Zelevinsky involution and ℓ-adic cohomology of the Rapoport-Zink tower

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Abstract. In this paper, we investigate how the Zelevinsky involution appears in the ℓ-adic cohomology of the Rapoport-Zink tower. We generalize the result of Fargues on the Drinfeld tower to the Rapoport-Zink towers for symplectic similitude groups.

1 Introduction

The non-abelian Lubin-Tate theory says that the local Langlands correspondence for $\mathrm{GL}(n)$ is geometrically realized in the ℓ-adic cohomology of the Lubin-Tate tower and the Drinfeld tower (*cf.* [Car90], [Har97], [HT01]). It urges us to consider how representation-theoretic operations are translated into geometry. In [Far06], Fargues found a relation between the Zelevinsky involution and the Poincaré duality of the ℓ-adic cohomology of the Drinfeld tower. Furthermore, by using Faltings' isomorphism between the Lubin-Tate tower and the Drinfeld tower (*cf.* [Fal02], [FGL08]), he obtained a similar result for the Lubin-Tate tower. This result is useful for study of the ℓ-adic cohomology itself. For example, it played a crucial role in Boyer's work [Boy09], which completely determined the ℓ-adic cohomology of the Lubin-Tate tower.

A Rapoport-Zink tower (*cf.* [RZ96]) is a natural generalization of the Lubin-Tate tower and the Drinfeld tower. It is a projective system of étale coverings of rigid spaces $\{M_K\}$ lying over the rigid generic fiber M of a formal scheme $\mathcal{M}$. The formal scheme $\mathcal{M}$, called a Rapoport-Zink space, is defined as a moduli space of deformations by quasi-isogenies of a p-divisible group over $\overline{\mathbb{F}}_p$ with additional structures. For a prime number $\ell \neq p$, consider the compactly supported ℓ-adic cohomology $H^i_c(M_\infty) = \varinjlim_K H^i_c(M_K, \overline{\mathbb{Q}}_\ell)$. It is naturally endowed with an action of $G \times J \times W$, where G and J are p-adic reductive groups and W is the Weil group of some p-adic field (a local analogue of a reflex field). The cohomology $H^i_c(M_\infty)$ is expected to be described by the local Langlands correspondence for G and J (*cf.* [Rap95]), but only few results are known.

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In this paper, we will give a generalization of Fargues' result mentioned above to Rapoport-Zink towers other than the Lubin-Tate tower and the Drinfeld tower. Although our method should be valid for many Rapoport-Zink towers (see Remark 5.14), here we restrict ourselves to the case of GSp($2n$) for the sake of simplicity. In this case, the Rapoport-Zink tower is a local analogue of the Siegel modular variety, and is also treated in [Mie12b]. The group G is equal to $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$, and J is an inner form $\mathrm{GU}(n,D)$ of G, where D is the quaternion division algebra over $\mathbb{Q}_p$.

The main difference between Fargues' case and ours is that the Rapoport-Zink space $\mathcal{M}$ is a p-adic formal scheme in the former, while not in the latter. Owing to this difference, we need to introduce a new kind of cohomology $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty)$. Contrary to the compactly supported cohomology $H^i_c(M_\infty)$, this cohomology depends on the formal model $\mathcal{M}$ of the base M of the Rapoport-Zink tower. Roughly speaking, it is a cohomology with compact support in the direction of the formal model $\mathcal{M}$ (for a precise definition, see Section 3.2 and Section 5.1). If $\mathcal{M}$ is a p-adic formal scheme, it coincides with the compactly supported cohomology. By using these two cohomology, our main theorem is stated as follows:

Theorem 1.1 (Theorem 5.6) *Let $\widetilde{K}$ be an open compact-mod-center subgroup of G and τ an irreducible smooth representation of $\widetilde{K}$. Denote by χ the central character of $\tau^\vee$. For a smooth G-representation V, put $V_\tau = \mathrm{Hom}_{\widetilde{K}}(\tau, V\otimes_{\mathcal{H}(Z_G)}\chi^{-1})$, where $\mathcal{H}(Z_G)$ is the Hecke algebra of the center Z_G of G. Let $\mathfrak{s}\in I_\chi$ be a Bernstein component of the category of smooth representations of J with central character χ. An integer $\iota(\mathfrak{s})$ is naturally attached to $\mathfrak{s}$; $\mathfrak{s}$ is supercuspidal if and only if $\iota(\mathfrak{s})=0$ (cf. Section 2).*

Assume that the $\mathfrak{s}$-component $H^q_c(M_\infty)_{\tau,\mathfrak{s}}$ of $H^q_c(M_\infty)_\tau$ is a finite length J-representation for every integer q. Then, for each integer i, we have an isomorphism of $J\times W_{\mathbb{Q}_p}$-representations

$$H^{2d+\iota(\mathfrak{s})-i}_{\mathcal{C}_\mathcal{M}}(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}(d)\cong \mathrm{Zel}\bigl(H^i_c(M_\infty)^\vee_{\tau,\mathfrak{s}}\bigr).$$

Here $W_{\mathbb{Q}_p}$ denotes the Weil group of $\mathbb{Q}_p$, $d=n(n+1)/2$ the dimension of M, and Zel *the Zelevinsky involution with respect to J (see Section 2).*

By applying this theorem to the case where $\text{c-Ind}^G_{\widetilde{K}}\tau$ becomes supercuspidal, we obtain the following consequence on the supercuspidal part of $H^i_c(M_\infty/p^\mathbb{Z})$ (here $p^\mathbb{Z}$ is regarded as a discrete subgroup of the center of J).

Corollary 1.2 (Corollary 5.11, Corollary 5.12) *Let π be an irreducible supercuspidal representation of G, ρ an irreducible non-supercuspidal representation of J, and σ an irreducible ℓ-adic representation of $W_{\mathbb{Q}_p}$. Under some technical assumption (Assumption 5.8), the following hold.*

i) *The representation $\pi\otimes\rho$ does not appear as a subquotient of $H^{d-\dim\mathcal{M}^{\mathrm{red}}}_c(M_\infty/p^\mathbb{Z})$. In particular, if $n=2$, $\pi\otimes\rho$ does not appear as a subquotient of $H^2_c(M_\infty/p^\mathbb{Z})$.*

ii) *If $n=2$, $\pi\otimes\rho\otimes\sigma$ appears as a subquotient of $H^3_c(M_\infty/p^\mathbb{Z})$ if and only if $\pi^\vee\otimes\mathrm{Zel}(\rho^\vee)\otimes\sigma^\vee(-3)$ appears as a subquotient of $H^4_c(M_\infty/p^\mathbb{Z})$.*

The proof of ii) also requires a result of [IM10], which measures the difference between the two cohomology groups $H^i_c(M_\infty/p^{\mathbb{Z}})$ and $H^i_{\mathcal{C}_{\mathcal{M}}}(M_\infty/p^{\mathbb{Z}})$. This corollary will be very useful to investigate how non-supercuspidal representations of J contribute to $H^i_c(M_\infty/p^{\mathbb{Z}})$ for each i.

The outline of this paper is as follows. In Section 2, we briefly recall the definition and properties of the Zelevinsky involution. The main purpose of this section is to fix notation, and most of the proofs are referred to [SS97] and [Far06]. Section 3 is devoted to give some preliminaries on algebraic and rigid geometry used in this article. The cohomology appearing in Theorem 1.1 is defined in this section. Because we prefer to use the theory of adic spaces (*cf.* [Hub94], [Hub96]) as a framework of rigid geometry, we need to adopt the theory of smooth equivariant sheaves for Berkovich spaces developed in [FGL08, §IV.9] to adic spaces. In fact proofs become simpler; especially the compactly supported cohomology and the Godement resolution can be easily treated. In Section 4, we prove a duality theorem under a general setting. Finally in Section 5, after introducing the Rapoport-Zink tower for $\mathrm{GSp}(2n)$, we deduce Theorem 1.1 from the duality theorem proved in Section 4. We also give the applications announced in Corollary 1.2.

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Notation For a field k, we denote its algebraic closure by $\overline{k}$.

2 Zelevinsky involution

In this section, we recall briefly about the Zelevinsky involution. See [Far06, §1] for details.

Let p be a prime number and G be a locally pro-p group; namely, G is a locally compact group which has an open pro-p subgroup. Fix a 0-dimensional Gorenstein local ring Λ in which p is invertible. We write $\mathbf{Rep}_\Lambda(G)$ for the category of smooth representations of G over Λ. Denote by G^{disc} the group G with the discrete topology. We have a natural functor $i_G\colon \mathbf{Rep}_\Lambda(G) \longrightarrow \mathbf{Rep}_\Lambda(G^{\mathrm{disc}})$, which has a right adjoint functor $\infty_G\colon \mathbf{Rep}_\Lambda(G^{\mathrm{disc}}) \longrightarrow \mathbf{Rep}_\Lambda(G)$; $V \longmapsto \varinjlim_K V^K$. Here K runs through compact open subgroups of G. The functor ∞_G is not exact in general.

Let $\mathcal{D}_c(G)$ be the convolution algebra of compactly supported Λ-valued distributions on G. It contains the Hecke algebra $\mathcal{H}(G)$ of G consisting of compactly supported distributions invariant under some compact open subgroup of G. For each open pro-p subgroup K of G, an idempotent e_K of $\mathcal{H}(G)$ is naturally attached. We denote by $\mathbf{Mod}(\mathcal{D}_c(G))$ the category of $\mathcal{D}_c(G)$-modules. We have a natural functor $i_{\mathcal{D}}\colon \mathbf{Rep}_\Lambda(G) \longrightarrow \mathbf{Mod}(\mathcal{D}_c(G))$. The right adjoint functor $\infty_{\mathcal{D}}\colon \mathbf{Mod}(\mathcal{D}_c(G)) \longrightarrow \mathbf{Rep}_\Lambda(G)$ of $i_{\mathcal{D}}$ is given by $M \longmapsto \varinjlim_K e_K M$, where K runs through compact open pro-p subgroups of G. Note that $\infty_{\mathcal{D}}$ is an exact functor.

For a compact open subgroup K of G, we have a functor

$$\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)}\colon \mathbf{Mod}(\mathcal{D}_c(K)) \longrightarrow \mathbf{Mod}(\mathcal{D}_c(G));\ M \longmapsto \mathcal{D}_c(G) \otimes_{\mathcal{D}_c(K)} M.$$

This functor is exact and the following diagrams are commutative:

$$\begin{tikzcd} \mathbf{Rep}_\Lambda(K) \arrow[r, "\text{c-Ind}_K^G"] \arrow[d, "i_{\mathcal{D}}"] & \mathbf{Rep}_\Lambda(G) \arrow[d, "i_{\mathcal{D}}"] \\ \mathbf{Mod}\bigl(\mathcal{D}_c(K)\bigr) \arrow[r, "\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)}"] & \mathbf{Mod}\bigl(\mathcal{D}_c(G)\bigr), \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Mod}\bigl(\mathcal{D}_c(K)\bigr) \arrow[r, "\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)}"] \arrow[d, "\infty_{\mathcal{D}}"] & \mathbf{Mod}\bigl(\mathcal{D}_c(G)\bigr) \arrow[d, "\infty_{\mathcal{D}}"] \\ \mathbf{Rep}_\Lambda(K) \arrow[r, "\text{c-Ind}_K^G"] & \mathbf{Rep}_\Lambda(G). \end{tikzcd}$$

Let us observe the commutativity of the right diagram, as it is not included in [Far06]. Take a system of representatives $\{g_i\}_{i\in I}$ of G/K. Then, as in [Far06, §1.4], we have $\mathcal{D}_c(G)=\bigoplus_{i\in I}\delta_{g_i^{-1}}*\mathcal{D}_c(K)$, where $\delta_{g_i^{-1}}$ denotes the Dirac distribution at $g_i^{-1}\in G$. Therefore, for a $\mathcal{D}_c(K)$-module M, an element x of $\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)} M$ can be written uniquely in the form $\sum_{i\in I}\delta_{g_i^{-1}}\otimes x_i$ with $x_i\in M$. Put $M_\infty=i_{\mathcal{D}}\infty_{\mathcal{D}}M$. It is the $\mathcal{D}_c(K)$-submodule of M consisting of $x\in M$ such that $e_{K'}x=x$ for some open pro-p subgroup K' of G. By the left diagram and the fact that $i_{\mathcal{D}}$ is fully faithful, it suffices to prove that $\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)} M_\infty=(\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)} M)_\infty$. First take $x=\sum_{i\in I}\delta_{g_i^{-1}}\otimes x_i$ in $\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)} M_\infty$. For each $i\in I$ with $x_i\neq 0$, take an open pro-p subgroup K_i of K such that $e_{K_i}x_i=x_i$. We can find an open pro-p subgroup K' of G such that $g_iK'g_i^{-1}\subset K_i$ for every $i\in I$ with $x_i\neq 0$. Then, we have $e_{K'}x=\sum_{i\in I,x_i\neq 0}\delta_{g_i^{-1}}\otimes e_{g_iK'g_i^{-1}}x_i=x$, and thus $x\in(\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)} M)_\infty$. Next, take $x=\sum_{i\in I}\delta_{g_i^{-1}}\otimes x_i$ in $(\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)} M)_\infty$. Then, there exists an open pro-p subgroup K' of G such that $e_{K'}x=x$. We may shrink K' so that $g_iK'g_i'^{-1}\subset K$ for every $i\in I$ with $x_i\neq 0$. Then, we have $x=e_{K'}x=\sum_{i\in I,x_i\neq 0}\delta_{g_i^{-1}}\otimes e_{g_iK'g_i^{-1}}x_i$. Hence $e_{g_iK'g_i^{-1}}x_i$ is equal to x_i for every $i\in I$ with $x_i\neq 0$, which implies $x_i\in M_\infty$. Thus $x\in\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)} M_\infty$. Now we conclude that $\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)} M_\infty=(\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)} M)_\infty$.

Definition 2.1 For $\pi\in\mathbf{Rep}_\Lambda(G)$, consider $\mathrm{Hom}_G(\pi,\mathcal{H}(G))$, where $\mathcal{H}(G)$ is regarded as a smooth representation of G by the left translation. As $\mathcal{H}(G)$ has another smooth G-action by the right translation, $\mathrm{Hom}_G(\pi,\mathcal{H}(G))$ has a structure of a $\mathcal{D}_c(G)$-module. Therefore we get a contravariant functor from $\mathbf{Rep}_\Lambda(G)$ to $\mathbf{Mod}(\mathcal{D}_c(G))$, for which we write D^m. Composing with $\infty_{\mathcal{D}}$, we obtain a contravariant functor $\mathrm{D}=\infty_{\mathcal{D}}\circ\mathrm{D}^m\colon \mathbf{Rep}_\Lambda(G)\longrightarrow\mathbf{Rep}_\Lambda(G)$. We denote by $R\mathrm{D}^m$ (resp. $R\mathrm{D}$) the right derived functor of D^m (resp. D). As $\infty_{\mathcal{D}}$ is exact, we have $R\mathrm{D}=\infty_{\mathcal{D}}\circ R\mathrm{D}^m$.

Proposition 2.2 *Let K be an open pro-p subgroup of G and ρ a smooth representation of K over Λ. Then there is a natural $\mathcal{D}_c(G)$-linear injection*

$$\text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)}(\rho^*)\hookrightarrow\mathrm{D}^m(\text{c-Ind}_K^G\rho).$$

Here $\rho^*=\mathrm{Hom}_K(\rho,\Lambda)$ denotes the algebraic dual, which is naturally equipped with a structure of a $\mathcal{D}_c(K)$-module. Applying $\infty_{\mathcal{D}}$ to the injection above, we obtain a G-equivariant injection

$$\text{c-Ind}_K^G(\rho^\vee)\hookrightarrow\mathrm{D}(\text{c-Ind}_K^G\rho),$$

where $\rho^\vee = \infty_{\mathcal{D}}(\rho^*) = \infty_K(\mathrm{Hom}_K(\rho,\Lambda))$ *denotes the contragredient representation of* ρ.

If moreover ρ *is a finitely generated* K*-representation (in other words,* ρ *has finite length as a* Λ*-module), then we have*

$$R\mathrm{D}^m\bigl(\text{c-Ind}_K^G(\rho)\bigr) = \mathrm{D}^m\bigl(\text{c-Ind}_K^G(\rho)\bigr) = \text{c-Ind}_{\mathcal{D}_c(K)}^{\mathcal{D}_c(G)}(\rho^*),$$
$$R\mathrm{D}\bigl(\text{c-Ind}_K^G(\rho)\bigr) = \mathrm{D}\bigl(\text{c-Ind}_K^G(\rho)\bigr) = \text{c-Ind}_K^G(\rho^\vee).$$

Proof. See [Far06, Lemme 1.10, Lemme 1.12, Lemme 1.13]. ∎

Corollary 2.3 *Assume that* Λ *is a field,* $\mathbf{Rep}_\Lambda(G)$ *is noetherian and has finite projective dimension.*

i) *Let* $D^b_{\mathrm{fg}}(\mathbf{Rep}_\Lambda(G))$ *be the full subcategory of* $D^b(\mathbf{Rep}_\Lambda(G))$ *consisting of complexes whose cohomology are finitely generated* G*-representations. Then the contravariant functor* $R\mathrm{D}$ *maps* $D^b_{\mathrm{fg}}(\mathbf{Rep}_\Lambda(G))$ *into itself.*

ii) *The contravariant functor*

$$R\mathrm{D}\colon D^b_{\mathrm{fg}}\bigl(\mathbf{Rep}_\Lambda(G)\bigr) \longrightarrow D^b_{\mathrm{fg}}\bigl(\mathbf{Rep}_\Lambda(G)\bigr)$$

satisfies $R\mathrm{D}\circ R\mathrm{D} \cong \mathrm{id}$.

iii) *For a field extension* Λ' *of* Λ*, the following diagram is* 2*-commutative:*

$$\begin{array}{ccc} D^b_{\mathrm{fg}}\bigl(\mathbf{Rep}_\Lambda(G)\bigr) & \xrightarrow{R\mathrm{D}} & D^b_{\mathrm{fg}}\bigl(\mathbf{Rep}_\Lambda(G)\bigr) \\ \downarrow & & \downarrow \\ D^b_{\mathrm{fg}}\bigl(\mathbf{Rep}_{\Lambda'}(G)\bigr) & \xrightarrow{R\mathrm{D}} & D^b_{\mathrm{fg}}\bigl(\mathbf{Rep}_{\Lambda'}(G)\bigr). \end{array}$$

Here the vertical arrows denote the base change functor.

Proof. For i) and ii), see [Far06, Proposition 1.18]. iii) follows immediately from Proposition 2.2. ∎

In the remaining part of this section, we assume that $\Lambda = \mathbb{C}$. Let F be a p-adic field and $\mathbf{G}$ a connected reductive group over F. Write G for $\mathbf{G}(F)$. Fix a discrete cocompact subgroup Γ of the center $Z_G = \mathbf{Z_G}(F)$ and put $G' = G/\Gamma$. We simply write $\mathbf{Rep}(G')$ for $\mathbf{Rep}_{\mathbb{C}}(G')$. Note that we have a natural decomposition of a category

$$\mathbf{Rep}(G') = \prod_{\chi\colon Z_G/\Gamma\to\mathbb{C}^\times} \mathbf{Rep}_\chi(G),$$

where χ runs through smooth characters of the compact group Z_G/Γ, and $\mathbf{Rep}_\chi(G)$ denotes the category of smooth representations of G with central character χ. We will apply the theory above to the group G'. In this case all the assumptions in Corollary 2.3 are satisfied (*cf.* [Ber84, Remarque 3.12], [Vig90, Proposition 37]).

Let I_Γ be the set of inertially equivalence classes of cuspidal data $(\mathbf{M},\sigma)$ such that $\sigma|_\Gamma$ is trivial. We have the Bernstein decomposition (*cf.* [Ren10, Théorème VI.7.2])

$$\mathbf{Rep}(G')=\prod_{\mathfrak{s}\in I_\Gamma}\mathbf{Rep}(G')_{\mathfrak{s}}.$$

For $V\in\mathbf{Rep}(G')$, we denote the corresponding decomposition by $V=\bigoplus_{\mathfrak{s}\in I_\Gamma}V_{\mathfrak{s}}$.

Proposition 2.4 *For $\mathfrak{s}=[(\mathbf{M},\sigma)]\in I_\Gamma$, put $\mathfrak{s}^\vee=[(\mathbf{M},\sigma^\vee)]\in I_\Gamma$. Then, $R\mathrm{D}$ induces a contravariant functor $D^b_{\mathrm{fg}}(\mathbf{Rep}(G')_{\mathfrak{s}})\longrightarrow D^b_{\mathrm{fg}}(\mathbf{Rep}(G')_{\mathfrak{s}^\vee})$.*

Proof. See [Far06, Remarque 1.5]. ∎

For $\mathfrak{s}=[(\mathbf{M},\sigma)]\in I_\Gamma$, put $\iota(\mathfrak{s})=r_{\mathbf{G}}-r_{\mathbf{M}}$, where $r_{\mathbf{G}}$ (resp. $r_{\mathbf{M}}$) denotes the split semisimple rank of $\mathbf{G}$ (resp. $\mathbf{M}$). The number $\iota(\mathfrak{s})$ is 0 if and only if $\mathbf{M}=\mathbf{G}$.

Theorem 2.5 ([SS97, Theorem III.3.1]) *Fix $\mathfrak{s}\in I_\Gamma$. Let $\mathbf{Rep}(G')^{\mathrm{fl}}_{\mathfrak{s}}$ be the full subcategory of $\mathbf{Rep}(G')_{\mathfrak{s}}$ consisting of representations of finite length. For $\pi\in\mathbf{Rep}(G')^{\mathrm{fl}}_{\mathfrak{s}}$, we have $R^i\mathrm{D}(\pi)=0$ if $i\neq\iota(\mathfrak{s})$. Moreover, $R^{\iota(\mathfrak{s})}\mathrm{D}(\pi)$ has finite length.*

Definition 2.6 For $\mathfrak{s}\in I_\Gamma$ and $\pi\in\mathbf{Rep}(G')^{\mathrm{fl}}_{\mathfrak{s}}$, put $\mathrm{Zel}(\pi)=R^{\iota(\mathfrak{s})}\mathrm{D}(\pi^\vee)$. The (covariant) functor $\mathrm{Zel}\colon\mathbf{Rep}(G')^{\mathrm{fl}}_{\mathfrak{s}}\longrightarrow\mathbf{Rep}(G')^{\mathrm{fl}}_{\mathfrak{s}}$ is called the Zelevinsky involution. It is an exact categorical equivalence. In particular, it preserves irreducibility.

Proposition 2.7 *For an irreducible smooth representation π of G', we have an isomorphism $\mathrm{Zel}(\pi^\vee)\cong\mathrm{Zel}(\pi)^\vee$. In particular, $\mathrm{Zel}(\mathrm{Zel}(\pi))\cong\pi$.*

Proof. It is an immediate consequence of [SS97, Proposition IV.5.4]. ∎

Let $\chi\colon Z_G\longrightarrow\mathbb{C}^\times$ be a smooth character, which is not necessarily unitary. We can consider a variant of Zel on $\mathbf{Rep}_\chi(G)$ as follows. Let $\mathcal{H}_\chi(G)$ be the set of locally constant $\mathbb{C}$-valued functions f such that

- $f(zg)=\chi(z)^{-1}f(g)$ for every $z\in Z_G$ and $g\in G$,
- and $\operatorname{supp}f$ is compact modulo Z_G.

Let $\mathrm{D}_\chi\colon\mathbf{Rep}_\chi(G)\longrightarrow\mathbf{Rep}_{\chi^{-1}}(G)$ be the contravariant functor defined by

$$\mathrm{D}_\chi(\pi)=\mathrm{Hom}_{\mathbf{Rep}_\chi(G)}\bigl(\pi,\mathcal{H}_\chi(G)\bigr),$$

and $R\mathrm{D}_\chi$ be the derived functor of D_χ. As in Corollary 2.3 i), $R\mathrm{D}_\chi$ induces a contravariant functor

$$R\mathrm{D}_\chi\colon D^b_{\mathrm{fg}}\bigl(\mathbf{Rep}_\chi(G)\bigr)\longrightarrow D^b_{\mathrm{fg}}\bigl(\mathbf{Rep}_{\chi^{-1}}(G)\bigr).$$

Let I_χ be the set of inertially equivalence classes of cuspidal data $(\mathbf{M},\sigma)$ such that $\sigma|_{Z_G}=\chi$. We have the Bernstein decomposition

$$\mathbf{Rep}_\chi(G)=\prod_{\mathfrak{s}\in I_\chi}\mathbf{Rep}_\chi(G)_{\mathfrak{s}}.$$

Let $\mathbf{Rep}_\chi(G)^{\mathrm{fl}}_{\mathfrak{s}}$ be the full subcategory of $\mathbf{Rep}_\chi(G)_{\mathfrak{s}}$ consisting of representations of finite length. By [SS97, Theorem III.3.1], for $\pi \in \mathbf{Rep}_\chi(G)^{\mathrm{fl}}_{\mathfrak{s}}$ we have $R^i\mathrm{D}_\chi(\pi) = 0$ if $i \neq \iota(\mathfrak{s})$ and $R^{\iota(\mathfrak{s})}\mathrm{D}_\chi(\pi)$ has finite length. Hence we can give the following definition.

Definition 2.8 For $\mathfrak{s} \in I_\chi$ and $\pi \in \mathbf{Rep}_\chi(G)^{\mathrm{fl}}_{\mathfrak{s}}$, put $\mathrm{Zel}_\chi(\pi) = R^{\iota(\mathfrak{s})}\mathrm{D}_{\chi^{-1}}(\pi^\vee)$. It induces an exact categorical equivalence $\mathrm{Zel}_\chi\colon \mathbf{Rep}_\chi(G)^{\mathrm{fl}}_{\mathfrak{s}} \longrightarrow \mathbf{Rep}_\chi(G)^{\mathrm{fl}}_{\mathfrak{s}}$ satisfying $\mathrm{Zel}_\chi^2 \cong \mathrm{id}$.

Lemma 2.9 i) *For $\pi \in \mathbf{Rep}_\chi(G)^{\mathrm{fl}}_{\mathfrak{s}}$ and a smooth character ω of G, we have $\mathrm{Zel}_{\chi\otimes\omega}(\pi\otimes\omega) \cong \mathrm{Zel}_\chi(\pi)\otimes\omega$.*

ii) *Let $\Gamma \subset Z_G$ and $G' = G/\Gamma$ be as above. If χ is trivial on Γ, then for every $\pi \in \mathbf{Rep}_\chi(G)^{\mathrm{fl}}_{\mathfrak{s}}$ we have $\mathrm{Zel}_\chi(\pi) \cong \mathrm{Zel}(\pi)$. In the right hand side, π is regarded as an object of $\mathbf{Rep}(G')^{\mathrm{fl}}_{\mathfrak{s}}$.*

Proof. i) is clear from definition. For ii), note that $\mathcal{H}(G') = \bigoplus_{\chi'} \mathcal{H}_{\chi'}(G)$, where χ' runs through smooth characters of Z_G which are trivial on Γ. By the decomposition $\mathbf{Rep}(G') = \prod_{\chi'} \mathbf{Rep}_{\chi'}(G)$, we have

$$R\mathrm{D}(\pi^\vee) = R\operatorname{Hom}_{\mathbf{Rep}(G')}\bigl(\pi^\vee, \mathcal{H}(G)\bigr) = R\operatorname{Hom}_{\mathbf{Rep}_{\chi^{-1}}(G)}\bigl(\pi^\vee, \mathcal{H}_{\chi^{-1}}(G)\bigr) = R\mathrm{D}_{\chi^{-1}}(\pi^\vee).$$

Hence we have $\mathrm{Zel}_\chi(\pi) \cong \mathrm{Zel}(\pi)$, as desired. ∎

By this lemma, we can simply write Zel for Zel_χ without any confusion.

3 Preliminaries

3.1 Compactly supported cohomology for partially proper schemes

In [Hub96, §5], Huber defined the compactly supported cohomology for adic spaces which are partially proper over a field as the derived functor of Γ_c. This construction is also applicable to schemes over a field.

Definition 3.1 Let $f\colon X \longrightarrow Y$ be a morphism between schemes.

i) The morphism f is said to be specializing if for every $x \in X$ and every specialization y' of $y = f(x)$, there exists a specialization x' of x such that $y' = f(x')$. If an arbitrary base change of f is specializing, f is said to be universally specializing.

ii) The morphism f is said to be partially proper if it is separated, locally of finite type and universally specializing.

Proposition 3.2 i) *A morphism of schemes is proper if and only if partially proper and quasi-compact.*

ii) *Partially properness can be checked by the valuative criterion.*

iii) *Let $f\colon X \longrightarrow Y$ be a partially proper morphism between schemes. Assume that Y is noetherian. Then, for every quasi-compact subset T of X, the closure $\overline{T}$ of T is quasi-compact.*

Proof. i) can be proved in the same way as [Hub96, Lemma 1.3.4]. ii) is straightforward and left to the reader. Let us prove iii). We may assume that T is open in X. Moreover we may assume that T and Y are affine. Put $T = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Let $\widetilde{B}$ be the integral closure of the image of $B \longrightarrow A$ in A, and consider the topological space $T_v = \operatorname{Spa}(A, \widetilde{B})$. Here we endow A with the discrete topology. As a set, T_v can be identified with the set of pairs (x, V_x) where

- $x \in T$, and
- V_x is a valuation ring of the residue field κ_x at x such that the composite $\operatorname{Spec} \kappa_x \longrightarrow T \longrightarrow Y$ can be extended to $\operatorname{Spec} V_x \longrightarrow Y$.

Therefore, by ii), we can construct a map $\phi\colon T_v \longrightarrow X$ as follows. For $(x, V_x) \in T_v$, the Y-morphism $\operatorname{Spec} \kappa_x \longrightarrow T$ uniquely extends to a Y-morphism $\operatorname{Spec} V_x \longrightarrow X$. We let $\phi(x, V_x)$ be the image of the closed point in $\operatorname{Spec} V_x$ under this morphism. Since X is quasi-separated locally spectral and T is quasi-compact open, each point in $\overline{T}$ is a specialization of some point in T ([Hoc69, Corollary of Theorem 1]). Thus $\overline{T}$ coincides with $\phi(T_v)$.

We will prove that ϕ is continuous. Fix $(x, V_x) \in T_v$ and take an affine neighborhood $U = \operatorname{Spec} C$ of $y = \phi(x, V_x)$. We can find $u \in A$ such that $T' = \operatorname{Spec} A[1/u]$ is an open neighborhood of x contained in U. On the other hand, as f is locally of finite type, C is a finitely generated B-algebra. Take a system of generators $c_1, \ldots, c_n \in C$ $(n \geq 1)$ and consider the images of them under the ring homomorphism $C \longrightarrow A[1/u]$ that comes from the inclusion $T' \hookrightarrow U$. There exist integers $k_1, \ldots, k_n$ and $a_1, \ldots, a_n \in A$ such that the image of a_i in $A[1/u]$ coincides with the image of $u^{k_i} c_i$ under $C \longrightarrow A[1/u]$. Let W be the open subset of T_v defined by the condition $v(a_i) \leq v(u^{k_i}) \neq 0$ $(i = 1, \ldots, n)$. Then it is easy to observe that (x, V_x) belongs to W and $\phi(W)$ is contained in U. This proves the continuity of ϕ.

By [Hub93, Theorem 3.5 (i)], T_v is quasi-compact. Therefore we conclude that $\overline{T} = \phi(T_v)$ is quasi-compact, as desired. ∎

Corollary 3.3 *Let X be a scheme which is partially proper over a noetherian scheme. Then X is a locally finite union of quasi-compact closed subsets.*

Proof. Let S be the set of minimal points in X. For $\eta \in S$, we denote by Z_η the closure of $\{\eta\}$. By Proposition 3.2 iii), Z_η is quasi-compact. It is easy to observe that $\{Z_\eta\}$ cover X. Take a quasi-compact open subset U of X. As U is noetherian, it contains finitely many minimal points, thus U intersects finitely many Z_η. This concludes that the closed covering $\{Z_\eta\}$ is locally finite. ∎

In the rest of this subsection, let k be a field and X a scheme which is partially proper over k.

Definition 3.4 i) For an (abelian étale) sheaf $\mathcal{F}$ on X, let $\Gamma_c(X,\mathcal{F})$ be the subset of $\Gamma(X,\mathcal{F})$ consisting of $s \in \Gamma(X,\mathcal{F})$ such that $\operatorname{supp} s$ is proper over k. As $\operatorname{supp} s$ is closed in X, this condition is equivalent to saying that $\operatorname{supp} s$ is quasi-compact (*cf.* Proposition 3.2 ii)).

ii) Let $H^i_c(X,-)$ be the ith derived functor of the left exact functor $\Gamma_c(X,-)$.

Proposition 3.5 *Let $\mathcal{F}$ be a sheaf on X.*

i) *We have $H^i_c(X,\mathcal{F}) \cong \varinjlim_Y H^i_Y(X,\mathcal{F})$, where Y runs through quasi-compact closed subsets of X.*

ii) *We have $H^i_c(X,\mathcal{F}) \cong \varinjlim_U H^i_c(U,\mathcal{F}|_U)$, where U runs through quasi-compact open subsets of X.*

Proof. i) If $i = 0$, then the claim follows immediately from the definition of Γ_c. On the other hand, if $\mathcal{F}$ is injective, then $\varinjlim_Y H^i_Y(X,\mathcal{F}) = 0$ for $i > 0$. Therefore we have the desired isomorphism.

ii) By Proposition 3.2 iii), for each quasi-compact open subset U of X, we can find a quasi-compact closed subset Y of X containing U. On the other hand, for such a Y, we can find a quasi-compact open subset U' of X containing Y. Under this situation, we have push-forward maps

$$H^i_c(U,\mathcal{F}|_U) \longrightarrow H^i_Y(X,\mathcal{F}) \longrightarrow H^i_c(U',\mathcal{F}|_{U'}).$$

These induce an isomorphism $\varinjlim_U H^i_c(U,\mathcal{F}|_U) \cong \varinjlim_Y H^i_Y(X,\mathcal{F})$. Hence the claim follows from i). ∎

Corollary 3.6 *The functor $H^i_c(X,-)$ commutes with filtered inductive limits.*

Proof. For a quasi-compact open subset U of X, $H^i_c(U,-)$ commutes with filtered inductive limits; indeed, for a compactification $j\colon U \hookrightarrow \overline{U}$, we have $H^i_c(U,-) = H^i(\overline{U}, j_!(-))$, and both $j_!$ and $H^i(\overline{U},-)$ commute with filtered inductive limits (for the later, note that $\overline{U}$ is quasi-compact and quasi-separated). On the other hand, the restriction functor to U also commutes with filtered inductive limits. Hence Proposition 3.5 ii) tells us that $H^i_c(X,-)$ commutes with filtered inductive limits. ∎

Corollary 3.7 *For $i \geq 0$, the functor $H^i_c(X,-)$ commutes with arbitrary direct sums. Namely, for a set Λ and sheaves $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on X indexed by Λ, we have an isomorphism $H^i_c(X,\bigoplus_{\lambda\in\Lambda}\mathcal{F}_\lambda) \cong \bigoplus_{\lambda\in\Lambda} H^i_c(X,\mathcal{F}_\lambda)$.*

Proof. For a finite subset Λ_0 of Λ, put $\mathcal{F}_{\Lambda_0} = \bigoplus_{\lambda\in\Lambda_0}\mathcal{F}_\lambda$. Then $\bigoplus_{\lambda\in\Lambda}\mathcal{F}_\lambda$ can be written as the filtered inductive limit $\varinjlim_{\Lambda_0\subset\Lambda}\mathcal{F}_{\Lambda_0}$. As $H^i_c(X,-)$ commutes with filtered inductive limits and finite direct sums, we obtain the desired result. ∎

Remark 3.8 By exactly the same method as in [Hub98b], we can extend the definitions and properties above to ℓ-adic coefficients.

3.2 Formal schemes and adic spaces

Let R be a complete discrete valuation ring with separably closed residue field κ, F the fraction field of R, and k a separable closure of F. We denote by k^+ the valuation ring of k. For a locally noetherian formal scheme $\mathcal{X}$ over $\operatorname{Spf} R$, we can associate an adic space $t(\mathcal{X})$ over $\operatorname{Spa}(R, R)$ (*cf.* [Hub94, Proposition 4.1]). Its open subset $t(\mathcal{X})_\eta = t(\mathcal{X}) \times_{\operatorname{Spa}(R,R)} \operatorname{Spa}(F, R)$ is called the rigid generic fiber of $\mathcal{X}$. In the following, we assume that $\mathcal{X}$ is special in the sense of Berkovich [Ber94, §1]. Then $t(\mathcal{X})_\eta$ is locally of finite type over $\operatorname{Spa}(F, R)$. Therefore, we can make the fiber product $t(\mathcal{X})_{\overline{\eta}} = t(\mathcal{X})_\eta \times_{\operatorname{Spa}(F,R)} \operatorname{Spa}(k, k^+)$, which we call the rigid geometric generic fiber of $\mathcal{X}$. The morphism $t(\mathcal{X}) \longrightarrow \mathcal{X}$ of locally ringed spaces induces a continuous map $t(\mathcal{X})_\eta \longrightarrow \mathcal{X} = \mathcal{X}^{\mathrm{red}}$. We also have morphisms of sites $t(\mathcal{X})_{\overline{\eta},\text{ét}} \longrightarrow t(\mathcal{X})_{\eta,\text{ét}} \longrightarrow \mathcal{X}_{\text{ét}} \cong (\mathcal{X}^{\mathrm{red}})_{\text{ét}}$.

Proposition 3.9 *Assume that $\mathcal{X}^{\mathrm{red}}$ is partially proper over κ. Then $t(\mathcal{X})_\eta$ is partially proper over* $\operatorname{Spa}(F, R)$.

Proof. In [Mie10, Proposition 4.23], we have obtained the same result under the assumption that $\mathcal{X}^{\mathrm{red}}$ is proper. In fact, the proof therein only uses the partially properness of $\mathcal{X}^{\mathrm{red}}$. ∎

For simplicity, we put $X = t(\mathcal{X})_{\overline{\eta}}$. We denote the composite $t(\mathcal{X})_{\overline{\eta}} \longrightarrow t(\mathcal{X})_\eta \longrightarrow \mathcal{X}^{\mathrm{red}}$ by sp. We also write sp for the morphism of étale sites $X_{\text{ét}} \longrightarrow (\mathcal{X}^{\mathrm{red}})_{\text{ét}}$. For a closed subset Z of $\mathcal{X}$, consider the interior $\mathrm{sp}^{-1}(Z)^\circ$ of $\mathrm{sp}^{-1}(Z)$ in X. It is called the tube of Z.

Proposition 3.10 i) *Let $\mathcal{Z}$ be the formal completion of $\mathcal{X}$ along Z. Then the natural morphism $t(\mathcal{Z})_{\overline{\eta}} \longrightarrow t(\mathcal{X})_{\overline{\eta}}$ induces an isomorphism $t(\mathcal{Z})_{\overline{\eta}} \cong \mathrm{sp}^{-1}(Z)^\circ$.*

ii) *If Z is quasi-compact, $\mathrm{sp}^{-1}(Z)^\circ$ is the union of countably many quasi-compact open subsets of X.*

iii) *Assume that $\mathcal{X}^{\mathrm{red}}$ is partially proper over κ. Then $\mathrm{sp}^{-1}(Z)^\circ$ is partially proper over* $\operatorname{Spa}(k, k^+)$.

iv) *Assume that $\mathcal{X}$ is locally algebraizable (cf. [Mie10, Definition 3.19]) and Z is quasi-compact. Then, for a noetherian torsion ring Λ whose characteristic is invertible in R, $H^i(\mathrm{sp}^{-1}(Z)^\circ, \Lambda)$ and $H^i_c(\mathrm{sp}^{-1}(Z)^\circ, \Lambda)$ are finitely generated Λ-modules.*

Proof. i) is proved in [Hub98a, Lemma 3.13 i)]. For ii), we may assume that $\mathcal{X}$ is affine. Then, the claim has been obtained in the proof of [Hub98a, Lemma 3.13 i)]. By i), to prove iii) and iv), we may assume that $Z = \mathcal{X}^{\mathrm{red}}$. Then iii) follows from Proposition 3.9. As for iv), we have $H^i(X, \Lambda) = H^i(\mathcal{X}^{\mathrm{red}}, R\,\mathrm{sp}_* \Lambda)$. By [Ber96, Theorem 3.1], $R\,\mathrm{sp}_* \Lambda$ is a constructible complex on $\mathcal{X}^{\mathrm{red}}$. Thus $H^i(X, \Lambda)$ is a finitely generated Λ-module. On the other hand, by [Mie10, Proposition 3.21, Theorem 4.35], $H^i_c(X, \Lambda)$ is a finitely generated Λ-module. ∎

Definition 3.11 Assume that $\mathcal{X}^{\mathrm{red}}$ is partially proper over κ. We write $\Gamma_{c,\mathcal{X}}(X,-)$ for the composite functor $\Gamma_c(\mathcal{X}^{\mathrm{red}},\mathrm{sp}_*(-))$. Denote the derived functor of $\Gamma_{c,\mathcal{X}}(X,-)$ by $R\Gamma_{c,\mathcal{X}}(X,-)$, and the ith cohomology of $R\Gamma_{c,\mathcal{X}}(X,-)$ by $H^i_{c,\mathcal{X}}(X,-)$.

Remark 3.12 If $\mathcal{X}$ is a p-adic formal scheme, then for a quasi-compact closed subset Z of $\mathcal{X}^{\mathrm{red}}$, $\mathrm{sp}^{-1}(Z)$ is quasi-compact. Hence we have $\Gamma_{c,\mathcal{X}}(X,-)=\Gamma_c(X,-)$, $R\Gamma_{c,\mathcal{X}}(X,-)=R\Gamma_c(X,-)$ and $H^i_{c,\mathcal{X}}(X,-)=H^i_c(X,-)$ in this case.

The cohomology $H^i_{c,\mathcal{X}}(X,-)$ will appear in our main result. We would like to discuss functoriality of this cohomology with respect to X. For this purpose, we give another interpretation of $H^i_{c,\mathcal{X}}(X,-)$. Here, more generally, let X denote an adic space which is locally of finite type and partially proper over $\mathrm{Spa}(k,k^+)$.

Definition 3.13 A support set $\mathcal{C}$ of X is a set consisting of closed subsets of X satisfying the following conditions:

- For $Z,Z'\in\mathcal{C}$, we have $Z\cup Z'\in\mathcal{C}$.
- For $Z\in\mathcal{C}$ and a closed subset Z' of Z, we have $Z'\in\mathcal{C}$.

For a support set $\mathcal{C}$, we define $\Gamma_{\mathcal{C}}(X,-)=\varinjlim_{Z\in\mathcal{C}}\Gamma_Z(X,-)$. Let $R\Gamma_{\mathcal{C}}(X,-)$ be the derived functor of $\Gamma_{\mathcal{C}}(X,-)$, and $H^i_{\mathcal{C}}(X,-)$ the ith cohomology of $R\Gamma_{\mathcal{C}}(X,-)$. It is easy to see that $H^i_{\mathcal{C}}(X,\mathcal{F})=\varinjlim_{Z\in\mathcal{C}}H^i_Z(X,\mathcal{F})$ for a sheaf $\mathcal{F}$ on X.

Definition 3.14 Let X and X' be adic spaces which are locally of finite type and partially proper over $\mathrm{Spa}(k,k^+)$.

i) For a support set $\mathcal{C}$ of X and a morphism of adic spaces $f\colon X'\longrightarrow X$, let $f^{-1}\mathcal{C}$ be the support set of X' consisting of closed subsets $Z'\subset X'$ which are contained in $f^{-1}(Z)$ for some $Z\in\mathcal{C}$.

ii) Let $\mathcal{C}$ and $\mathcal{C}'$ be support sets of X and X', respectively. A morphism of pairs $f\colon(X',\mathcal{C}')\longrightarrow(X,\mathcal{C})$ is a morphism $f\colon X'\longrightarrow X$ satisfying $f^{-1}\mathcal{C}\subset\mathcal{C}'$. Such a morphism induces a morphism $f^*\colon R\Gamma_{\mathcal{C}}(X,\mathcal{F})\longrightarrow R\Gamma_{\mathcal{C}'}(X',f^*\mathcal{F})$. If moreover f is an isomorphism and $f^{-1}\mathcal{C}=\mathcal{C}'$, f is said to be an isomorphism of pairs.

A formal model naturally gives a support set of the rigid generic fiber.

Definition 3.15 Let $\mathcal{X}$ be as in the beginning of this subsection, and assume that $\mathcal{X}^{\mathrm{red}}$ is partially proper over κ. We define a support set $\mathcal{C}_{\mathcal{X}}$ of $X=t(\mathcal{X})_{\overline{\eta}}$ as follows: a closed set of X belongs to $\mathcal{C}_{\mathcal{X}}$ if it is contained in $\mathrm{sp}^{-1}(Z)$ for some quasi-compact closed subset of $\mathcal{X}^{\mathrm{red}}$.

Proposition 3.16 *Let $\mathcal{X}$ be as in the previous definition. Then we have an isomorphism $R\Gamma_{c,\mathcal{X}}(X,-)\cong R\Gamma_{\mathcal{C}_{\mathcal{X}}}(X,-)$.*

Proof. For a sheaf $\mathcal{F}$ on X, we have

$$\Gamma_{c,\mathcal{X}}(X,\mathcal{F})=\varinjlim_{Z\subset\mathcal{X}^{\mathrm{red}}}\Gamma_Z(\mathcal{X}^{\mathrm{red}},\mathrm{sp}_*\mathcal{F})=\varinjlim_{Z\subset\mathcal{X}^{\mathrm{red}}}\Gamma_{\mathrm{sp}^{-1}(Z)}(X,\mathcal{F})=\Gamma_{\mathcal{C}_{\mathcal{X}}}(X,\mathcal{F}),$$

where Z runs through quasi-compact closed subsets of $\mathcal{X}^{\mathrm{red}}$. This concludes the proof. ∎

Lemma 3.17 *Let $\mathcal{X}$ and $\mathcal{X}'$ be special formal schemes over* $\operatorname{Spf} R$ *such that $\mathcal{X}^{\mathrm{red}}$ and $\mathcal{X}'^{\mathrm{red}}$ are partially proper over κ. Put $X = t(\mathcal{X})_{\overline{\eta}}$ and $X' = t(\mathcal{X}')_{\overline{\eta}}$, respectively. Let $f\colon \mathcal{X}' \longrightarrow \mathcal{X}$ be a morphism of finite type over* $\operatorname{Spf} R$ *and write $f_{\overline{\eta}}$ for the induced morphism $X' \longrightarrow X$. Then, we have $f_{\overline{\eta}}^{-1}\mathcal{C}_{\mathcal{X}} = \mathcal{C}_{\mathcal{X}'}$. In particular, if moreover $f_{\overline{\eta}}$ is an isomorphism, then $f_{\overline{\eta}}$ induces an isomorphism of pairs $(X', \mathcal{C}_{\mathcal{X}'}) \longrightarrow (X, \mathcal{C}_{\mathcal{X}})$.*

Proof. For a quasi-compact closed subset Z of $\mathcal{X}^{\mathrm{red}}$, $Z' = f^{-1}(Z)$ is a quasi-compact closed subset of $\mathcal{X}'^{\mathrm{red}}$. Therefore $f_{\overline{\eta}}^{-1}(\mathrm{sp}^{-1}(Z)) = \mathrm{sp}^{-1}(Z')$ lies in $\mathcal{C}_{\mathcal{X}'}$. This implies that $f^{-1}\mathcal{C}_{\mathcal{X}}$ is contained in $\mathcal{C}_{\mathcal{X}'}$. Conversely, let Z' be a quasi-compact closed subset of $\mathcal{X}'^{\mathrm{red}}$. As $\mathcal{X}^{\mathrm{red}}$ and $\mathcal{X}'^{\mathrm{red}}$ are partially proper over κ and f is of finite type, the induced morphism $f\colon \mathcal{X}'^{\mathrm{red}} \longrightarrow \mathcal{X}^{\mathrm{red}}$ is proper. Therefore $Z = f(Z')$ is a quasi-compact closed subset of $\mathcal{X}^{\mathrm{red}}$. Hence, $\mathrm{sp}^{-1}(Z')$, being contained in $f_{\overline{\eta}}^{-1}(\mathrm{sp}^{-1}(Z))$, lies in $f_{\overline{\eta}}^{-1}\mathcal{C}_{\mathcal{X}}$. Thus we have $\mathcal{C}_{\mathcal{X}'} \subset f_{\overline{\eta}}^{-1}\mathcal{C}_{\mathcal{X}}$. ∎

3.3 Smooth equivariant sheaves on adic spaces

In [FGL08, §IV.9], the theory of smooth equivariant sheaves on Berkovich spaces is developed. In this subsection, we will adapt it to the framework of adic spaces.

3.3.1 Basic definitions

Let k be a non-archimedean field and X an adic space locally of finite type over $\operatorname{Spa}(k, k^+)$. We denote by $X_{\text{ét}}$ the étale site of X, and $X_{\text{qcét}}$ the full subcategory consisting of étale morphisms $f\colon Y \longrightarrow X$ where Y is quasi-compact. The category $X_{\text{qcét}}$ has a natural induced structure of a site and the associated topos $\widetilde{X}_{\text{qcét}}$ can be identified with the étale topos $\widetilde{X}_{\text{ét}}$ for X.

We write p for the residue characteristic of k^+. Fix a prime $\ell \neq p$ and a truncated discrete valuation ring Λ with residue characteristic ℓ. Namely, Λ can be written as $\mathcal{O}/(\lambda^n)$, where $\mathcal{O}$ is a discrete valuation ring with residue characteristic ℓ, λ is a uniformizer of $\mathcal{O}$ and $n \geq 1$ is an integer. Such a ring has the following property:

Lemma 3.18 *For a Λ-module M, the following are equivalent:*

i) *M is an injective Λ-module.*
ii) *M is a flat Λ-module.*
iii) *M is a free Λ-module.*

Proof. We write $\Lambda = \mathcal{O}/(\lambda^n)$ as above. For every $0 \leq k \leq n$, we have an exact sequence $\Lambda \xrightarrow{\times\lambda^{n-k}} \Lambda \xrightarrow{\times\lambda^k} \Lambda \longrightarrow \Lambda/(\lambda^k) \longrightarrow 0$. Thus we obtain

$$\operatorname{Tor}_1^{\Lambda}\bigl(\Lambda/(\lambda^k), M\bigr) = \operatorname{Ker}(M \xrightarrow{\times\lambda^k} M)/\operatorname{Im}(M \xrightarrow{\times\lambda^{n-k}} M) = \operatorname{Ext}^1_{\Lambda}\bigl(\Lambda/(\lambda^{n-k}), M\bigr).$$

Therefore, if M is injective, then $\operatorname{Tor}_1^{\Lambda}(\Lambda/(\lambda), M) = 0$ and M is flat. Conversely, if M is flat, then $\operatorname{Ext}^1_{\Lambda}(\Lambda/(\lambda^k), M) = 0$ for every $0 \leq k \leq n$. Hence Baer's criterion on injectivity tells us that M is an injective Λ-module.

On the other hand, as Λ is an Artinian local ring, ii) and iii) are equivalent. ∎

We write $\Lambda\text{-}\widetilde{X}_{\text{ét}}$ for the category of Λ-sheaves on $X_{\text{ét}}$.

Let G be a locally pro-p group, and assume that X is equipped with a continuous action of G in the following sense.

Definition 3.19 An action of G on X is said to be continuous if the following conditions are satisfied:

> For every affinoid open subspace $U = \mathrm{Spa}(A, A^+)$ of X, $f \in A$ and $a \in k^\times$, there exists an open subgroup $G' \subset G$ such that each element $g \in G'$ satisfies $gU = U$ and $|(g^*f - f)(x)| \le |a(x)|$ for $x \in U$.

Under the continuity condition, we know the following result due to Berkovich:

Theorem 3.20 ([Ber94, Key Lemma 7.2]) *For every object $f\colon Y \longrightarrow X$ of $X_{\text{qcét}}$, there exists a compact open subgroup K_Y of G such that the action of K_Y on X lifts canonically and continuously to Y.*

Definition 3.21 A G-equivariant sheaf $\mathcal{F}$ on $X_{\text{ét}}$ is said to be smooth if for every object $Y \longrightarrow X$ of $X_{\text{qcét}}$ the action of K_Y on $\Gamma(Y, \mathcal{F})$ is smooth. We write $\widetilde{X}_{\text{ét}}/G$ for the category of smooth G-equivariant sheaves on $X_{\text{ét}}$, and $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$ for the category of smooth G-equivariant Λ-sheaves on $X_{\text{ét}}$.

As $\widetilde{X}_{\text{ét}} = \widetilde{X}_{\text{qcét}}$, the definition above is a special case of [FGL08, Définition IV.8.1]. In particular, $\widetilde{X}_{\text{ét}}/G$ is a topos (*cf.* [FGL08, Proposition IV.8.12]). The forgetful functor $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G \longrightarrow \Lambda\text{-}\widetilde{X}_{\text{ét}}$ is exact (*cf.* [FGL08, Corollaire IV.8.6]).

Definition 3.22 Let H be a closed subgroup of G.

i) A smooth G-equivariant Λ-sheaf on $X_{\text{ét}}$ can be obviously regarded as a smooth H-equivariant Λ-sheaf on $X_{\text{ét}}$. Therefore we get a functor $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G \longrightarrow \Lambda\text{-}\widetilde{X}_{\text{ét}}/H$, which is denoted by $\mathrm{Res}^{X/G}_{X/H}$.

ii) Let Ω_H be a system of representatives of $H\backslash G$. For a smooth H-equivariant Λ-sheaf $\mathcal{F}$ on $X_{\text{ét}}$, the Λ-sheaf $\prod_{g\in\Omega_H} g^*\mathcal{F}$ has a natural G-equivariant structure. We put $\mathrm{Ind}^{X/G}_{X/H}\mathcal{F} = (\prod_{g\in\Omega_H} g^*\mathcal{F})^\infty$, where $(-)^\infty$ denotes the smoothification functor (*cf.* [FGL08, §IV.8.3.3]). This gives a functor $\mathrm{Ind}^{X/G}_{X/H}\colon \Lambda\text{-}\widetilde{X}_{\text{ét}}/H \longrightarrow \Lambda\text{-}\widetilde{X}_{\text{ét}}/G$.

iii) If H is an open subgroup of G, for $\mathcal{F} \in \Lambda\text{-}\widetilde{X}_{\text{ét}}/H$ we put

$$\text{c-Ind}^{X/G}_{X/H}\mathcal{F} = \bigoplus_{g\in\Omega_H} g^*\mathcal{F}.$$

It can be naturally regarded as an object of $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$. This gives a functor $\text{c-Ind}^{X/G}_{X/H}\colon \Lambda\text{-}\widetilde{X}_{\text{ét}}/H \longrightarrow \Lambda\text{-}\widetilde{X}_{\text{ét}}/G$.

The following proposition is obvious.

Proposition 3.23 *The functor* $\mathrm{Ind}_{X/H}^{X/G}$ *is the right adjoint of* $\mathrm{Res}_{X/H}^{X/G}$*, and* $\text{c-Ind}_{X/H}^{X/G}$ *is the left adjoint of* $\mathrm{Res}_{X/H}^{X/G}$*. The functors* $\mathrm{Res}_{X/H}^{X/G}$ *and* $\text{c-Ind}_{X/H}^{X/G}$ *are exact, and* $\mathrm{Ind}_{X/H}^{X/G}$ *is left exact.*

Let $f\colon U \longrightarrow X$ be an étale morphism. Assume that there exists a compact open subgroup K of G whose action on X lifts to U continuously (if U is quasi-compact, this is always the case). We fix such a compact open subgroup K.

Lemma 3.24 *For* $\mathcal{F} \in \Lambda\text{-}\widetilde{U}_{\text{ét}}/K$*, the induced* K*-equivariant structure on* $f_!\mathcal{F}$ *is smooth.*

Proof. Let $\mathcal{G}$ be the presheaf on $X_{\text{qcét}}$ defined as follows: for an object $Y \longrightarrow X$ of $X_{\text{qcét}}$, we put

$$\Gamma(Y,\mathcal{G}) = \bigoplus_{\phi\in \mathrm{Hom}_X(Y,U)} \Gamma(Y \xrightarrow{\phi} U, \mathcal{F}).$$

We will prove that the action of $K \cap K_Y$ on $\Gamma(Y,\mathcal{G})$ is smooth. Applying [Ber94, Key Lemma 7.2] to $\phi \in \mathrm{Hom}_X(Y,U)$, we obtain a compact open subgroup K_ϕ of $K \cap K_Y$ such that $g \in K_\phi$ satisfies $g\circ\phi = \phi\circ g$. Such K_ϕ acts on $\Gamma(Y \xrightarrow{\phi} U, \mathcal{F})$, and by the smoothness of $\mathcal{F}$, this action is smooth. This concludes the smoothness of the action of $K \cap K_Y$ on $\Gamma(Y,\mathcal{G})$.

In other words, $\mathcal{G}$ is a smooth K-equivariant Λ-presheaf on $X_{\text{qcét}}$ (*cf.* [FGL08, Définition IV.8.1]). Hence [FGL08, Lemme IV.8.4] tells us that the sheafification $f_!\mathcal{F}$ of $\mathcal{G}$ is smooth. ∎

Definition 3.25 i) Let $\mathrm{Res}_{U/K}^{X/K}\colon \Lambda\text{-}\widetilde{X}_{\text{ét}}/K \longrightarrow \Lambda\text{-}\widetilde{U}_{\text{ét}}/K$ be the functor $\mathcal{F} \longmapsto f^*\mathcal{F}$. In fact, it is easy to see that the induced K-equivariant structure on $f^*\mathcal{F}$ is smooth. We put $\mathrm{Res}_{U/K}^{X/G} = \mathrm{Res}_{U/K}^{X/K} \circ \mathrm{Res}_{X/K}^{X/G}$, which is a functor from $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$ to $\Lambda\text{-}\widetilde{U}_{\text{ét}}/K$.

ii) Let $\mathrm{Ind}_{U/K}^{X/K}\colon \Lambda\text{-}\widetilde{U}_{\text{ét}}/K \longrightarrow \Lambda\text{-}\widetilde{X}_{\text{ét}}/K$ be the functor $\mathcal{F} \longmapsto (f_*\mathcal{F})^\infty$; note that $f_*\mathcal{F}$ carries a K-equivariant structure, but it is not necessarily smooth. We put $\mathrm{Ind}_{U/K}^{X/G} = \mathrm{Ind}_{X/K}^{X/G} \circ \mathrm{Ind}_{U/K}^{X/K}$, which is a functor from $\Lambda\text{-}\widetilde{U}_{\text{ét}}/K$ to $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$.

iii) Let $\text{c-Ind}_{U/K}^{X/K}\colon \Lambda\text{-}\widetilde{U}_{\text{ét}}/K \longrightarrow \Lambda\text{-}\widetilde{X}_{\text{ét}}/K$ be the functor $\mathcal{F} \longmapsto f_!\mathcal{F}$; (*cf.* the lemma above). We put $\text{c-Ind}_{U/K}^{X/G} = \text{c-Ind}_{X/K}^{X/G} \circ \text{c-Ind}_{U/K}^{X/K}$, which is a functor from $\Lambda\text{-}\widetilde{U}_{\text{ét}}/K$ to $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$.

The following proposition is also immediate:

Proposition 3.26 *The functor* $\mathrm{Ind}_{U/K}^{X/G}$ *is the right adjoint of* $\mathrm{Res}_{U/K}^{X/G}$*, and* $\text{c-Ind}_{U/K}^{X/G}$ *is the left adjoint of* $\mathrm{Res}_{U/K}^{X/G}$*. The functors* $\mathrm{Res}_{U/K}^{X/G}$ *and* $\text{c-Ind}_{U/K}^{X/G}$ *are exact, and* $\mathrm{Ind}_{U/K}^{X/G}$ *is left exact. In particular,* $\mathrm{Res}_{U/K}^{X/G}$ *and* $\mathrm{Ind}_{U/K}^{X/G}$ *preserve injective objects.*

Let U be an open subset of X which is stable under a compact open subgroup K of G. If $\{gU\}_{g\in G}$ covers X, we can construct canonical resolutions of a smooth G-equivariant Λ-sheaf by using the functors $\mathrm{Res}^{X/G}_{U/K}$, $\mathrm{Ind}^{X/G}_{U/K}$ and $\text{c-Ind}^{X/G}_{U/K}$ (*cf.* [FGL08, Théorème IV.9.31]).

Proposition 3.27 *Let U be an open subset of X which is stable under a compact open subgroup K of G. Assume that $X=\bigcup_{g\in G} gU$. For $\alpha=(\overline{g}_1,\ldots,\overline{g}_s)\in (G/K)^s$, we put $U_\alpha=g_1U\cap\cdots\cap g_sU$ and $K_\alpha=g_1Kg_1^{-1}\cap\cdots\cap g_sKg_s^{-1}$. Let $\mathcal{F}$ be an object of $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$.*

i) *We have a functorial resolution $C^\bullet(\mathcal{F})\longrightarrow\mathcal{F}$ in $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$ where $C^m(\mathcal{F})$ is given by*
$$C^m(\mathcal{F})=\bigoplus_{\alpha\in G\backslash(G/K)^{-m+1}}\text{c-Ind}^{X/G}_{U_\alpha/K_\alpha}\mathrm{Res}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}.$$
Here we abuse notation: U_α and K_α depends on the choice of a lift of $\alpha\in G\backslash(G/K)^{-m+1}$ to $(G/K)^{-m+1}$, but $\text{c-Ind}^{X/G}_{U_\alpha/K_\alpha}\mathrm{Res}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}$ does not.

ii) *Assume moreover that the covering $\{gU\}_{g\in G/K}$ is locally finite; namely, each point $x\in X$ has an open neighborhood which intersects only finitely many of gU with $g\in G/K$. Then we have a functorial resolution $\mathcal{F}\longrightarrow D^\bullet(\mathcal{F})$ in $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$ where $D^m(\mathcal{F})$ is given by*
$$D^m(\mathcal{F})=\bigoplus_{\alpha\in G\backslash(G/K)^{m+1}}\mathrm{Ind}^{X/G}_{U_\alpha/K_\alpha}\mathrm{Res}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}.$$

Proof. For $\lambda\in(G/K)^s$, we write j_λ for the natural open immersion $U_\lambda\hookrightarrow X$.

i) We have a well-known exact sequence
$$\cdots\longrightarrow\bigoplus_{\lambda\in(G/K)^2} j_{\lambda!}j_\lambda^*\mathcal{F}\longrightarrow\bigoplus_{\lambda\in G/K} j_{\lambda!}j_\lambda^*\mathcal{F}\longrightarrow\mathcal{F}\longrightarrow 0$$
of Λ-sheaves over $X_{\text{ét}}$. It is easy to see that $\bigoplus_{\lambda\in(G/K)^{-m+1}} j_{\lambda!}j_\lambda^*\mathcal{F}$ coincides with the underlying Λ-sheaf of $\bigoplus_{\alpha\in G\backslash(G/K)^{-m+1}}\text{c-Ind}^{X/G}_{U_\alpha/K_\alpha}\mathrm{Res}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}$, and each homomorphism in the complex above is G-equivariant. Hence we have a resolution
$$\bigoplus_{\alpha\in G\backslash(G/K)^{-\bullet+1}}\text{c-Ind}^{X/G}_{U_\alpha/K_\alpha}\mathrm{Res}^{X/G}_{U_\alpha/K_\alpha}\mathcal{F}\longrightarrow\mathcal{F}\longrightarrow 0.$$

ii) Consider the complex
$$0\longrightarrow\mathcal{F}\longrightarrow\prod_{\lambda\in G/K} j_{\lambda*}j_\lambda^*\mathcal{F}\longrightarrow\prod_{\lambda\in(G/K)^2} j_{\lambda*}j_\lambda^*\mathcal{F}\longrightarrow\cdots\qquad(*)$$
of Λ-sheaves over $X_{\text{ét}}$. Each term has a G-equivariant structure and each homomorphism is G-equivariant. For each $g\in G$ and $m\ge 0$, the map
$$\Bigl(\prod_{\lambda\in(G/K)^{m+1}} j_{\lambda*}j_\lambda^*\mathcal{F}\Bigr)\Big|_{gU}\longrightarrow\Bigl(\prod_{\lambda\in(G/K)^{m}} j_{\lambda*}j_\lambda^*\mathcal{F}\Bigr)\Big|_{gU};\ (s_\lambda)_{\lambda\in(G/K)^{m+1}}\longmapsto(s_{(g,\lambda)})_{\lambda\in(G/K)^m},$$

where s_λ is a local section of $(j_{\lambda*}j_\lambda^*\mathcal{F})|_{gU}$, is gKg^{-1}-equivariant and gives a homotopy between id and 0 on the complex $(*)$ restricted on gU. In particular, the smoothification

$$0 \longrightarrow \mathcal{F} \longrightarrow \Big(\prod_{\lambda\in G/K} j_{\lambda*}j_\lambda^*\mathcal{F}\Big)^\infty \longrightarrow \Big(\prod_{\lambda\in (G/K)^2} j_{\lambda*}j_\lambda^*\mathcal{F}\Big)^\infty \longrightarrow \cdots$$

of the complex $(*)$ is exact.

Now, by the assumption on the covering $\{gU\}_{g\in G/K}$, we have

$$\prod_{\lambda\in (G/K)^{m+1}} j_{\lambda*}j_\lambda^*\mathcal{F} = \bigoplus_{\alpha\in G\backslash (G/K)^{m+1}} \prod_{g\in G} g^*j_{\alpha*}j_\alpha^*\mathcal{F}.$$

Therefore the smoothification of the G-equivariant sheaf $\prod_{\lambda\in(G/K)^{m+1}} j_{\lambda*}j_\lambda^*\mathcal{F}$ coincides with $\bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \mathrm{Ind}_{U_\alpha/K_\alpha}^{X/G} \mathrm{Res}_{U_\alpha/K_\alpha}^{X/G} \mathcal{F}$. Hence we have a resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{\alpha\in G\backslash (G/K)^{\bullet+1}} \mathrm{Ind}_{U_\alpha/K_\alpha}^{X/G} \mathrm{Res}_{U_\alpha/K_\alpha}^{X/G} \mathcal{F}.$$

∎

Remark 3.28 In this paper, we only use the part i) of the proposition above. Later we will consider a variant of ii) (see Lemma 4.4).

3.3.2 Acyclicity

In the following we will give several acyclicity results for injective objects in $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$.

Proposition 3.29 *For an injective object $\mathcal{F}$ in $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$ and an object $Y \longrightarrow X$ of $X_{\text{ét}}$ such that Y is quasi-compact and quasi-separated, we have $H^i(Y,\mathcal{F}) = 0$ for $i \geq 1$.*

Proof. Let $X_{\text{cohét}}$ be the full subcategory of $X_{\text{ét}}$ consisting of étale morphisms $Y \longrightarrow X$ where Y is quasi-compact and quasi-separated. It is naturally equipped with a structure of a site, and the natural morphism of sites $X_{\text{ét}} \longrightarrow X_{\text{cohét}}$ induces an isomorphism of toposes $\widetilde{X}_{\text{ét}} \cong \widetilde{X}_{\text{cohét}}$.

Let $\mathcal{F}$ be a G-equivariant sheaf on $X_{\text{ét}}$ such that for every object $Y \longrightarrow X$ in $X_{\text{cohét}}$ the action of $K_Y \subset G$ on $\Gamma(Y,\mathcal{F})$ is smooth. Then $\mathcal{F}$ is a smooth G-equivariant sheaf; note that every object $Y \longrightarrow X$ in $X_{\text{qcét}}$ can be covered by finitely many objects $(Y_\alpha \longrightarrow X)_\alpha$ in $X_{\text{cohét}}$, and then $\Gamma(Y,\mathcal{F}) \longrightarrow \prod_\alpha \Gamma(Y_\alpha,\mathcal{F})$ is injective. Therefore we have an isomorphism of toposes

$$\widetilde{X}_{\text{ét}}/G = \widetilde{X}_{\text{qcét}}/G \cong \widetilde{X}_{\text{cohét}}/G$$

(for the definition of $\widetilde{X}_{\text{cohét}}/G$, see [FGL08, Définition IV.8.2]).

We prove that fiber products exist in the category $X_{\text{cohét}}$. Let $Y \longrightarrow X$ and $Z_i \longrightarrow X$ $(i = 1,2)$ be objects in $X_{\text{cohét}}$, and assume that we are given morphisms

$Z_1 \longrightarrow Y$ and $Z_2 \longrightarrow Y$ over X. Then, these morphisms are quasi-compact quasi-separated, and so is $Z_1 \times_Y Z_2 \longrightarrow Z_2$. (In the case of schemes, see [EGA, IV, §1.1, §1.2]. The arguments there can be applied to adic spaces.) Hence $Z_1 \times_Y Z_2$ is quasi-compact and quasi-separated, as desired.

Now we can apply [FGL08, Théorème IV.8.15] (or [FGL08, Théorème IV.8.17]) to conclude the proposition. ∎

Proposition 3.30 *Let $Y \longrightarrow X$ be an object of $X_{\mathrm{qcét}}$, and U a quasi-compact open subset of Y. For an injective object $\mathcal{F}$ in $\Lambda\text{-}\widetilde{X}_{\mathrm{ét}}/G$, the homomorphism $\Gamma(Y,\mathcal{F}) \longrightarrow \Gamma(U,\mathcal{F})$ is surjective.*

Proof. Let K be a compact open subgroup of K_Y which stabilizes U. We write Λ_Y (resp. Λ_U) for the constant sheaf on Y (resp. U) with values in Λ. They can be regarded as a smooth K-equivariant sheaves by the trivial K-actions. Therefore we can form $\text{c-Ind}_{Y/K}^{X/G}\Lambda_Y$ and $\text{c-Ind}_{U/K}^{X/G}\Lambda_U$. By Proposition 3.26, we have

$$\mathrm{Hom}(\text{c-Ind}_{Y/K}^{X/G}\Lambda_Y,\mathcal{F}) = \Gamma(Y,\mathcal{F})^K, \quad \mathrm{Hom}(\text{c-Ind}_{U/K}^{X/G}\Lambda_U,\mathcal{F}) = \Gamma(U,\mathcal{F})^K.$$

If we denote by j the natural open immersion $U \hookrightarrow Y$, we have an injection $j_!\Lambda_U \longrightarrow \Lambda_Y$. This gives an injection $\text{c-Ind}_{U/K}^{Y/K}\Lambda_U \longrightarrow \Lambda_Y$ in $\Lambda\text{-}\widetilde{Y}_{\mathrm{ét}}/K$, and thus an injection $\text{c-Ind}_{U/K}^{X/G}\Lambda_U \longrightarrow \text{c-Ind}_{Y/K}^{X/G}\Lambda_Y$ in $\Lambda\text{-}\widetilde{X}_{\mathrm{ét}}/G$. Since $\mathcal{F}$ is an injective object of $\Lambda\text{-}\widetilde{X}_{\mathrm{ét}}/G$, the induced homomorphism

$$\mathrm{Hom}(\text{c-Ind}_{Y/K}^{X/G}\Lambda_Y,\mathcal{F}) \longrightarrow \mathrm{Hom}(\text{c-Ind}_{U/K}^{X/G}\Lambda_U,\mathcal{F})$$

is surjective. Therefore the map $\Gamma(Y,\mathcal{F})^K \longrightarrow \Gamma(U,\mathcal{F})^K$ is surjective.

As $\mathcal{F}$ is smooth, we have $\Gamma(U,\mathcal{F}) = \varinjlim_K \Gamma(U,\mathcal{F})^K$. Hence the homomorphism $\Gamma(Y,\mathcal{F}) \longrightarrow \Gamma(U,\mathcal{F})$ is also surjective. ∎

Corollary 3.31 *Let U and V be quasi-compact quasi-separated open subsets of X such that $V \subset U$. For an injective object $\mathcal{F}$ in $\Lambda\text{-}\widetilde{X}_{\mathrm{ét}}/G$, we have $H^i_{U\setminus V}(U,\mathcal{F}) = 0$ for $i \geq 1$.*

Proof. By Proposition 3.29 and Proposition 3.30, we have $H^i_{U\setminus V}(U,\mathcal{F}) = 0$ for $i \geq 2$, and $H^1_{U\setminus V}(U,\mathcal{F}) = \mathrm{Coker}(\Gamma(U,\mathcal{F}) \to \Gamma(V,\mathcal{F})) = 0$. ∎

Proposition 3.32 *Let $Y \longrightarrow X$ be an étale morphism. Assume that Y is quasi-separated, and is the union of countably many quasi-compact open subsets of Y. Then, for an injective object $\mathcal{F}$ in $\Lambda\text{-}\widetilde{X}_{\mathrm{ét}}/G$, we have $H^i(Y,\mathcal{F}) = 0$ for $i \geq 1$.*

Proof. By the assumption on Y, there exists an increasing series $U_1 \subset U_2 \subset \cdots$ of quasi-compact open subsets of Y such that $Y = \bigcup_{n=1}^{\infty} U_n$. For a Λ-sheaf $\mathcal{G}$ on $X_{\mathrm{ét}}$, we will construct the following spectral sequence:

$$E_2^{i,j} = \varprojlim_n{}^i H^j(U_n,\mathcal{G}) \longrightarrow H^{i+j}(Y,\mathcal{G}).$$

To show the existence of this spectral sequence, it suffices to show that for an injective object $\mathcal{I}$ in Λ-$\widetilde{X}_{\text{ét}}$, the projective system $(\Gamma(U_n,\mathcal{I}))_n$ is an injective object in the category of projective systems of Λ-modules. By [Jan88, Proposition 1.1], we should prove the following two properties:

(a) $\Gamma(U_n,\mathcal{I})$ is an injective Λ-module for every $n\geq 1$.

(b) The transition map $\Gamma(U_{n+1},\mathcal{I})\longrightarrow\Gamma(U_n,\mathcal{I})$ is a split surjection for every n.

(a) is easy. For (b), note the following exact sequence:

$$0\longrightarrow\Gamma_{U_{n+1}\setminus U_n}(U_{n+1},\mathcal{I})\longrightarrow\Gamma(U_{n+1},\mathcal{I})\longrightarrow\Gamma(U_n,\mathcal{I})\longrightarrow 0.$$

As $\Gamma_{U_{n+1}\setminus U_n}(U_{n+1},\mathcal{I})$ is an injective Λ-module, the map $\Gamma(U_{n+1},\mathcal{I})\longrightarrow\Gamma(U_n,\mathcal{I})$ is a split surjection, as desired.

As $\varprojlim_n^i=0$ for $i\geq 2$, we obtain the following exact sequence:

$$0\longrightarrow\varprojlim_n{}^1 H^{i-1}(U_n,\mathcal{G})\longrightarrow H^i(Y,\mathcal{G})\longrightarrow\varprojlim_n H^i(U_n,\mathcal{G})\longrightarrow 0.$$

If $i\geq 1$, Proposition 3.29 tells us that $H^i(U_n,\mathcal{F})=0$ for every n. Therefore we have $H^i(Y,\mathcal{F})=0$ for $i\geq 2$, and $H^1(Y,\mathcal{F})=\varprojlim_n^1\Gamma(U_n,\mathcal{F})$. By Proposition 3.30, the map $\Gamma(U_{n+1},\mathcal{F})\longrightarrow\Gamma(U_n,\mathcal{F})$ is a surjection. Hence we have $H^1(Y,\mathcal{F})=\varprojlim_n^1\Gamma(U_n,\mathcal{F})=0$. This completes the proof. ∎

Corollary 3.33 *Assume that X is quasi-separated. Let U be an open subset of X which is the union of countably many quasi-compact open subsets of X. We write j for the natural open immersion $U\hookrightarrow X$. Then, for an injective object $\mathcal{F}$ in Λ-$\widetilde{X}_{\text{ét}}/G$, we have $R^ij_*j^*\mathcal{F}=0$ for $i\geq 1$.*

Proof. Note that $R^ij_*j^*\mathcal{F}$ is the sheafification of the presheaf

$$(Y\longrightarrow X)\longmapsto H^i(Y\times_X U,\mathcal{F})$$

on $X_{\text{ét}}$. Thus it suffices to show that $H^i(Y\times_X U,\mathcal{F})=0$ for $i\geq 1$ and an object $Y\longrightarrow X$ of $X_{\text{cohét}}$ (*cf.* the proof of Proposition 3.29). Take an increasing series $U_1\subset U_2\subset\cdots$ of quasi-compact open subsets of U such that $U=\bigcup_{n=1}^\infty U_n$. Then, as X is quasi-separated, $Y\times_X U_n$ is a quasi-compact open subset of Y. Therefore, $Y\times_X U$ is the union of countably many quasi-compact open subsets $(Y\times_X U_n)_{n\geq 1}$. Hence Proposition 3.32 tells us that $H^i(Y\times_X U,\mathcal{F})=0$ for $i\geq 1$. This concludes the proof. ∎

Recall the result in [Far06, Lemme 2.3]:

Lemma 3.34 *Let U be an open subset of X which is stable under a compact open subgroup K of G. For a smooth G-equivariant Λ-sheaf $\mathcal{F}$, $\Gamma(U,\mathcal{F})$ has a natural structure of a $\mathcal{D}_c(K)$-module.*

Proof. We have $\Gamma(U,\mathcal{F}) = \varprojlim_{V\subset U}\Gamma(V,\mathcal{F})$, where V runs through quasi-compact open subsets of U which are stable under K. As the action of K on $\Gamma(V,\mathcal{F})$ is smooth, $\Gamma(V,\mathcal{F})$ has a structure of a $\mathcal{D}_c(K)$-module, and the transition maps of the projective system $(\Gamma(V,\mathcal{F}))_V$ are compatible with the actions of $\mathcal{D}_c(K)$. Hence $\Gamma(U,\mathcal{F})$ has a structure of a $\mathcal{D}_c(K)$-module. ∎

Definition 3.35 Let U be an open subset of X which is stable under a compact open subgroup K of G. By the lemma above, a left exact functor

$$\Gamma(U/K,-)\colon \Lambda\text{-}\widetilde{X}_{\text{ét}}/G \longrightarrow \mathbf{Mod}\bigl(\mathcal{D}_c(K)\bigr);\quad \mathcal{F}\longmapsto \Gamma(U,\mathcal{F})$$

is induced. We denote by $R\Gamma(U/K,-)$ the right derived functor of $\Gamma(U/K,-)$, and by $H^i(U/K,-)$ the ith cohomology of $R\Gamma(U/K,-)$.

Remark 3.36 As the functors

$$i_{\mathcal{D}}\colon\ \mathbf{Rep}_\Lambda(K)\longrightarrow \mathbf{Mod}\bigl(\mathcal{D}_c(K)\bigr),\quad \infty_{\mathcal{D}}\colon\ \mathbf{Mod}\bigl(\mathcal{D}_c(K)\bigr)\longrightarrow \mathbf{Rep}_\Lambda(K)$$

are exact, they induce functors between derived categories

$$\begin{aligned} i_{\mathcal{D}}\colon D^+\bigl(\mathbf{Rep}_\Lambda(K)\bigr) &\longrightarrow D^+\bigl(\mathbf{Mod}(\mathcal{D}_c(K))\bigr),\\ \infty_{\mathcal{D}}\colon D^+\bigl(\mathbf{Mod}(\mathcal{D}_c(K))\bigr) &\longrightarrow D^+\bigl(\mathbf{Rep}_\Lambda(K)\bigr).\end{aligned}$$

These two functors are adjoint to each other, and satisfy $\infty_{\mathcal{D}}\circ i_{\mathcal{D}}=\mathrm{id}$. Therefore $D^+(\mathbf{Rep}_\Lambda(K))$ can be regarded as a full subcategory of $D^+(\mathbf{Mod}(\mathcal{D}_c(K)))$ by $i_{\mathcal{D}}$.

Under this setting, if U in the previous definition is quasi-compact, the image of $R\Gamma(U/K,-)$ lies in $D^+(\mathbf{Rep}_\Lambda(K))$.

Corollary 3.37 *Let U and K be as in the definition above. Assume that U is quasi-separated, and is the union of countably many quasi-compact open subsets. Then the following diagram is 2-commutative:*

$$\begin{tikzcd} D^+(\Lambda\text{-}\widetilde{X}_{\text{ét}}/G) \arrow[r, "R\Gamma(U/K{,}-)"] \arrow[d] & D^+\bigl(\mathbf{Mod}(\mathcal{D}_c(K))\bigr) \arrow[d] \\ D^+(\Lambda\text{-}\widetilde{X}_{\text{ét}}) \arrow[r, "R\Gamma(U{,}-)"] & D^+\bigl(\mathbf{Mod}(\Lambda)\bigr). \end{tikzcd}$$

Here $\mathbf{Mod}(\Lambda)$ denotes the category of Λ-modules, and the vertical arrows denote the forgetful functors.

Proof. Clear from Proposition 3.32. ∎

Proposition 3.38 *Let U be an open subset of X which is stable under a compact open subgroup K of G. Let $\mathcal{F}$ be an object of $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$ and $i\geq 0$ an integer. Assume the following conditions:*

- *the base field k is separably closed.*
- *the characteristic of k is zero, or X is smooth over* $\mathrm{Spa}(k,k^+)$.
- *U is quasi-separated, and is the union of countably many quasi-compact open subsets of X.*
- *$\mathcal{F}$ is constructible as a Λ-sheaf.*
- *$H^i(U,\mathcal{F})$ is a finitely generated Λ-module.*

Then the action of K on $H^i(U,\mathcal{F})$ is smooth and the $\mathcal{D}_c(K)$-module structure on $H^i(U/K,\mathcal{F})$ can be identified with that on $i_{\mathcal{D}}H^i(U,\mathcal{F})$. In particular, we have $i_{\mathcal{D}}\infty_{\mathcal{D}}H^i(U/K,\mathcal{F}) = H^i(U/K,\mathcal{F})$.

Proof. We can take an increasing sequence $U_1 \subset U_2 \subset \cdots$ of quasi-compact open subsets of X which are stable under K such that $U = \bigcup_{n=1}^{\infty} U_n$. As in the proof of Proposition 3.32, we have the following exact sequence:

$$0 \longrightarrow \varprojlim_n{}^1 H^{i-1}(U_n,\mathcal{F}) \longrightarrow H^i(U,\mathcal{F}) \longrightarrow \varprojlim_n H^i(U_n,\mathcal{F}) \longrightarrow 0.$$

As $\mathcal{F}$ is constructible, $H^{i-1}(U_n,\mathcal{F})$ is a finitely generated Λ-module for every n by [Hub98a, Proposition 3.1] (in the case where the characteristic of k is 0) or [Hub96, Proposition 6.1.1, (1.7.7)] (in the case where X is smooth over $\mathrm{Spa}(k,k^+)$). Therefore, as Λ is Artinian, the projective system $(H^{i-1}(U_n,\mathcal{F}))_n$ satisfies the Mittag-Leffler condition, and thus $\varprojlim^1_n H^{i-1}(U_n,\mathcal{F}) = 0$. Hence we have an isomorphism $H^i(U,\mathcal{F}) \xrightarrow{\cong} \varprojlim_n H^i(U_n,\mathcal{F})$.

On the other hand, by the assumption, $H^i(U,\mathcal{F})$ is an Artinian Λ-module. Therefore the decreasing series of Λ-submodules $(\mathrm{Ker}(H^i(U,\mathcal{F}) \to H^i(U_n,\mathcal{F})))_n$ is stationary. Hence, for a large enough n, the map $H^i(U,\mathcal{F}) \longrightarrow H^i(U_n,\mathcal{F})$ is injective. By Corollary 3.37 and Remark 3.36, the action of K on $H^i(U_n,\mathcal{F}) = H^i(U_n/K,\mathcal{F})$ is smooth. Thus, the action of K on $H^i(U,\mathcal{F})$ is also smooth.

Consider a $\mathcal{D}_c(K)$-homomorphism $H^i(U/K,\mathcal{F}) \longrightarrow H^i(U_n/K,\mathcal{F})$, which fits into the following diagram by Corollary 3.37:

$$\begin{tikzcd} H^i(U/K,\mathcal{F}) \arrow[r] \arrow[d, equal] & H^i(U_n/K,\mathcal{F}) \arrow[d, equal] \\ H^i(U,\mathcal{F}) \arrow[r] & H^i(U_n,\mathcal{F}). \end{tikzcd}$$

This map is injective if n is large enough. Hence $H^i(U/K,\mathcal{F})$ satisfies

$$\infty_{\mathcal{D}}H^i(U/K,\mathcal{F}) = H^i(U,\mathcal{F}), \quad i_{\mathcal{D}}\infty_{\mathcal{D}}H^i(U/K,\mathcal{F}) = H^i(U/K,\mathcal{F}).$$

This concludes the proof. ∎

Proposition 3.39 *Let U be an open subspace of X which is partially proper over* $\mathrm{Spa}(k,k^+)$. *For an injective object $\mathcal{F}$ in Λ-$\widetilde{X}_{\text{ét}}/G$, we have $H^i_c(U,\mathcal{F}) = 0$ for $i \geq 1$.*

Proof. By the same way as in the proof of Proposition 3.5 i), we can prove that $H^i_c(U,\mathcal{F})\cong\varinjlim_{Z\subset U}H^i_Z(U,\mathcal{F})$, where Z runs through quasi-compact closed subsets of U. Let $\mathcal{C}$ be the set consisting of closed subsets of U of the form $V\setminus W$ where V and W are quasi-compact open subsets of U with $W\subset V$. We prove that this set is cofinal in the set of all quasi-compact closed subsets of U. Let Z be an arbitrary quasi-compact closed subset of U. We can take a quasi-compact open subset V of U containing Z. As U is partially proper over $\mathrm{Spa}(k,k^+)$, the closure $\overline{V}$ of V in U is quasi-compact (*cf.* [Hub96, Lemma 1.3.13]). Since Z is closed, for each point $x\in\overline{V}\setminus V$, there exists a quasi-compact open neighborhood W_x of x in U such that $W_x\cap Z=\varnothing$. As $\overline{V}\setminus V$ is quasi-compact, we can find finitely many $x_1,\ldots,x_n\in\overline{V}\setminus V$ so that $W_{x_1},\ldots,W_{x_n}$ cover $\overline{V}\setminus V$. Put $W=V\cap(W_{x_1}\cup\cdots\cup W_{x_n})$, which is a quasi-compact open subset of V. Since $V\setminus W=\overline{V}\setminus(W_{x_1}\cup\cdots\cup W_{x_n})$ is closed in U, it gives an element of $\mathcal{C}$ containing Z.

Therefore we have $H^i_c(U,\mathcal{F})\cong\varinjlim_{Z\in\mathcal{C}}H^i_Z(U,\mathcal{F})$. On the other hand, Proposition 3.31 tells us that $H^i_Z(U,\mathcal{F})=0$ for $Z\in\mathcal{C}$ and $i\geq 1$. This concludes the proof. ∎

Lemma 3.40 *Assume that X is partially proper over* $\mathrm{Spa}(k,k^+)$. *For an object $\mathcal{F}$ in Λ-$\widetilde{X}_{\text{ét}}/G$, the induced G-action on $\Gamma_c(X,\mathcal{F})$ is smooth.*

Proof. We have $\Gamma_c(X,\mathcal{F})=\varinjlim_{V\subset X}\Gamma_{\overline{V}}(X,\mathcal{F})$, where V runs through quasi-compact open subsets of X and $\overline{V}$ denotes the closure of V (note that $\overline{V}$ is quasi-compact). Therefore it suffices to show that the action of K_V on $\Gamma_{\overline{V}}(X,\mathcal{F})$ is smooth. As $\overline{V}$ is quasi-compact, we can take a quasi-compact open subset U of X containing $\overline{V}$. Put $K=K_U\cap K_V$. Then the homomorphisms $\Gamma_{\overline{V}}(X,\mathcal{F})\xrightarrow{\cong}\Gamma_{\overline{V}}(U,\mathcal{F})\hookrightarrow\Gamma(U,\mathcal{F})$ are K-equivariant. Since the action of K on $\Gamma(U,\mathcal{F})$ is smooth, so is the action on $\Gamma_{\overline{V}}(X,\mathcal{F})$. ∎

Definition 3.41 Assume that X is partially proper over $\mathrm{Spa}(k,k^+)$. By the lemma above, a left exact functor

$$\Gamma_c(X/G,-)\colon\Lambda\text{-}\widetilde{X}_{\text{ét}}/G\longrightarrow\mathbf{Rep}_\Lambda(G);\quad \mathcal{F}\longmapsto\Gamma_c(X,\mathcal{F})$$

is induced. We denote by $R\Gamma_c(X/G,-)$ the right derived functor of $\Gamma_c(X/G,-)$, and by $H^i_c(X/G,-)$ the ith cohomology of $R\Gamma_c(X/G,-)$.

Corollary 3.42 *Assume that X is partially proper over* $\mathrm{Spa}(k,k^+)$. *The following diagram is 2-commutative:*

$$\begin{tikzcd}
D^+(\Lambda\text{-}\widetilde{X}_{\text{ét}}/G) \arrow[r, "R\Gamma_c(X/G{,}-)"] \arrow[d] & D^+\big(\mathbf{Rep}_\Lambda(G)\big) \arrow[d] \\
D^+(\Lambda\text{-}\widetilde{X}_{\text{ét}}) \arrow[r, "R\Gamma_c(X{,}-)"] & D^+\big(\mathbf{Mod}(\Lambda)\big).
\end{tikzcd}$$

Here the vertical arrows denote the forgetful functors.

Proof. It follows immediately from Proposition 3.39. ∎

3.3.3 The Godement resolution

Here we introduce the Godement resolution for a smooth G-equivariant sheaf.

Definition 3.43 For each $x \in X$, fix a geometric point $i_{\overline{x}}\colon \overline{x} \longrightarrow X$ lying over x. For an arbitrary Λ-sheaf $\mathcal{F}$ on $X_{\text{ét}}$, consider the smooth G-equivariant Λ-sheaf

$$\mathcal{C}(\mathcal{F}) = \operatorname{Ind}_{X/\{1\}}^{X/G}\Bigl(\prod_{x\in X} i_{\overline{x}*}\mathcal{F}_{\overline{x}}\Bigr).$$

If $\mathcal{F}$ is an object of $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$, the canonical morphism $\mathcal{F} \longrightarrow \prod_{x\in X} i_{\overline{x}*}\mathcal{F}_{\overline{x}}$ in $\Lambda\text{-}\widetilde{X}_{\text{ét}}$ induces a morphism $\mathcal{F} \longrightarrow \mathcal{C}(\mathcal{F})$ in $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$. It is an injection; indeed, $\mathcal{C}(\mathcal{F})$ is a subsheaf of $\prod_{g\in G}\prod_{x\in X} g^*i_{\overline{x}*}\mathcal{F}_{\overline{x}}$, and the natural morphism $\mathcal{F} \longrightarrow \prod_{g\in G}\prod_{x\in X} g^*i_{\overline{x}*}\mathcal{F}_{\overline{x}}$ is obviously injective.

By repeating this construction, we have the following functorial resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{F}) \longrightarrow \cdots,$$

which is called the Godement resolution of $\mathcal{F}$.

Proposition 3.44 *Let $\mathcal{F}$ be an object of $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$ which is flat as a Λ-sheaf. For every $i \ge 0$, $\mathcal{C}^i(\mathcal{F})$ is flat and injective in the category $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$. Moreover, for the maximal ideal $\mathfrak{m}$ of Λ, we have $\mathcal{C}^i(\mathcal{F}) \otimes_\Lambda \Lambda/\mathfrak{m} \cong \mathcal{C}^i(\mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m})$.*

Proof. By Lemma 3.18, $\mathcal{F}_{\overline{x}}$ is an injective Λ-module. Therefore $\prod_{x\in X} i_{\overline{x}*}\mathcal{F}_{\overline{x}}$ is an injective Λ-sheaf. As $\operatorname{Ind}_{X/\{1\}}^{X/G}$ preserves injective objects, $\mathcal{C}(\mathcal{F})$ is an injective object in the category $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$.

Next we prove that $\mathcal{C}(\mathcal{F})$ is flat. For an object $Y \longrightarrow X$ in $\widetilde{X}_{\text{qcét}}$, we have

$$\Gamma\bigl(Y, \mathcal{C}(\mathcal{F})\bigr) = \varinjlim_{K\subset K_Y}\Bigl(\prod_{g\in G}\prod_{x\in X}\Gamma(Y, g^*i_{\overline{x}*}\mathcal{F}_{\overline{x}})\Bigr)^K = \varinjlim_{K\subset K_Y}\Bigl(\prod_{g\in \Omega_K}\prod_{x\in X}\Gamma(Y, g^*i_{\overline{x}*}\mathcal{F}_{\overline{x}})\Bigr),$$

where K runs through compact open subgroups of K_Y and Ω_K is a system of representatives of G/K. As $\Gamma(Y, g^*i_{\overline{x}*}\mathcal{F}_{\overline{x}})$ is a finite direct sum of $\mathcal{F}_{\overline{x}}$, it is a flat Λ-module. Since flatness of Λ-modules is preserved by arbitrary direct products and filtered inductive limits (*cf.* Lemma 3.18), we can conclude that $\Gamma(Y, \mathcal{C}(\mathcal{F}))$ is flat. Therefore each stalk of $\mathcal{C}(\mathcal{F})$ is also flat.

By the description of $\Gamma(Y, \mathcal{C}(\mathcal{F}))$ above, the functor $\mathcal{F} \longmapsto \mathcal{C}(\mathcal{F})$ is exact. If we take a generator λ of $\mathfrak{m}$, we have an exact sequence $\mathcal{F} \xrightarrow{\times\lambda} \mathcal{F} \longrightarrow \mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m} \longrightarrow 0$. Therefore the sequence $\mathcal{C}(\mathcal{F}) \xrightarrow{\times\lambda} \mathcal{C}(\mathcal{F}) \longrightarrow \mathcal{C}(\mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m}) \longrightarrow 0$ is exact, and we obtain $\mathcal{C}(\mathcal{F}) \otimes_\Lambda \Lambda/\mathfrak{m} \cong \mathcal{C}(\mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m})$.

Put $\mathcal{G} = \operatorname{Coker}(\mathcal{F} \to \mathcal{C}(\mathcal{F}))$. For each $x \in X$, we have an exact sequence $0 \longrightarrow \mathcal{F}_{\overline{x}} \longrightarrow \mathcal{C}(\mathcal{F})_{\overline{x}} \longrightarrow \mathcal{G}_{\overline{x}} \longrightarrow 0$. As $\mathcal{F}_{\overline{x}}$ and $\mathcal{C}(\mathcal{F})_{\overline{x}}$ are flat (= injective), so is $\mathcal{G}_{\overline{x}}$. Thus $\mathcal{G}$ is flat. Moreover, we have $\mathcal{G} \otimes_\Lambda \Lambda/\mathfrak{m} = \operatorname{Coker}(\mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m} \to \mathcal{C}(\mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m}))$. Now we can repeat the same argument. ∎

The following proposition is also needed in the next section (*cf.* [Far06, Lemme 2.7]).

Proposition 3.45 *Assume that k is separably closed and X is d-dimensional. Let $\mathcal{F}$ be an object in Λ-$\widetilde{X}_{\text{ét}}/G$ which is flat as a Λ-sheaf, and $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^\bullet(\mathcal{F})$ the Godement resolution of $\mathcal{F}$. Put $\mathcal{G} = \operatorname{Ker}(\mathcal{C}^{2d}(\mathcal{F}) \to \mathcal{C}^{2d+1}(\mathcal{F}))$.*

Then, for every open subspace U of X which is partially proper over $\operatorname{Spa}(k, k^+)$, $\Gamma_c(U, \mathcal{C}^m(\mathcal{F}))$ and $\Gamma_c(U, \mathcal{G})$ are free Λ-modules and $H^i_c(U, \mathcal{G}) = 0$ for $i \geq 1$.

To prove this proposition, we use the following lemma.

Lemma 3.46 *Assume that k is separably closed and X is finite-dimensional. Let U be an open subspace of X which is partially proper over $\operatorname{Spa}(k, k^+)$. Let $\mathcal{F}$ be a flat Λ-sheaf on $X_{\text{ét}}$ satisfying $H^i_c(U, \mathcal{F}) = 0$ for every $i \geq 1$. Then the following are equivalent:*

i) $\Gamma_c(U, \mathcal{F})$ *is a free Λ-module.*

ii) $H^1_c(U, \mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m}) = 0$, *where $\mathfrak{m}$ is the maximal ideal of Λ.*

iii) $H^i_c(U, \mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m}) = 0$ *for every $i \geq 1$.*

Proof. As $R\Gamma_c(U, -)$ is bounded, we have

$$R\Gamma_c(U, \mathcal{F}) \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/\mathfrak{m} \cong R\Gamma_c(U, \mathcal{F} \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/\mathfrak{m}).$$

By the conditions on $\mathcal{F}$, the left hand side is equal to $\Gamma_c(U, \mathcal{F}) \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda/\mathfrak{m}$, and the right hand side is equal to $R\Gamma_c(U, \mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m})$. In particular we have

$$\operatorname{Tor}^\Lambda_i\bigl(\Lambda/\mathfrak{m}, \Gamma_c(U, \mathcal{F})\bigr) \cong H^i_c(U, \mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m})$$

for every i. Therefore we have i) $\Longrightarrow$ iii) $\Longrightarrow$ ii) $\Longrightarrow$ i), as desired. ∎

Proof of Proposition 3.45. First we prove that $\Gamma_c(U, \mathcal{C}^m(\mathcal{F}))$ is free. By Proposition 3.44, $\mathcal{C}^m(\mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m})$ is an injective object of $(\Lambda/\mathfrak{m})$-$\widetilde{X}_{\text{ét}}/G$. Therefore, by Proposition 3.39 and Proposition 3.44 we have

$$H^i_c\bigl(U, \mathcal{C}^m(\mathcal{F}) \otimes_\Lambda \Lambda/\mathfrak{m}\bigr) \cong H^i_c\bigl(U, \mathcal{C}^m(\mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m})\bigr) = 0$$

for $i \geq 1$. Hence Lemma 3.46 tells us that $\Gamma_c(U, \mathcal{C}^m(\mathcal{F}))$ is free.

Next we show that $\Gamma_c(U, \mathcal{G})$ is free and $H^i_c(U, \mathcal{G}) = 0$ for $i \geq 1$. For simplicity we put $\mathcal{J}^m = \mathcal{C}^m(\mathcal{F})$ for $0 \leq m \leq 2d-1$. We have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}^0 \longrightarrow \cdots \longrightarrow \mathcal{J}^{2d-1} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Thus, by Proposition 3.44 and Proposition 3.39 we have $H^i_c(U, \mathcal{G}) \cong H^{i+2d}_c(U, \mathcal{F}) = 0$ for every $i \geq 1$ (*cf.* [Hub96, Proposition 5.3.11]). On the other hand, by the proof of Proposition 3.44, we know that $\mathcal{G}$ is a flat Λ-sheaf. Therefore we have an exact sequence

$$0 \to \mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m} \to \mathcal{J}^0 \otimes_\Lambda \Lambda/\mathfrak{m} \to \cdots \to \mathcal{J}^{2d-1} \otimes_\Lambda \Lambda/\mathfrak{m} \to \mathcal{G} \otimes_\Lambda \Lambda/\mathfrak{m} \to 0.$$

In the beginning of the proof, we have proved that $H^i_c(U, \mathcal{J}^m \otimes_\Lambda \Lambda/\mathfrak{m}) = 0$ for $i \geq 1$ and $0 \leq m \leq 2d-1$. This implies that $H^1_c(U, \mathcal{G} \otimes_\Lambda \Lambda/\mathfrak{m}) \cong H^{1+2d}_c(U, \mathcal{F} \otimes_\Lambda \Lambda/\mathfrak{m}) = 0$. Thus Lemma 3.46 tells us that $\Gamma_c(U, \mathcal{G})$ is a free Λ-module. ∎

4 Duality theorem

Let R, κ, F and k be as in Section 3.2. We denote the residue characteristic of R by p. Fix a truncated discrete valuation ring Λ with residue characteristic $\neq p$ and a locally pro-p group G.

Theorem 4.1 *Let $\mathcal{X}$ be a special formal scheme over* $\operatorname{Spf} R$ *equipped with a continuous action of G in the sense of [Far04, Définition 2.3.10]. Assume the following:*

(a) *The rigid geometric generic fiber $X = t(\mathcal{X})_{\overline{\eta}}$ of $\mathcal{X}$ is purely d-dimensional and smooth over* $\operatorname{Spa}(k, k^+)$.

(b) *$\mathcal{X}$ is locally algebraizable (cf. [Mie10, Definition 3.19]).*

(c) *$\mathcal{X}^{\mathrm{red}}$ is partially proper over κ.*

(d) *There exists a quasi-compact open subset V of $\mathcal{X}^{\mathrm{red}}$ such that $\mathcal{X}^{\mathrm{red}} = \bigcup_{g\in G} gV$ and $\{g \in G \mid g\overline{V} \cap \overline{V} \neq \varnothing\}$ is compact.*

Then, for each integer i, we have a G-equivariant isomorphism

$$H^{2d+i}_{c,\mathcal{X}}(X, \Lambda)(d) \xrightarrow{\cong} R^i\mathrm{D}\bigl(R\Gamma_c(X/G, \Lambda)\bigr).$$

Note that the condition (c) ensures that X is partially proper over $\operatorname{Spa}(k, k^+)$ *(cf. Proposition 3.9), and thus Corollary 3.42 and [Hub96, Proposition 5.3.11] imply that $R\Gamma_c(X/G, \Lambda)$ lies in the category $D^b(\mathbf{Rep}_\Lambda(G))$.*

Fix V satisfying the condition (d) in the theorem above and put $Z = \overline{V}$, $U = \mathrm{sp}^{-1}(Z)^\circ$. Take a pro-$p$ open subgroup K which stabilizes V. Then Z and U are also stable under K. The condition (d) tells us that $\{gZ\}_{g\in G/K}$ is a locally finite covering of $\mathcal{X}^{\mathrm{red}}$. Note also that $\{gU\}_{g\in G/K}$ is a locally finite covering of X. By Proposition 3.10 iii), U is partially proper over $\operatorname{Spa}(k, k^+)$.

Let $0 \longrightarrow \Lambda \longrightarrow \mathcal{I}^\bullet$ be the Godement resolution introduced in Definition 3.43 for the smooth G-equivariant constant sheaf Λ on $X_{\text{ét}}$. By Proposition 3.44, this is an injective resolution in the category $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$. Put $\mathcal{J}^\bullet = \tau_{\le 2d}\mathcal{I}^\bullet$. For $\alpha = (\overline{g}_1, \ldots, \overline{g}_m) \in (G/K)^m$, let U_α and K_α be as in Proposition 3.27. We write j_α for the natural open immersion $U_\alpha \hookrightarrow X$.

The following lemma gives a complex which computes $R\mathrm{D}(R\Gamma_c(X/G, \Lambda))$.

Lemma 4.2 *Let $C^{\bullet\bullet}$ denote the the double complex*

$$C^{\bullet\bullet} = \Gamma_c\bigl(X/G, C^\bullet(\mathcal{J}^\bullet)\bigr) = \bigoplus_{\alpha\in G\backslash(G/K)^{-\bullet+1}} \text{c-Ind}^G_{K_\alpha} \Gamma_c(U_\alpha, \mathcal{J}^\bullet)$$

in $\mathbf{Rep}_\Lambda(G)$ (cf. Proposition 3.27 i)). Then we have an isomorphism

$$R\Gamma_c(X/G, \Lambda) \cong \operatorname{Tot} C^{\bullet\bullet}$$

in $D^-(\mathbf{Rep}_\Lambda(G))$. Moreover, we have an isomorphism

$$R\mathrm{D}\bigl(R\Gamma_c(X/G, \Lambda)\bigr) \cong \mathrm{D}(\operatorname{Tot} C^{\bullet\bullet})$$

in $D^+(\mathbf{Rep}_\Lambda(G))$.

Proof. By Proposition 3.27 i), $\mathrm{Tot}\, C^\bullet(\mathcal{J}^\bullet)$ gives a resolution of Λ in $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$. By Corollary 3.42 and Proposition 3.45, for integers $i \geq 1$ and $m, n \geq 0$ we have

$$\begin{aligned} H^i_c\bigl(X/G, C^m(\mathcal{J}^n)\bigr) &= \bigoplus_{\alpha \in G\backslash (G/K)^{-m+1}} \text{c-Ind}^G_{K_\alpha} H^i_c(X, j_{\alpha !} j^*_\alpha \mathcal{J}^n) \\ &= \bigoplus_{\alpha \in G\backslash (G/K)^{-m+1}} \text{c-Ind}^G_{K_\alpha} H^i_c(U_\alpha, \mathcal{J}^n) \\ &= 0. \end{aligned}$$

Therefore, each component of the complex $\mathrm{Tot}\, C^\bullet(\mathcal{J}^\bullet)$ is acyclic with respect to $\Gamma_c(X/G, -)$. Hence we have

$$R\Gamma_c(X/G, \Lambda) \cong \Gamma_c\bigl(X/G, \mathrm{Tot}\, C^\bullet(\mathcal{J}^\bullet)\bigr) = \mathrm{Tot}\, \Gamma_c\bigl(X/G, C^\bullet(\mathcal{J}^\bullet)\bigr) = \mathrm{Tot}\, C^{\bullet\bullet}$$

in $D^-(\mathbf{Rep}_\Lambda(G))$.

Furthermore, Proposition 3.45 tells us that $\Gamma_c(U_\alpha, \mathcal{J}^n)$ is a projective object in $\mathbf{Rep}_\Lambda(K_\alpha)$. Therefore $C^{mn} = \bigoplus_{\alpha \in G\backslash (G/K)^{-m+1}} \text{c-Ind}^G_{K_\alpha} \Gamma_c(U_\alpha, \mathcal{J}^n)$ is a projective object in $\mathbf{Rep}_\Lambda(G)$. Hence we have $R\mathrm{D}(R\Gamma_c(X/G, \Lambda)) \cong \mathrm{D}(\mathrm{Tot}\, C^{\bullet\bullet})$. ∎

Lemma 4.3 *For an injective object $\mathcal{F}$ of $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$, we have*

$$R\Gamma_{c,\mathcal{X}}(X, j_{\alpha *} j^*_\alpha \mathcal{F}) = \Gamma(U_\alpha, \mathcal{F}).$$

In particular, $j_{\alpha *} j^*_\alpha \mathcal{F}$ is acyclic with respect to $\Gamma_{c,\mathcal{X}}(X, -)$.

Proof. For $\alpha = (\overline{g}_1, \ldots, \overline{g}_m) \in (G/K)^m$, we set $Z_\alpha = g_1 Z \cap \cdots \cap g_m Z$. Then we have $U_\alpha = \mathrm{sp}^{-1}(Z_\alpha)^\circ$. Therefore U_α is the union of countably many quasi-compact open subsets by Proposition 3.10 ii).

Note that $\Gamma_{c,\mathcal{X}}(X, j_{\alpha *}(-)) = \Gamma_c(\mathcal{X}^{\mathrm{red}}, -) \circ \mathrm{sp}_* \circ j_{\alpha *} = \Gamma(\mathcal{X}^{\mathrm{red}}, -) \circ \mathrm{sp}_* \circ j_{\alpha *}$; indeed, for a sheaf $\mathcal{G}$ on $(U_\alpha)_{\text{ét}}$, all elements of $\Gamma(\mathcal{X}^{\mathrm{red}}, \mathrm{sp}_*\, j_{\alpha *}\mathcal{G})$ are supported on the quasi-compact closed subset $Z_\alpha \subset \mathcal{X}^{\mathrm{red}}$. On the other hand, by Proposition 3.32 and Corollary 3.33, we have $R\Gamma(U_\alpha, \mathcal{F}) = \Gamma(U_\alpha, \mathcal{F})$ and $Rj_{\alpha *} j^*_\alpha \mathcal{F} = j_{\alpha *} j^*_\alpha \mathcal{F}$, respectively. Hence we have

$$\begin{aligned} R\Gamma_{c,\mathcal{X}}(X, j_{\alpha *} j^*_\alpha \mathcal{F}) &= R\Gamma_c(\mathcal{X}^{\mathrm{red}}, R\,\mathrm{sp}_*\, j_{\alpha *} j^*_\alpha \mathcal{F}) = R\Gamma_c(\mathcal{X}^{\mathrm{red}}, R\,\mathrm{sp}_*\, Rj_{\alpha *} j^*_\alpha \mathcal{F}) \\ &= R\Gamma(\mathcal{X}^{\mathrm{red}}, R\,\mathrm{sp}_*\, Rj_{\alpha *} j^*_\alpha \mathcal{F}) = R\Gamma(U_\alpha, \mathcal{F}) = \Gamma(U_\alpha, \mathcal{F}), \end{aligned}$$

as desired. ∎

Lemma 4.4 *For an object $\mathcal{F}$ of $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$, we have a functorial resolution $\mathcal{F} \longrightarrow D^\bullet(\mathcal{F})$ in $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G^{\mathrm{disc}}$ where $D^m(\mathcal{F})$ is given by*

$$D^m(\mathcal{F}) = \prod_{\alpha \in G\backslash (G/K)^{m+1}} \prod_{g \in K_\alpha \backslash G} g^* j_{\alpha *} j^*_\alpha \mathcal{F}.$$

Recall that G^{disc} denotes the group G with discrete topology.

Moreover, $\Gamma_{c,\mathcal{X}}(X, D^m(\mathcal{F}))$ is naturally equipped with a structure of a $\mathcal{D}_c(G)$-module under which

$$\Gamma_{c,\mathcal{X}}\bigl(X, D^m(\mathcal{F})\bigr) = \bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} \Gamma(U_\alpha, \mathcal{F})$$

and $\Gamma_{c,\mathcal{X}}(X, D^\bullet(\mathcal{F}))$ is a complex in $\mathbf{Mod}(\mathcal{D}_c(G))$.

Proof. Consider the complex

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_{\lambda\in G/K} j_{\lambda*}j_\lambda^*\mathcal{F} \longrightarrow \prod_{\lambda\in (G/K)^2} j_{\lambda*}j_\lambda^*\mathcal{F} \longrightarrow \cdots \qquad (*)$$

of Λ-sheaves over $X_{\text{ét}}$. It is well-known that this complex is exact (*cf.* the proof of Proposition 3.27 ii)). Each term has a G-equivariant structure and each homomorphism is G-equivariant. Clearly we have

$$\prod_{\lambda\in (G/K)^{m+1}} j_{\lambda*}j_\lambda^*\mathcal{F} = \prod_{\alpha\in G\backslash(G/K)^{m+1}} \prod_{g\in K_\alpha\backslash G} g^* j_{\alpha*}j_\alpha^*\mathcal{F}.$$

Hence we have a desired resolution.

As $\{gZ\}_{g\in G/K}$ is a locally finite covering of $\mathcal{X}_{\mathrm{red}}$, we have

$$\begin{aligned}\mathrm{sp}_*\Bigl(\prod_{\alpha\in G\backslash(G/K)^{m+1}} \prod_{g\in K_\alpha\backslash G} g^* j_{\alpha*}j_\alpha^*\mathcal{F}\Bigr) &= \prod_{\alpha\in G\backslash(G/K)^{m+1}} \prod_{g\in K_\alpha\backslash G} \mathrm{sp}_*\, g^* j_{\alpha*}j_\alpha^*\mathcal{F}\\ &= \bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \bigoplus_{g\in K_\alpha\backslash G} \mathrm{sp}_*\, g^* j_{\alpha*}j_\alpha^*\mathcal{F}.\end{aligned}$$

Therefore, by Corollary 3.7 and Lemma 4.3, we obtain

$$\begin{aligned}\Gamma_{c,\mathcal{X}}\bigl(X, D^m(\mathcal{F})\bigr) &= \Gamma_c\bigl(\mathcal{X}_{\mathrm{red}}, \mathrm{sp}_*\, D^m(\mathcal{F})\bigr)\\ &= \bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \bigoplus_{g\in K_\alpha\backslash G} \Gamma_c(\mathcal{X}_{\mathrm{red}}, \mathrm{sp}_*\, g^* j_{\alpha*}j_\alpha^*\mathcal{F})\\ &= \bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \bigoplus_{g\in K_\alpha\backslash G} \Gamma(g^{-1}U_\alpha, \mathcal{F}).\end{aligned}$$

By Lemma 3.34, $\Gamma(U_\alpha, \mathcal{F})$ has a natural structure of a $\mathcal{D}_c(K_\alpha)$-module and we have

$$\Gamma_{c,\mathcal{X}}\bigl(X, D^m(\mathcal{F})\bigr) = \bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} \Gamma(U_\alpha, \mathcal{F})$$

as G-modules. Hence $\Gamma_{c,\mathcal{X}}(X, D^m(\mathcal{F}))$ can be regarded as a $\mathcal{D}_c(G)$-module.

It is easy to see that the homomorphism

$$\Gamma_{c,\mathcal{X}}\bigl(X, D^m(\mathcal{F})\bigr) \longrightarrow \Gamma_{c,\mathcal{X}}(X, D^{m+1}(\mathcal{F}))$$

is $\mathcal{D}_c(G)$-linear. Indeed, it follows from the following simple fact:

for $\alpha = (\overline{g}_1, \dots, \overline{g}_{m+2}) \in (G/K)^{m+2}$ and $\beta \in (G/K)^{m+1}$ which can be obtained by removing one of the entries $\overline{g}_1, \dots, \overline{g}_{m+2}$ from α, the restriction map $\Gamma(U_\beta, \mathcal{F}) \longrightarrow \Gamma(U_\alpha, \mathcal{F})$ is $\mathcal{D}_c(K_\alpha)$-linear.

This completes the proof. ∎

By these lemmas, we can give a complex which represents $R\Gamma_{c,\mathcal{X}}(X, \Lambda)$.

Corollary 4.5 *We write $D_0^{\bullet\bullet}$ for the double complex*

$$\Gamma_{c,\mathcal{X}}\bigl(X, D^\bullet(\mathcal{I}^\bullet)\bigr) = \bigoplus_{\alpha \in G\backslash (G/K)^{\bullet+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} \Gamma(U_\alpha, \mathcal{I}^\bullet)$$

in $\mathbf{Mod}(\mathcal{D}_c(G))$. Then we have a G-equivariant isomorphism

$$R\Gamma_{c,\mathcal{X}}(X, \Lambda) \cong \operatorname{Tot} D_0^{\bullet\bullet}$$

in $D^+(\mathbf{Mod}(\Lambda))$.

Proof. By Lemma 4.4, $\operatorname{Tot} D^\bullet(\mathcal{I}^\bullet)$ gives a resolution of Λ in $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G^{\text{disc}}$. In the same way as in the proof of Lemma 4.4, we can deduce from Lemma 4.3 that

$$R\Gamma_{c,\mathcal{X}}\bigl(X, D^m(\mathcal{I}^n)\bigr) = \bigoplus_{\alpha \in G\backslash (G/K)^{m+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} \Gamma(U_\alpha, \mathcal{I}^n)$$

(see also Corollary 3.7). Therefore, each component of $\operatorname{Tot} D^\bullet(\mathcal{I}^\bullet)$ is acyclic with respect to $\Gamma_{c,\mathcal{X}}(X, -)$. Hence we have

$$R\Gamma_{c,\mathcal{X}}(X, \Lambda) \cong \Gamma_{c,\mathcal{X}}\bigl(X, \operatorname{Tot} D^\bullet(\mathcal{I}^\bullet)\bigr) = \operatorname{Tot} D_0^{\bullet\bullet}$$

in $D^+(\mathbf{Mod}(\Lambda))$. ∎

Lemma 4.6 *We put $D^{\bullet\bullet} = \infty_{\mathcal{D}} D_0^{\bullet\bullet}$, which is a double complex in $\mathbf{Rep}_\Lambda(G)$. Then we have a G-equivariant isomorphism*

$$R\Gamma_{c,\mathcal{X}}(X, \Lambda) \cong \operatorname{Tot} D^{\bullet\bullet}$$

in $D^+(\mathbf{Mod}(\Lambda))$.

Proof. We need to prove that the natural homomorphism of complexes $\operatorname{Tot} D^{\bullet\bullet} \longrightarrow \operatorname{Tot} D_0^{\bullet\bullet}$ is a quasi-isomorphism. Consider the following morphism of spectral sequences:

$$\begin{array}{ccc} E_1^{m,n} = H^n(D^{m\bullet}) & \Longrightarrow & H^{m+n}(\operatorname{Tot} D^{\bullet\bullet}) \\ \downarrow & & \downarrow \\ E_1^{m,n} = H^n(D_0^{m\bullet}) & \Longrightarrow & H^{m+n}(\operatorname{Tot} D_0^{\bullet\bullet}). \end{array}$$

It suffices to show that $H^n(D^{m\bullet}) \longrightarrow H^n(D_0^{m\bullet})$ is an isomorphism. By definition, the ith cohomology of $\Gamma(U_\alpha, \mathcal{I}^\bullet)$ is $H^i(U_\alpha/K_\alpha, \Lambda)$. Therefore we have

$$\begin{aligned}
H^n(D_0^{m\bullet}) &= \bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} H^n(U_\alpha/K_\alpha, \Lambda),\\
H^n(D^{m\bullet}) &= \bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \infty_{\mathcal{D}}\bigl(\text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} H^n(U_\alpha/K_\alpha, \Lambda)\bigr)\\
&= \bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \text{c-Ind}_{K_\alpha}^{G}\bigl(\infty_{\mathcal{D}} H^n(U_\alpha/K_\alpha, \Lambda)\bigr)\\
&= \bigoplus_{\alpha\in G\backslash(G/K)^{m+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)}\bigl(i_{\mathcal{D}}\infty_{\mathcal{D}} H^n(U_\alpha/K_\alpha, \Lambda)\bigr).
\end{aligned}$$

By Proposition 3.38 and Proposition 3.10 iv), we have $i_{\mathcal{D}}\infty_{\mathcal{D}} H^n(U_\alpha/K_\alpha, \Lambda) = H^n(U_\alpha/K_\alpha, \Lambda)$. This concludes the proof. ∎

Lemma 4.7 *Let $\widetilde{C}^{\bullet\bullet}$ be the double complex in $\mathbf{Mod}(\mathcal{D}_c(G))$ given by*

$$\bigoplus_{\alpha\in G\backslash(G/K)^{\bullet+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} \Gamma_c(U_\alpha, \mathcal{J}^\bullet)^*.$$

Then there exists a natural morphism of double complexes $\widetilde{C}^{\bullet\bullet} \longrightarrow \mathrm{D}^m(C^{\bullet\bullet})$ that induces a quasi-isomorphism $\mathrm{Tot}\,\widetilde{C}^{\bullet\bullet} \longrightarrow \mathrm{D}^m(\mathrm{Tot}\,C^{\bullet\bullet})$.

Proof. By Proposition 2.2, we have a natural morphism $\widetilde{C}^{\bullet\bullet} \longrightarrow \mathrm{D}^m(C^{\bullet\bullet})$. To observe that $\mathrm{Tot}\,\widetilde{C}^{\bullet\bullet} \longrightarrow \mathrm{D}^m(\mathrm{Tot}\,C^{\bullet\bullet})$ is a quasi-isomorphism, it suffices to show that the morphism $\widetilde{C}^{n\bullet} \longrightarrow \mathrm{D}^m(C^{-n,\bullet})$ is a quasi-isomorphism for each n. We have

$$H^{-i}(\widetilde{C}^{n\bullet}) = \bigoplus_{\alpha\in G\backslash(G/K)^{n+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} H_c^i(U_\alpha, \Lambda)^*.$$

To compute the cohomology of $\mathrm{D}^m(C^{-n,\bullet})$, note the following points:

- Since the set $\{\overline{g} \in G/K \mid U \cap gU \neq \varnothing\}$ is finite, for a fixed n there exist only finitely many $\alpha \in G\backslash(G/K)^{n+1}$ such that $U_\alpha \neq \varnothing$. Therefore the direct sum $\bigoplus_{\alpha\in G\backslash(G/K)^{n+1}}$ in $C^{-n,\bullet}$ is finite and commutes with D^m.
- By Proposition 3.45, we have
$$\mathrm{D}^m\bigl(\text{c-Ind}_{K_\alpha}^{G} \Gamma_c(U_\alpha, \mathcal{J}^\bullet)\bigr) = R\mathrm{D}^m\bigl(\text{c-Ind}_{K_\alpha}^{G} \Gamma_c(U_\alpha, \mathcal{J}^\bullet)\bigr).$$
- By Proposition 3.10 iv), $H_c^i(U_\alpha, \Lambda)$ is a finitely generated Λ-module. Therefore, by Proposition 2.2, $\text{c-Ind}_{K_\alpha}^{G} H_c^i(U_\alpha, \Lambda)$ is acyclic with respect to D^m.

Therefore we have

$$\begin{aligned}
H^{-i}\bigl(\mathrm{D}^m(C^{-n,\bullet})\bigr) &= \bigoplus_{\alpha\in G\backslash(G/K)^{n+1}} R^{-i}\mathrm{D}^m\bigl(\text{c-Ind}_{K_\alpha}^{G} \Gamma_c(U_\alpha, \mathcal{J}^\bullet)\bigr)\\
&= \bigoplus_{\alpha\in G\backslash(G/K)^{n+1}} \mathrm{D}^m\bigl(\text{c-Ind}_{K_\alpha}^{G} H_c^i(U_\alpha, \Lambda)\bigr).
\end{aligned}$$

Since $H_c^i(U_\alpha, \Lambda)$ is a finitely generated Λ-module, Proposition 2.2 tells us that the homomorphism

$$\text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} H_c^i(U_\alpha, \Lambda)^* \longrightarrow \mathrm{D}^m\bigl(\text{c-Ind}_{K_\alpha}^G H_c^i(U_\alpha, \Lambda)\bigr)$$

is an isomorphism. Hence the homomorphism $H^{-i}(\widetilde{C}^{n\bullet}) \longrightarrow H^{-i}(\mathrm{D}^m(C^{-n,\bullet}))$ is an isomorphism, as desired. ∎

Lemma 4.8 *Let $D_0^{\bullet\bullet}$ be as in Corollary 4.5, and $\widetilde{C}^{\bullet\bullet}$ as in Lemma 4.7. We have a natural isomorphism*

$$\mathrm{Tot}(D_0^{\bullet\bullet})(d)[2d] \xrightarrow{\cong} \mathrm{Tot}(\widetilde{C}^{\bullet\bullet})$$

in $D^+(\mathbf{Mod}(\mathcal{D}_c(G)))$.

Proof. First we fix $\alpha \in (G/K)^{m+1}$, and construct a morphism

$$\Gamma(U_\alpha, \mathcal{I}^\bullet)(d)[2d] \longrightarrow \Gamma_c(U_\alpha, \mathcal{J}^\bullet)^*$$

in $D^+(\mathbf{Mod}(\mathcal{D}_c(K_\alpha)))$. Since $\mathcal{I}^n$ is flat for each n, the complex $\mathrm{Tot}(\mathcal{J}^\bullet \otimes \mathcal{I}^\bullet)$ gives a resolution of Λ. Therefore we have a morphism of complexes $\mathrm{Tot}(\mathcal{J}^\bullet \otimes \mathcal{I}^\bullet) \longrightarrow \mathcal{I}^\bullet$, which is determined up to homotopy. The cup product

$$\Gamma_c(U_\alpha, \mathcal{J}^\bullet) \otimes_\Lambda \Gamma(U_\alpha, \mathcal{I}^\bullet)(d)[2d] \longrightarrow \Gamma_c(U_\alpha, \mathcal{J}^\bullet \otimes \mathcal{I}^\bullet)(d)[2d]$$

gives a morphism of complexes

$$\Gamma(U_\alpha, \mathcal{I}^\bullet)(d)[2d] \longrightarrow \mathrm{Hom}\Bigl(\Gamma_c(U_\alpha, \mathcal{J}^\bullet), \Gamma_c\bigl(U_\alpha, \mathrm{Tot}(\mathcal{J}^\bullet \otimes \mathcal{I}^\bullet)\bigr)(d)[2d]\Bigr).$$

Consider the following morphisms of complexes:

$$\begin{aligned}\Gamma_c\bigl(U_\alpha, \mathrm{Tot}(\mathcal{J}^\bullet \otimes \mathcal{I}^\bullet)\bigr)(d)[2d] &\longrightarrow \Gamma_c(U_\alpha, \mathcal{I}^\bullet)(d)[2d] \longrightarrow \Gamma_c(X, \mathcal{I}^\bullet)(d)[2d] \\ &\longrightarrow \tau_{\geq 0}\Bigl(\Gamma_c(X, \mathcal{I}^\bullet)(d)[2d]\Bigr) \xleftarrow{(*)} H_c^{2d}(X, \Lambda)(d) \xrightarrow{\mathrm{Tr}_X} \Lambda.\end{aligned}$$

The morphism $(*)$ is a quasi-isomorphism, as $H_c^i(X, \Lambda)(d) = 0$ for $i > 2d$. By composing these morphisms, we get morphisms of complexes

$$\begin{aligned}\Gamma(U_\alpha, \mathcal{I}^\bullet)(d)[2d] &\longrightarrow \mathrm{Hom}\bigl(\Gamma_c(U_\alpha, \mathcal{J}^\bullet), \tau_{\geq 0}(\Gamma_c(X, \mathcal{I}^\bullet)(d)[2d])\bigr) \\ &\xleftarrow{(*)} \mathrm{Hom}\bigl(\Gamma_c(U_\alpha, \mathcal{J}^\bullet), H_c^{2d}(X, \Lambda)(d)\bigr) \\ &\longrightarrow \mathrm{Hom}\bigl(\Gamma_c(U_\alpha, \mathcal{J}^\bullet), \Lambda\bigr) = \Gamma_c(U_\alpha, \mathcal{J}^\bullet)^*.\end{aligned}$$

Since $\Gamma_c(U_\alpha, \mathcal{J}^\bullet)$ consists of free Λ-modules (*cf.* Proposition 3.45), $(*)$ is a quasi-isomorphism. As in [Far06, Lemme 2.6], it is easy to show that these morphisms are $\mathcal{D}_c(K_\alpha)$-linear. Moreover, by Corollary 3.37 and Proposition 3.45, the $(-i)$th cohomology of the composite morphism $\Gamma(U_\alpha, \mathcal{I}^\bullet)(d)[2d] \longrightarrow \Gamma_c(U_\alpha, \mathcal{J}^\bullet)^*$ in the derived category $D^+(\mathbf{Mod}(\mathcal{D}_c(K_\alpha)))$ is by definition the isomorphism of the Poincaré duality

$$H^{2d-i}(U_\alpha, \Lambda)(d) \xrightarrow{\cong} H^i_c(U_\alpha, \Lambda)^*.$$

Hence we obtain an isomorphism

$$\Gamma(U_\alpha, \mathcal{I}^\bullet)(d)[2d] \xrightarrow{\cong} \Gamma_c(U_\alpha, \mathcal{J}^\bullet)^*$$

in $D^+(\mathbf{Mod}(\mathcal{D}_c(K_\alpha)))$.

Put

$$\widetilde{C}_1^{\bullet\bullet} = \bigoplus_{\alpha \in G\backslash (G/K)^{\bullet+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} \operatorname{Hom}\bigl(\Gamma_c(U_\alpha, \mathcal{J}^\bullet), \tau_{\geq 0}(\Gamma_c(X, \mathcal{I}^\bullet)(d)[2d])\bigr),$$
$$\widetilde{C}_2^{\bullet\bullet} = \bigoplus_{\alpha \in G\backslash (G/K)^{\bullet+1}} \text{c-Ind}_{\mathcal{D}_c(K_\alpha)}^{\mathcal{D}_c(G)} \operatorname{Hom}\bigl(\Gamma_c(U_\alpha, \mathcal{J}^\bullet), H_c^{2d}(X, \Lambda)(d)\bigr),$$

which are double complexes in $\mathbf{Mod}(\mathcal{D}_c(G))$. Then, by the construction above, we obtain morphisms of double complexes

$$D_0^{\bullet\bullet}(d)[2d] \longrightarrow \widetilde{C}_1^{\bullet\bullet} \xleftarrow{(*)} \widetilde{C}_2^{\bullet\bullet} \longrightarrow \widetilde{C}^{\bullet\bullet}.$$

As $(*)$ induces a quasi-isomorphism $\widetilde{C}_1^{n\bullet} \longleftarrow \widetilde{C}_2^{n\bullet}$ for each n, the morphism $\operatorname{Tot}(*)$ is also a quasi-isomorphism. Similarly, we can conclude that the composite morphism

$$\operatorname{Tot} D_0^{\bullet\bullet}(d)[2d] \longrightarrow \operatorname{Tot} \widetilde{C}^{\bullet\bullet}$$

in $D^+(\mathbf{Mod}(\mathcal{D}_c(G)))$ is an isomorphism. ∎

Proof of Theorem 4.1. By Lemma 4.7 and Lemma 4.8, we have isomorphisms

$$\operatorname{Tot} D^{\bullet\bullet}(d)[2d] \cong \infty_{\mathcal{D}}(\operatorname{Tot} \widetilde{C}^{\bullet\bullet}) \cong \infty_{\mathcal{D}}\bigl(\mathrm{D}^m(\operatorname{Tot} C^{\bullet\bullet})\bigr) = \mathrm{D}(\operatorname{Tot} C^{\bullet\bullet})$$

in $D^+(\mathbf{Rep}_\Lambda(G))$. Therefore, by Lemma 4.2 and Lemma 4.6, we have a G-equivariant isomorphism

$$R\Gamma_{c,\mathcal{X}}(X, \Lambda)(d)[2d] \cong R\mathrm{D}\bigl(R\Gamma_c(X/G, \Lambda)\bigr)$$

in $D^+(\mathbf{Mod}(\Lambda))$. By taking cohomology, we get the desired isomorphism

$$H^{2d+i}_{c,\mathcal{X}}(X, \Lambda)(d) \xrightarrow{\cong} R^i\mathrm{D}\bigl(R\Gamma_c(X/G, \Lambda)\bigr).$$ ∎

To construct the isomorphism above, we chose V and K. Next we prove that the isomorphism is independent of these choices.

Proposition 4.9 *The isomorphism*

$$H^{2d+i}_{c,\mathcal{X}}(X,\Lambda)(d) \xrightarrow{\cong} R^i\mathrm{D}\bigl(R\Gamma_c(X/G,\Lambda)\bigr)$$

in Theorem 4.1 is independent of the choice of V and K.

Proof. We denote the isomorphism attached to V and K by $f_{V,K}$. Let V' and K' be another choice. We should prove that $f_{V,K} = f_{V',K'}$. We may assume either $V = V'$ or $K = K'$. Indeed, if we obtain the equality in this case, then in general we have

$$f_{V,K} = f_{V,K\cap K'} = f_{V\cup V',K\cap K'} = f_{V',K\cap K'} = f_{V',K'}$$

as desired. In particular, we may assume that $V \subset V'$ and $K \subset K'$.

We put $Z' = \overline{V'}$ and $U' = \mathrm{sp}^{-1}(Z')^\circ$. For $\beta \in (G/K')^m$, we define U'_β and K'_β in the same way as in Proposition 3.27. Since the open covering $\{gU\}_{g\in G/K}$ is a refinement of $\{gU'\}_{g\in G/K'}$, there exists a natural morphism of double complexes

$$\prod_{\beta\in(G/K')^{\bullet+1}} j'_{\beta*}j'^*_\beta\mathcal{I}^\bullet \longrightarrow \prod_{\alpha\in(G/K)^{\bullet+1}} j_{\alpha*}j^*_\alpha\mathcal{I}^\bullet.$$

Here j'_β denotes the open immersion $U'_\beta \hookrightarrow X$. This morphism turns out to be a G-equivariant morphism of double complexes $D'^\bullet(\mathcal{I}^\bullet) \longrightarrow D^\bullet(\mathcal{I}^\bullet)$, where $D'^\bullet(\mathcal{I}^n)$ denotes the complex $D^\bullet(\mathcal{I}^n)$ in Lemma 4.4 attached to V' and K'. Put $D'^{\bullet\bullet}_0 = \Gamma_{c,\mathcal{X}}(X, D'^\bullet(\mathcal{I}^\bullet))$, which is a double complex in $\mathbf{Mod}(\mathcal{D}_c(G))$ by Lemma 4.4. It is easy to see that the induced morphism $D'^{\bullet\bullet}_0 \longrightarrow D^{\bullet\bullet}_0$ is $\mathcal{D}_c(G)$-equivariant. Note that the following diagram is commutative, where $(*)$ (resp. $(**)$) is induced from the augmentation morphism $\mathcal{I}^\bullet \longrightarrow D'^\bullet(\mathcal{I}^\bullet)$ (resp. $\mathcal{I}^\bullet \longrightarrow D^\bullet(\mathcal{I}^\bullet)$) in Lemma 4.4:

$$\begin{tikzcd} & \Gamma_{c,\mathcal{X}}(X,\mathcal{I}^\bullet) \arrow[dl, "(*)"'] \arrow[dr, "(**)"] & \\ \mathrm{Tot}\, D'^{\bullet\bullet}_0 \arrow[rr] & & \mathrm{Tot}\, D^{\bullet\bullet}_0. \end{tikzcd}$$

Hence we obtain the commutative diagram below, where we put $D'^{\bullet\bullet} = \infty_{\mathcal{D}} D'^{\bullet\bullet}_0$ (*cf.* Lemma 4.6):

$$\begin{tikzcd} & H^i_{c,\mathcal{X}}(X,\Lambda) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H^i(\mathrm{Tot}\, D'^{\bullet\bullet}) \arrow[rr] & & H^i(\mathrm{Tot}\, D^{\bullet\bullet}). \end{tikzcd}$$

Let $C'^\bullet(\mathcal{J}^\bullet)$ denote the double complex

$$\bigoplus_{\beta\in G\backslash(G/K')^{-\bullet+1}} \text{c-Ind}^{X/G}_{U'_\beta/K'_\beta} \mathrm{Res}^{X/G}_{U'_\beta/K'_\beta} \mathcal{J}^\bullet$$

attached to U' and K' defined in Proposition 3.27 i). Then we have a morphism $C^\bullet(\mathcal{J}^\bullet) \longrightarrow C'^\bullet(\mathcal{J}^\bullet)$ of double complexes in $\Lambda\text{-}\widetilde{X}_{\text{ét}}/G$. This morphism is clearly compatible with the augmentation morphisms $C^\bullet(\mathcal{J}^\bullet) \longrightarrow \mathcal{J}^\bullet$ and $C'^\bullet(\mathcal{J}^\bullet) \longrightarrow \mathcal{J}^\bullet$ (*cf.* Proposition 3.27 i)). Put

$$C'^{\bullet\bullet} = \Gamma_c\bigl(X/G, C'^\bullet(\mathcal{J}^\bullet)\bigr) = \bigoplus_{\beta\in G\backslash(G/K')^{-\bullet+1}} \text{c-Ind}_{K'_\beta}^G \Gamma_c(U'_\beta, \mathcal{J}^\bullet).$$

Then, a G-equivariant morphism $C'^{\bullet\bullet} \longrightarrow C^{\bullet\bullet}$ is induced, and the following diagrams are commutative (*cf.* Lemma 4.2):

$$\begin{tikzcd}
\operatorname{Tot} C^{\bullet\bullet} \arrow[rr] \arrow[dr, "\cong"'] & & \operatorname{Tot} C'^{\bullet\bullet} \arrow[dl, "\cong"] \\
& R\Gamma_c(X/G, \Lambda), &
\end{tikzcd}$$

$$\begin{tikzcd}
& R\mathrm{D}\bigl(R\Gamma_c(X/G,\Lambda)\bigr) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\mathrm{D}(\operatorname{Tot} C'^{\bullet\bullet}) \arrow[rr] & & \mathrm{D}(\operatorname{Tot} C^{\bullet\bullet}).
\end{tikzcd}$$

As in Lemma 4.7 and the proof of Lemma 4.8, we put

$$\begin{aligned}
\widetilde{C}'^{\bullet\bullet} &= \bigoplus_{\beta\in G\backslash(G/K')^{\bullet+1}} \text{c-Ind}_{\mathcal{D}_c(K'_\beta)}^{\mathcal{D}_c(G)} \Gamma_c(U'_\beta, \mathcal{J}^\bullet)^*,\\
\widetilde{C}'^{\bullet\bullet}_1 &= \bigoplus_{\beta\in G\backslash(G/K')^{\bullet+1}} \text{c-Ind}_{\mathcal{D}_c(K'_\beta)}^{\mathcal{D}_c(G)} \operatorname{Hom}\bigl(\Gamma_c(U'_\beta, \mathcal{J}^\bullet), \tau_{\ge 0}(\Gamma_c(X,\mathcal{I}^\bullet)(d)[2d])\bigr),\\
\widetilde{C}'^{\bullet\bullet}_2 &= \bigoplus_{\beta\in G\backslash(G/K')^{\bullet+1}} \text{c-Ind}_{\mathcal{D}_c(K'_\beta)}^{\mathcal{D}_c(G)} \operatorname{Hom}\bigl(\Gamma_c(U'_\beta, \mathcal{J}^\bullet), H^{2d}_c(X,\Lambda)(d)\bigr).
\end{aligned}$$

Then, $\mathcal{D}_c(G)$-equivariant morphisms $\widetilde{C}'^{\bullet\bullet} \longrightarrow \widetilde{C}^{\bullet\bullet}$, $\widetilde{C}'^{\bullet\bullet}_1 \longrightarrow \widetilde{C}^{\bullet\bullet}_1$ and $\widetilde{C}'^{\bullet\bullet}_2 \longrightarrow \widetilde{C}^{\bullet\bullet}_2$ are naturally induced. Furthermore, we can easily check that the following diagram is commutative (the horizontal arrows are the morphisms appeared in Lemma 4.7 and the proof of Lemma 4.8):

$$\begin{tikzcd}
D'^{\bullet\bullet}_0(d)[2d] \arrow[r] \arrow[d] & \widetilde{C}'^{\bullet\bullet}_1 \arrow[d] & \widetilde{C}'^{\bullet\bullet}_2 \arrow[l] \arrow[r] \arrow[d] & \widetilde{C}'^{\bullet\bullet} \arrow[r] \arrow[d] & \mathrm{D}^m(C'^{\bullet\bullet}) \arrow[d] \\
D^{\bullet\bullet}_0(d)[2d] \arrow[r] & \widetilde{C}^{\bullet\bullet}_1 & \widetilde{C}^{\bullet\bullet}_2 \arrow[l] \arrow[r] & \widetilde{C}^{\bullet\bullet} \arrow[r] & \mathrm{D}^m(C^{\bullet\bullet}).
\end{tikzcd}$$

Putting all together, we obtain the commutative diagram

$$\begin{tikzcd}
H^{2d+i}_{c,\mathcal{X}}(X,\Lambda)(d) \arrow[d, "\cong"] \arrow[dd, bend right=60, "\cong"'] & R^i\mathrm{D}\bigl(R\Gamma_c(X/G,\Lambda)\bigr) \arrow[d, "\cong"'] \arrow[dd, bend left=60, "\cong"] \\
H^i(\operatorname{Tot} D'^{\bullet\bullet}(d)[2d]) \arrow[r] \arrow[d] & H^i\bigl(\mathrm{D}(\operatorname{Tot} C'^{\bullet\bullet})\bigr) \arrow[d] \\
H^i(\operatorname{Tot} D^{\bullet\bullet}(d)[2d]) \arrow[r] & H^i\bigl(\mathrm{D}(\operatorname{Tot} C^{\bullet\bullet})\bigr),
\end{tikzcd}$$

which gives the desired equality $f_{V,K} = f_{V',K'}$. ∎

Corollary 4.10 *For an isomorphism $\varphi\colon \mathcal{X} \xrightarrow{\cong} \mathcal{X}$ of formal schemes which is compatible with the action of G on $\mathcal{X}$, we have the following commutative diagram:*

$$\begin{array}{ccc} H^{2d+i}_{c,\mathcal{X}}(X,\Lambda)(d) & \xrightarrow{\cong} & R^i\mathrm{D}\bigl(R\Gamma_c(X/G,\Lambda)\bigr) \\ \cong\Big\downarrow\varphi^* & & \cong\Big\downarrow R^i\mathrm{D}(\varphi_*) \\ H^{2d+i}_{c,\mathcal{X}}(X,\Lambda)(d) & \xrightarrow{\cong} & R^i\mathrm{D}\bigl(R\Gamma_c(X/G,\Lambda)\bigr). \end{array}$$

Proof. We use the notation $f_{V,K}$ in the proof of Proposition 4.9. It is immediate to see that the following diagram is commutative:

$$\begin{array}{ccc} H^{2d+i}_{c,\mathcal{X}}(X,\Lambda)(d) & \xrightarrow[\cong]{f_{V,K}} & R^i\mathrm{D}\bigl(R\Gamma_c(X/G,\Lambda)\bigr) \\ \cong\Big\downarrow\varphi^* & & \cong\Big\downarrow R^i\mathrm{D}(\varphi_*) \\ H^{2d+i}_{c,\mathcal{X}}(X,\Lambda)(d) & \xrightarrow[\cong]{f_{\varphi^{-1}(V),K}} & R^i\mathrm{D}\bigl(R\Gamma_c(X/G,\Lambda)\bigr). \end{array}$$

Hence the corollary follows from Proposition 4.9. ∎

By standard argument as in [Far06, §3], we can show the similar results for ℓ-adic coefficients.

Theorem 4.11 *Let $\mathcal{X}$ and G be as in Theorem 4.1, and L_λ a finite extension of $\mathbb{Q}_\ell$. Then, for each integer i, we have a G-equivariant isomorphism*

$$H^{2d+i}_{c,\mathcal{X}}(X,L_\lambda)(d) \xrightarrow{\cong} R^i\mathrm{D}\bigl(R\Gamma_c(X/G,L_\lambda)\bigr).$$

This isomorphism is functorial with respect to an automorphism of $\mathcal{X}$ which is compatible with the action of G on $\mathcal{X}$.

Remark 4.12 Assume that G is a quotient of the group $\mathbf{H}(\mathbb{Q}_p)$ for some connected reductive group $\mathbf{H}$ over $\mathbb{Q}_p$. Then, for every field L of characteristic 0, the category $\mathbf{Rep}_L(G)$ is noetherian and has finite projective dimension.

In this case, for a finite extension L_λ of $\mathbb{Q}_\ell$, the Čech spectral sequence

$$E_1^{i,j} = \bigoplus_{\alpha\in G\backslash(G/K)^{-i+1}} \text{c-Ind}_{K_\alpha}^G H^j_c(U_\alpha,L_\lambda) \Longrightarrow H^{i+j}_c(X,L_\lambda)$$

(*cf.* Lemma 4.2) and Proposition 3.10 iv) tell us that $R\Gamma_c(X/G,L_\lambda)$ is an object of $D^b_{\mathrm{fg}}(\mathbf{Rep}_{L_\lambda}(G))$ (recall that $U_\alpha = \varnothing$ for all but finitely many $\alpha \in G\backslash(G/K)^{-i+1}$). Hence, by Corollary 2.3 iii), we have a G-equivariant functorial isomorphism

$$H^{2d+i}_{c,\mathcal{X}}(X,\overline{\mathbb{Q}}_\ell)(d) \xrightarrow{\cong} R^i\mathrm{D}\bigl(R\Gamma_c(X/G,\overline{\mathbb{Q}}_\ell)\bigr).$$

5 Application to the Rapoport-Zink tower

5.1 Rapoport-Zink tower for $\mathrm{GSp}(2n)$

Let $n\ge 1$ be an integer. For a ring A, let $\langle\ ,\ \rangle\colon A^{2n}\times A^{2n}\longrightarrow A$ be the symplectic pairing defined by $\langle (x_i),(y_i)\rangle = x_1y_{2n}+x_2y_{2n-1}+\cdots+x_ny_{n+1}-x_{n+1}y_n-\cdots-x_{2n}y_1$, and $\mathrm{GSp}_{2n}(A)$ the symplectic similitude group with respect to $\langle\ ,\ \rangle$.

Here we briefly recall the definition of the Rapoport-Zink tower for $\mathrm{GSp}(2n)$. See [Mie12b, §3.1] for details. In this section, we assume that $p\neq 2$.

We fix a n-dimensional p-divisible group $\mathbb{X}$ over $\overline{\mathbb{F}}_p$ which is isoclinic of slope $1/2$ and a polarization $\lambda_0\colon \mathbb{X}\xrightarrow{\cong}\mathbb{X}^\vee$. We write $\mathbb{Z}_{p^\infty}$ for the completion of the maximal unramified extension of $\mathbb{Z}_p$ and **Nilp** for the category of $\mathbb{Z}_{p^\infty}$-schemes on which p is locally nilpotent. Let $\mathcal{M}\colon \mathbf{Nilp}\longrightarrow \mathbf{Set}$ be the moduli functor of deformations by quasi-isogenies (X,ρ) of $(\mathbb{X},\lambda_0)$ (for the precise definition, see [Mie12b, §3.1]). It is known that $\mathcal{M}$ is represented by a special formal scheme over $\operatorname{Spf}\mathbb{Z}_{p^\infty}$, which is also denoted by $\mathcal{M}$. By [RZ96, Proposition 2.32], every irreducible component of $\mathcal{M}^{\mathrm{red}}$ is projective over $\overline{\mathbb{F}}_p$. In particular, $\mathcal{M}^{\mathrm{red}}$ is partially proper over $\overline{\mathbb{F}}_p$. We write M for the rigid generic fiber $t(\mathcal{M})_\eta$ of $\mathcal{M}$. The adic space M is purely $n(n+1)/2$-dimensional and smooth over $\operatorname{Spa}(\mathbb{Q}_{p^\infty},\mathbb{Z}_{p^\infty})$, where $\mathbb{Q}_{p^\infty}=\operatorname{Frac}\mathbb{Z}_{p^\infty}$. By Proposition 3.9, M is partially proper over $\operatorname{Spa}(\mathbb{Q}_{p^\infty},\mathbb{Z}_{p^\infty})$.

By adding level structures on the universal polarized p-divisible group on M, we can construct a projective system of étale coverings $\{M_K\}_{K\subset K_0}$ of M, where K runs through open subgroups of $K_0=\mathrm{GSp}_{2n}(\mathbb{Z}_p)$. This projective system is called the Rapoport-Zink tower for $\mathrm{GSp}(2n)$. For each K and $g\in G=\mathrm{GSp}_{2n}(\mathbb{Q}_p)$ with $g^{-1}Kg\subset K_0$, we have a natural isomorphism $[g]\colon M_K\xrightarrow{\cong}M_{g^{-1}Kg}$ called the Hecke operator. In particular, the group G acts on the pro-object $\{M_K\}_{K\subset K_0}$ on the right.

Let J be the group of self-quasi-isogenies of $\mathbb{X}$ preserving λ_0 up to multiplication by $\mathbb{Q}_p^\times$. We can construct a connected reductive algebraic group $\mathbf{J}$ over $\mathbb{Q}_p$ in a natural way such that $\mathbf{J}(\mathbb{Q}_p)=J$. In particular, J is naturally equipped with a topology. Concretely, $\mathbf{J}$ is isomorphic to $\mathrm{GU}(n,D)$, where D is the quaternion division algebra over $\mathbb{Q}_p$ (*cf.* [Mie12b, Remark 3.11]). By definition, J acts on $\mathcal{M}$ and M. This action naturally extends to M_K for each K and transition maps in the projective system $\{M_K\}_{K\subset K_0}$ are compatible with the actions of J. Further, the Hecke operators also commute with the actions of J. By [Far04, Corollaire 4.4.1], the action of J on M_K is continuous in the sense of Definition 3.19. Sometimes it is convenient to consider the quotient $M_K/p^{\mathbb{Z}}$ of M_K by the discrete subgroup $p^{\mathbb{Z}}$ of the center of J.

Fix a prime number ℓ which is different from p. Put

$$H^i_c(M_K)=H^i_c(M_K\otimes_{\mathbb{Q}_{p^\infty}}\overline{\mathbb{Q}}_{p^\infty},\overline{\mathbb{Q}}_\ell),\qquad H^i_c(M_\infty)=\varinjlim_{K\subset K_0}H^i_c(M_K),$$
$$H^i_c(M_K/p^{\mathbb{Z}})=H^i_c\bigl((M_K/p^{\mathbb{Z}})\otimes_{\mathbb{Q}_{p^\infty}}\overline{\mathbb{Q}}_{p^\infty},\overline{\mathbb{Q}}_\ell\bigr),\quad H^i_c(M_\infty/p^{\mathbb{Z}})=\varinjlim_{K\subset K_0}H^i_c(M_K/p^{\mathbb{Z}}).$$

As $G\times J$ acts on the tower $\{M_K\}_{K\subset K_0}$, the $\overline{\mathbb{Q}}_\ell$-vector spaces $H^i_c(M_\infty)$ and $H^i_c(M_\infty/p^{\mathbb{Z}})$ are equipped with actions of $G\times J$. The actions of G are obviously smooth. The actions of J are also smooth, as the action of J on M_K is smooth (*cf.* [Far04, Corollaire 4.4.7]). By [RZ96, Lemma 5.36], the action of G on $H^i_c(M_\infty/p^{\mathbb{Z}})$ factors through $G/p^{\mathbb{Z}}$, where $p^{\mathbb{Z}}\subset\mathbb{Q}_p^\times\subset G$ is a discrete subgroup of the center of G. For an open subgroup K of K_0, we have

$$H^i_c(M_\infty)^K=H^i_c(M_K),\quad H^i_c(M_\infty/p^{\mathbb{Z}})^K=H^i_c(M_K/p^{\mathbb{Z}}).$$

By using the Weil descent datum on $\mathcal{M}$ (*cf.* [RZ96, 3.48]), we can define actions of the Weil group $W_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ on $H^i_c(M_K)$, $H^i_c(M_\infty)$, $H^i_c(M_K/p^{\mathbb{Z}})$, and $H^i_c(M_\infty/p^{\mathbb{Z}})$. Hence, $H^i_c(M_\infty)$ is a representation of $G\times J\times W_{\mathbb{Q}_p}$, and $H^i_c(M_\infty/p^{\mathbb{Z}})$ is a representation of $G/p^{\mathbb{Z}}\times J/p^{\mathbb{Z}}\times W_{\mathbb{Q}_p}$.

In the following, we also write M and M_K for $M\otimes_{\mathbb{Q}_{p^\infty}}\overline{\mathbb{Q}}_{p^\infty}$ and $M_K\otimes_{\mathbb{Q}_{p^\infty}}\overline{\mathbb{Q}}_{p^\infty}$ by abuse of notation. Recall that the formal model $\mathcal{M}$ of M gives a support set $\mathcal{C}_{\mathcal{M}}$ of M (*cf.* Definition 3.15). We denote by the same symbol $\mathcal{C}_{\mathcal{M}}$ the support set of M_K induced by the morphism $M_K\longrightarrow M$ (*cf.* Definition 3.14 i)). Similarly, the formal model $\mathcal{M}/p^{\mathbb{Z}}$ of $M/p^{\mathbb{Z}}$ determines a support set of $M_K/p^{\mathbb{Z}}$, which is also denoted by $\mathcal{C}_{\mathcal{M}}$ for simplicity.

Proposition 5.1 *Let K be an open subgroup of K_0, and g an element of G such that $g^{-1}Kg\subset K_0$. Then, the Hecke operator $[g]\colon M_K\xrightarrow{\cong}M_{g^{-1}Kg}$ induces an isomorphism of pairs $(M_K,\mathcal{C}_{\mathcal{M}})\xrightarrow{\cong}(M_{g^{-1}Kg},\mathcal{C}_{\mathcal{M}})$ (cf. Definition 3.14 ii)). Similarly, we have an isomorphism of pairs $[g]\colon(M_K/p^{\mathbb{Z}},\mathcal{C}_{\mathcal{M}})\xrightarrow{\cong}(M_{g^{-1}Kg}/p^{\mathbb{Z}},\mathcal{C}_{\mathcal{M}})$.*

Proof. It suffices to show that if $Z\subset M_K$ belongs to $\mathcal{C}_{\mathcal{M}}$, then $[g](Z)$ belongs to $\mathcal{C}_{\mathcal{M}}$. If g lies in the center Z_G of G, then the claim holds, because [RZ96, Lemma 5.36] tells us that the Hecke action $[g]$ on $M=M_{K_0}$ extends to an automorphism of $\mathcal{M}$. Therefore, replacing g by zg with a suitable element $z\in p^{\mathbb{Z}}$, we may assume that $\mathbb{Z}_p^{2n}\subset g\mathbb{Z}_p^{2n}$. Take an integer $N\geq 0$ such that $g\mathbb{Z}_p^{2n}\subset p^{-N}\mathbb{Z}_p^{2n}$.

For an integer $k\geq 0$, let $\mathcal{M}^{\mathrm{red},k}$ be the subfunctor of $\mathcal{M}^{\mathrm{red}}$ consisting of (X,ρ) such that $p^k\rho$ and $p^k\rho^{-1}$ are isogenies. By [RZ96, Proposition 2.9], it is represented by a closed subscheme of $\mathcal{M}^{\mathrm{red}}$, which is quasi-compact (*cf.* [RZ96, Corollary 2.31]). Clearly we have $\mathcal{M}^{\mathrm{red}}=\bigcup_{k\geq 0}\mathcal{M}^{\mathrm{red},k}$. Since $Z\in\mathcal{C}_{\mathcal{M}}$, we can find $k\geq 0$ such that $Z\subset\mathrm{sp}_K^{-1}(\mathcal{M}^{\mathrm{red},k})$, where sp_K denotes the composite $M_K\xrightarrow{p_K}M\xrightarrow{\mathrm{sp}}\mathcal{M}^{\mathrm{red}}$. It suffices to prove that $[g](Z)\subset\mathrm{sp}_{g^{-1}Kg}^{-1}(\mathcal{M}^{\mathrm{red},k+N})$.

Take a geometric point x in Z. It corresponds to a triple (X,ρ,α), where X is a p-divisible group over the valuation ring $\kappa(x)^+$, $\rho\colon\mathbb{X}\otimes_{\overline{\mathbb{F}}_p}(\kappa(x)^+/p)\longrightarrow X\otimes_{\kappa(x)^+}(\kappa(x)^+/p)$ is a quasi-isogeny, and α is a level structure $\mathbb{Z}_p^{2d}\xrightarrow{\cong}T_p(X\otimes_{\kappa(x)^+}\kappa(x))$ (mod K). Put $y=[g](x)$. The pair (X',ρ') corresponding to the point $p_{g^{-1}Kg}(y)$ in M can be described as follows. Note that α gives a homomorphisms $(\mathbb{Q}_p/\mathbb{Z}_p)^{2n}\longrightarrow X\otimes_{\kappa(x)^+}\kappa(x)$, which is well-defined up to K-action. Let H be the scheme-theoretic closure in X of the image of $g\mathbb{Z}_p^{2n}/\mathbb{Z}_p^{2n}$ under this homomorphism. Then, it is a finite locally free subgroup scheme of $X[p^N]$, since $\kappa(x)^+$ is a valuation ring. By the definition of Hecke operators (*cf.* [RZ96, 5.34]), we have $X' = X/H$ and $\rho' = (\phi \bmod p) \circ \rho$, where $\phi\colon X \longrightarrow X'$ is the canonical isogeny. As $\mathrm{sp}_K(x)$ lies in $\mathcal{M}^{\mathrm{red},k}$, $p^k\rho$ and $p^k\rho^{-1}$ are isogenies. Therefore, so is $p^k\rho'$. On the other hand, since $H \subset X[p^N]$, $\operatorname{Ker}\phi$ is killed by p^N, and thus $p^N\phi^{-1}$ is an isogeny. Hence $p^{k+N}\rho'^{-1}$ is also an isogeny. Namely, $y = [g](x)$ lies in $\mathrm{sp}_{g^{-1}Kg}^{-1}(\mathcal{M}^{\mathrm{red},k+N})$. Now we conclude that $[g](Z) \subset \mathrm{sp}_{g^{-1}Kg}^{-1}(\mathcal{M}^{\mathrm{red},k+N})$. ∎

Put $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty) = \varinjlim_{K\subset K_0} H^i_{\mathcal{C}_\mathcal{M}}(M_K)$ and $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty/p^\mathbb{Z}) = \varinjlim_{K\subset K_0} H^i_{\mathcal{C}_\mathcal{M}}(M_K/p^\mathbb{Z})$. By the previous proposition, G acts naturally on $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty)$ and $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty/p^\mathbb{Z})$. For an open subgroup K of K_0, we have

$$H^i_{\mathcal{C}_\mathcal{M}}(M_\infty)^K = H^i_{\mathcal{C}_\mathcal{M}}(M_K), \quad H^i_{\mathcal{C}_\mathcal{M}}(M_\infty/p^\mathbb{Z})^K = H^i_{\mathcal{C}_\mathcal{M}}(M_K/p^\mathbb{Z}).$$

Obviously the groups J and $W_{\mathbb{Q}_p}$ act on $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty)$ and $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty/p^\mathbb{Z})$. We have natural homomorphisms $H^i_c(M_\infty) \longrightarrow H^i_{\mathcal{C}_\mathcal{M}}(M_\infty)$ and $H^i_c(M_\infty/p^\mathbb{Z}) \longrightarrow H^i_{\mathcal{C}_\mathcal{M}}(M_\infty/p^\mathbb{Z})$, which are $G \times J \times W_{\mathbb{Q}_p}$-equivariant.

In the sequel, we will describe the action of G on $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty)$ by using some formal models (*cf.* [IM10, §5.2]). For an integer $m \geq 0$, let $\mathcal{M}_m$ be the formal scheme classifying Drinfeld m-level structures on the universal p-divisible group on $\mathcal{M}$ (for a precise definition, see [IM10, §3.2]). The formal scheme $\mathcal{M}_m$ is finite over $\mathcal{M}$, and satisfies $t(\mathcal{M}_m)_\eta \cong M_{K_m}$, where K_m is the kernel of $\mathrm{GSp}_{2n}(\mathbb{Z}_p) \longrightarrow \mathrm{GSp}_{2n}(\mathbb{Z}/p^m\mathbb{Z})$.

Lemma 5.2 *There exists a natural isomorphism*

$$H^i_{\mathcal{C}_\mathcal{M}}(M_{K_m}) \cong H^i_c(\mathcal{M}_m^{\mathrm{red}}, R\Psi\overline{\mathbb{Q}}_\ell),$$

where $R\Psi\overline{\mathbb{Q}}_\ell = R\,\mathrm{sp}_*\,\overline{\mathbb{Q}}_\ell$ denotes the formal nearby cycle complex (see [Ber96]).

Proof. By Proposition 3.16, we have

$$H^i_c(\mathcal{M}_m^{\mathrm{red}}, R\Psi\overline{\mathbb{Q}}_\ell) = H^i_{c,\mathcal{M}_m}(M_{K_m}) = H^i_{\mathcal{C}_{\mathcal{M}_m}}(M_{K_m}).$$

Hence it suffices to show the equality $\mathcal{C}_{\mathcal{M}_m} = \mathcal{C}_\mathcal{M}$ of support sets of M_{K_m}. This immediately follows from Lemma 3.17 and the fact that $\mathcal{M}_m \longrightarrow \mathcal{M}$ is finite. ∎

Let G^+ denote the submonoid $\{g \in G \mid \mathbb{Z}_p^{2n} \subset g\mathbb{Z}_p^{2n}\}$ of G. For $g \in G^+$, let $e(g)$ be the minimal non-negative integer such that $g\mathbb{Z}_p^{2n} \subset p^{-e(g)}\mathbb{Z}_p^{2n}$. Following [Man05, §6], for $g \in G^+$ with $m \geq e = e(g)$, we can define a formal scheme $\mathcal{M}_{m,g}$ over $\mathcal{M}_m$ satisfying the following properties (*cf.* [IM10, §5.2]):

- The structure morphism $\mathrm{pr}\colon \mathcal{M}_{m,g} \longrightarrow \mathcal{M}_m$ is proper, and the induced morphism on rigid generic fibers $t(\mathcal{M}_{m,g})_\eta \longrightarrow t(\mathcal{M}_m)_\eta$ is an isomorphism.

– There exists a proper morphism $[g]\colon \mathcal{M}_{m,g} \longrightarrow \mathcal{M}_{m-e}$ such that the composite

$$M_{K_m} = t(\mathcal{M}_m)_\eta \xrightarrow{\mathrm{pr}^{-1}} t(\mathcal{M}_{m,g})_\eta \xrightarrow{[g]} t(\mathcal{M}_{m-e})_\eta = M_{K_{m-e}}$$

coincides with the Hecke operator $M_{K_m} \xrightarrow{[g]} M_{g^{-1}K_m g} \longrightarrow M_{K_{m-e}}$ attached to $g \in G$ (note that $\mathbb{Z}_p^{2n} \subset g\mathbb{Z}_p^{2n} \subset p^{-e}\mathbb{Z}_p^{2n}$ implies that $g^{-1}K_m g \subset K_{m-e}$).

Lemma 5.3 *For $g \in G^+$ and $m \geq e = e(g)$, the composite of*

$$H_c^i(\mathcal{M}_{m-e}^{\mathrm{red}}, R\Psi\overline{\mathbb{Q}}_\ell) \xrightarrow{[g]^*} H_c^i(\mathcal{M}_{m,g}^{\mathrm{red}}, R\Psi\overline{\mathbb{Q}}_\ell) \xrightarrow[\cong]{\mathrm{pr}_*} H_c^i(\mathcal{M}_m^{\mathrm{red}}, R\Psi\overline{\mathbb{Q}}_\ell)$$

corresponds to the composite of

$$H^i_{\mathcal{C}_\mathcal{M}}(M_{K_{m-e}}) \longrightarrow H^i_{\mathcal{C}_\mathcal{M}}(M_{g^{-1}K_m g}) \xrightarrow{[g]^*} H^i_{\mathcal{C}_\mathcal{M}}(M_{K_m})$$

under the isomorphism in Lemma 5.2.

Proof. As in Lemma 5.2, we can see that

$$H_c^i(\mathcal{M}_{m,g}^{\mathrm{red}}, R\Psi\overline{\mathbb{Q}}_\ell) \cong H^i_{\mathcal{C}_\mathcal{M}}\bigl(t(\mathcal{M}_{m,g})_{\overline{\eta}}\bigr) \xrightarrow[\cong]{(\mathrm{pr}^*)^{-1}} H^i_{\mathcal{C}_\mathcal{M}}(M_{K_m}).$$

Hence the lemma immediately follows from the property of $[g]\colon \mathcal{M}_{m,g} \longrightarrow \mathcal{M}_{m-e}$. ∎

By this description and the main theorem in [IM10], we can compare the cuspidal parts of $H_c^i(M_\infty)$ and $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty)$ in the case $n = 2$.

Corollary 5.4 *Assume that $n = 2$. Then, no supercuspidal representation of G appears as a subquotient of the kernel and the cokernel of the map $H_c^i(M_\infty) \longrightarrow H^i_{\mathcal{C}_\mathcal{M}}(M_\infty)$. The same holds for the map $H_c^i(M_\infty/p^\mathbb{Z}) \longrightarrow H^i_{\mathcal{C}_\mathcal{M}}(M_\infty/p^\mathbb{Z})$.*

Proof. We will use the notation in [IM10, §5] freely. By [IM10, Proposition 5.11, Proposition 5.18], we have a G-equivariant isomorphism $H_c^i(M_\infty) \cong H_c^i(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{[0]})_{\overline{\mathbb{Q}}_\ell}$. On the other hand, [Ber96, Theorem 3.1] tells us that $H^i_{\mathcal{C}_\mathcal{M}}(M_\infty) \cong H_c^i(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{[2]})_{\overline{\mathbb{Q}}_\ell}$. The right hand side $H_c^i(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{[2]})_{\overline{\mathbb{Q}}_\ell}$ is endowed with an action of G (*cf.* [IM10, §5.2]), and Lemma 5.3 ensures that the isomorphism above is G-equivariant. By [IM10, Proposition 5.6 i)], for each $h \in \{1, 2\}$ we have an exact sequence of smooth G-representations

$$H_c^{i-1}(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell} \to H_c^i(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{[h-1]})_{\overline{\mathbb{Q}}_\ell} \to H_c^i(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{[h]})_{\overline{\mathbb{Q}}_\ell} \to H_c^i(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell}.$$

Hence the kernel and the cokernel of $H_c^i(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{[0]})_{\overline{\mathbb{Q}}_\ell} \longrightarrow H_c^i(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{[2]})_{\overline{\mathbb{Q}}_\ell}$ is a successive extension of subquotients of $H_c^j(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell}$ for $i - 1 \leq j \leq i$ and $1 \leq h \leq 2$. On the other hand, by [IM10, Theorem 5.21], $H_c^j(\mathcal{M}_\infty^{\mathrm{red}}, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell}$ has no supercuspidal subquotient. Therefore the kernel and the cokernel of $H_c^i(M_\infty) \longrightarrow H^i_{\mathcal{C}_\mathcal{M}}(M_\infty)$ have no supercuspidal subquotient. Similar argument can be applied to the map $H_c^i(M_\infty/p^\mathbb{Z}) \longrightarrow H^i_{\mathcal{C}_\mathcal{M}}(M_\infty/p^\mathbb{Z})$. ∎

The following result on the vanishing of cohomology is also essentially obtained in [IM10].

Proposition 5.5 i) *For an integer* $i < \dim M - \dim \mathcal{M}^{\mathrm{red}}$, *we have* $H^i_c(M_\infty) = 0$.
ii) *For an integer* $i > \dim M + \dim \mathcal{M}^{\mathrm{red}}$, *we have* $H^i_{\mathcal{C}_{\mathcal{M}}}(M_\infty) = 0$.

Proof. Fix an integer $m \geq 0$. Let $\mathcal{U}$ be a quasi-compact open formal subscheme of $\mathcal{M}_m$. By the p-adic uniformization theorem, there exist a scheme U which is separated of finite type over $\mathbb{Z}_{p^\infty}$ and a closed subscheme Z of the special fiber U_s of U such that $\mathcal{U}$ is isomorphic to the formal completion of U along Z (*cf.* [IM10, Corollary 4.4]; we can take U as an open subscheme of a suitable integral model of the Siegel modular variety). We denote the closed immersion $Z \hookrightarrow U_s$ by ι. By [Mie10, Theorem 4.35] and [Mie10, Proposition 3.13], we have

$$H^i_c\bigl(t(\mathcal{U})_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell\bigr) \cong H^i_c(Z, R\Psi_{\mathcal{U},c}\overline{\mathbb{Q}}_\ell) \cong H^i_c(Z, R\iota^! R\psi_U \overline{\mathbb{Q}}_\ell).$$

Therefore, [IM10, Lemma 5.26] tells us that $H^i_c(t(\mathcal{U})_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) = 0$ if $i < \dim U_\eta - \dim Z = \dim M - \dim \mathcal{M}^{\mathrm{red}}$. Since $H^i_c(M_\infty) = \varinjlim_m \varinjlim_{\mathcal{U}\subset\mathcal{M}_m} H^i_c(t(\mathcal{U})_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell)$, we conclude i).

We prove ii). By [Ber96, Theorem 3.1] and [IM10, Lemma 5.26], we have

$$H^i_c(\mathcal{U}^{\mathrm{red}}, R\Psi\overline{\mathbb{Q}}_\ell) \cong H^i_c(Z, \iota^* R\psi_U \overline{\mathbb{Q}}_\ell) = 0$$

for $i > \dim M - \dim \mathcal{M}^{\mathrm{red}}$. Therefore, by Lemma 5.2 and Proposition 3.5 ii), we have

$$H^i_{\mathcal{C}_{\mathcal{M}}}(M_{K_m}) \cong H^i_c(\mathcal{M}^{\mathrm{red}}_m, R\Psi\overline{\mathbb{Q}}_\ell) \cong \varinjlim_{\mathcal{U}\subset\mathcal{M}_m} H^i_c(\mathcal{U}^{\mathrm{red}}, R\Psi\overline{\mathbb{Q}}_\ell) = 0$$

for $i > \dim M - \dim \mathcal{M}^{\mathrm{red}}$. Hence $H^i_{\mathcal{C}_{\mathcal{M}}}(M_\infty) = \varinjlim_m H^i_{\mathcal{C}_{\mathcal{M}}}(M_{K_m})$ also vanishes for $i > \dim M - \dim \mathcal{M}^{\mathrm{red}}$. ∎

5.2 Application of the duality theorem

Fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and identify them. Every representation in this subsection is considered over $\mathbb{C}$. Let $\widetilde{K}$ be an open compact-mod-center subgroup of G and τ an irreducible smooth representation of $\widetilde{K}$. Denote by $\chi\colon Z_G \longrightarrow \mathbb{C}^\times$ the central character of $\tau^\vee$. For a smooth G-representation V, put $V_\tau = \mathrm{Hom}_{\widetilde{K}}(\tau, V \otimes_{\mathcal{H}(Z_G)} \chi^{-1})$. Then, $H^i_c(M_\infty)_\tau$ and $H^i_{\mathcal{C}_{\mathcal{M}}}(M_\infty)_\tau$ are representations of $J \times W_{\mathbb{Q}_p}$. By [RZ96, Lemma 5.36], the actions of the center $Z_J = Z_G$ of J on $H^i_c(M_\infty)_\tau$ and $H^i_{\mathcal{C}_{\mathcal{M}}}(M_\infty)_\tau$ are given by χ. Hence we can consider the Bernstein decomposition with respect to the action of J (*cf.* Section 2):

$$H^i_c(M_\infty)_\tau = \bigoplus_{\mathfrak{s}\in I_\chi} H^i_c(M_\infty)_{\tau,\mathfrak{s}}, \qquad H^i_{\mathcal{C}_{\mathcal{M}}}(M_\infty)_\tau = \bigoplus_{\mathfrak{s}\in I_\chi} H^i_{\mathcal{C}_{\mathcal{M}}}(M_\infty)_{\tau,\mathfrak{s}}.$$

Theorem 5.6 *Fix $\mathfrak{s} \in I_\chi$. Assume that $H_c^q(M_\infty)_{\tau,\mathfrak{s}}$ is a finite length J-representation for every integer q. Then, for each integer i, we have an isomorphism of $J \times W_{\mathbb{Q}_p}$-representations*
$$H_{\mathcal{C}_{\mathcal{M}}}^{2d+\iota(\mathfrak{s})-i}(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}(d) \cong \mathrm{Zel}\bigl(H_c^i(M_\infty)_{\tau,\mathfrak{s}}^\vee\bigr),$$
where $d = n(n+1)/2$ is the dimension of M.

Remark 5.7 For the Drinfeld tower, a similar result is proved by Fargues [Far06, Théorème 4.6]. In that case $\mathcal{M}$ is a p-adic formal scheme, thus $H_{\mathcal{C}_{\mathcal{M}}}^i(M_\infty)$ coincides with $H_c^i(M_\infty)$ (*cf.* Remark 3.12).

Proof of Theorem 5.6. First we will reduce the theorem to the case where τ is trivial on $p^{\mathbb{Z}}$. Take $c \in \mathbb{C}$ such that $c^2 = \chi(p)$ and define the character $\omega\colon \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$ by $\omega(a) = c^{v_p(a)}$, where v_p is the p-adic valuation. We denote the composite of the similitude character $G \longrightarrow \mathbb{Q}_p^\times$ (resp. $J \longrightarrow \mathbb{Q}_p^\times$) and ω by ω_G (resp. ω_J). Then, $\tau' = \tau \otimes (\omega_G|_{\widetilde{K}})$ is trivial on $p^{\mathbb{Z}}$. Moreover, as in the proof of [Mie12b, Lemma 3.5], we have $H_c^q(M_\infty) \otimes \omega_G \otimes \omega_J^{-1} \cong H_c^q(M_\infty)$ as $G \times J \times W_{\mathbb{Q}_p}$-representations for every integer q. Therefore we obtain
$$\begin{aligned} H_c^q(M_\infty)_\tau &= \mathrm{Hom}_{\widetilde{K}}\bigl(\tau, H_c^q(M_\infty) \otimes_{\mathcal{H}(Z_G)} \chi^{-1}\bigr)\\ &= \mathrm{Hom}_{\widetilde{K}}\bigl(\tau', (H_c^q(M_\infty) \otimes_{\mathcal{H}(Z_G)} \chi^{-1}) \otimes \omega_G\bigr)\\ &= \mathrm{Hom}_{\widetilde{K}}\bigl(\tau', (H_c^q(M_\infty) \otimes \omega_G) \otimes_{\mathcal{H}(Z_G)} \omega^2\chi^{-1}\bigr)\\ &\cong \mathrm{Hom}_{\widetilde{K}}\bigl(\tau', (H_c^q(M_\infty) \otimes \omega_J) \otimes_{\mathcal{H}(Z_G)} \omega^2\chi^{-1}\bigr) = H_c^q(M_\infty)_{\tau'} \otimes \omega_J. \end{aligned}$$
For $\mathfrak{s} = [(\mathbf{M}, \sigma)] \in I_\chi$, put $\mathfrak{s}' = [(\mathbf{M}, \sigma \otimes (\omega_J^{-1}|_{\mathbf{M}(\mathbb{Q}_p)}))]$. Then, we conclude that $H_c^q(M_\infty)_{\tau,\mathfrak{s}} \cong H_c^q(M_\infty)_{\tau',\mathfrak{s}'} \otimes \omega_J$ as $J \times W_{\mathbb{Q}_p}$-representations. In particular, the J-representation $H_c^q(M_\infty)_{\tau',\mathfrak{s}'}$ has finite length. Similarly we have $H_{\mathcal{C}_{\mathcal{M}}}^q(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee} \cong H_{\mathcal{C}_{\mathcal{M}}}^q(M_\infty)_{\tau'^\vee,\mathfrak{s}'^\vee} \otimes \omega_J^{-1}$. Suppose that the theorem holds for τ' and $\mathfrak{s}'$. Then, by the isomorphisms above, we have
$$\begin{aligned} H_{\mathcal{C}_{\mathcal{M}}}^{2d+\iota(\mathfrak{s})-i}(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}(d) &\cong H_{\mathcal{C}_{\mathcal{M}}}^{2d+\iota(\mathfrak{s}')-i}(M_\infty)_{\tau'^\vee,\mathfrak{s}'^\vee}(d) \otimes \omega_J^{-1} \cong \mathrm{Zel}\bigl(H_c^i(M_\infty)_{\tau',\mathfrak{s}'}^\vee\bigr) \otimes \omega_J^{-1}\\ &\cong \mathrm{Zel}\bigl(H_c^i(M_\infty)_{\tau,\mathfrak{s}}^\vee \otimes \omega_J\bigr) \otimes \omega_J^{-1} \overset{(*)}{\cong} \mathrm{Zel}\bigl(H_c^i(M_\infty)_{\tau,\mathfrak{s}}^\vee\bigr). \end{aligned}$$
For the isomorphism $(*)$, see Lemma 2.9 i).

Thus, in the following we may assume that τ is trivial on $p^{\mathbb{Z}}$. Then, we have
$$\begin{aligned} H_c^q(M_\infty)_{\tau,\mathfrak{s}} &= H_c^q(M_\infty/p^{\mathbb{Z}})_{\tau,\mathfrak{s}} = \mathrm{Hom}_{\widetilde{K}}\bigl(\tau, H_c^q(M_\infty/p^{\mathbb{Z}})_{\mathfrak{s}}\bigr),\\ H_{\mathcal{C}_{\mathcal{M}}}^q(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee} &= H_{\mathcal{C}_{\mathcal{M}}}^q(M_\infty/p^{\mathbb{Z}})_{\tau^\vee,\mathfrak{s}^\vee} = \mathrm{Hom}_{\widetilde{K}}\bigl(\tau^\vee, H_{\mathcal{C}_{\mathcal{M}}}^q(M_\infty/p^{\mathbb{Z}})_{\mathfrak{s}^\vee}\bigr). \end{aligned}$$
Since $\widetilde{K}$ is compact-mod-center, there exists a self-dual chain of lattices $\mathscr{L}$ of $\mathbb{Q}_p^{2d}$ (*cf.* [RZ96, Definition 3.1, Definition 3.13]) such that every $g \in \widetilde{K}$ and $L \in \mathscr{L}$ satisfy $gL \in \mathscr{L}$. As in [Mie12b, §4.1], we write $K_{\mathscr{L}}$ for the stabilizer of $\mathscr{L}$ in G. For an integer $m \geq 0$, we put
$$K_{\mathscr{L},m} = \{g \in K_{\mathscr{L}} \mid \text{for every } L \in \mathscr{L},\ g \text{ acts trivially on } L/p^mL\}.$$

It is an open normal subgroup of $K_{\mathscr{L}}$. We denote by $N_{\mathscr{L}}$ the subgroup of G consisting of $g \in G$ satisfying $g\mathscr{L} = \mathscr{L}$. We have $K_{\mathscr{L},m} \subset K_{\mathscr{L}} \subset N_{\mathscr{L}}$ and $K_{\mathscr{L},m}$ is normal in $N_{\mathscr{L}}$. By definition, $\widetilde{K}$ is contained in $N_{\mathscr{L}}$.

Take an integer $m \geq 0$ large enough so that $K_{\mathscr{L},m} \subset K_0$ and $\tau|_{K_{\mathscr{L},m}}$ is trivial. In [Mie12b, Definition 4.3], the author constructed a formal scheme $\mathcal{M}^{\flat}_{\mathscr{L},m}$ over $\operatorname{Spf} \mathbb{Z}_{p^\infty}$ satisfying the following:

- $t(\mathcal{M}^{\flat}_{\mathscr{L},m})_\eta$ is naturally isomorphic to $M_{K_{\mathscr{L},m}}$.
- $\mathcal{M}^{\flat}_{\mathscr{L},m}$ is naturally endowed with an action of $\widetilde{K} \times J$ and a Weil descent datum, and they are compatible with those structures on $M_{K_{\mathscr{L},m}}$ under the isomorphism above.

We shall apply Theorem 4.11 to $\mathcal{M}^{\flat}_{\mathscr{L},m}/p^{\mathbb{Z}}$ and $J/p^{\mathbb{Z}}$. We should verify the conditions in Theorem 4.1. The conditions (a) and (c) are satisfied, as explained in the previous subsection. The condition (b) is satisfied by [Mie12b, Remark 4.12]. For the condition (d), let $\mathcal{I}$ denote the set of irreducible components of $\mathcal{M}^{\flat,\mathrm{red}}_{\mathscr{L},m}/p^{\mathbb{Z}}$. For $\alpha \in \mathcal{I}$, put $V_\alpha = (\mathcal{M}^{\flat,\mathrm{red}}_{\mathscr{L},m}/p^{\mathbb{Z}}) \setminus \bigcup_{\beta\in\mathcal{I},\alpha\cap\beta=\varnothing}\beta$. It is a quasi-compact open subset of $\mathcal{M}^{\flat,\mathrm{red}}_{\mathscr{L},m}/p^{\mathbb{Z}}$. By [Mie12b, Lemma 5.1 ii)], the action of $J/p^{\mathbb{Z}}$ on $\mathcal{I}$ has finite orbits. Take $\alpha_1,\ldots,\alpha_k \in \mathcal{I}$ so that $\mathcal{I} = \bigcup_{j=1}^{k}(J/p^{\mathbb{Z}})\alpha_j$, and put $V = \bigcup_{j=1}^{k} V_{\alpha_j}$. Clearly we have $\mathcal{M}^{\flat,\mathrm{red}}_{\mathscr{L},m}/p^{\mathbb{Z}} = \bigcup_{h\in J/p^{\mathbb{Z}}} hV$. The closure $\overline{V}$ is the union of finitely many irreducible components of $\mathcal{M}^{\flat,\mathrm{red}}_{\mathscr{L},m}/p^{\mathbb{Z}}$. Therefore, by the same way as in [Mie13, Corollary 4.3 ii)], we can prove that the set $\{g \in J/p^{\mathbb{Z}} \mid g\overline{V}\cap\overline{V} \neq \varnothing\}$ is compact. Thus the condition (d) is satisfied. Now, by Theorem 4.11 and Remark 4.12, we have a J-equivariant isomorphism

$$H^{2d+i}_{c,\mathcal{M}^{\flat}_{\mathscr{L},m}}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})(d) \cong R^i\mathrm{D}\bigl(R\Gamma_c\bigl((M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})/(J/p^{\mathbb{Z}}),\overline{\mathbb{Q}}_\ell\bigr)\bigr).$$

By Corollary 4.10, this isomorphism is also $\widetilde{K} \times W_{\mathbb{Q}_p}$-equivariant.

We prove that the left hand side is equal to $H^{2d+i}_{\mathcal{C}_{\mathcal{M}}}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})(d)$. For simplicity, we denote the support set $\mathcal{C}_{\mathcal{M}^{\flat}_{\mathscr{L},m}}$ of $M_{K_{\mathscr{L},m}}$ by $\mathcal{C}_{\mathscr{L},m}$. Let $\mathscr{L}_{\mathrm{Iw}}$ be the self-dual chain of lattices

$$\bigl\{(p^m\mathbb{Z}_p)^{\oplus j} \oplus (p^{m+1}\mathbb{Z}_p)^{\oplus(2d-j)}\bigr\}_{0\leq j\leq 2d,m\in\mathbb{Z}}.$$

The group $K_{\mathscr{L}_{\mathrm{Iw}}}$ attached to $\mathscr{L}_{\mathrm{Iw}}$ is an Iwahori subgroup of G. There exists $g_0 \in G$ such that $\mathscr{L} \subset g_0\mathscr{L}_{\mathrm{Iw}}$. The following morphisms of formal schemes are naturally induced:

$$\mathcal{M}^{\flat}_{\mathscr{L},m} \xleftarrow{(1)} \mathcal{M}^{\flat}_{g_0\mathscr{L}_{\mathrm{Iw}},m} \xrightarrow[\cong]{[g_0]} \mathcal{M}^{\flat}_{\mathscr{L}_{\mathrm{Iw}},m} \xrightarrow{(2)} \mathcal{M}_m \xrightarrow{(3)} \mathcal{M}.$$

The morphisms (1), (2) and (3) are proper. The rigid generic fiber of the diagram above is identified with the following diagram:

$$M_{K_{\mathscr{L},m}} \xleftarrow{\pi} M_{g_0K_{\mathscr{L}_{\mathrm{Iw}},m}g_0^{-1}} \xrightarrow[\cong]{[g_0]} M_{K_{\mathscr{L}_{\mathrm{Iw}},m}} \longrightarrow M_{K_m} \longrightarrow M.$$

Therefore, by Lemma 3.17 and Proposition 5.1, we have equalities

$$\pi^{-1}\mathcal{C}_{\mathscr{L},m} = \mathcal{C}_{g_0\mathscr{L}_{\mathrm{Iw}},m} = [g_0]^{-1}\mathcal{C}_{\mathscr{L}_{\mathrm{Iw}},m} = [g_0]^{-1}\mathcal{C}_{\mathcal{M}} = \mathcal{C}_{\mathcal{M}} = \pi^{-1}\mathcal{C}_{\mathcal{M}}$$

of support sets of $M_{g_0K_{\mathscr{L}_{\mathrm{Iw}},m}g_0^{-1}}$ (recall that we denote by $\mathcal{C}_{\mathcal{M}}$ the support set of M_K induced from the support set $\mathcal{C}_{\mathcal{M}}$ of M for various $K \subset K_0$). Since π is finite and surjective, we conclude that $\mathcal{C}_{\mathscr{L},m} = \mathcal{C}_{\mathcal{M}}$. Hence $H^{2d+i}_{c,\mathcal{M}^\flat_{\mathscr{L},m}}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})(d) = H^{2d+i}_{\mathcal{C}_{\mathcal{M}}}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})(d)$, and thus we have a $\widetilde{K} \times J \times W_{\mathbb{Q}_p}$-equivariant isomorphism

$$H^{2d+i}_{\mathcal{C}_{\mathcal{M}}}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})(d) \cong R^i\mathrm{D}\bigl(R\Gamma_c\bigl((M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})/(J/p^{\mathbb{Z}}), \overline{\mathbb{Q}}_\ell\bigr)\bigr).$$

The $(\tau^\vee, \mathfrak{s}^\vee)$-part of the left hand side is equal to $H^{2d+i}_{\mathcal{C}_{\mathcal{M}}}(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}(d)$. We will consider the $(\tau^\vee, \mathfrak{s}^\vee)$-part of the right hand side. For simplicity, we put

$$A = R\Gamma_c\bigl((M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})/(J/p^{\mathbb{Z}}), \overline{\mathbb{Q}}_\ell\bigr),$$

which is an object of $D^b(\mathbf{Rep}(J/p^{\mathbb{Z}}))$ endowed with an action of $\widetilde{K} \times W_{\mathbb{Q}_p}$. The action of $\widetilde{K}$ factors through the finite quotient $H = \widetilde{K}/K_{\mathscr{L},m}p^{\mathbb{Z}}$. By Corollary 3.42, $H^q(A) = H^q_c(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})$. Therefore, we have a spectral sequence

$$E_2^{s,t} = R^s\mathrm{D}\bigl(H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})\bigr) \Longrightarrow R^{s+t}\mathrm{D}(A).$$

Take the $(\tau^\vee, \mathfrak{s}^\vee)$-part of this spectral sequence:

$$E_2^{s,t} = R^s\mathrm{D}\bigl(H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})\bigr)_{\tau^\vee,\mathfrak{s}^\vee} \Longrightarrow R^{s+t}\mathrm{D}(A)_{\tau^\vee,\mathfrak{s}^\vee}.$$

We can observe that

$$R^s\mathrm{D}\bigl(H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})\bigr)_{\tau^\vee,\mathfrak{s}^\vee} = R^s\mathrm{D}\bigl(H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})_{\tau,\mathfrak{s}}\bigr) = R^s\mathrm{D}\bigl(H_c^{-t}(M_\infty)_{\tau,\mathfrak{s}}\bigr).$$

Indeed, for the $\tau^\vee$-part, notice the isotypic decomposition

$$H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}}) = \bigoplus_{\sigma} H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})_\sigma \otimes \sigma,$$

where σ runs through irreducible representations of H. Since each σ is finite-dimensional and there are only finitely many such σ, we have

$$R^s\mathrm{D}\bigl(H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})\bigr) = \bigoplus_{\sigma} R^s\mathrm{D}\bigl(H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})_\sigma\bigr) \otimes \sigma^\vee,$$

and thus $R^s\mathrm{D}(H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}}))_{\tau^\vee} = R^s\mathrm{D}(H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})_\tau)$. For the $\mathfrak{s}$-part, see [Far06, Remarque 1.5].

By the assumption, $H_c^{-t}(M_\infty)_{\tau,\mathfrak{s}}$ has finite length as a J-representation. Therefore, Theorem 2.5 tells us that $R^s\mathrm{D}\bigl(H_c^{-t}(M_{K_{\mathscr{L},m}}/p^{\mathbb{Z}})\bigr)_{\tau^\vee,\mathfrak{s}^\vee} = 0$ unless $s = \iota(\mathfrak{s})$. Thus, we have

$$R^i\mathrm{D}(A)_{\tau^\vee,\mathfrak{s}^\vee} \cong R^{\iota(\mathfrak{s})}\mathrm{D}\bigl(H_c^{-i+\iota(\mathfrak{s})}(M_\infty)_{\tau,\mathfrak{s}}\bigr) = \mathrm{Zel}\bigl(H_c^{-i+\iota(\mathfrak{s})}(M_\infty)^\vee_{\tau,\mathfrak{s}}\bigr).$$

Hence we obtain a $J \times W_{\mathbb{Q}_p}$-equivariant isomorphism

$$H^{2d+i}_{\mathcal{C}_{\mathcal{M}}}(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}(d) \cong \mathrm{Zel}\bigl(H_c^{-i+\iota(\mathfrak{s})}(M_\infty)^\vee_{\tau,\mathfrak{s}}\bigr).$$

Replacing i by $\iota(\mathfrak{s}) - i$, we conclude the theorem. ∎

To apply Theorem 5.6, we need the following technical assumption.

Assumption 5.8 For each integer q and each compact open subgroup H of J, the G-representation $H_c^q(M_\infty)^H$ is finitely generated.

Remark 5.9 We know that for every compact open subgroup K of G, the J-representation $H_c^q(M_\infty)^K$ is finitely generated (*cf.* [Far04, Proposition 4.4.13]; see also [Mie13, Theorem 4.4]). Therefore, if we can establish an analogue of Faltings' isomorphism for our tower $\{M_K\}_K$, we can prove Assumption 5.8 by switching the role of G and J.

Lemma 5.10 i) *For an irreducible supercuspidal representation π of G, there exists a compact-mod-center open subgroup $\widetilde{K}$ of G and an irreducible smooth representation τ of $\widetilde{K}$ such that* $\text{c-Ind}_{\widetilde{K}}^G \tau$ *is admissible (hence supercuspidal) and π is a direct summand of* $\text{c-Ind}_{\widetilde{K}}^G \tau$.

ii) *If $n = 2$, under the same setting as i), we can take $(\widetilde{K}, \tau)$ so that* $\pi = \text{c-Ind}_{\widetilde{K}}^G \tau$.

iii) *Let $(\widetilde{K}, \tau)$ be as in i). Under Assumption 5.8, $H_c^q(M_\infty)_\tau$ is a finite length J-representation for every integer q.*

Proof. i) Put $G_1 = \mathrm{Sp}_{2n}(\mathbb{Q}_p)$ and write G' for the image of $\mathbb{Q}_p^\times \times G_1 \longrightarrow G$; $(c, g) \longmapsto cg$. Let π' be the restriction of π to G'. Since G' is a normal subgroup of index 2 in G, π' has finite length. Take an irreducible subrepresentation π_1 of π', and regard it as a representation of $\mathbb{Q}_p^\times \times G_1$. It is irreducible and supercuspidal. By [Ste08, Theorem 7.14], there exist a compact open subgroup K of G_1, an irreducible smooth representation σ of K and a smooth character χ of $\mathbb{Q}_p^\times$ such that $\pi_1 \cong \chi \otimes \text{c-Ind}_K^{G_1} \sigma$. Let $\widetilde{K}$ be the image of $\mathbb{Q}_p^\times \times K$ in G_1. Then, $\chi \otimes \sigma$ descends to an irreducible smooth representation τ of $\widetilde{K}$ and we have $\text{c-Ind}_{\widetilde{K}}^{G'} \tau \cong \pi_1$. Then, $\text{c-Ind}_{\widetilde{K}}^G \tau \cong \text{c-Ind}_{G'}^G \pi_1$ is admissible. If $\pi_1 \neq \pi'$, $\text{c-Ind}_{\widetilde{K}}^G \tau$ is isomorphic to π; otherwise $\text{c-Ind}_{\widetilde{K}}^G \tau$ is the direct sum of two supercuspidal representations, one of which is isomorphic to π.

ii) is proved in [Moy88]; see the final comment in [Moy88, p. 328].

iii) Let χ be the central character of $\tau^\vee$. By the Frobenius reciprocity, we have $H_c^q(M_\infty)_\tau \cong \mathrm{Hom}_G(\text{c-Ind}_{\widetilde{K}}^G \tau, H_c^q(M_\infty) \otimes_{\mathcal{H}(Z_G)} \chi^{-1})$. As $\text{c-Ind}_{\widetilde{K}}^G \tau$ is a supercuspidal representation of finite length with central character χ^{-1}, it is a finite direct sum of irreducible supercuspidal representations of G. Therefore, it suffices to show that for an irreducible supercuspidal representation π'' of G with central character χ^{-1}, $\mathrm{Hom}_G(\pi'', H_c^q(M_\infty) \otimes_{\mathcal{H}(Z_G)} \chi^{-1})$ is a J-representation of finite length. We apply [Mie12a, Lemma 5.2] to $H_c^q(M_\infty) \otimes_{\mathcal{H}(Z_G)} \chi^{-1}$. By [Far04, Proposition 4.4.13] and Assumption 5.8, all the conditions in [Mie12a, Lemma 5.2] are satisfied. Hence we conclude that $\mathrm{Hom}_G(\pi'', H_c^q(M_\infty) \otimes_{\mathcal{H}(Z_G)} \chi^{-1})$ has finite length (see the final paragraph of the proof of [Mie12a, Lemma 5.2]). ∎

Now we will give some consequences of Theorem 5.6. For simplicity, we focus on the cohomology $H_c^i(M_\infty/p^{\mathbb{Z}})$.

Corollary 5.11 *Suppose Assumption 5.8. Let π be an irreducible supercuspidal representation of G and ρ an irreducible non-supercuspidal representation of J. Then, the representation $\pi\otimes\rho$ of $G\times J$ does not appear as a subquotient of $H_c^{d-\dim\mathcal{M}^{\mathrm{red}}}(M_\infty/p^{\mathbb{Z}})$.*

Proof. Clearly we may assume that π (resp. ρ) is trivial on $p^{\mathbb{Z}}\subset G$ (resp. $p^{\mathbb{Z}}\subset J$). Take $(\widetilde{K},\tau)$ as in Lemma 5.10 i). Then τ is trivial on $p^{\mathbb{Z}}\subset G$. Let $\mathfrak{s}\in I_{p^{\mathbb{Z}}}$ be the inertially equivalence class of cuspidal data for J such that $\rho\in\mathbf{Rep}(J/p^{\mathbb{Z}})_{\mathfrak{s}}$. As ρ is non-supercuspidal, we have $\iota(\mathfrak{s})\ge 1$.

Put $d_0=\dim\mathcal{M}^{\mathrm{red}}$. As π is projective in the category $\mathbf{Rep}(G/p^{\mathbb{Z}})$, $\pi\otimes\rho$ appears in $H_c^{d-d_0}(M_\infty/p^{\mathbb{Z}})$ if and only if ρ appears in $\mathrm{Hom}_G(\pi,H_c^{d-d_0}(M_\infty/p^{\mathbb{Z}}))$. Since $\mathrm{Hom}_G(\pi,H_c^{d-d_0}(M_\infty/p^{\mathbb{Z}}))$ is embedded into

$$\mathrm{Hom}_G\bigl(\text{c-Ind}_{\widetilde{K}}^G\tau,H_c^{d-d_0}(M_\infty/p^{\mathbb{Z}})\bigr)=H_c^{d-d_0}(M_\infty)_\tau,$$

it suffices to show that $H_c^{d-d_0}(M_\infty)_{\tau,\mathfrak{s}}=0$. By Theorem 5.6 and Lemma 5.10 iii), we have $\mathrm{Zel}(H_c^{d-d_0}(M_\infty)_{\tau,\mathfrak{s}}^\vee)\cong H_{\mathcal{C}_{\mathcal{M}}}^{d+d_0+\iota(\mathfrak{s})}(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}(d)$. By Proposition 5.5 ii), the right hand side equals 0. Hence we conclude that $H_c^{d-d_0}(M_\infty)_{\tau,\mathfrak{s}}=0$. ∎

Corollary 5.12 *In addition to Assumption 5.8, assume that $n=2$. Let π and ρ be as in Corollary 5.11, and σ an irreducible ℓ-adic representation of $W_{\mathbb{Q}_p}$. Then, $\pi\otimes\rho\otimes\sigma$ appears as a subquotient of $H_c^3(M_\infty/p^{\mathbb{Z}})$ if and only if $\pi^\vee\otimes\mathrm{Zel}(\rho^\vee)\otimes\sigma^\vee(-3)$ appears as a subquotient of $H_c^4(M_\infty/p^{\mathbb{Z}})$.*

Proof. Again we may assume that π (resp. ρ) is trivial on $p^{\mathbb{Z}}\subset G$ (resp. $p^{\mathbb{Z}}\subset J$). Let $\mathfrak{s}\in I_{p^{\mathbb{Z}}}$ be as in the proof of Corollary 5.11. In this case we have $\iota(\mathfrak{s})=1$. Take $(\widetilde{K},\tau)$ as in Lemma 5.10 ii). Note that, since $\pi=\text{c-Ind}_{\widetilde{K}}^G\tau$ is irreducible, we have $\pi^\vee=(\text{c-Ind}_{\widetilde{K}}^G\tau)^\vee=\mathrm{Ind}_{\widetilde{K}}^G\tau^\vee=\text{c-Ind}_{\widetilde{K}}^G\tau^\vee$. In the same way as in the proof of Corollary 5.11, we can see that $\pi\otimes\rho\otimes\sigma$ appears as a subquotient of $H_c^3(M_\infty/p^{\mathbb{Z}})$ if and only if $\rho\otimes\sigma$ appears as a subquotient of $H_c^3(M_\infty)_{\tau,\mathfrak{s}}$. Similarly, $\pi^\vee\otimes\mathrm{Zel}(\rho^\vee)\otimes\sigma^\vee(-3)$ appears as a subquotient of $H_c^4(M_\infty/p^{\mathbb{Z}})$ if and only if $\mathrm{Zel}(\rho^\vee)\otimes\sigma^\vee(-3)$ appears as a subquotient of $H_c^4(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}$.

On the other hand, by Theorem 5.6 and Lemma 5.10 iii), we have

$$\mathrm{Zel}\bigl(H_c^3(M_\infty)_{\tau,\mathfrak{s}}^\vee\bigr)\cong H_{\mathcal{C}_{\mathcal{M}}}^4(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}(3).$$

Corollary 5.4 tells us that

$$\begin{aligned}H_{\mathcal{C}_{\mathcal{M}}}^4(M_\infty)_{\tau^\vee}&=\mathrm{Hom}_{\widetilde{K}}\bigl(\tau^\vee,H_{\mathcal{C}_{\mathcal{M}}}^4(M_\infty/p^{\mathbb{Z}})\bigr)=\mathrm{Hom}_G\bigl(\pi^\vee,H_{\mathcal{C}_{\mathcal{M}}}^4(M_\infty/p^{\mathbb{Z}})\bigr)\\&\cong\mathrm{Hom}_G\bigl(\pi^\vee,H_c^4(M_\infty/p^{\mathbb{Z}})\bigr)=H_c^4(M_\infty)_{\tau^\vee}.\end{aligned}$$

Thus we have an isomorphism of $J \times W_{\mathbb{Q}_p}$-representations

$$\mathrm{Zel}\bigl(H_c^3(M_\infty)_{\tau,\mathfrak{s}}^\vee\bigr) \cong H_c^4(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}(3).$$

In particular, $\rho\otimes\sigma$ appears in $H_c^3(M_\infty)_{\tau,\mathfrak{s}}$ if and only if $\mathrm{Zel}(\rho^\vee)\otimes\sigma^\vee(-3)$ appears in $H_c^4(M_\infty)_{\tau^\vee,\mathfrak{s}^\vee}$. This concludes the proof. ∎

Remark 5.13 i) In the case $n=2$, by a global method, Tetsushi Ito and the author proved that for a supercuspidal representation π of G and a supercuspidal representation ρ of J, $\pi\otimes\rho$ does not appear in $H_c^i(M_\infty/p^{\mathbb{Z}})$ unless $i=3$. Together with Corollary 5.11, we obtain the vanishing of the G-cuspidal part $H_c^2(M_\infty/p^{\mathbb{Z}})_{G\text{-cusp}}=0$ (note that in this case $\dim\mathcal{M}^{\mathrm{red}}=1$).

ii) Consider the case $n=2$. Let π be an irreducible supercuspidal representation of G which is trivial on $p^{\mathbb{Z}}$, ϕ the L-parameter of π (*cf.* [GT11]), and Π_ϕ^J the L-packet of J attached to ϕ (*cf.* [GT]). Suppose that there exists a non-supercuspidal representation ρ in Π_ϕ^J. Then, Π_ϕ^J consists of two representations ρ and ρ', where ρ' is supercuspidal. In this case, $\{\mathrm{Zel}(\rho),\rho'\}$ is known to be a non-tempered A-packet.

Motivated by Corollary 5.12 and the results in [Mie12b], the author expects the following:

- $\pi\otimes\rho^\vee$ and $\pi\otimes\rho'^\vee$ appear in $H_c^3(M_\infty/p^{\mathbb{Z}})$, and
- $\pi\otimes\mathrm{Zel}(\rho)^\vee$ appears in $H_c^4(M_\infty/p^{\mathbb{Z}})$.

This problem will be considered in a forthcoming joint paper with Tetsushi Ito. The speculation above suggests the existence of some relation between local A-packets and the cohomology of the Rapoport-Zink tower. In the case of $\mathrm{GL}(n)$, a result in this direction is obtained by Dat [Dat12].

Remark 5.14 i) The arguments here can be applied to more general Rapoport-Zink towers. In [Mie13], the geometric properties used in the proof of Theorem 5.6 are obtained for many Rapoport-Zink spaces. See [Mie13, Theorem 2.6, Corollary 4.3, Theorem 4.4] especially.

ii) There is another way to generalize the result in [Far06]; the author expects some relation between $\mathrm{Zel}_{G\times J}(H_c^i(M_\infty)_{\mathfrak{s}}^\vee)$ and $H_c^{2d+\iota(\mathfrak{s})-i}(M_\infty)_{\mathfrak{s}^\vee}(d)$, where $\mathfrak{s}$ is an inertially equivalence class of cuspidal data for $G\times J$. In a subsequent paper, the author plans to work on this problem in the case of $\mathrm{GSp}(4)$. However, the case where $n\geq 3$ seems much more difficult, and Theorem 5.6 has advantage that it is valid for all the $\mathrm{GSp}(2n)$ cases (and many other cases).

References

[Ber84] J. N. Bernstein, *Le “centre” de Bernstein*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, pp. 1–32.

[Ber94] V. G. Berkovich, *Vanishing cycles for formal schemes*, Invent. Math. **115** (1994), no. 3, 539–571.

[Ber96] ———, *Vanishing cycles for formal schemes. II*, Invent. Math. **125** (1996), no. 2, 367–390.

[Boy09] P. Boyer, *Monodromie du faisceau pervers des cycles évanescents de quelques variétés de Shimura simples*, Invent. Math. **177** (2009), no. 2, 239–280.

[Car90] H. Carayol, *Nonabelian Lubin-Tate theory*, Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), Perspect. Math., vol. 11, Academic Press, Boston, MA, 1990, pp. 15–39.

[Dat12] J.-F. Dat, *Opérateur de Lefschetz sur les tours de Drinfeld et Lubin-Tate*, Compos. Math. **148** (2012), no. 2, 507–530.

[Fal02] G. Faltings, *A relation between two moduli spaces studied by V. G. Drinfeld*, Algebraic number theory and algebraic geometry, Contemp. Math., vol. 300, Amer. Math. Soc., Providence, RI, 2002, pp. 115–129.

[Far04] L. Fargues, *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales*, Astérisque (2004), no. 291, 1–199, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales.

[Far06] ———, *Dualité de Poincaré et involution de Zelevinsky dans la cohomologie étale équivariante des espaces analytiques rigides*, preprint, `http://www.math.jussieu.fr/~fargues/Prepublications.html`, 2006.

[FGL08] L. Fargues, A. Genestier, and V. Lafforgue, *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, Progress in Mathematics, vol. 262, Birkhäuser Verlag, Basel, 2008.

[GT] W. T. Gan and W. Tantono, *The local Langlands conjecture for* GSp(4) *II: the case of inner forms*, preprint.

[GT11] W. T. Gan and S. Takeda, *The local Langlands conjecture for* GSp(4), Ann. of Math. (2) **173** (2011), no. 3, 1841–1882.

[Har97] M. Harris, *Supercuspidal representations in the cohomology of Drinfel'd upper half spaces; elaboration of Carayol's program*, Invent. Math. **129** (1997), no. 1, 75–119.

[Hoc69] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969), 43–60.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[Hub93] R. Huber, *Continuous valuations*, Math. Z. **212** (1993), no. 3, 455–477.

[Hub94] ———, *A generalization of formal schemes and rigid analytic varieties*, Math. Z. **217** (1994), no. 4, 513–551.

[Hub96] ______, *Étale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30, Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Hub98a] ______, *A finiteness result for direct image sheaves on the étale site of rigid analytic varieties*, J. Algebraic Geom. **7** (1998), no. 2, 359–403.

[Hub98b] ______, *A comparison theorem for l-adic cohomology*, Compositio Math. **112** (1998), no. 2, 217–235.

[IM10] T. Ito and Y. Mieda, *Cuspidal representations in the ℓ-adic cohomology of the Rapoport-Zink space for* $\mathrm{GSp}(4)$, preprint, arXiv:1005.5619, 2010.

[Jan88] U. Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), no. 2, 207–245.

[Man05] E. Mantovan, *On the cohomology of certain PEL-type Shimura varieties*, Duke Math. J. **129** (2005), no. 3, 573–610.

[Mie10] Y. Mieda, *Variants of formal nearby cycles*, preprint, arXiv:1005.5616, to appear in J. Inst. Math. Jussieu, 2010.

[Mie12a] ______, *Lefschetz trace formula and ℓ-adic cohomology of Lubin-Tate tower*, Math. Res. Lett. **19** (2012), no. 1, 95–107.

[Mie12b] ______, *Lefschetz trace formula and ℓ-adic cohomology of Rapoport-Zink tower for* $\mathrm{GSp}(4)$, preprint, arXiv:1212.4922, 2012.

[Mie13] ______, *On irreducible components of Rapoport-Zink spaces*, preprint, arXiv:1312.4784, 2013.

[Moy88] A Moy, *Representations of $G\,\mathrm{Sp}(4)$ over a p-adic field. I, II*, Compositio Math. **66** (1988), no. 3, 237–284, 285–328.

[Rap95] M. Rapoport, *Non-Archimedean period domains*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 423–434.

[Ren10] D. Renard, *Représentations des groupes réductifs p-adiques*, Cours Spécialisés, vol. 17, Société Mathématique de France, Paris, 2010.

[RZ96] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.

[SS97] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 85, 97–191.

[Ste08] S. Stevens, *The supercuspidal representations of p-adic classical groups*, Invent. Math. **172** (2008), no. 2, 289–352.

[Vig90] M.-F. Vignéras, *On formal dimensions for reductive p-adic groups*, Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), Israel Math. Conf. Proc., vol. 2, Weizmann, Jerusalem, 1990, pp. 225–266.

[EGA] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. (1961–1967), no. 4, 8, 11, 17, 20, 24, 28, 32.