \varpi & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_n(F).$$

It normalizes (Iw_+, ψ).

Definition 6.1 For a character $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $c \in \mathbb{C}^\times$, let $\Lambda_{\chi,c}\colon \mathcal{O}_F^\times \varphi^{\mathbb{Z}} \mathrm{Iw}_+ \to \mathbb{C}^\times$ be the character defined as follows:

$$\Lambda_{\chi,c}(a) = \chi(\overline{a}) \ (a \in \mathcal{O}_F^\times), \quad \Lambda_{\chi,c}(\varphi) = c, \quad \Lambda_{\chi,c}|_{\mathrm{Iw}_+} = \psi.$$

We put $\pi_{\chi,c} = \text{c-Ind}_{\mathcal{O}_F^\times \varphi^{\mathbb{Z}} \mathrm{Iw}_+}^{\mathrm{GL}_n(F)} \Lambda_{\chi,c}$, which is known to be an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$.

Next we consider the group $D^\times$. Recall that F_n denotes the degree n unramified extension of F. We write $\tau \in \mathrm{Gal}(F_n/F)$ for the arithmetic Frobenius lift. Since the invariant of D is $1/n$, there exists an isomorphism $D \cong F_n[\Pi]$, where $\Pi^n = \varpi$ and $\Pi a = \tau(a)\Pi$ for $a \in F_n$. In the following, we fix this isomorphism and identify them.

Definition 6.2 For a character $\chi\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ and $c \in \mathbb{C}^\times$, let $\Lambda^D_{\chi,c}\colon \mathcal{O}_F^\times \Pi^{\mathbb{Z}}(1 + \Pi\mathcal{O}_D) \to \mathbb{C}^\times$ be the character defined as follows:

$$\Lambda^D_{\chi,c}(a) = \chi(\overline{a}) \ (a \in \mathcal{O}_F^\times), \quad \Lambda^D_{\chi,c}(\Pi) = c, \quad \Lambda^D_{\chi,c}(1 + \Pi d) = \psi(\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\overline{d})).$$

We put $\rho_{\chi,c} = \text{c-Ind}_{\mathcal{O}_F^\times \Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)}^{D^\times} \Lambda^D_{\chi,c}$, which is known to be an irreducible smooth representation of $D^\times$. Note that the isomorphism class of $\rho_{\chi,c}$ is independent of the choice of an isomorphism $D \cong F_n[\Pi]$, as every automorphism of D is inner.

Remark 6.3 (i) As in [Mie16, §4.1], simple supercuspidal representations $\{\pi_{\zeta,\chi,c}\}$ of $\mathrm{GL}_n(F)$ are in fact parameterized by $\zeta \in \mathbb{F}_q^\times$ and (χ, c) as above. The representation $\pi_{\chi,c}$ in Definition 6.1 corresponds to $\pi_{1,\chi,c}$. On the other hand, $\pi_{\zeta,\chi,c}$ equals $\pi_{\chi,c}$ for the uniformizer $\widetilde{\zeta}\varpi$, where $\widetilde{\zeta} \in \mathcal{O}_F^\times$ is the Teichmüller lift of ζ. Therefore, if we vary the fixed uniformizer ϖ, the representations of the form $\pi_{\chi,c}$ cover all simple supercuspidal representations. Similar remark holds for $\rho_{\chi,c}$.

(ii) Let $\xi\colon F^\times \to \mathbb{C}^\times$ be an unramified character. Then, we can easily see that $\pi_{\chi,c} \otimes (\xi \circ \det) \cong \pi_{\chi,\xi(\varpi)c}$ and $\rho_{\chi,c} \otimes (\xi \circ \mathrm{Nrd}) \cong \rho_{\chi,\xi(\varpi)c}$. Since rec_F and JL are compatible with character twist, to determine $\mathrm{rec}_F(\pi_{\chi,c})$ and $\mathrm{JL}(\pi_{\chi,c})$, we may assume that $c = 1$. In the following we put $\Lambda_\chi = \Lambda_{\chi,1}$, $\pi_\chi = \pi_{\chi,1}$, $\Lambda_\chi^D = \Lambda^D_{\chi,(-1)^{n-1}}$ and $\rho_\chi = \rho_{\chi,(-1)^{n-1}}$.

We write $n = n'p^e$ with $p \nmid n'$. Take an nth root ϖ_L of ϖ and put $L = F(\varpi_L)$, $\varpi_E = \varpi_L^{p^e}$ and $E = F(\varpi_E)$. For $x \in E$, we write

$$x = \varpi_E^{v_E(x)}\zeta_x(1 + u_x\varpi_E),$$

where $\zeta_x \in \mu_{q-1}(E) = \{a \in E \mid a^{q-1} = 1\} \subset \mathcal{O}_F$ and $u_x \in \mathcal{O}_E$. The goal of this section is as follows:

Theorem 6.4 *Let χ be a character of $\mathbb{F}_q^\times$.*

(i) *We have $\mathrm{JL}(\pi_\chi) = \rho_\chi$.*

(ii) *Assume that $p \nmid n$. We define two characters $\xi_\chi, \mu\colon E^\times \to \mathbb{C}^\times$ by*

$$\xi_\chi(x) = \chi(\overline{\zeta}_x)\psi(\overline{u}_x)^n, \quad \mu(x) = \lambda^{v_E(x)}\delta(\overline{\zeta}_x)^{n-1},$$

where

$$\lambda = \begin{cases} q^{-\frac{n-1}{2}} \sum_{y_1,\dots,y_{n-1}\in\mathbb{F}_q} \psi\bigl(\sum_{1\le i\le j\le n-1} y_iy_j\bigr) & p \neq 2, \\ \left(\frac{q}{n}\right) & p = 2, \end{cases}$$

and $\delta\colon \mathbb{F}_q^\times \to \mathbb{C}^\times$ is the quadratic character. Then we have

$$\mathrm{rec}_F(\pi_\chi) = \mathrm{Ind}_{W_E}^{W_F}\mu^{-1}\xi_\chi.$$

(iii) *Assume that $p \mid n$. We define two characters $\nu_\chi, \phi\colon E^\times \to \mathbb{C}$ by*

$$\nu_\chi(x) = \chi(\overline{\zeta}_x^{p^{-e}}), \quad \phi(x) = (-1)^{(n-1)v_E(x)}.$$

Put $f = [\mathbb{F}_q : \mathbb{F}_p]$ and $m = \gcd(e, f)$. Let IT_{w} be the affine algebraic variety over $\overline{\mathbb{F}}_q$ defined by the equation

$$z^{p^m} - z = y^{p^e+1} - \frac{1}{n'}\sum_{1\le i\le j\le n-2} y_iy_j.$$

As defined in [IT16, (3.17), (3.18)], the group W_E acts on $\mathrm{IT_w}$. On the other hand, the group $\mathbb{F}_{p^m}$ acts on $\mathrm{IT_w}$ by $a\colon z \mapsto z+a$, $y \mapsto y$, $y_i \mapsto y_i$ ($a \in \mathbb{F}_{p^m}$). These two actions commute. Let ψ' denote the additive character $\psi_0^{n'} \circ \mathrm{Tr}_{\mathbb{F}_{p^m}/\mathbb{F}_p}$ of $\mathbb{F}_{p^m}$. We write $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'}$ for the subspace of $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)$ on which $\mathbb{F}_{p^m}$ acts by ψ'. It is a smooth representation of W_E. Put $\tau = H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'}(\frac{n-1}{2})$.
Then we have

$$\mathrm{rec}_F(\pi_\chi) = \mathrm{Ind}_{W_E}^{W_F}(\tau \otimes \nu_\chi \otimes \phi).$$

Remark 6.5 (i) Theorem 6.4 (i) was proved in [IT14] by explicit computation of characters.

(ii) Theorem 6.4 (ii), (iii) were obtained in [BH05b], [IT15b], respectively (for (ii), see also [BH05b, Theorem 2.1 (1)] and [IT15a, §5]).

The proof of Theorem 6.4 is divided into the cases $p \nmid n$ and $p \mid n$. In this paper we only treat the case $p \mid n$; the other is similar but easier.

Definition 6.6 We put

$$J = \mathcal{O}_F^\times \varphi^{\mathbb{Z}} \mathrm{Iw}_+ \times (\mathcal{O}_F^\times \Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}}) \times W_E \subset G, \quad J^1 = G^1 \cap J.$$

Let J_{s}^1 be the subgroup of J^1 generated by the following elements:

- $(g, d, 1)$ for $g \in \mathrm{Iw}_+$ and $d \in 1+\Pi\mathcal{O}_D$ with $\det g = \mathrm{Nrd}\, d$,
- $(a, a, 1)$ for $a \in \mathcal{O}_F^\times$,
- $(\varphi, \Pi, 1)$, and
- $(g_\sigma, \Pi^{-n_\sigma}, \sigma)$ for $\sigma \in W_E$, where $g_\sigma \in \mathcal{O}_F^\times \mathrm{Iw}_+$ is the element in [IT16, (3.14)], and $n_\sigma = v_E(\mathrm{Art}_E^{-1}(\sigma))$.

Lemma 6.7 (i) *The group J_{s}^1 is an open normal subgroup of J^1.*

(ii) *Put $n_1' = \gcd(n', q-1)$ and $\mu = \mu_{n_1'}(\mathcal{O}_F)$. We regard μ as a subgroup of $D^\times/\varpi^{\mathbb{Z}}$. Then, we have $J^1 = \bigcup_{\zeta\in\mu} \zeta J_{\mathrm{s}}^1$ (later we will also prove that the union in the right hand side is disjoint).*

(iii) *The composite $J_{\mathrm{s}}^1 \hookrightarrow J \twoheadrightarrow \mathcal{O}_F^\times \varphi^{\mathbb{Z}} \mathrm{Iw}_+ \times W_E$ is surjective.*

Proof. Let I be the open subgroup of W_E consisting of σ such that $\mathrm{Art}_E^{-1}(\sigma) \in 1+\mathfrak{p}_E$. If $\sigma \in I$, then g_σ belongs to Iw_+ and $n_\sigma = 0$. Therefore, we can easily check that J_{s}^1 contains $J^1 \cap (\mathrm{Iw}_+ \times (1+\Pi\mathcal{O}_D) \times I)$. Hence J_{s}^1 is open in J^1.

We prove (ii). Take $(g, d, \sigma) \in J^1$. If we put $g' = gg_\sigma^{-1}$ and $d' = d\Pi^{n_\sigma}$, then $(g, d, \sigma) \in (g', d', 1)J_{\mathrm{s}}^1$. Since $\det g' = \mathrm{Nrd}\, d'$, there exists an integer m such that $\varphi^m g' \in \mathcal{O}_F^\times \mathrm{Iw}_+$ and $\Pi^m d' \in \mathcal{O}_F^\times(1+\Pi\mathcal{O}_D)$. Write $\varphi^m g' = ag''$ and $\Pi^m d' = a'd''$, where $a, a' \in \mu_{q-1}(\mathcal{O}_F)$, $g'' \in \mathrm{Iw}_+$ and $d'' \in 1+\Pi\mathcal{O}_D$. Since $\det(\varphi^m g') = \mathrm{Nrd}(\Pi^m d')$ and $\det g'', \mathrm{Nrd}\, d'' \in 1+\mathfrak{p}_F$, we have $a^n = a'^n$ and $\det g'' = \mathrm{Nrd}\, d''$. Put $\zeta = a'/a$, which is an element of μ. Now we have

$$(g', d', 1) = (1, \zeta, 1)(a, a, 1)(\varphi, \Pi, 1)^{-m}(g'', d'', 1) \in \zeta J_{\mathrm{s}}^1$$

and $(g,d,\sigma)\in\zeta J^1_{\mathrm{s}}$. This concludes (ii). As μ is contained in the center of J^1, J^1_{s} is normal in J^1, hence (i).

Finally consider (iii). Take an element (g,σ) of $\mathcal{O}_F^\times\varphi^{\mathbb{Z}}\operatorname{Iw}_+\times W_E$. As $gg_\sigma^{-1}\in\mathcal{O}_F^\times\varphi^{\mathbb{Z}}\operatorname{Iw}_+$, there exist $a\in\mathcal{O}_F^\times$, $m\in\mathbb{Z}$ and $g'\in\operatorname{Iw}_+$ such that $gg_\sigma^{-1}=a\varphi^m g'$. Since $\operatorname{Nrd}\colon 1+\Pi\mathcal{O}_D\to 1+\mathfrak{p}_F$ is surjective (see [Rie70, Lemma 5]), we can take $d\in 1+\Pi\mathcal{O}_D$ such that $\operatorname{Nrd} d=\det g'$. Then, the element

$$(a,a,1)(\varphi,\Pi,1)^m(g',d,1)(g_\sigma,\Pi^{-n_\sigma},\sigma)\in J^1_{\mathrm{s}}$$

is mapped to (g,σ) under $J^1_{\mathrm{s}}\hookrightarrow J\twoheadrightarrow\mathcal{O}_F^\times\varphi^{\mathbb{Z}}\operatorname{Iw}_+\times W_E$. ■

Now we use results in [IT16]. We may assume that $\varphi_{D,1}$ in [IT16, §2.1] is equal to Π. Let $U\subset M^{(0)}_{\infty,C}$ be the affinoid and $\mathcal{X}$ the formal model of U constructed in [IT16, §2] (in [IT16], U and $\mathcal{X}$ are denoted by $\mathcal{X}_1$ and $\mathfrak{X}_1$, respectively). By construction, U is a rational subset of $t(\mathcal{M}^{(0)}_{\infty,\mathcal{O}_C})$.

Proposition 6.8 (i) *The action of $J^1_{\mathrm{s}}\subset G^1$ on $M^{(0)}_{\infty,C}$ stabilizes U and extends to $\mathcal{X}$.*

(ii) *The formal scheme $\mathcal{X}$ is flat over k° and its reduction $\mathcal{X}_s$ is isomorphic to the perfection of IT_{w}.*

(iii) *The induced action of J^1_{s} on $\mathcal{X}_s$ comes from an action on IT_{w}, which is described as follows:*
- *$(g,d,1)$ for $g=(g_{ij})\in\operatorname{Iw}_+$ and $d=1+\Pi d'\in 1+\Pi\mathcal{O}_D$ with $\det g=\operatorname{Nrd} d$ acts on IT_{w} as the element*
$$\frac{1}{n'}\operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_{p^m}}\bigl(\operatorname{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\overline{d'})-(\overline{g_{12}}+\cdots+\overline{g_{n-1,n}}+\overline{\varpi^{-1}g_{n1}})\bigr)\in\mathbb{F}_{p^m},$$
- *$(a,a,1)$ for $a\in\mathcal{O}_F^\times$ acts on IT_{w} trivially,*
- *$(\varphi,\Pi,1)$ acts on IT_{w} as in [IT16, (3.2)], and*
- *$(g_\sigma,\Pi^{-n_\sigma},\sigma)$ for $\sigma\in W_E$ acts on IT_{w} as mentioned in Theorem 6.4 (iii).*

(iv) *For $\zeta\in\mu\setminus\{1\}$, we have $U\zeta\cap U=\varnothing$.*

(v) *We have $J^1=\coprod_{\zeta\in\mu}\zeta J^1_{\mathrm{s}}$ and $J^1_{\mathrm{s}}\cap(\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})=(1+\Pi\mathcal{O}_D)^1$, where $(1+\Pi\mathcal{O}_D)^1$ denotes the subgroup of $1+\Pi\mathcal{O}_D$ consisting of elements with reduced norm 1.*

Proof. The assertion (i) is obtained in [IT16, §3]. Consider (ii). The reduction $\mathcal{X}_s$ is computed in [IT16, Theorem 2.5]. In particular, $\dim\mathcal{X}_s=\dim\mathrm{IT}_{\mathrm{w}}=n-1$. By Corollary A.6 and the presentation of $\mathcal{X}$ in [IT16, §2], we conclude that $\mathcal{X}$ is flat over k°. The claim (iii) is also included in [IT16, §3].

We prove (iv). We use the notation in [IT16]. Recall that U is defined by the inequalities [IT16, (2.5)]:

$$v\Bigl(\frac{\boldsymbol{x}_i}{\boldsymbol{x}_{i+1}}-\Bigl(\frac{\boldsymbol{x}_{n-1}}{\boldsymbol{x}_n}\Bigr)^{q^{n-1-i}}\Bigr)\geq\frac{1}{2nq^i}\quad\text{for } 1\leq i\leq n-2,$$
$$v(\boldsymbol{x}_i-1)\geq\frac{1}{nq^{n-1}(p^e+1)}\quad\text{for } n-1\leq i\leq n.$$

By [IT16, (1.7)], ζ acts by $\boldsymbol{x}_i \mapsto \zeta^{-1}\boldsymbol{x}_i$. Since n_1' is prime to p and $\zeta \neq 1$, we have $\zeta^{-1} - 1 \in \mathcal{O}_F^\times$. Therefore, if $v(\boldsymbol{x}_n - 1) > 0$, we have $v(\boldsymbol{x}_n) = 0$ and

$$v(\zeta^{-1}\boldsymbol{x}_n - 1) = v((\zeta^{-1} - 1)\boldsymbol{x}_n + \boldsymbol{x}_n - 1) = 0.$$

Hence we have $U \cap U\zeta = \varnothing$, as desired.

Finally we prove (v). By (i) and (iv), we have $\mu \cap J_{\mathrm{s}}^1 = \{1\}$. Hence the cosets $\{\zeta J_{\mathrm{s}}^1\}_{\zeta\in\mu}$ are disjoint to each other. Together with Lemma 6.7 (ii), we obtain $J^1 = \coprod_{\zeta\in\mu} \zeta J_{\mathrm{s}}^1$. On the other hand, clearly we have

$$\begin{aligned}(1+\Pi\mathcal{O}_D)^1 &\subset J_{\mathrm{s}}^1 \cap (\mathcal{O}_F^\times \Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}}) \\ &\subset J^1 \cap (\mathcal{O}_F^\times \Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}}) = \mu(1+\Pi\mathcal{O}_D)^1.\end{aligned}$$

Since $\mu \cap J_{\mathrm{s}}^1 = \{1\}$, we conclude that $(1+\Pi\mathcal{O}_D)^1 = J_{\mathrm{s}}^1 \cap (\mathcal{O}_F^\times \Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})$.■

Corollary 6.9 *The subspace $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'}$ of $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)$ in Theorem 6.4 (iii) is J_{s}^1-stable. Moreover, we have an isomorphism*

$$\mathrm{Hom}_{J_{\mathrm{s}}^1 \cap (\mathcal{O}_F^\times \Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})}\big(\Lambda_\chi^D, H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'}\big) \cong \Lambda_\chi^{-1} \boxtimes (\tau \otimes \nu_\chi \otimes \phi)(\tfrac{1-n}{2})$$

of representations of $\mathcal{O}_F^\times \varphi^{\mathbb{Z}} \mathrm{Iw}_+ \times W_E$. Note that Lemma 6.7 (iii) enables us to define an action of $\mathcal{O}_F^\times \varphi^{\mathbb{Z}} \mathrm{Iw}_+ \times W_E$ on the left hand side as in Lemma 4.8 (i).

Proof. Since the action of $\mathbb{F}_{p^m}$ on $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)$ commutes with the action of J_{s}^1, the subspace $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'} \subset H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)$ is J_{s}^1-stable.

Recall that in Proposition 6.8 (v) we proved that $J_{\mathrm{s}}^1 \cap (\mathcal{O}_F^\times \Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}}) = (1+\Pi\mathcal{O}_D)^1$. For $1+\Pi d \in (1+\Pi\mathcal{O}_D)^1$, we have $\Lambda_\chi^D(1+\Pi d) = \psi(\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\overline{d}))$. On the other hand, by Proposition 6.8 (iii), $1+\Pi d$ acts on $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'}$ by $\psi'(n'^{-1}\,\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_{p^m}}(\overline{d})) = \psi(\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(\overline{d}))$. Therefore, as a vector space we have

$$\mathrm{Hom}_{J_{\mathrm{s}}^1 \cap (\mathcal{O}_F^\times \Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})}\big(\Lambda_\chi^D, H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'}\big) = H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'}. \qquad (*)$$

Under this identification, the action of $\mathcal{O}_F^\times \varphi^{\mathbb{Z}} \mathrm{Iw}_+ \times W_E$ on the left hand side is described as follows:

(a) $(a, 1)$ for $a \in \mathcal{O}_F^\times$ acts by the scalar $\chi(\overline{a})^{-1} = \Lambda_\chi^{-1}(a)$,

(b) $(\varphi, 1)$ acts trivially,

(c) $(g, 1)$ for $g \in \mathrm{Iw}_+$ acts by the scalar $\psi(g)^{-1} = \Lambda_\chi^{-1}(g)$ and

(d) (g_σ, σ) for $\sigma \in W_E$ acts on $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'}$ as $(-1)^{(n-1)n_\sigma}\sigma$ on $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)_{\psi'}$.

These are consequences of Proposition 6.8 (iii). For (b), note that (φ, Π) acts on $H_c^{n-1}(\mathrm{IT_w}, \overline{\mathbb{Q}}_\ell)$ by the scalar $(-1)^{n-1}$ (see [IT16, Proposition 3.2.2, Corollary 4.5] and [IT15a, Proposition 4.2.3]).

For $\sigma \in W_E$, put $u_\sigma = \varpi_E^{-n_\sigma}\,\mathrm{Art}_E^{-1}(\sigma)$ and take $b_\sigma \in \mu_{q-1}(\mathcal{O}_F)$ such that $\overline{b}_\sigma^{p^e} = \overline{u}_\sigma$. Recall that $g_\sigma = b_\sigma \mathrm{diag}(1, \ldots, 1, c_\sigma)$ for some $c_\sigma \in 1 + \mathfrak{p}_F$ (see [IT16, (3.14)]). Therefore, by (a) and (c), $(g_\sigma, 1)$ acts by the scalar $\chi(\overline{b}_\sigma)^{-1} = \nu_\chi(\mathrm{Art}_E^{-1}(\sigma))^{-1}$.

Hence the action of $(1,\sigma)$ on the left hand side of $(*)$ is $\nu_\chi(\mathrm{Art}_E^{-1}(\sigma))\phi(\mathrm{Art}_E^{-1}(\sigma))\sigma$. Now we conclude that

$$\mathrm{Hom}_{J_{\mathrm{s}}^1\cap(\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})}\bigl(\Lambda_\chi^D, H_c^{n-1}(\mathrm{IT}_{\mathrm{w}},\overline{\mathbb{Q}}_\ell)_{\psi'}\bigr)\cong\Lambda_\chi^{-1}\boxtimes(\tau\otimes\nu_\chi\otimes\phi)(\tfrac{1-n}{2}).\qquad\blacksquare$$

Proposition 6.10 *Let $\mathcal{L}_{\psi'}$ denote the rank 1 sheaf over $\mathbb{A}^1_{\overline{\mathbb{F}}_q}=\mathrm{Spec}\,\overline{\mathbb{F}}_q[t]$ defined by the $\mathbb{F}_{p^m}$-torsor $z^{p^m}-z=t$ and the additive character $\psi'\colon\mathbb{F}_{p^m}\to\mathbb{C}^\times$, and $\mathcal{L}_{\psi'}^{\mathrm{IT}_{\mathrm{w}}}$ the pull-back of $\mathcal{L}_{\psi'}$ under*

$$\mathbb{A}^{n-1}_{\overline{\mathbb{F}}_q}\to\mathbb{A}^1_{\overline{\mathbb{F}}_q};\quad(y,y_1,\dots,y_{n-2})\mapsto y^{p^e+1}-\frac{1}{n'}\sum_{1\le i\le j\le n-2}y_iy_j.$$

Then, $\dim H_c^{n-1}(\mathbb{A}^{n-1}_{\overline{\mathbb{F}}_q},\mathcal{L}_{\psi'}^{\mathrm{IT}_{\mathrm{w}}})=p^e$ and the map

$$H_c^{n-1}(\mathbb{A}^{n-1}_{\overline{\mathbb{F}}_q},\mathcal{L}_{\psi'}^{\mathrm{IT}_{\mathrm{w}}})\to H^{n-1}(\mathbb{A}^{n-1}_{\overline{\mathbb{F}}_q},\mathcal{L}_{\psi'}^{\mathrm{IT}_{\mathrm{w}}})$$

is an isomorphism.

In particular, $\dim H_c^{n-1}(\mathrm{IT}_{\mathrm{w}},\overline{\mathbb{Q}}_\ell)_{\psi'}=p^e$ and the composite

$$H_c^{n-1}(\mathrm{IT}_{\mathrm{w}},\overline{\mathbb{Q}}_\ell)_{\psi'}\hookrightarrow H_c^{n-1}(\mathrm{IT}_{\mathrm{w}},\overline{\mathbb{Q}}_\ell)\to H^{n-1}(\mathrm{IT}_{\mathrm{w}},\overline{\mathbb{Q}}_\ell)$$

is injective.

Proof. We denote by $\mathcal{L}^1_{\psi'}$ (resp. $\mathcal{L}^2_{\psi'}$) the pull-back of $\mathcal{L}_{\psi'}$ under $\mathbb{A}^1_{\overline{\mathbb{F}}_q}\to\mathbb{A}^1_{\overline{\mathbb{F}}_q}$; $y\mapsto y^{p^e+1}$ (resp. $\mathbb{A}^{n-2}_{\overline{\mathbb{F}}_q}\to\mathbb{A}^1_{\overline{\mathbb{F}}_q}$; $(y_1,\dots,y_{n-2})\mapsto -n'^{-1}\sum_{1\le i\le j\le n-2}y_iy_j$). Then we have $\mathcal{L}_{\psi'}^{\mathrm{IT}_{\mathrm{w}}}=\mathcal{L}^1_{\psi'}\boxtimes\mathcal{L}^2_{\psi'}$.

We use results in Appendix B. By Remark B.1 and Lemma B.2, we have

- $\dim H_c^1(\mathbb{A}^1_{\overline{\mathbb{F}}_q},\mathcal{L}^1_{\psi'})=p^e$,
- $H_c^i(\mathbb{A}^1_{\overline{\mathbb{F}}_q},\mathcal{L}^1_{\psi'})=H^i(\mathbb{A}^1_{\overline{\mathbb{F}}_q},\mathcal{L}^1_{\psi'})=0$ for $i\neq 1$, and
- $H_c^1(\mathbb{A}^1_{\overline{\mathbb{F}}_q},\mathcal{L}^1_{\psi'})\to H^1(\mathbb{A}^1_{\overline{\mathbb{F}}_q},\mathcal{L}^1_{\psi'})$ is an isomorphism.

By Remark B.1, Lemma B.3 and Example B.4, we have

- $\dim H_c^{n-2}(\mathbb{A}^{n-2}_{\overline{\mathbb{F}}_q},\mathcal{L}^2_{\psi'})=1$ and
- $H_c^{n-2}(\mathbb{A}^{n-2}_{\overline{\mathbb{F}}_q},\mathcal{L}^2_{\psi'})\to H^{n-2}(\mathbb{A}^{n-2}_{\overline{\mathbb{F}}_q},\mathcal{L}^2_{\psi'})$ is an isomorphism.

Hence we conclude the proposition by the Künneth formula. $\blacksquare$

Corollary 6.11 *We have a J^1-equivariant injection $\mathrm{Ind}_{J_{\mathrm{s}}^1}^{J^1}H_c^{n-1}(\mathrm{IT}_{\mathrm{w}},\overline{\mathbb{Q}}_\ell)_{\psi'}\to H'_{\mathrm{LT}}$.*

Proof. By Proposition 6.8 (i), (ii), the pair $(U,\mathcal{X})$ satisfies the condition in Proposition 4.5 (we may take $K_U=J_{\mathrm{s}}^1\cap\mathrm{SL}_n(\mathcal{O}_F)$). It is easy to see that IT_{w} is purely $n-1$-dimensional and smooth. By Proposition 6.8 (i), (iv) and (v), for $g\in J^1\setminus J_{\mathrm{s}}^1$ we have $U\cap Ug=\varnothing$. Hence, by Corollary 4.6 (iii) and Proposition 6.10, we have J^1-equivariant homomorphisms

$$\mathrm{Ind}_{J_{\mathrm{s}}^1}^{J^1}H_c^{n-1}(\mathrm{IT}_{\mathrm{w}},\overline{\mathbb{Q}}_\ell)_{\psi'}\hookrightarrow\mathrm{Ind}_{J_{\mathrm{s}}^1}^{J^1}H_c^{n-1}(\mathrm{IT}_{\mathrm{w}},\overline{\mathbb{Q}}_\ell)\to H'_{\mathrm{LT}},$$

whose composite is injective. $\blacksquare$

Proof of Theorem 6.4. By Corollary 6.11, we have an injection

$$\begin{aligned}&\mathrm{Hom}_{J^1\cap(\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})}\big(\Lambda_\chi^D, \mathrm{Ind}_{J_{\mathrm{s}}^1}^{J^1} H_c^{n-1}(\mathrm{IT}_{\mathrm{w}}, \overline{\mathbb{Q}}_\ell)_{\psi'}\big)\\ &\quad\hookrightarrow \mathrm{Hom}_{J^1\cap(\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})}(\Lambda_\chi^D, H'_{\mathrm{LT}}),\end{aligned}$$

which is $\mathcal{O}_F^\times\varphi^{\mathbb{Z}}\,\mathrm{Iw}_+\times W_E$-equivariant. Taking the $\mathcal{O}_F^\times\varphi^{\mathbb{Z}}\,\mathrm{Iw}_+\times W_E$-smooth part $(-)^{\mathrm{sm}}$, we obtain an $\mathcal{O}_F^\times\varphi^{\mathbb{Z}}\,\mathrm{Iw}_+\times W_E$-equivariant injection

$$\begin{aligned}&\mathrm{Hom}_{J^1\cap(\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})}\big(\Lambda_\chi^D, \mathrm{Ind}_{J_{\mathrm{s}}^1}^{J^1} H_c^{n-1}(\mathrm{IT}_{\mathrm{w}}, \overline{\mathbb{Q}}_\ell)_{\psi'}\big)^{\mathrm{sm}}\\ &\quad\hookrightarrow \mathrm{Hom}_{J^1\cap(\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})}(\Lambda_\chi^D, H'_{\mathrm{LT}})^{\mathrm{sm}}.\end{aligned}$$

By Lemma 4.8 (iii), the left hand side is identified with

$$\mathrm{Hom}_{J_{\mathrm{s}}^1\cap(\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})}\big(\Lambda_\chi^D, H_c^{n-1}(\mathrm{IT}_{\mathrm{w}}, \overline{\mathbb{Q}}_\ell)_{\psi'}\big)^{\mathrm{sm}}.$$

On the other hand, in the same way as in the proof of Theorem 5.3, we have

$$\begin{aligned}\mathrm{Hom}_{J^1\cap(\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}})}(\Lambda_\chi^D, H'_{\mathrm{LT}})^{\mathrm{sm}} &\cong \mathrm{Hom}_{\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}}}(\Lambda_\chi^D, \mathrm{Ind}_{J^1}^{J} H'_{\mathrm{LT}})\\ &= \mathrm{Hom}_{\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}}}(\Lambda_\chi^D, \mathrm{Ind}_{G^1J}^{G^1J} H'_{\mathrm{LT}})\\ &\hookrightarrow \mathrm{Hom}_{\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}}}(\Lambda_\chi^D, \mathrm{Ind}_{G^1}^{G} H'_{\mathrm{LT}})\\ &= \mathrm{Hom}_{\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}}}(\Lambda_\chi^D, H_{\mathrm{LT}}).\end{aligned}$$

Therefore, by Corollary 6.9 we have an $\mathcal{O}_F^\times\varphi^{\mathbb{Z}}\,\mathrm{Iw}_+\times W_E$-equivariant injection

$$\Lambda_\chi^{-1}\boxtimes(\tau\otimes\nu_\chi\otimes\phi)(\tfrac{1-n}{2})\hookrightarrow \mathrm{Hom}_{\mathcal{O}_F^\times\Pi^{\mathbb{Z}}(1+\Pi\mathcal{O}_D)/\varpi^{\mathbb{Z}}}(\Lambda_\chi^D, H_{\mathrm{LT}}).$$

This induces a non-zero J-equivariant map

$$\Lambda_\chi^{-1}\boxtimes\Lambda_\chi^D\boxtimes(\tau\otimes\nu_\chi\otimes\phi)(\tfrac{1-n}{2})\to H_{\mathrm{LT}},$$

which corresponds to a non-zero G-equivariant map

$$\pi_\chi^\vee\boxtimes\rho_\chi\boxtimes\mathrm{Ind}_{W_E}^{W_F}(\tau\otimes\nu_\chi\otimes\phi)(\tfrac{1-n}{2})\to H_{\mathrm{LT}}$$

by the Frobenius reciprocity. Since $\pi_\chi^\vee$ is supercuspidal and its central character is trivial on $\varpi^{\mathbb{Z}}$, Theorem 4.2 tells us that there exists a non-zero map $\rho_\chi\boxtimes\mathrm{Ind}_{W_E}^{W_F}(\tau\otimes\nu_\chi\otimes\phi)\to\mathrm{JL}(\pi_\chi)\boxtimes\mathrm{rec}_F(\pi_\chi)$. As ρ_χ and $\mathrm{JL}(\pi_\chi)$ are irreducible, we have $\rho_\chi=\mathrm{JL}(\pi_\chi)$. As $\mathrm{rec}_F(\pi_\chi)$ is irreducible and $\dim\mathrm{Ind}_{W_E}^{W_F}(\tau\otimes\nu_\chi\otimes\phi)=n'\dim\tau=n'p^e=n=\dim\mathrm{rec}_F(\pi_\chi)$ (see Proposition 6.10), we conclude that $\mathrm{rec}_F(\pi_\chi)=\mathrm{Ind}_{W_E}^{W_F}(\tau\otimes\nu_\chi\otimes\phi)$.■

Remark 6.12 In the proof above, we do not need any information about the representation τ except its dimension. The irreducibility of $\mathrm{Ind}_{W_E}^{W_F}(\tau\otimes\nu_\chi\otimes\phi)$ is also a consequence of the proof.

A Some ring theory over k°

Let k be a complete non-archimedean field. As in Section 2, k° denotes the valuation ring of k, $\mathfrak{m}$ the maximal ideal of k°, and κ the residue field of k°. Choose a non-zero element $\varpi \in \mathfrak{m}$. In this appendix, we collect some useful facts on algebras over k°.

For a k°-algebra A, we consider the following three properties:

(a) A is ϖ-torsion free (in other words, flat over k°).

(b) $A/\mathfrak{m}A$ is an integral domain.

(c) For every non-zero element $f \in A$, there exist $c_f \in k^\circ$ and $f' \in A \setminus \mathfrak{m}A$ such that $f = c_f f'$.

Proposition A.1 *Assume that A satisfies the properties (a), (b) and (c). Then the following hold.*

(i) *The ring A is an integral domain.*

(ii) *For $f \in A \setminus \mathfrak{m}A$, $A/(f)$ is flat over k°.*

Proof. We prove (i). First, by the property (b), we have $A \neq 0$. Let f and g be non-zero elements in A. Take decompositions $f = c_f f'$ and $g = c_g g'$ as in the property (c). Clearly $c_f c_g \neq 0$ in k°. By (b), we have $f'g' \in A \setminus \mathfrak{m}A$, and then $f'g' \neq 0$. Hence, by (a) we conclude that $fg = (c_f c_g)(f'g') \neq 0$, as desired.

Next consider (ii). Take $a \in A$ such that $\varpi a \in (f)$ and prove $a \in (f)$. Write $\varpi a = fg$ with $g \in A$. We may assume that $a \neq 0$, which implies that $g \neq 0$ by (a). Take a decomposition $g = c_g g'$ as in (c). Since $c_g \neq 0$, we have either $c_g/\varpi \in k^\circ$ or $\varpi/c_g \in \mathfrak{m}$. If the latter holds, then $fg' = (\varpi/c_g)a$ lies in $\mathfrak{m}A$. As $f, g' \in A \setminus \mathfrak{m}A$, this contradicts the property (b). Therefore c_g/ϖ lies in k°, and $a = (c_g/\varpi)fg'$ lies in (f). ■

Corollary A.2 *Assume that A satisfies the properties (a), (b) and (c). Fix an integer $q > 1$. Let $(f_m)_{m\geq 1}$ and $(g_m)_{m\geq 1}$ be sequences of elements in A satisfying the following:*

$$f_{m+1}^q = f_m, \quad g_{m+1}^q = g_m, \quad f_1 - g_1 \in A \setminus \mathfrak{m}A.$$

Then the k°-algebra $A/(f_m - g_m \mid m \geq 1)$ is flat over k°.

Proof. Put $I = (f_m - g_m \mid m \geq 1)$ and $I_m = (f_m - g_m)$. Since $f_m - g_m = f_{m+1}^q - g_{m+1}^q = (f_{m+1} - g_{m+1})(f_{m+1}^{q-1} + \cdots + g_{m+1}^{q-1}) \in I_{m+1}$, we have $I_1 \subset I_2 \subset \cdots$ and $I = \bigcup_{m\geq 1} I_m$. Therefore $A/I \cong \varinjlim_m A/I_m$, and it suffices to prove that A/I_m is flat over k° for each m.

If $f_m - g_m$ belongs to $\mathfrak{m}A$, I_m is contained in $\mathfrak{m}A$, and then $f_1 - g_1 \in I_m$ lies in $\mathfrak{m}A$. Hence the assumption $f_1 - g_1 \in A \setminus \mathfrak{m}A$ implies that $f_m - g_m \in A \setminus \mathfrak{m}A$. Now Proposition A.1 (ii) tells us that A/I_m is flat over k°. ■

Proposition A.3 *Let $\widehat{A} = \varprojlim_m A/\varpi^m A$ denote the ϖ-adic completion of A. If A satisfies the property (a) (resp. (b), resp. (c)), so does $\widehat{A}$.*

Proof. Assume that A satisfies the property (a). Then, for every integer $m \geq 1$, the map $A/\varpi^m A \xrightarrow{\times\varpi} A/\varpi^{m+1}A$ is an injection. By taking projective limit, we obtain the injectivity of $\widehat{A} \xrightarrow{\times\varpi} \widehat{A}$. Hence $\widehat{A}$ satisfies (a).

Next assume that A satisfies (b). By [FK13, Chapter 0, Lemma 7.2.8, Corollary 7.2.9, Proposition 7.2.15], we have $\widehat{A}/\varpi\widehat{A} \cong A/\varpi A$. By taking base change $(-) \otimes_{k^\circ/\varpi k^\circ} k^\circ/\mathfrak{m}$, we have $\widehat{A}/\mathfrak{m}\widehat{A} \cong A/\mathfrak{m}A$. Therefore $\widehat{A}/\mathfrak{m}\widehat{A}$ is also an integral domain, and $\widehat{A}$ satisfies the property (b).

Finally assume that A satisfies (c). We write i for the natural homomorphism $A \to \widehat{A}$. Take an arbitrary non-zero element $a \in \widehat{A}$. By [FK13, Chapter 0, Proposition 7.2.15], $\widehat{A}$ is ϖ-adically complete, hence ϖ-adically separated. Thus there exists an integer $m \geq 0$ such that $a \notin \varpi^m\widehat{A}$. Since $\widehat{A}/\varpi^m\widehat{A} \cong A/\varpi^m A$, we can find $f \in A$ such that $a - i(f) \in \varpi^m\widehat{A}$. Since $a \notin \varpi^m\widehat{A}$, we have $i(f) \notin \varpi^m\widehat{A}$. In particular $f \neq 0$, therefore, by the property (c) for A, we have a decomposition $f = c_f f'$ with $c_f \in k^\circ$ and $f' \in A \setminus \mathfrak{m}A$. As $i(f) \notin \varpi^m\widehat{A}$, c_f does not lie in $\varpi^m k^\circ$. Hence we have $\varpi^m/c_f \in \mathfrak{m}$.

If we write $a - i(f) = \varpi^m a_1$ with $a_1 \in \widehat{A}$, then

$$a = i(f) + \varpi^m a_1 = c_f\big(i(f') + (\varpi^m/c_f)a_1\big).$$

Since $A/\mathfrak{m}A \cong \widehat{A}/\mathfrak{m}\widehat{A}$, we have $i(f') \in \widehat{A} \setminus \mathfrak{m}\widehat{A}$. Together with $(\varpi^m/c_f)a_1 \in \mathfrak{m}\widehat{A}$, we obtain $i(f') + (\varpi^m/c_f)a_1 \in \widehat{A} \setminus \mathfrak{m}\widehat{A}$. Therefore $a = c_f(i(f') + (\varpi^m/c_f)a_1)$ gives the desired decomposition. ■

Corollary A.4 *Assume that A is ϖ-adically complete and satisfies the properties (a), (b) and (c). Fix an integer $q > 1$ and let $(f_m)_{m\geq 1}$ and $(g_m)_{m\geq 1}$ be as in Corollary A.2. We denote by $\overline{I}$ the closure in A of the ideal $I = (f_m - g_m \mid m \geq 1)$. Then, $A/\overline{I}$ is flat over k°.*

Proof. By Corollary A.2, A/I is ϖ-torsion free. By [FK13, Chapter 0, Proposition 7.4.5], the ϖ-adic completion of A/I equals $A/\overline{I}$. Hence Proposition A.3 tells us that $A/\overline{I}$ is also ϖ-torsion free, that is, flat over k°. ■

Example A.5 Let $n \geq 0$ be an integer.

(i) The polynomial ring $k^\circ[T_1, \ldots, T_n]$ obviously satisfies the properties (a), (b), (c).

(ii) Suppose that the characteristic of $\kappa = k^\circ/\mathfrak{m}$ is $p > 0$. Let q be a power of p and put $k^\circ[T_1^{q^{-\infty}}, \ldots, T_n^{q^{-\infty}}] = \varinjlim_{T_i \mapsto T_i^q} k^\circ[T_1, \ldots, T_n]$. Then, $k^\circ[T_1^{q^{-\infty}}, \ldots, T_n^{q^{-\infty}}]$ satisfies the properties (a), (b), (c). Indeed, (a) and (b) follow from (i), since these are preserved by filtered inductive limits. The property (c) follows from that on $k^\circ[T_1, \ldots, T_n]$ and the fact that $\kappa[T_1, \ldots, T_n] \to \kappa[T_1, \ldots, T_n]$; $T_i \mapsto T_i^q$ is an injection.

(iii) Under the same assumption as in (ii), let $k^\circ\langle T_1^{q^{-\infty}},\dots,T_n^{q^{-\infty}}\rangle$ denote the ϖ-adic completion of $k^\circ[T_1^{q^{-\infty}},\dots,T_n^{q^{-\infty}}]$. Then, Proposition A.3 tells us that $k^\circ\langle T_1^{q^{-\infty}},\dots,T_n^{q^{-\infty}}\rangle$ satisfies the properties (a), (b), (c).

Corollary A.6 *Let $A=k^\circ\langle T_1^{q^{-\infty}},\dots,T_n^{q^{-\infty}}\rangle$ be as in Example A.5 (iii), and $(f_m)_{m\geq 1}$, $(g_m)_{m\geq 1}$ sequences of elements in A satisfying $f_{m+1}^q=f_m$ and $g_{m+1}^q=g_m$. We denote by $\overline{I}$ the closure in A of the ideal $I=(f_m-g_m\mid m\geq 1)$. Assume that $\dim(A/\overline{I}\otimes_{k^\circ}\kappa)<n$. Then, $A/\overline{I}$ is flat over k°.*

Proof. Since $A/\overline{I}$ is the ϖ-adic completion of A/I, as in the proof of Proposition A.3 we have $A/\overline{I}\otimes_{k^\circ}\kappa=A/I\otimes_{k^\circ}\kappa=\kappa[T_1^{q^{-\infty}},\dots,T_n^{q^{-\infty}}]/(\overline{f_m}-\overline{g_m}\mid m\geq 1)$, where $\overline{f_m}$ and $\overline{g_m}$ denote the images of f_m and g_m in $\kappa[T_1^{q^{-\infty}},\dots,T_n^{q^{-\infty}}]$, respectively. Since $\kappa[T_1^{q^{-\infty}},\dots,T_n^{q^{-\infty}}]$ is n-dimensional, the condition $\dim(A/\overline{I}\otimes_{k^\circ}\kappa)<n$ implies that $\overline{f_m}-\overline{g_m}\neq 0$ for some $m\geq 1$. Since $\kappa[T_1^{q^{-\infty}},\dots,T_n^{q^{-\infty}}]$ is an integral domain, we have $\overline{f_1}-\overline{g_1}=(\overline{f_m}-\overline{g_m})^{q^{m-1}}\neq 0$, in other words, $f_1-g_1\notin\mathfrak{m}A$. Hence Corollary A.4 tells us that $A/\overline{I}$ is flat over k°, as desired. ■

B Cohomology of Artin-Schreier sheaves

Let k be an algebraically closed field of characteristic $p>0$. Fix a non-trivial additive character $\psi\colon\mathbb{F}_p\to\mathbb{C}^\times$. We write $\mathcal{L}_\psi$ for the rank 1 sheaf defined by ψ and the $\mathbb{F}_p$-torsor $a^p-a=t$ over $\mathbb{A}^1=\operatorname{Spec}k[t]$.

Remark B.1 For $m\geq 1$, let ψ_m be the additive character $\psi\circ\operatorname{Tr}_{\mathbb{F}_{p^m}/\mathbb{F}_p}$ of $\mathbb{F}_{p^m}$. Then, $\mathcal{L}_\psi$ is the rank 1 sheaf defined by ψ_m and the $\mathbb{F}_{p^m}$-torsor $a^{p^m}-a=t$ over $\mathbb{A}^1$.

Lemma B.2 *Let $d\geq 1$ be an integer prime to p, and ϕ denote the morphism $\mathbb{A}^1\to\mathbb{A}^1$; $t\mapsto t^d$. Then the following hold.*

(i) *We have $\dim H_c^1(\mathbb{A}^1,\phi^*\mathcal{L}_\psi)=d-1$ and $H_c^i(\mathbb{A}^1,\phi^*\mathcal{L}_\psi)=H^i(\mathbb{A}^1,\phi^*\mathcal{L}_\psi)=0$ for $i\neq 1$.*

(ii) *The canonical map $H_c^1(\mathbb{A}^1,\phi^*\mathcal{L}_\psi)\to H^1(\mathbb{A}^1,\phi^*\mathcal{L}_\psi)$ is an isomorphism.*

Proof. In [Del74, (8.11)], (i) and the following is proved:

(ii′) the cup product pairing

$$H_c^1(\mathbb{A}^1,\phi^*\mathcal{L}_\psi)\times H_c^1(\mathbb{A}^1,\phi^*\mathcal{L}_{\psi^{-1}})\to H_c^2(\mathbb{A}^1,\overline{\mathbb{Q}}_\ell)=\overline{\mathbb{Q}}_\ell(-1)$$

is perfect.

The assertion (ii) follows from (ii′) and the Poincaré duality. ■

Lemma B.3 *Fix an integer $n\geq 0$ and consider a non-degenerate quadratic form with n variables $Q\colon\mathbb{A}^n\to\mathbb{A}^1$. If $p=2$, we assume that n is even. Then, we have $\dim H_c^n(\mathbb{A}^n,Q^*\mathcal{L}_\psi)=1$ and the canonical map $H_c^n(\mathbb{A}^n,Q^*\mathcal{L}_\psi)\to H^n(\mathbb{A}^n,Q^*\mathcal{L}_\psi)$ is bijective.*

Proof. First assume that $p \neq 2$. Then, Q is diagonalizable, hence the claim follows from Lemma B.2 and the Künneth formula.

Next consider the case $p = 2$. We may assume that $n > 0$. Let $X \subset \mathbb{A}^{n+1}$ be the affine variety defined by $y^2 - y = Q(x_1, \ldots, x_n)$. We have

$$H_c^n(X, \overline{\mathbb{Q}}_\ell) = H_c^n(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell) \oplus H_c^n(\mathbb{A}^n, Q^*\mathcal{L}_\psi) = H_c^n(\mathbb{A}^n, Q^*\mathcal{L}_\psi),$$
$$H^n(X, \overline{\mathbb{Q}}_\ell) = H^n(\mathbb{A}^n, \overline{\mathbb{Q}}_\ell) \oplus H^n(\mathbb{A}^n, Q^*\mathcal{L}_\psi) = H^n(\mathbb{A}^n, Q^*\mathcal{L}_\psi).$$

Therefore, it suffices to show that $\dim H_c^n(X, \overline{\mathbb{Q}}_\ell) = 1$ and the map $H_c^n(X, \overline{\mathbb{Q}}_\ell) \to H^n(X, \overline{\mathbb{Q}}_\ell)$ is a bijection. Consider the quadric $\overline{X}$ in $\mathbb{P}^{n+1}$ defined by the homogeneous equation $y^2 - yz = Q(x_1, \ldots, x_n)$, which includes X as the complement of the hyperplane section $z = 0$. We will show that $\overline{X}$ is smooth over k. Since Q is non-degenerate and n is even, it is ordinary in the sense of [SGA7, Exposé XII]. Therefore, by changing variables, we may assume that $Q(x_1, \ldots, x_n) = \sum_{i=1}^m x_i x_{i+m}$, where we put $m = n/2$ (see [SGA7, Exposé XII, Proposition 1.2]). In this case, the quadratic form $y(y - z) - \sum_{i=1}^m x_i x_{i+m}$ is obviously ordinary; in other words, the quadric $\overline{X}$ is smooth over k. Now [SGA7, Exposé XII, §3.6 and Table 3.7] tells us that $\dim H_c^n(X, \overline{\mathbb{Q}}_\ell) = 1$ and $H_c^n(X, \overline{\mathbb{Q}}_\ell) \to H^n(X, \overline{\mathbb{Q}}_\ell)$ is an isomorphism. This concludes the proof. ■

Example B.4 For an integer $n \geq 0$ and $a \in k^\times$, the quadratic form

$$\nu(x_1, \ldots, x_n) = a \sum_{1 \leq i \leq j \leq n} x_i x_j$$

is non-degenerate if $p \neq 2$ or n is even.

Proof. If $p \neq 2$, this is proved in [IT15a, Lemma 4.1]. Assume that $p = 2$ and n is even. For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we have

$$\Phi(x, y) := \nu(x + y) - \nu(x) - \nu(y) = a \sum_{i \neq j} x_i y_j.$$

This satisfies $\Phi(e_i, e_i) = 0$ and $\Phi(e_i, e_j) = a$ for $i \neq j$, where $e_1, \ldots, e_n$ denote the standard basis. It suffices to show that Φ is non-degenerate. Take a non-zero element $x = \sum_{i=1}^n a_i e_i$ with $a_i \in k$. Put $s = a_1 + \cdots + a_n$. If $s - a_i = 0$ for every $1 \leq i \leq n$, we have $0 = \sum_{i=1}^n (s - a_i) = ns - s = s$, $a_i = s = 0$ and $x = 0$. Hence there exists $1 \leq j \leq n$ such that $s - a_j \neq 0$. This j satisfies $\Phi(x, e_j) = a(s - a_j) \neq 0$, which concludes that Φ is non-degenerate. ■

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