Note on weight-monodromy conjecture for p-adically uniformized varieties

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Abstract. We prove the weight-monodromy conjecture for varieties which are p-adically uniformized by a product of the Drinfeld upper half spaces. It is an easy consequence of Dat's work on the cohomology complex of the Drinfeld upper half space.

1 Introduction

Let X be a proper smooth variety over a p-adic field F. For a prime number $\ell \neq p$ and an integer i, the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ acts on the ith ℓ-adic étale cohomology $H^i(X \otimes_F \overline{F}, \overline{\mathbb{Q}}_\ell)$. This action determines two filtrations on the cohomology; the weight filtration and the monodromy filtration. The weight-monodromy conjecture predicts that these two filtrations coincide up to shift by i. This conjecture, due to Deligne [Del71], is widely open. It is known in the following cases:

(i) X has good reduction over $\mathcal{O}_F$ ([Del74], [Del80]).

(ii) X is an abelian variety ([SGA7, Exposé IX]).

(iii) $i \leq 2$ ([RZ82], [dJ96]).

(iv) X is uniformized by the covering of the Drinfeld upper half space ([Ito05], [Dat06], [Dat07]).

(v) X is a set-theoretic complete intersection in a projective smooth toric variety ([Sch12]).

In this short note, we will slightly generalize the case (iv); we will consider a variety X which is uniformized by a product of the Drinfeld upper half spaces. Interesting examples of such varieties are given by some unitary Shimura varieties (see [RZ96, Theorem 6.50]). By using our result, we can compute the ℓ-adic cohomology and the local Hasse-Weil zeta functions of such Shimura varieties without any effort.

Our setting is as follows. Let F, F' be p-adic fields and F'' a p-adic field containing F and F'. Fix integers $d, d' \geq 1$ and put $G = \mathrm{PGL}_d(F)$, $G' = \mathrm{PGL}_d(F')$,

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respectively. Let $\Omega_F = \Omega_F^{d-1}$ (resp. $\Omega_{F'} = \Omega_{F'}^{d'-1}$) denote the $d-1$-dimensional (resp. $d'-1$-dimensional) Drinfeld upper half space. To simplify the notation, we write $\Omega_F \times_{F''} \Omega_{F'} = (\Omega_F \otimes_F F'') \times_{F''} (\Omega_{F'} \otimes_{F'} F'')$. For a discrete torsion-free cocompact subgroup $\Gamma \subset G \times G'$, the quotient $\Omega_F \times_{F''} \Omega_{F'}/\Gamma$ becomes a projective smooth variety over F''. Such a variety is said to be uniformized by $\Omega_F \times_{F''} \Omega_{F'}$.

The main theorem of this article is the following:

Theorem 1.1 *Let X be a projective smooth variety over F'' which is uniformized by $\Omega_F \times_{F''} \Omega_{F'}$. Then, the weight-monodromy conjecture holds for X.*

Our strategy is the same as that in [Dat06]. First we determine the monodromy operator on the cohomology complex $R\Gamma_c((\Omega_F \times_{F''} \Omega_{F'}) \otimes_{F''} \overline{F''}, \overline{\mathbb{Q}}_\ell)$, which is an object of the derived category of smooth $G \times G'$-representations. Using this result, one can easily compute the cohomology of X, from which the weight monodromy conjecture is deduced.

Although Theorem 1.1 is stated for the product of two Drinfeld upper half spaces, our argument also works for the product of more than two Drinfeld upper half spaces. Further, as in [Dat07], we may replace the Drinfeld upper half space by its covering introduced by Drinfeld [Dri76]. See Theorem 2.5.

The structure of this paper is as follows. In Section 2, we give a proof of Theorem 1.1. A short remark on applications to some unitary Shimura varieties are included in the end of this section. In Appendix A, we prove an equivariant version of the Künneth formula, which is needed in Section 2.

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2 Proof of the main theorem

In this section, we continue to use the notation in Introduction. We fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and identify them. All representations are considered over this field. In the notation of ℓ-adic étale cohomology, we omit the coefficient $\overline{\mathbb{Q}}_\ell$ and the base change to a separable closure. For example, $H^i(X \otimes_F \overline{F}, \overline{\mathbb{Q}}_\ell)$ is written as $H^i(X)$.

For a subset I of $\{1, \dots, d-1\}$, an irreducible smooth representation π_I of G is naturally attached (see [Dat06, 2.1.3]). For example, $\pi_\varnothing$ is the Steinberg representation $\mathbf{St}_d$ and $\pi_{\{1,\dots,d-1\}}$ the trivial representation $\mathbf{1}$. For $0 \le i \le d-1$, we write $\pi_{\le i}$ for $\pi_{\{1,\dots,i\}}$. Similarly, for $J \subset \{1, \dots, d'-1\}$, consider an irreducible smooth representation π'_J of G'.

Lemma 2.1 (i) *For $I_1, I_2 \subset \{1, \dots, d-1\}$ and $J_1, J_2 \subset \{1, \dots, d'-1\}$, we have*

$$\mathrm{Ext}^i_{G\times G'}(\pi_{I_1} \boxtimes \pi'_{J_1}, \pi_{I_2} \boxtimes \pi'_{J_2}) = \begin{cases} \overline{\mathbb{Q}}_\ell & \text{if } i = \delta(I_1, I_2) + \delta(J_1, J_2), \\ 0 & \text{otherwise.} \end{cases}$$

Here $\delta(I_1, I_2) = \#(I_1 \cup I_2) - \#(I_1 \cap I_2)$.

(ii) *Let I_1, I_2, I_3 be subsets of $\{1,\ldots,d-1\}$ satisfying $\delta(I_1,I_2)+\delta(I_2,I_3)=\delta(I_1,I_3)$. Take a non-zero element $\beta\in\operatorname{Ext}_G^{\delta(I_1,I_2)}(\pi_{I_1},\pi_{I_2})$. For $J_1,J_2\subset\{1,\ldots,d'-1\}$, the homomorphism*

$$\operatorname{Ext}_{G\times G'}^{\delta(I_2,I_3)+\delta(J_1,J_2)}(\pi_{I_2}\boxtimes\pi'_{J_1},\pi_{I_3}\boxtimes\pi'_{J_2}) \xrightarrow{(\beta\cup-)\boxtimes\mathrm{id}} \operatorname{Ext}_{G\times G'}^{\delta(I_1,I_3)+\delta(J_1,J_2)}(\pi_{I_1}\boxtimes\pi'_{J_1},\pi_{I_3}\boxtimes\pi'_{J_2})$$

is an isomorphism.

(iii) *Let $\pi\boxtimes\pi'$ be an irreducible smooth representation of $G\times G'$. If it is not of the form $\pi_{I_0}\boxtimes\pi'_{J_0}$ with $I_0\subset\{1,\ldots,d-1\}$ and $J_0\subset\{1,\ldots,d'-1\}$, then we have $\operatorname{Ext}^i_{G\times G'}(\pi_I\boxtimes\pi'_J,\pi\boxtimes\pi')=0$ for every $I\subset\{1,\ldots,d-1\}$, $J\subset\{1,\ldots,d'-1\}$ and i.*

Proof. For (i) and (ii), apply [Dat06, Théorème 1.3] to the semisimple group $G\times G'$. The claim (iii) follows from [Vig97, Theorem 6.1], since the cuspidal supports of $\pi_I\boxtimes\pi'_J$ and $\pi\boxtimes\pi'$ are different. ■

Now we recall a result of Dat, which is crucial for our work. In [Dat06], he studied the cohomology complex $R\Gamma_c(\Omega_F)=R\Gamma_c(\Omega_F\otimes_F\overline{F},\overline{\mathbb{Q}}_\ell)$, which is an object of the bounded derived category of smooth representations of G (see Definition A.2 and Proposition A.3 (ii)). The Weil group W_F of F acts on $R\Gamma_c(\Omega_F)$.

Theorem 2.2 ([Dat06]) *Fix a Frobenius lift $\varphi\in W_F$.*

(i) *([Dat06, Proposition 4.2.2]) There exists an isomorphism*

$$\alpha\colon R\Gamma_c(\Omega_F)\xrightarrow{\cong}\bigoplus_{i=0}^{d-1}\pi_{\le i}(-i)[-d+1-i]$$

compatible with the actions of φ. In the following, we fix such an α. It induces an isomorphism

$$\operatorname{End}\bigl(R\Gamma_c(\Omega_F)\bigr)\cong\bigoplus_{0\le i<j\le d-1}\operatorname{Ext}_G^{j-i}(\pi_{\le j},\pi_{\le i})(j-i).$$

(ii) *([Dat06, Lemme 4.2.1]) The monodromy operator $N\in\operatorname{End}(R\Gamma_c(\Omega_F))(-1)$ on $R\Gamma_c(\Omega_F)$ is naturally determined. The image of N under the isomorphism α in (i) belongs to $\bigoplus_{0\le i\le d-2}\operatorname{Ext}^1_G(\pi_{\le i+1},\pi_{\le i})$. We denote it by $(\beta_i)_i$.*

(iii) *([Dat06, Proposition 4.2.7]) For each i with $0\le i\le d-2$, $\beta_i\ne 0$.*

The following theorem is an analogue of [Dat06, Théorème 1.1].

Theorem 2.3 *For subsets $I\subset\{1,\ldots,d-1\}$ and $J\subset\{1,\ldots,d'-1\}$, we have an isomorphism of Weil-Deligne representations of F'':*

$$\mathcal{H}^*\bigl(R\operatorname{Hom}\bigl(R\Gamma_c(\Omega_F\times_{F''}\Omega_{F'}),\pi_I\boxtimes\pi'_J\bigr)\bigr)\cong\operatorname{rec}_F(\pi_I)\bigl(\tfrac{d-1}{2}\bigr)|_{W_{F''}}\otimes\operatorname{rec}_{F'}(\pi'_J)\bigl(\tfrac{d'-1}{2}\bigr)|_{W_{F''}},$$

where rec_F *(resp.* $\mathrm{rec}_{F'}$*) denotes the local Langlands correspondence for* F *(resp.* F'*). The functor* $\mathcal{H}^*$ *from* $D^b(\overline{\mathbb{Q}}_\ell)$ *to the category of* $\mathbb{Z}$*-graded* $\overline{\mathbb{Q}}_\ell$*-vector spaces is given by* $L^\bullet \mapsto \bigoplus_{i\in\mathbb{Z}} H^i(L^\bullet)$.

If an irreducible smooth representation $\pi\boxtimes\pi'$ *of* $G\times G'$ *is not of the form* $\pi_I\boxtimes\pi'_J$*, then* $R\operatorname{Hom}(R\Gamma_c(\Omega_F\times_{F''}\Omega_{F'}),\pi\boxtimes\pi')=0$.

Proof. For simplicity, we only consider the cases $(I,J)=(\varnothing,\varnothing),(\varnothing,\{1,\ldots,d'-1\})$. Other cases can be treated similarly.

First consider the case $(I,J)=(\varnothing,\varnothing)$. By Theorem 2.2 (i) and the Künneth formula (See Theorem A.5), we have

$$R\Gamma_c(\Omega_F\times_{F''}\Omega_{F'})\xrightarrow{\cong}\bigoplus_{i=0}^{d-1}\bigoplus_{j=0}^{d'-1}(\pi_{\le i}\boxtimes\pi'_{\le j})(-i-j)[-d-d'+2-i-j].$$

By Lemma 2.1 (i), we have

$$R\operatorname{Hom}\bigl(R\Gamma_c(\Omega_F\times_{F''}\Omega_{F'}),\pi_\varnothing\boxtimes\pi'_\varnothing\bigr)\cong\bigoplus_{i=0}^{d-1}\bigoplus_{j=0}^{d'-1}\operatorname{Ext}^{i+j}_{G\times G'}(\pi_{\le i}\boxtimes\pi'_{\le j},\pi_\varnothing\boxtimes\pi'_\varnothing)(i+j),$$

where $\operatorname{Ext}^{i+j}_{G\times G'}(\pi_{\le i}\boxtimes\pi'_{\le j},\pi_\varnothing\boxtimes\pi'_\varnothing)$ is a one-dimensional vector space for each i and j. Let $e_{0,0}\in\operatorname{Hom}_{G\times G'}(\pi_\varnothing\boxtimes\pi'_\varnothing,\pi_\varnothing\boxtimes\pi'_\varnothing)$ be the identity. Define $e_{i,j}\in\operatorname{Ext}^{i+j}_{G\times G'}(\pi_{\le i}\boxtimes\pi'_{\le j},\pi_\varnothing\boxtimes\pi'_\varnothing)$ as the image of $e_{0,0}$ under the map

$$(\beta_{i-1}\cup\cdots\cup\beta_0\cup -)\boxtimes(\beta'_{j-1}\cup\cdots\cup\beta'_0\cup -).$$

Here, $(\beta'_j)\in\bigoplus_{0\le j\le d'-2}\operatorname{Ext}^1_{G'}(\pi'_{j+1},\pi'_j)$ denotes the image of $N\in\operatorname{End}(R\Gamma_c(\Omega_{F'}))(-1)$. By Lemma 2.1 (ii) and Theorem 2.2 (iii), $e_{i,j}$ is a basis of $\operatorname{Ext}^{i+j}_{G\times G'}(\pi_{\le i}\boxtimes\pi'_{\le j},\pi_\varnothing\boxtimes\pi'_\varnothing)$. Now the monodromy operator on $R\operatorname{Hom}(R\Gamma_c(\Omega_F\times_{F''}\Omega_{F'}),\pi_\varnothing\boxtimes\pi'_\varnothing)$ can be described explicitly:

$$Ne_{i,j}=e_{i+1,j}+e_{i,j+1}.$$

Therefore we conclude that

$$\begin{aligned}\mathcal{H}^*\bigl(R\operatorname{Hom}(R\Gamma_c(\Omega_F\times_{F''}\Omega_{F'}),\pi_\varnothing\boxtimes\pi'_\varnothing)\bigr)&\cong\mathbf{Sp}_d(\tfrac{d-1}{2})\otimes\mathbf{Sp}_{d'}(\tfrac{d'-1}{2})\\&=\mathrm{rec}_F(\pi_\varnothing)(\tfrac{d-1}{2})|_{W_{F''}}\otimes\mathrm{rec}_{F'}(\pi'_\varnothing)(\tfrac{d'-1}{2})|_{W_{F''}}.\end{aligned}$$

We note that the right hand side concentrates in the degree $-d-d'+2$.

Next assume that $I=\varnothing$ and $J=\{1,\ldots,d'-1\}$. Then, Lemma 2.1 (i) tells us that

$$\begin{aligned}&R\operatorname{Hom}\bigl(R\Gamma_c(\Omega_F\times_{F''}\Omega_{F'}),\pi_\varnothing\boxtimes\pi'_{\le d'-1}\bigr)\\&\quad\cong\bigoplus_{i=0}^{d-1}\bigoplus_{j=0}^{d'-1}\operatorname{Ext}^{i+d'-1-j}_{G\times G'}(\pi_{\le i}\boxtimes\pi'_{\le j},\pi_\varnothing\boxtimes\pi'_{\le d'-1})(i+j).\end{aligned}$$

The (i,j)-component on the right hand side has degree $-d+1-2j$. Note that $\mathrm{id}\boxtimes(\beta'_j\cup -)$ induces the zero map on the right hand side. On the other hand, $(\beta_i\cup -)\boxtimes\mathrm{id}$ gives an isomorphism

$$\mathrm{Ext}^{i+d'-1-j}_{G\times G'}(\pi_{\le i}\boxtimes\pi'_{\le j},\pi_\varnothing\boxtimes\pi'_{\le d'-1})\to \mathrm{Ext}^{i+d'-j}_{G\times G'}(\pi_{\le i+1}\boxtimes\pi'_{\le j},\pi_\varnothing\boxtimes\pi'_{\le d'-1}).$$

Therefore, $\mathcal{H}^*(R\,\mathrm{Hom}(R\Gamma_c(\Omega_F\times_{F''}\Omega_{F'}),\pi_\varnothing\boxtimes\pi'_{\le d'-1}))$ is isomorphic to

$$\begin{aligned}&\mathbf{Sp}_d(\tfrac{d-1}{2})\otimes\big(\overline{\mathbb{Q}}_\ell\oplus\overline{\mathbb{Q}}_\ell(1)\oplus\cdots\oplus\overline{\mathbb{Q}}_\ell(d'-1)\big)\\&\qquad=\mathrm{rec}_F(\pi_\varnothing)(\tfrac{d-1}{2})|_{W_{F''}}\otimes\mathrm{rec}_{F'}(\pi'_{\le d'-1})(\tfrac{d'-1}{2})|_{W_{F''}}.\end{aligned}$$

Finally, if $\pi\boxtimes\pi'$ is not of the form $\pi_I\boxtimes\pi'_J$,

$$\begin{aligned}&R\,\mathrm{Hom}\big(R\Gamma_c(\Omega_F\times_{F''}\Omega_{F'}),\pi\boxtimes\pi'\big)\\&\quad=\bigoplus_{i=0}^{d-1}\bigoplus_{j=0}^{d'-1}R\,\mathrm{Hom}(\pi_{\le i}\boxtimes\pi'_{\le j},\pi\boxtimes\pi')(i+j)[d+d'-2+i+j]=0\end{aligned}$$

by Lemma 2.1 (iii). ■

Corollary 2.4 *Let Γ be a discrete torsion-free cocompact subgroup of $G\times G'$. Let $m_{1,0}$ (resp. $m_{0,1}$, resp. $m_{1,1}$) be the multiplicity of $\mathbf{St}_d\boxtimes\mathbf{1}$ (resp. $\mathbf{1}\boxtimes\mathbf{St}_{d'}$, resp. $\mathbf{St}_d\boxtimes\mathbf{St}_{d'}$) in the representation $C^\infty(G\times G'/\Gamma)$ of $G\times G'$. Then, we have a $W_{F''}$-equivariant isomorphism*

$$\begin{aligned}R\Gamma(\Omega_F\times_{F''}\Omega_{F'}/\Gamma)\cong{}&\Big(\bigoplus_{i=0}^{d-1}\bigoplus_{j=0}^{d'-1}\overline{\mathbb{Q}}_\ell(-i-j)[-2i-2j]\Big)\\&\oplus\left(\mathbf{Sp}_d(\tfrac{1-d}{2})[1-d]\otimes\Big(\bigoplus_{j=0}^{d'-1}\overline{\mathbb{Q}}_\ell(-j)[-2j]\Big)\right)^{m_{1,0}}\\&\oplus\left(\Big(\bigoplus_{i=0}^{d-1}\overline{\mathbb{Q}}_\ell(-i)[-2i]\Big)\otimes\mathbf{Sp}_{d'}(\tfrac{1-d'}{2})[1-d']\right)^{m_{0,1}}\\&\oplus(\mathbf{Sp}_d\otimes\mathbf{Sp}_{d'})^{m_{1,1}}(\tfrac{2-d-d'}{2})[2-d-d']\end{aligned}$$

preserving monodromy operators. Moreover, the weight-monodromy conjecture holds for $\Omega_F\times_{F''}\Omega_{F'}/\Gamma$.

Proof. The proof is the same as [Dat06, Corollaire 4.5.1]. By the fixed isomorphism $\overline{\mathbb{Q}}_\ell\cong\mathbb{C}$, we regard $C^\infty(G\times G'/\Gamma)$ as a representation over $\mathbb{C}$. Then, it is a unitary representation, and decomposes into the direct sum of irreducible smooth unitary representations of $G\times G'$ with finite multiplicities. Assume that $\pi_I\boxtimes\pi'_J$ appears in $C^\infty(G\times G'/\Gamma)$. Then it is unitary, and thus π_I and π'_J are unitary. Hence we conclude that I (resp. J) is either $\varnothing$ or $\{1,\dots,d-1\}$ (resp. $\{1,\dots,d'-1\}$). Note that the multiplicity of the trivial representation $\mathbf{1}\boxtimes\mathbf{1}$ in $C^\infty(G\times G'/\Gamma)$ equals 1.

As in the proof of [Dat06, Corollaire 4.5.1], we have

$$\begin{aligned} R\Gamma(\Omega_F \times_{F''} \Omega_{F'}/\Gamma)^\vee &\cong \bigl(R\Gamma_c(\Omega_F \times_{F''} \Omega_{F'}) \overset{\mathbb{L}}{\otimes}_{\overline{\mathbb{Q}}_\ell[\Gamma]} \overline{\mathbb{Q}}_\ell\bigr)^\vee \\ &\cong R\operatorname{Hom}\bigl(R\Gamma_c(\Omega_F \times_{F''} \Omega_{F'}), C^\infty(G \times G'/\Gamma)\bigr) \\ &= R\operatorname{Hom}\bigl(R\Gamma_c(\Omega_F \times_{F''} \Omega_{F'}), (\mathbf{1} \boxtimes \mathbf{1}) \oplus (\mathbf{St}_d \boxtimes \mathbf{1})^{m_{1,0}} \\ &\qquad\qquad \oplus (\mathbf{1} \boxtimes \mathbf{St}_{d'})^{m_{0,1}} \oplus (\mathbf{St}_d \boxtimes \mathbf{St}_{d'})^{m_{1,1}}\bigr). \end{aligned}$$

By using Theorem 2.3 and taking dual, we obtain the desired description of $R\Gamma(\Omega_F \times_{F''} \Omega_{F'}/\Gamma)$.

For the weight-monodromy conjecture, we note that

$$\mathbf{Sp}_d \otimes \mathbf{Sp}_{d'} \cong \bigoplus_{\substack{|d-d'|+1 \le j \le d+d'-1, \\ j \equiv d+d'-1 \bmod 2}} \mathbf{Sp}_j$$

(see [Del80, (1.6.11.2)]; note that S_d in [Del80, (1.6.11)] is $d+1$-dimensional). Therefore, for every integer $m \ge 0$, $H^m(\Omega_F \times_{F''} \Omega_{F'}/\Gamma)$ is the direct sum of $W_{F''}$-representations of the following forms:

- $\overline{\mathbb{Q}}_\ell(-\frac{m}{2})$ if m is even,
- $\mathbf{Sp}_d(-\frac{m}{2})$ if $m+1-d$ is even,
- $\mathbf{Sp}_{d'}(-\frac{m}{2})$ if $m+1-d'$ is even,
- and $\mathbf{Sp}_j(-\frac{m}{2})$ for some $j \ge 1$ if $m = d+d'-2$.

As mentioned in [TY07, p. 471], all of these representations are pure of weight m (note that $\mathbf{Sp}_j = \mathrm{rec}_{F''}(\mathbf{St}_j)$ here is equal to $\mathbf{Sp}_j(\frac{1-j}{2})$ in [TY07, p. 471]). Therefore, $H^m(\Omega_F \times_{F''} \Omega_{F'}/\Gamma)$ is also pure of weight m. That is to say, the weight monodromy conjecture holds for $\Omega_F \times_{F''} \Omega_{F'}/\Gamma$. ■

The argument above applies to the product of more than two Drinfeld upper half spaces without any difficulty. Furthermore, it is also valid even if we replace the Drinfeld upper half spaces by its coverings introduced in [Dri76]. Namely, the following theorem holds.

Theorem 2.5 *Let D (resp. D') be the central division algebra over F (resp. F') with invariant $1/d$ (resp. $1/d'$). We denote $\mathcal{M} = \{\mathcal{M}_n\}$ (resp. $\mathcal{M}' = \{\mathcal{M}'_{n'}\}$) the Drinfeld tower on which $D^\times$ (resp. $D'^\times$) acts.*

(i) *Fix irreducible smooth representations ρ, ρ' of $D^\times$, $D'^\times$, respectively. Let π (resp. π') be an irreducible smooth representation of $\mathrm{GL}_n(F)$ (resp. $\mathrm{GL}_n(F')$) with the same central character as ρ (resp. ρ').*

(a) *If $\rho = \mathrm{LJ}_d(\pi)$ and $\rho' = \mathrm{LJ}_{d'}(\pi')$ (for the definition of* LJ, *see [Dat07, §2]), then we have*

$$\begin{aligned} &\mathcal{H}^*\bigl(R\operatorname{Hom}(R\Gamma_c(\mathcal{M} \times_{F''} \mathcal{M}')[\rho \boxtimes \rho']), \pi \boxtimes \pi'\bigr) \\ &\qquad\qquad \cong \mathrm{rec}_F(\pi)(\tfrac{d-1}{2})|_{W_{F''}} \oplus \mathrm{rec}_{F'}(\pi')(\tfrac{d'-1}{2})|_{W_{F''}}. \end{aligned}$$

(b) *Otherwise* $R\operatorname{Hom}(R\Gamma_c(\mathcal{M}\times_{F''}\mathcal{M}')[\rho\boxtimes\rho'']),\pi\boxtimes\pi')=0$.

(See also [Dat07, Lemme 4.4.1].)

(ii) *Let* Γ *be a discrete torsion-free cocompact subgroup of* $\mathrm{GL}_d(F)\times\mathrm{GL}_{d'}(F')$, *and* $n,n'\geq 0$ *integers. Then,* $R\Gamma(\mathcal{M}_n\times_{F''}\mathcal{M}'_{n'}/\Gamma)$ *can be computed as in [Dat07, p. 139–140]. In particular, the weight-monodromy conjecture holds for* $\mathcal{M}_n\times_{F''}\mathcal{M}'_{n'}/\Gamma$.

Proof. Use the result in [Dat07] in place of Theorem 2.2. ■

We may apply Theorem 2.5 to the unitary Shimura varieties appearing in [RZ96, Theorem 6.50]. By the same method as in [She16, §3], one can compute the ℓ-adic cohomology of them using Theorem 2.5 (i). This considerably simplifies the proof of the main result of [She16]; the study on test functions in [She16, §4–7] is no longer needed. The local Hasse-Weil zeta functions of such Shimura varieties can be computed directly. The result is the same as in [She16, Corollary 7.4] (but we do not need the assumption $r=1$).

A Künneth formula

In this appendix, we will prove the Künneth formula used in the proof of Theorem 2.3. Here let k be an algebraically closed non-archimedean field with residue characteristic p and ℓ a prime number different from p.

We shall use the notation in [Mie14b, §3.3]. Let G be a locally pro-p group. For a ring Λ, we write $\mathbf{Rep}_\Lambda(G)$ (resp. $\mathbf{Mod}(\Lambda)$) for the category of smooth G-representations over Λ (resp. Λ-modules). Let X be an adic space locally of finite type over $\mathrm{Spa}(k,k^+)$ equipped with a continuous G-action. For an integer $m\geq 1$, we write $\mathbb{Z}/\ell^m\mathbb{Z}\text{-}\widetilde{X}_{\text{ét}}/G$ for the category of smooth G-equivariant $\mathbb{Z}/\ell^m\mathbb{Z}$-sheaves on $X_{\text{ét}}$.

Lemma A.1 (i) *Let* Y *be an adic space locally of finite type over* $\mathrm{Spa}(k,k^+)$ *and* $f\colon Y\to X$ *a* G*-equivariant morphism over* $\mathrm{Spa}(k,k^+)$*. Then, for every object* $\mathcal{F}$ *of* $\mathbb{Z}/\ell^m\mathbb{Z}\text{-}\widetilde{X}_{\text{ét}}/G$*, the pull-back* $f^*\mathcal{F}$ *belongs to the category* $\mathbb{Z}/\ell^m\mathbb{Z}\text{-}\widetilde{Y}_{\text{ét}}/G$.

(ii) *Let* G' *be another locally pro-*p *group and* X' *an adic space locally of finite type over* $\mathrm{Spa}(k,k^+)$ *equipped with a continuous* G'*-action. We write* pr_1 *(resp.* pr_2*) for the first (resp. second) projection* $X\times_k X'\to X$ *(resp.* $X\times_k X'\to X'$*). Then, for every objects* $\mathcal{F}$ *of* $\mathbb{Z}/\ell^m\mathbb{Z}\text{-}\widetilde{X}_{\text{ét}}/G$ *and* $\mathcal{G}$ *of* $\mathbb{Z}/\ell^m\mathbb{Z}\text{-}\widetilde{X}'_{\text{ét}}/G'$*, the exterior tensor product* $\mathcal{F}\boxtimes\mathcal{G}=\mathrm{pr}_1^*\mathcal{F}\otimes\mathrm{pr}_2^*\mathcal{G}$ *belongs to the category* $\mathbb{Z}/\ell^m\mathbb{Z}\text{-}(X\times X')^{\sim}_{\text{ét}}/(G\times G')$.

Proof. First we consider (i). We can directly observe that the pull-back of $\mathcal{F}$ as a presheaf is a smooth G-equivariant presheaf on $Y_{\text{ét}}$. As the sheafification preserves smoothness (see [Far08, Lemme IV.8.4]), we conclude that $f^*\mathcal{F}$ is a smooth G-equivariant sheaf on $Y_{\text{ét}}$.

Next we prove (ii). By (i), $\mathrm{pr}_1^*\mathcal{F}$ and $\mathrm{pr}_2^*\mathcal{G}$ are smooth $G\times G'$-equivariant sheaves on $(X\times_k X')_{\text{ét}}$. By using [Far08, Lemme IV.8.4], we can easily verify that the tensor product of them is also a smooth $G\times G'$-equivariant sheaf. ■

Let $\mathbb{Z}_\ell\text{-}\widetilde{X}_{\text{ét}}/G$ be the category of projective systems $(\mathcal{F}_m)_{m\ge 1}$ where $\mathcal{F}_m$ is an object of $\mathbb{Z}/\ell^m\mathbb{Z}\text{-}\widetilde{X}_{\text{ét}}/G$ and the transition maps are G-equivariant.

In the following, we assume that X is partially proper over $\mathrm{Spa}(k,k^+)$. The following definition is due to [Dat06, §B.2.4].

Definition A.2 We define the functor $\Gamma_{c,G\text{-eq}}(X,-)\colon \mathbb{Z}_\ell\text{-}\widetilde{X}_{\text{ét}}/G\to \mathbf{Rep}_{\mathbb{Z}_\ell}(G)$ by

$$\Gamma_{c,G\text{-eq}}(X,(\mathcal{F}_m))=\Gamma_c(X,\varprojlim_m \mathcal{F}_m)^{\mathrm{sm}},$$

where $(-)^{\mathrm{sm}}$ denotes the G-smooth part. We write $R\Gamma_{c,G\text{-eq}}(X,-)$ for the right derived functor of $\Gamma_{c,G\text{-eq}}(X,-)$, and $H^i_{c,G\text{-eq}}(X,-)$ for the ith cohomology of $R\Gamma_{c,G\text{-eq}}(X,-)$. We set $R\Gamma_{c,G\text{-eq}}(X,\mathbb{Z}_\ell)=R\Gamma_{c,G\text{-eq}}(X,(\mathbb{Z}/\ell^m\mathbb{Z})_{m\ge1})$, $R\Gamma_{c,G\text{-eq}}(X,\mathbb{Q}_\ell)=R\Gamma_{c,G\text{-eq}}(X,\mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ and $R\Gamma_{c,G\text{-eq}}(X,\overline{\mathbb{Q}}_\ell)=R\Gamma_{c,G\text{-eq}}(X,\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\overline{\mathbb{Q}}_\ell$, as usual.

The following proposition has been obtained in [Dat06, Proposition B.2.5].

Proposition A.3 (i) *For an object $\mathcal{F}=(\mathcal{F}_m)$ of $\mathbb{Z}_\ell\text{-}\widetilde{X}_{\text{ét}}/G$, we have $H^i_{c,G\text{-eq}}(X,\mathcal{F})\cong H^i_c(X,\mathcal{F})^{\mathrm{sm}}$.*

(ii) *Assume that X is locally algebraic (it is satisfied when X is smooth over $\mathrm{Spa}(k,k^+)$). Then, the image of $R\Gamma_{c,G\text{-eq}}(X,\mathbb{Z}_\ell)$ under the forgetful functor $D^+(\mathbf{Rep}_{\mathbb{Z}_\ell}(G))\to D^+(\mathbf{Mod}(\mathbb{Z}_\ell))$ is isomorphic to $R\Gamma_c(X,\mathbb{Z}_\ell)$.*

The following lemma is a key to construct the cup product.

Lemma A.4 *For each $m\ge 1$, let $0\to\mathbb{Z}/\ell^m\mathbb{Z}\to\mathcal{C}^\bullet(\mathbb{Z}/\ell^m\mathbb{Z})$ be the Godement resolution constructed in [Mie14b, §3.3.3]. Then, for each $i\ge0$, $\mathcal{C}^i(\mathbb{Z}_\ell)=(\mathcal{C}^i(\mathbb{Z}/\ell^m\mathbb{Z}))_{m\ge1}$ is an object of $\mathbb{Z}_\ell\text{-}\widetilde{X}_{\text{ét}}/G$. Further, it is acyclic with respect to $R\Gamma_c(X,-)$ and $R\Gamma_{c,G\text{-eq}}(X,-)$. In particular, we have $R\Gamma_{c,G\text{-eq}}(X,\mathbb{Z}_\ell)=\Gamma_{c,G\text{-eq}}(X,\mathcal{C}^\bullet(\mathbb{Z}_\ell))$.*

Proof. By Proposition A.3 (i), we have only to show that $H^j_c(X,\mathcal{C}^i(\mathbb{Z}_\ell))=0$ for $j\ge1$. We will write $\mathbb{Z}_\ell\text{-}\widetilde{X}_{\text{ét}}$ for $\mathbb{Z}_\ell\text{-}\widetilde{X}_{\text{ét}}/\{1\}$. For a closed subset Z of X, let $H^j_Z(X,-)$ be the jth derived functor of the functor $\mathbb{Z}_\ell\text{-}\widetilde{X}_{\text{ét}}\to\mathbf{Mod}(\mathbb{Z}_\ell)$; $(\mathcal{F}_m)\mapsto\Gamma_Z(X,\varprojlim_m\mathcal{F}_m)=\varprojlim_m\Gamma_Z(X,\mathcal{F}_m)$. Then, as in the proof of [Mie14b, Proposition 3.39], we can prove that $H^j_c(X,(\mathcal{F}_m))=\varinjlim_Z H^j_Z(X,(\mathcal{F}_m))$, where $Z\subset X$ runs through quasi-compact closed subsets of the form $V\setminus W$ with quasi-compact open subsets V, W of X. Since $H^j_Z(X,(\mathcal{F}_m))=H^j_{V\setminus W}(V,(\mathcal{F}_m))$, it suffices to prove that $H^j_{V\setminus W}(V,\mathcal{C}^i(\mathbb{Z}_\ell))=0$ for every $j\ge1$ and every quasi-compact open subsets V, W of X. As in [Jan88, (3.7)], we have an exact sequence

$$H^j_{V\setminus W}(V,\mathcal{C}^i(\mathbb{Z}_\ell))\to H^j(V,\mathcal{C}^i(\mathbb{Z}_\ell))\to H^j(V\cap W,\mathcal{C}^i(\mathbb{Z}_\ell)).$$

Therefore, it suffices to prove that $H^j(V,\mathcal{C}^i(\mathbb{Z}_\ell)) = 0$ for every $j \geq 1$ and every quasi-compact open subset V. As in [Jan88, (3.1)], we have an exact sequence

$$0 \to \varprojlim_m{}^1 H^{j-1}(V,\mathcal{C}^i(\mathbb{Z}/\ell^m\mathbb{Z})) \to H^j(V,\mathcal{C}^i(\mathbb{Z}_\ell)) \to \varprojlim_m H^j(V,\mathcal{C}^i(\mathbb{Z}/\ell^m\mathbb{Z})) \to 0.$$

By [Mie14b, Proposition 3.44], $\mathcal{C}^i(\mathbb{Z}/\ell^m\mathbb{Z})$ is an injective object of $\mathbb{Z}_\ell\text{-}\widetilde{X}_{\text{ét}}/G$. Hence $H^j(V,\mathcal{C}^i(\mathbb{Z}/\ell^m\mathbb{Z})) = 0$ for $j \geq 1$ by [Mie14b, Proposition 3.29]. Further, by the same method as in [Mie14b, Proposition 3.44], we can observe that the sequence

$$0 \to \mathcal{C}^i(\mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{\times \ell^m} \mathcal{C}^i(\mathbb{Z}/\ell^{m+1}\mathbb{Z}) \to \mathcal{C}^i(\mathbb{Z}/\ell^m\mathbb{Z}) \to 0$$

is exact. Therefore the map $H^0(V,\mathcal{C}^i(\mathbb{Z}/\ell^{m+1}\mathbb{Z})) \to H^0(V,\mathcal{C}^i(\mathbb{Z}/\ell^m\mathbb{Z}))$ is surjective, and hence $\varprojlim_m^1 H^0(V,\mathcal{C}^i(\mathbb{Z}/\ell^m\mathbb{Z})) = 0$. Now we conclude that $H^j(V,\mathcal{C}^i(\mathbb{Z}_\ell)) = 0$ for $j \geq 1$, as desired. ■

Let G and X be as before, and let G' and X' be as in Lemma A.1 (ii) such that X' is partially proper over $\operatorname{Spa}(k,k^+)$. By the definition of $\Gamma_{c,G\text{-eq}}(X,-)$, we have natural morphisms

$$\begin{aligned}
&\operatorname{Tot}(\Gamma_{c,G\text{-eq}}(X,\mathcal{C}^\bullet(\mathbb{Z}_\ell)) \otimes \Gamma_{c,G'\text{-eq}}(X',\mathcal{C}^\bullet(\mathbb{Z}_\ell)))\\
&\quad\to \Gamma_{c,G\times G'\text{-eq}}(X\times_k X', \operatorname{Tot}(\mathcal{C}^\bullet(\mathbb{Z}_\ell)\boxtimes\mathcal{C}^\bullet(\mathbb{Z}_\ell)))\\
&\quad\to R\Gamma_{c,G\times G'\text{-eq}}(X\times_k X', \operatorname{Tot}(\mathcal{C}^\bullet(\mathbb{Z}_\ell)\boxtimes\mathcal{C}^\bullet(\mathbb{Z}_\ell)))\\
&\quad\overset{(*)}{\cong} R\Gamma_{c,G\times G'\text{-eq}}(X\times_k X',\mathbb{Z}_\ell),
\end{aligned}$$

where the isomorphism $(*)$ comes from the fact that $\operatorname{Tot}(\mathcal{C}^\bullet(\mathbb{Z}_\ell)\boxtimes\mathcal{C}^\bullet(\mathbb{Z}_\ell))$ gives a resolution of $(\mathbb{Z}/\ell^m\mathbb{Z})_{m\geq 1}$ in $\mathbb{Z}_\ell\text{-}(X\times_k X')^{\sim}_{\text{ét}}/(G\times G')$ (see Lemma A.1 (ii) and [Mie14b, Proposition 3.44]). On the other hand, Lemma A.4 gives us an isomorphism

$$\operatorname{Tot}(\Gamma_{c,G\text{-eq}}(X,\mathcal{C}^\bullet(\mathbb{Z}_\ell))\otimes\Gamma_{c,G'\text{-eq}}(X',\mathcal{C}^\bullet(\mathbb{Z}_\ell)))\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell \cong R\Gamma_{c,G\text{-eq}}(X,\mathbb{Q}_\ell)\otimes R\Gamma_{c,G'\text{-eq}}(X',\mathbb{Q}_\ell)$$

in $D^+(\mathbf{Rep}_{\mathbb{Q}_\ell}(G\times G'))$. Therefore, by tensoring $\mathbb{Q}_\ell$ to the morphism above, we obtain a morphism

$$R\Gamma_{c,G\text{-eq}}(X,\mathbb{Q}_\ell)\otimes R\Gamma_{c,G'\text{-eq}}(X',\mathbb{Q}_\ell) \to R\Gamma_{c,G\times G'\text{-eq}}(X\times_k X',\mathbb{Q}_\ell)$$

in $D^+(\mathbf{Rep}_{\mathbb{Q}_\ell}(G\times G'))$, which we call the cup product.

Now we can state the main result in this appendix.

Theorem A.5 *Assume that X and X' are locally algebraic. Then, the cup product*

$$R\Gamma_{c,G\text{-eq}}(X,\mathbb{Q}_\ell)\otimes R\Gamma_{c,G'\text{-eq}}(X',\mathbb{Q}_\ell) \to R\Gamma_{c,G\times G'\text{-eq}}(X\times_k X',\mathbb{Q}_\ell)$$

constructed above is an isomorphism. By tensoring $\overline{\mathbb{Q}}_\ell$, we also obtain a similar isomorphism for $\overline{\mathbb{Q}}_\ell$-coefficients.

Proof. By using Lemma A.4, we can easily check that the cup product above is mapped to the usual cup product

$$R\Gamma_c(X, \mathbb{Q}_\ell) \otimes R\Gamma_c(X', \mathbb{Q}_\ell) \to R\Gamma_c(X \times_k X', \mathbb{Q}_\ell)$$

under the forgetful functor $D^+(\mathbf{Rep}_{\mathbb{Q}_\ell}(G \times G')) \to D^+(\mathbf{Mod}(\mathbb{Q}_\ell))$. Therefore, it suffices to prove that the usual cup product is an isomorphism. Let $\mathbb{U}$ (resp. $\mathbb{U}'$) be the set consisting of all quasi-compact open subsets of X (resp. X'). Then, we have $H^i_c(X, \mathbb{Q}_\ell) = \varinjlim_{U \in \mathbb{U}} H^i_c(U, \mathbb{Q}_\ell)$, $H^i_c(X', \mathbb{Q}_\ell) = \varinjlim_{U' \in \mathbb{U}'} H^i_c(U', \mathbb{Q}_\ell)$ and $H^i_c(X \times_k X', \mathbb{Q}_\ell) = \varinjlim_{(U,U') \in \mathbb{U} \times \mathbb{U}'} H^i_c(U \times_k U', \mathbb{Q}_\ell)$ by [Hub98, Proposition 2.1 (iv)]. Since the cup product is compatible with open immersions, we may suppose that X and X' are quasi-compact. Exactly as in the scheme case (for example, see [Eke90]), the Künneth formula in this case is a consequence of the following two properties for torsion coefficients:

- $R\Gamma_c(X, \mathbb{Z}/\ell^{m+1}\mathbb{Z}) \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/\ell^{m+1}\mathbb{Z}} \mathbb{Z}/\ell^m\mathbb{Z} \cong R\Gamma_c(X, \mathbb{Z}/\ell^m\mathbb{Z})$,
- $R\Gamma_c(X, \mathbb{Z}/\ell^m\mathbb{Z}) \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/\ell^m\mathbb{Z}} R\Gamma_c(X', \mathbb{Z}/\ell^m\mathbb{Z}) \xrightarrow{\cong} R\Gamma_c(X \times_k X', \mathbb{Z}/\ell^m\mathbb{Z})$.

These are proved in [Mie14a, Lemma 3.2, Proposition 3.6]. ■

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