Lefschetz trace formula and ℓ-adic cohomology of Rapoport-Zink tower for GSp(4)

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ABSTRACT. We investigate the alternating sum of the ℓ-adic cohomology of the Rapoport-Zink tower for GSp(4) by the Lefschetz trace formula. Under some assumptions on L-packets of GSp(4) and its inner form, we observe that the local Jacquet-Langlands correspondence appears in the cohomology.

1 Introduction

A Rapoport-Zink space is a certain moduli space of deformations by quasi-isogenies of a p-divisible group with additional structures. By using level structures on the universal p-divisible group, we can construct a projective system of étale coverings over the rigid generic fiber of the Rapoport-Zink space. This projective system is called the Rapoport-Zink tower. It can be regarded as a local analogue of a tower of Shimura varieties of PEL type.

By taking a compactly supported ℓ-adic cohomology of the tower, we obtain a representation H_{RZ}^i of $\mathbf{G}(\mathbb{Q}_p) \times \mathbf{J}(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$, where $\mathbf{G}$ is the reductive group which is naturally attached to the local Shimura datum defining the Rapoport-Zink space, $\mathbf{J}$ is an inner form of $\mathbf{G}$, and $W_{\mathbb{Q}_p}$ is the Weil group of $\mathbb{Q}_p$. It is expected that the alternating sum $H_{\mathrm{RZ}} = \sum_i (-1)^i H_{\mathrm{RZ}}^i$ of H_{RZ}^i can be described by the (still conjectural) local Langlands correspondence of $\mathbf{G}$ and $\mathbf{J}$ (cf. [Rap95]).

The most classical examples of the Rapoport-Zink tower are the Lubin-Tate tower and the Drinfeld tower. In these cases, the expectation above is called the non-abelian Lubin-Tate theory (cf. [Car90]) and has been already proven ([Har97], [HT01]). There are more precise studies on the individual cohomology H_{RZ}^i; see [Boy09] and [Dat07].

In this paper, we consider the case where $\mathbf{G} = \mathrm{GSp}_4$. In this case, the Rapoport-Zink space $\breve{\mathscr{M}}$ is the moduli space of deformations by quasi-isogenies of a principally polarized 2-dimensional p-divisible group with slope $1/2$. We ignore the action of the Weil group $W_{\mathbb{Q}_p}$ and concentrate on the action of $\mathbf{G}(\mathbb{Q}_p) \times \mathbf{J}(\mathbb{Q}_p)$ on H_{RZ}^i. Our main result can be summarized as follows:

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Theorem 1.1 (Theorem 7.8, Corollary 7.9) *For an irreducible smooth representation ρ of $\mathbf{J}(\mathbb{Q}_p)$, we put $H_{\mathrm{RZ}}[\rho] = \sum_{i,j\geq 0}(-1)^{i+j}\operatorname{Ext}^j_{\mathbf{J}(\mathbb{Q}_p)}(H^i_{\mathrm{RZ}}, \rho)^{\mathrm{sm}}$, where $\operatorname{Ext}^j_{\mathbf{J}(\mathbb{Q}_p)}$ is taken in the category of smooth $\mathbf{J}(\mathbb{Q}_p)$-representations and $(-)^{\mathrm{sm}}$ denotes the set of $\mathbf{G}(\mathbb{Q}_p)$-smooth vectors. Let $\phi\colon W_{\mathbb{Q}_p}\times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C})$ be an L-parameter which is relevant for $\mathbf{J}(\mathbb{Q}_p)$. Assume that the L-packets $\Pi^{\mathbf{G}(\mathbb{Q}_p)}_\phi$ and $\Pi^{\mathbf{J}(\mathbb{Q}_p)}_\phi$ corresponding to ϕ are stable and satisfy the character relation (see Section 7.1 for notation on L-parameters and L-packets).*

Then, for an element of the Hecke algebra $f \in \mathcal{H}(\mathbf{G}(\mathbb{Q}_p))$ supported on regular elliptic elements, we have

$$\sum_{\rho\in\Pi^{\mathbf{J}(\mathbb{Q}_p)}_\phi}\operatorname{Tr}(f; H_{\mathrm{RZ}}[\rho]) = -4\sum_{\pi\in\Pi^{\mathbf{G}(\mathbb{Q}_p)}_\phi}\operatorname{Tr}(f; \pi).$$

Moreover, if the $\mathbf{G}(\mathbb{Q}_p)$-representation $\operatorname{Ext}^j_{\mathbf{J}(\mathbb{Q}_p)}(H^i_{\mathrm{RZ}}, \rho)^{\mathrm{sm}}$ has finite length for every $i, j \geq 0$ and $\rho \in \Pi^{\mathbf{J}(\mathbb{Q}_p)}_\phi$, we have

$$\sum_{\rho\in\Pi^{\mathbf{J}(\mathbb{Q}_p)}_\phi}\theta_{H_{\mathrm{RZ}}[\rho]}(g) = -4\sum_{\pi\in\Pi^{\mathbf{G}(\mathbb{Q}_p)}_\phi}\theta_\pi(g)$$

for every regular elliptic element g of $\mathbf{G}(\mathbb{Q}_p)$. Here $\theta_{H_{\mathrm{RZ}}[\rho]}$ and θ_π denote the distribution characters of $H_{\mathrm{RZ}}[\rho]$ and π respectively, which are locally constant functions over regular elements of $\mathbf{G}(\mathbb{Q}_p)$.

Very roughly speaking, this theorem says that the local Jacquet-Langlands correspondence $\Pi^{\mathbf{G}(\mathbb{Q}_p)}_\phi \leftrightarrow \Pi^{\mathbf{J}(\mathbb{Q}_p)}_\phi$ appears in H_{RZ}.

To prove the theorem above, we will apply the Lefschetz trace formula for adic spaces developed in [Mie10a]; we count fixed points on the Rapoport-Zink space under the action of elements in $\mathbf{G}(\mathbb{Q}_p) \times \mathbf{J}(\mathbb{Q}_p)$ to compute the trace on the cohomology H_{RZ}. Such a method goes back to a pioneering work of Faltings [Fal94], in which he treated the Drinfeld tower. A similar study for the Lubin-Tate tower has been carried out by Strauch [Str08]. Needless to say, our work is strongly inspired by these two works. However, our case is more difficult than the classical cases in the following two points. First, any connected component of our Rapoport-Zink space $\breve{\mathcal{M}}$ is neither quasi-compact nor p-adic, therefore harder to deal with. This is related to the fact that neither $\mathbf{G}(\mathbb{Q}_p)$ nor $\mathbf{J}(\mathbb{Q}_p)$ is compact modulo center (in the classical cases, $\mathbf{G}(\mathbb{Q}_p)$ or $\mathbf{J}(\mathbb{Q}_p)$ is the multiplicative group of a division algebra). Recall that, in the Lubin-Tate case the connected component is the formal spectrum of a complete local ring. This fact makes the approximation arguments in [Str08, §2.3, §3.2, §3.3] possible. In our case it is impossible to apply the same method. To avoid this problem, we require the Lefschetz trace formula proved in [Mie10a]. The other point is representation-theoretic one; the local Langlands correspondences for $\mathbf{G}(\mathbb{Q}_p)$ and $\mathbf{J}(\mathbb{Q}_p)$ are not bijective. Under the "dictionary" between irreducible

representations and conjugacy classes, this corresponds to the fact that conjugacy and stable conjugacy are different in $\mathbf{G}(\mathbb{Q}_p)$ and $\mathbf{J}(\mathbb{Q}_p)$. This difference makes our argument on counting points and harmonic analysis more subtle.

We sketch the outline of this paper. In Section 2, we introduce some notation on algebraic groups and stable orbital integrals, which will be used throughout this paper. In section 3, after recalling basic definitions on the Rapoport-Zink tower for GSp_{2d}, we count fixed points on the Rapoport-Zink space under the action of an element $(g, h) \in \mathbf{G}(\mathbb{Q}_p) \times \mathbf{J}(\mathbb{Q}_p)$. Our method of counting is similar to [Str08, §2.6]; we use the period map introduced in [RZ96, Chapter 5] and the p-adic Hodge theory for p-divisible groups. In Section 4, we construct formal models of the Rapoport-Zink spaces with some higher levels (more precisely, levels which are open normal subgroups of parahoric subgroups of $\mathbf{G}(\mathbb{Q}_p)$). Moreover, we introduce "boundary strata" of these formal models and investigate group actions on them. These constructions are extremely important for applying the Lefschetz trace formula such as [Mie10a, Theorem 4.5]. Basically, the content of this section (especially Proposition 4.11) forces us to assume that $d = 2$. In Section 5, we construct a nice open covering of the Rapoport-Zink space with parahoric level. The construction is similar to the case of the Drinfeld upper half space, which has an open covering indexed by vertices of the Bruhat-Tits building for PGL_n. In Section 6, we apply the Lefschetz trace formula to a finite union of open subsets belonging to the open covering constructed in Section 5. Finally in Section 7, we briefly review the local Langlands correspondence for $\mathbf{G}(\mathbb{Q}_p)$ and $\mathbf{J}(\mathbb{Q}_p)$ due to Gan-Takeda [GT11a] and Gan-Tantono [GT] respectively, and give a proof of the main theorem. We use the harmonic-analytic method introduced in [Mie12].

During writing this paper, the author found a preprint by Xu Shen [She12a] in which a related topic was studied. Our works are totally independent, and our methods are also different. Moreover, the main result in [She12a] does not seem completely sufficient for deducing representation-theoretic results such as Theorem 1.1. See Remark 7.13 for detailed comments.

Notation

Let $d \geq 1$ be an integer. For a ring A, let $\langle \ , \ \rangle \colon A^{2d} \times A^{2d} \longrightarrow A$ be the symplectic pairing defined as follows: for $x = (x_i), y = (y_i) \in A^{2d}$,

$$\langle x, y \rangle = x_1 y_{2d} + \cdots + x_d y_{d+1} - x_{d+1} y_d - \cdots - x_{2d} y_1.$$

We denote by $\mathrm{GSp}_{2d}(A)$ the symplectic similitude group with respect to the symplectic pairing $\langle \ , \ \rangle$.

For a field k, we denote its algebraic closure by $\overline{k}$. Fix a prime number p. For an integer $m \geq 1$, we denote by $\mathbb{Q}_{p^m}$ the unique degree m unramified extension of $\mathbb{Q}_p$,

and by $\mathbb{Z}_{p^m}$ the ring of integers of $\mathbb{Q}_{p^m}$. We denote by $\mathbb{Q}_{p^\infty}$ the completion of the maximal unramified extension of $\mathbb{Q}_p$, and by $\mathbb{Z}_{p^\infty}$ the ring of integers of $\mathbb{Q}_{p^\infty}$. Let ℓ be a prime number distinct from p. We fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and identify them. Every representation is considered over $\mathbb{C}$, and every function is $\mathbb{C}$-valued.

For a totally disconnected locally compact group G with a fixed Haar measure, we denote by $\mathcal{H}(G)$ the Hecke algebra of G, namely, the abelian group of locally constant compactly supported functions on G with convolution product. Put $\overline{\mathcal{H}}(G) = \mathcal{H}(G)/[\mathcal{H}(G), \mathcal{H}(G)] = \mathcal{H}(G)_G$ (the G-coinvariant quotient). For smooth representations π_1, π_2 of G, we denote by $\mathrm{Ext}^i_G(\pi_1, \pi_2)$ the ith Ext group in the category of smooth G-representations.

2 Notation on stable conjugacy classes

In this section, we introduce some basic notation on algebraic groups and harmonic analysis. Here we will work on slightly general situation; let F be a p-adic field and $\mathbf{G}$ a connected reductive group over F. Put $G = \mathbf{G}(F)$. Assume for simplicity that the derived group of $\mathbf{G}$ is simply connected. We denote by $\mathbf{Z}_{\mathbf{G}}$ the center of $\mathbf{G}$ and put $Z_G = \mathbf{Z}_{\mathbf{G}}(F)$. For $g \in G$, let $\mathbf{Z}(g)$ denote the centralizer of g and put $Z(g) = \mathbf{Z}(g)(F)$. Since we assume that the derived group of $\mathbf{G}$ is simply connected, $\mathbf{Z}(g)$ is connected for a semisimple g. We say that g is regular if $\mathbf{Z}(g)$ is a maximal torus of $\mathbf{G}$ (note that a regular element is assumed to be semisimple). We write G^{reg} for the set of regular elements of G. For a maximal torus $\mathbf{T}$ of $\mathbf{G}$, we put $T^{\mathrm{reg}} = \mathbf{T}(F) \cap G^{\mathrm{reg}}$. We say that g is elliptic if it is contained in an elliptic maximal torus. If g is regular, this is equivalent to saying that $\mathbf{Z}(g)$ is an elliptic maximal torus. We write G^{ell} for the set of regular elliptic elements of G.

Two elements $g_1, g_2 \in G = \mathbf{G}(F)$ is said to be stably conjugate if they are conjugate in $\mathbf{G}(\overline{F})$. For $g \in G$, we write $\{g\}$ (resp. $\{g\}_{\mathrm{st}}$) for the conjugacy class (resp. stable conjugacy class) of g. It is well-known that $\{g\}_{\mathrm{st}}/\sim$, the set of conjugacy classes in $\{g\}_{\mathrm{st}}$, is a finite set if g is regular.

Two maximal tori $\mathbf{T}_1$, $\mathbf{T}_2$ of $\mathbf{G}$ are said to be stably conjugate if $T_1 = \mathbf{T}_1(F)$ and $T_2 = \mathbf{T}_2(F)$ are conjugate in $\mathbf{G}(\overline{F})$. For such tori $\mathbf{T}_1$, $\mathbf{T}_2$ and elements $g_1 \in T_1^{\mathrm{reg}}$, $g_2 \in T_2^{\mathrm{reg}}$ which are stably conjugate, we can construct an isomorphism $\iota_{g_1,g_2} \colon \mathbf{T}_1 \xrightarrow{\cong} \mathbf{T}_2$ as follows. Take $h \in \mathbf{G}(\overline{F})$ such that $g_2 = h^{-1}g_1 h$. Since g_1 and g_2 are F-valued points, such h satisfies $h\sigma(h)^{-1} \in \mathbf{T}_1(\overline{F})$ for every $\sigma \in \mathrm{Gal}(\overline{F}/F)$. By this fact, it is easy to see that $\mathbf{T}_1 \otimes_F \overline{F} \longrightarrow \mathbf{T}_2 \otimes_F \overline{F}$; $g \longmapsto h^{-1}gh$ descends to an isomorphism $\iota_{g_1,g_2} \colon \mathbf{T}_1 \xrightarrow{\cong} \mathbf{T}_2$ over F. It does not depend on the choice of h. In particular, stably conjugate maximal tori are isomorphic, and thus a maximal tori which is stably conjugate to an elliptic torus is elliptic.

For a maximal torus $\mathbf{T}$, we write $\{\mathbf{T}\}$ (resp. $\{\mathbf{T}\}_{\mathrm{st}}$) for its conjugacy class (resp. stable conjugacy class). We denote the set of conjugacy classes of maximal tori (resp. elliptic maximal tori) of $\mathbf{G}$ by $\mathcal{T}_{\mathbf{G}}$ (resp. $\mathcal{T}_{\mathbf{G}}^{\mathrm{ell}}$), and the set of stable conjugacy classes of maximal tori (resp. elliptic maximal tori) of $\mathbf{G}$ by $\mathcal{T}_{\mathbf{G},\mathrm{st}}$ (resp. $\mathcal{T}_{\mathbf{G},\mathrm{st}}^{\mathrm{ell}}$).

Fix a Haar measure on G. For an element $g \in G^{\mathrm{reg}}$, we also choose a Haar measure on $Z(g)$. Then, for each $g' \in \{g\}_{\mathrm{st}}$, $Z(g')$ is naturally equipped with a Haar measure induced by the isomorphism $\iota_{g,g'} \colon Z(g) \longrightarrow Z(g')$. For a locally constant function f on G whose support is compact modulo Z_G, we set

$$O_g(f) = \int_{Z(g)\backslash G} f(h^{-1}gh)dh, \quad SO_g(f) = \sum_{g' \in \{g\}_{\mathrm{st}}/\sim} O_{g'}(f),$$

and call them the orbital integral and the stable orbital integral of f, respectively. It is well-known that $O_g(f)$ always converges ([RR72]).

Next we compare stable conjugacy classes between inner forms. Let $\mathbf{G}'$ be an inner form of $\mathbf{G}$, and fix an inner twist $\xi \colon \mathbf{G}' \otimes_F \overline{F} \xrightarrow{\cong} \mathbf{G} \otimes_F \overline{F}$. For $g \in G$ and $g' \in G' = \mathbf{G}'(F)$, g is said to be a transfer of g' with respect to ξ if g and $\xi(g')$ are conjugate in $\mathbf{G}(\overline{F})$. We also say that g and g' match, and write $g \leftrightarrow g'$. For $g \in G^{\mathrm{reg}}$ and $g' \in G'^{\mathrm{reg}}$ with $g \leftrightarrow g'$, we can construct an isomorphism $\iota_{g',g} \colon \mathbf{Z}(g') \xrightarrow{\cong} \mathbf{Z}(g)$ in the same way as above. In particular, g is elliptic if and only if g' is elliptic.

By [Kot86, Lemma 10.2], there is a natural bijection $\mathcal{T}^{\mathrm{ell}}_{\mathbf{G}',\mathrm{st}} \xrightarrow{\cong} \mathcal{T}^{\mathrm{ell}}_{\mathbf{G},\mathrm{st}}$ such that $\{\mathbf{T}'\}_{\mathrm{st}}$ corresponds to $\{\mathbf{T}\}_{\mathrm{st}}$ if and only if $\xi(\mathbf{T}'(F))$ and $\mathbf{T}(F)$ are conjugate in $\mathbf{G}(\overline{F})$. Therefore, for every $g' \in G'^{\mathrm{ell}}$ (resp. $g \in G^{\mathrm{ell}}$), we can always find $g \in G^{\mathrm{ell}}$ (resp. $g' \in G'^{\mathrm{ell}}$) with $g \leftrightarrow g'$. In particular, stable conjugacy classes of regular elliptic elements of G are in bijection with those of G'.

The following lemma, which is used in Section 7, would be well-known.

Lemma 2.1 *Let $\mathbf{T}$ be an elliptic maximal torus of $\mathbf{G}$, and $\mathbf{T}'$ that of $\mathbf{G}'$ such that $\{\mathbf{T}'\}_{\mathrm{st}}$ corresponds to $\{\mathbf{T}\}_{\mathrm{st}}$ under the bijection above. Let $W_{\mathbf{T}}$ denote the Weyl group $N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ of $\mathbf{T}$, which is an algebraic group over F. Similarly we define $W_{\mathbf{T}'}$. Then, we have an isomorphism $W_{\mathbf{T}} \cong W_{\mathbf{T}'}$. In particular, $\#W_{\mathbf{T}}(F) = \#W_{\mathbf{T}'}(F)$.*

Proof. Take $t \in T^{\mathrm{reg}}$ and $t' \in T'^{\mathrm{reg}}$ such that $t \leftrightarrow t'$, and $h \in \mathbf{G}(\overline{F})$ such that $t = h^{-1}\xi(t')h$. Let $\xi_h \colon \mathbf{G}' \otimes_F \overline{F} \xrightarrow{\cong} \mathbf{G} \otimes_F \overline{F}$ be the composite $\mathrm{Ad}(h^{-1}) \circ \xi$. It satisfies $\xi_h(\mathbf{T}') = \mathbf{T}$, thus induces $W_{\mathbf{T}'} \otimes_F \overline{F} \xrightarrow{\cong} W_{\mathbf{T}} \otimes_F \overline{F}$. It suffices to check that this isomorphism descends to an isomorphism over F. Take $\sigma \in \mathrm{Gal}(\overline{F}/F)$. Since ξ is an inner twist, there exists $c_\sigma \in \mathbf{G}(\overline{F})$ such that $\sigma \circ \xi = \mathrm{Ad}(c_\sigma) \circ \xi \circ \sigma$. Then $t = h^{-1}\xi(t')h$ implies that $t = \mathrm{Ad}(\sigma(h)^{-1})\mathrm{Ad}(c_\sigma)\xi(t')$ (note that t and t' are rational), and thus $\sigma(h)^{-1}c_\sigma h \in \mathbf{T}(\overline{F})$. Therefore, for $g' \in N_{\mathbf{G}'}(\mathbf{T}')(\overline{F})$ we have

$$\sigma\big(\xi_h(g')\big) = \mathrm{Ad}\big(\sigma(h)^{-1}c_\sigma h\big)\xi_h\big(\sigma(g')\big) \in \xi_h\big(\sigma(g')\big)\mathbf{T}(\overline{F}).$$

This means that $\xi_h \colon W_{\mathbf{T}'} \otimes_F \overline{F} \xrightarrow{\cong} W_{\mathbf{T}} \otimes_F \overline{F}$ commutes with the action of σ, as desired. $\blacksquare$

3 Rapoport-Zink tower for GSp($2d$)

3.1 Definition of the Rapoport-Zink tower

In this subsection, we recall basic notions on the Rapoport-Zink tower. General definitions are given in [RZ96], but here we restrict ourselves to the Siegel case, namely, the case for GSp($2d$).

Fix a d-dimensional isoclinic p-divisible group $\mathbb{X}$ over $\overline{\mathbb{F}}_p$ with slope $1/2$, and a (principal) polarization $\lambda_0 \colon \mathbb{X} \xrightarrow{\cong} \mathbb{X}^\vee$ of $\mathbb{X}$, namely, an isomorphism satisfying $\lambda_0^\vee = -\lambda_0$. Let **Nilp** be the category of $\mathbb{Z}_{p^\infty}$-schemes on which p is locally nilpotent. For an object S of **Nilp**, we put $\overline{S} = S \otimes_{\mathbb{Z}_{p^\infty}} \overline{\mathbb{F}}_p$. Consider the contravariant functor $\breve{\mathcal{M}} \colon$ **Nilp** $\longrightarrow$ **Set** that associates S with the set of isomorphism classes of pairs (X, ρ) consisting of

  − a d-dimensional p-divisible group X over S,
  − and a quasi-isogeny (*cf.* [RZ96, Definition 2.8]) $\rho \colon \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{S} \longrightarrow X \otimes_S \overline{S}$,

such that there exists an isomorphism $\lambda \colon X \longrightarrow X^\vee$ which makes the following diagram commutative up to multiplication by $\mathbb{Q}_p^\times$:

$$
\begin{array}{ccc}
\mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{S} & \xrightarrow{\ \rho\ } & X \otimes_S \overline{S} \\
\downarrow{\scriptstyle \lambda_0 \otimes \mathrm{id}} & & \downarrow{\scriptstyle \lambda \otimes \mathrm{id}} \\
\mathbb{X}^\vee \otimes_{\overline{\mathbb{F}}_p} \overline{S} & \xleftarrow{\ \rho^\vee\ } & X^\vee \otimes_S \overline{S}.
\end{array}
$$

Note that such λ is uniquely determined by (X, ρ) up to multiplication by $\mathbb{Z}_p^\times$ and gives a polarization of X. It is proved by Rapoport-Zink that $\breve{\mathcal{M}}$ is represented by a special formal scheme (*cf.* [Ber96]) over $\mathrm{Spf}\,\mathbb{Z}_{p^\infty}$. Moreover, $\breve{\mathcal{M}}$ is separated over $\mathrm{Spf}\,\mathbb{Z}_{p^\infty}$ ([Far04, Lemme 2.3.23]). However, each connected component of $\breve{\mathcal{M}}$ is neither quasi-compact nor p-adic. It is known that $\dim \breve{\mathcal{M}}^{\mathrm{red}} = \lfloor d^2/4 \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x (for example, see [Vie08]), and every irreducible component of $\breve{\mathcal{M}}^{\mathrm{red}}$ is projective over $\overline{\mathbb{F}}_p$ ([RZ96, Proposition 2.32]).

Let J be the group consisting of self-quasi-isogenies h on $\mathbb{X}$ which makes the following diagram commutative up to multiplication by $\mathbb{Q}_p^\times$:

$$
\begin{array}{ccc}
\mathbb{X} & \xrightarrow{\ h\ } & \mathbb{X} \\
\downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle \lambda_0} \\
\mathbb{X}^\vee & \xleftarrow{\ h^\vee\ } & \mathbb{X}^\vee.
\end{array}
$$

Then, we can define a right action of J on $\breve{\mathcal{M}}$ by $h \colon \breve{\mathcal{M}}(S) \longrightarrow \breve{\mathcal{M}}(S); (X, \rho) \longmapsto (X, \rho \circ h)$. It is known that J is the group of $\mathbb{Q}_p$-valued points of an inner form **J** of GSp_{2d} (see the next subsection). In particular, J is naturally endowed with a topology.

We denote the rigid generic fiber $\breve{\mathscr{M}}^{\mathrm{rig}}$ of $\breve{\mathscr{M}}$ by M. It is defined as $t(\breve{\mathscr{M}}) \setminus V(p)$, where $t(\breve{\mathscr{M}})$ is the adic space associated with $\breve{\mathscr{M}}$ (*cf.* [Hub94, Proposition 4.1]). It is locally of finite type, partially proper and smooth over $\mathrm{Spa}(\mathbb{Q}_{p^\infty}, \mathbb{Z}_{p^\infty})$ ([Far04, Lemme 2.3.24]). Moreover, we know that $\dim M = d(d+1)/2$; it can be proved by using étaleness of the period map ([RZ96, Proposition 5.17]) or the p-adic uniformization theorem ([RZ96, Theorem 6.30]).

Let $\widetilde{X}$ be the universal p-divisible group over $\breve{\mathscr{M}}$ and $\widetilde{X}^{\mathrm{rig}}$ the induced p-divisible group over M. For each geometric point x of M, the rational Tate module $V_p(\widetilde{X}_x^{\mathrm{rig}})$ is endowed with a non-degenerate alternating pairing $V_p(\widetilde{X}_x^{\mathrm{rig}}) \times V_p(\widetilde{X}_x^{\mathrm{rig}}) \longrightarrow \mathbb{Q}_p(1)$ induced by a polarization on $\widetilde{X}$. It is well-defined up to $\mathbb{Z}_p^\times$-multiplication. Therefore, by taking a trivialization of the Tate twist, we get a non-degenerate symplectic form $V_p(\widetilde{X}_x^{\mathrm{rig}}) \times V_p(\widetilde{X}_x^{\mathrm{rig}}) \longrightarrow \mathbb{Q}_p$ which is well-defined up to $\mathbb{Q}_p^\times$-multiplication. By considering K-level structures on $\widetilde{X}^{\mathrm{rig}}$ for each compact open subgroup $K \subset K_0 = \mathrm{GSp}_{2d}(\mathbb{Z}_p)$, we can construct a projective system $\{M_K\}_{K \subset K_0}$ of finite étale coverings of M, which is called the Rapoport-Zink tower. If K is a normal subgroup of K_0, M_K is a finite étale Galois covering of M with Galois group K_0/K. In particular, M_{K_0} is nothing but M. For more precise description, see [RZ96, 5.34] or [IM10, §3.1].

The group J naturally acts on the projective system $\{M_K\}_{K \subset K_0}$. On the other hand, for $g \in G = \mathrm{GSp}_{2d}(\mathbb{Q}_p)$ and a compact open subgroup $K \subset K_0$ satisfying $g^{-1}Kg \subset K_0$, we can define a natural morphism $M_K \longrightarrow M_{g^{-1}Kg}$ over $\mathbb{Q}_{p^\infty}$. Therefore, we have a right action of G on the pro-object "$\varprojlim$" M_K.

Definition 3.1 For an integer i, we put

$$H_c^i(M_K) = H_c^i(M_K \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell, \qquad H_{\mathrm{RZ}}^i = \varinjlim_{K \subset K_0} H_c^i(M_K).$$

Here $H_c^i(M_K \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell)$ denotes the compactly supported ℓ-adic cohomology introduced in [Hub98]. The group $G \times J$ naturally acts on H_{RZ}^i. It is known that this action is smooth ([Ber94, Corollary 7.7], [Far04, Corollaire 4.4.7]).

Remark 3.2 We can also define a natural action of the Weil group $W_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ on H_{RZ}^i. This action is expected to be very interesting, but in this article we do not consider it.

Definition 3.3 For an irreducible smooth representation ρ of J and integers $i, j \geq 0$, we put

$$H_{\mathrm{RZ}}^{i,j}[\rho] = \mathrm{Ext}_J^j(H_{\mathrm{RZ}}^i, \rho)^{\mathcal{D}_c\text{-sm}}.$$

The definition of $(-)^{\mathcal{D}_c\text{-sm}}$ is as follows. Let $\mathcal{D}_c(G)$ denotes the convolution algebra of compactly supported distributions on G. For a (left or right) $\mathcal{D}_c(G)$-module V, we put $V^{\mathcal{D}_c\text{-sm}} = \varinjlim_K e_K V$, where K runs through compact open subgroups of G and $e_K \in \mathcal{D}_c(G)$ denotes the idempotent corresponding to K. It is a smooth representation of G. Note that $\mathrm{Ext}_J^j(H_{\mathrm{RZ}}^i, \rho)$ has a structure of a right $\mathcal{D}_c(G)$-module which comes from the left $\mathcal{D}_c(G)$-module structure on H_{RZ}^i.

Remark 3.4 By the same argument as in [Mie11, Lemma 3.1], we can show that $H_{\mathrm{RZ}}^{i,j}[\rho]^K = \mathrm{Ext}_J^j((H_{\mathrm{RZ}}^i)^K, \rho) = \mathrm{Ext}_J^j(H_c^i(M_K), \rho)$ for a compact open subgroup K of K_0.

Lemma 3.5 Let $\chi \colon \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ be an unramified character, that is, a character which is trivial on $\mathbb{Z}_p^\times$. Denote the composite $G \xrightarrow{\mathrm{sim}} \mathbb{Q}_p^\times \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$ (resp. $J \xrightarrow{\mathrm{sim}} \mathbb{Q}_p^\times \xrightarrow{\chi} \overline{\mathbb{Q}}_\ell^\times$) by χ_G (resp. χ_J), where sim denotes the similitude character. Then, we have $H_{\mathrm{RZ}}^{i,j}[\rho \otimes \chi_J] \cong H_{\mathrm{RZ}}^{i,j}[\rho] \otimes \chi_G$.

Proof. First let us recall the natural partition of $\breve{\mathscr{M}}$ into open and closed formal subschemes introduced in [RZ96, 3.52]. For an integer $\delta \in \mathbb{Z}$, let $\breve{\mathscr{M}}^{(\delta)}$ be the open and closed subscheme consisting of (X, ρ) such that $d^{-1} \cdot \mathrm{height}(\rho) = \delta$. Note that the left hand side is always an integer. Indeed, by the definition of $\breve{\mathscr{M}}$, there exist a polarization $\lambda \colon X \longrightarrow X^\vee$ and an element $a \in \mathbb{Q}_p^\times$ such that $a\lambda_0 = \rho^\vee \circ (\lambda \bmod p) \circ \rho$. Taking the heights of both sides, we obtain $d^{-1} \cdot \mathrm{height}(\rho) = v_p(a) \in \mathbb{Z}$, where v_p is the p-adic valuation. Denote by $M^{(\delta)}$ the rigid generic fiber of $\breve{\mathscr{M}}^{(\delta)}$. For a compact open subgroup K of K_0, let $M_K^{(\delta)}$ be the inverse image of $M^{(\delta)}$ under the map $M_K \longrightarrow M$. We have $M_K = \coprod_{\delta \in \mathbb{Z}} M_K^{(\delta)}$. Put $H_{\mathrm{RZ},\delta}^i = \varinjlim_{K \subset K_0} H_c^i(M_K^{(\delta)})$. Then $H_{\mathrm{RZ}}^i = \bigoplus_{\delta \in \mathbb{Z}} H_{\mathrm{RZ},\delta}^i$.

For $(g, h) \in G \times J$ with $g^{-1}Kg \subset K_0$, it is known that $(g, h) \colon M_K \longrightarrow M_{g^{-1}Kg}$ maps $M_K^{(\delta)}$ to $M_K^{(\delta - v_p(\mathrm{sim}\, g) + v_p(\mathrm{sim}\, h))}$. In particular, if we denote by $(G \times J)^0$ the kernel of the homomorphism $G \times J \longrightarrow \mathbb{Z}$; $(g, h) \longmapsto v_p(\mathrm{sim}\, g) - v_p(\mathrm{sim}\, h)$, $H_{\mathrm{RZ},0}^i$ is a smooth representation of $(G \times J)^0$ and H_{RZ}^i is isomorphic to c-$\mathrm{Ind}_{(G \times J)^0}^{G \times J} H_{\mathrm{RZ},0}^i$ (*cf.* [Far04, §4.4.2]). Since the character $\chi_G \otimes \chi_J^{-1} \colon G \times J \longrightarrow \overline{\mathbb{Q}}_\ell^\times$; $(g, h) \longmapsto \chi_G(g)\chi_J(h)^{-1}$ is trivial on $(G \times J)^0$, we have a natural isomorphism $H_{\mathrm{RZ}}^i \otimes \chi_G \otimes \chi_J^{-1} \cong H_{\mathrm{RZ}}^i$ of $G \times J$-representations. Hence we have

$$\mathrm{Ext}_J^j(H_{\mathrm{RZ}}^i, \rho \otimes \chi_J) \cong \mathrm{Ext}_J^j(H_{\mathrm{RZ}}^i \otimes \chi_J^{-1}, \rho) \cong \mathrm{Ext}_J^j(H_{\mathrm{RZ}}^i \otimes \chi_G^{-1}, \rho) \cong \mathrm{Ext}_J^j(H_{\mathrm{RZ}}^i, \rho) \otimes \chi_G,$$

and thus $H_{\mathrm{RZ}}^{i,j}[\rho \otimes \chi_J] \cong H_{\mathrm{RZ}}^{i,j}[\rho] \otimes \chi_G$. ∎

Sometimes it is convenient to work on the quotient $M_K/p^{\mathbb{Z}}$ of M_K by the discrete subgroup $p^{\mathbb{Z}}$ of J. The cohomology of $M_K/p^{\mathbb{Z}}$ and $H_{\mathrm{RZ}}^{i,j}[\rho]$ are related by the following lemma.

Lemma 3.6 Assume that an irreducible smooth representation ρ of J is trivial on the subgroup $p^{\mathbb{Z}} \subset J$. Then we have $\mathrm{Ext}_J^j(H_c^i(M_K), \rho) \cong \mathrm{Ext}_{J/p^{\mathbb{Z}}}^j(H_c^i(M_K/p^{\mathbb{Z}}), \rho)$ for each compact open subgroup $K \subset K_0$.

Proof. We will use the notation in the proof of Lemma 3.5. Since $p \in J$ maps $M_K^{(\delta)}$ isomorphically onto $M_K^{(\delta+2)}$ for every integer δ, we have $M_K/p^{\mathbb{Z}} \cong M_K^{(0)} \amalg M_K^{(1)}$. Under this isomorphism, the natural morphism from M_K to $M_K/p^{\mathbb{Z}}$ is described as follows:

– If $\delta = 2\delta'$ is even, the restriction to $M_K^{(\delta)}$ is given by $p^{-\delta'}\colon M_K^{(\delta)} \longrightarrow M_K^{(0)}$.

– If $\delta = 2\delta' + 1$ is odd, the restriction to $M_K^{(\delta)}$ is given by $p^{-\delta'}\colon M_K^{(\delta)} \longrightarrow M_K^{(1)}$.

From this description we deduce that the natural push-forward map $H_c^i(M_K) \longrightarrow H_c^i(M_K/p^{\mathbb{Z}})$ induces a J-equivariant isomorphism $H_c^i(M_K)_{p^{\mathbb{Z}}} \cong H_c^i(M_K/p^{\mathbb{Z}})$. On the other hand, it is immediate to see that $H_c^i(M_K) = \bigoplus_{\delta \in \mathbb{Z}} H_c^i(M_K^{(\delta)})$ is a free $\overline{\mathbb{Q}}_p[p^{\mathbb{Z}}]$-module, where $\overline{\mathbb{Q}}_\ell[p^{\mathbb{Z}}]$ denotes the group algebra of $p^{\mathbb{Z}}$. In particular, $H_c^i(M_K)$ is acyclic for $(-)_{p^{\mathbb{Z}}}$ (namely, the higher left derived functor of $(-)_{p^{\mathbb{Z}}}$ vanishes). Therefore we have

$$\mathrm{Ext}_J^j\big(H_c^i(M_K), \rho\big) = \mathrm{Ext}_{J/p^{\mathbb{Z}}}^j\big(H_c^i(M_K)_{p^{\mathbb{Z}}}, \rho\big) \cong \mathrm{Ext}_{J/p^{\mathbb{Z}}}^j\big(H_c^i(M_K/p^{\mathbb{Z}}), \rho\big),$$

as desired. ∎

Corollary 3.7 *Let ρ be an irreducible smooth representation of J.*

  i) *For integers $i, j \geq 0$, $H_{\mathrm{RZ}}^{i,j}[\rho]$ is an admissible representation of G.*

  ii) *If $j > d - 1$, we have $H_{\mathrm{RZ}}^{i,j}[\rho] = 0$.*

Proof. First assume that ρ is trivial on $p^{\mathbb{Z}} \subset J$. Then, for a compact open subgroup $K \subset K_0$, we have

$$H_{\mathrm{RZ}}^{i,j}[\rho]^K = \mathrm{Ext}_{J/p^{\mathbb{Z}}}^j\big(H_c^i(M_K/p^{\mathbb{Z}}), \rho\big)$$

by Remark 3.4 and Lemma 3.6. As in [Far04, Proposition 4.4.13], $H_c^i(M_K/p^{\mathbb{Z}})$ is a finitely generated $J/p^{\mathbb{Z}}$-module, and thus [SS97, Corollary II.3.2] tells us that $\mathrm{Ext}_{J/p^{\mathbb{Z}}}^j(H_c^i(M_K/p^{\mathbb{Z}}), \rho)$ is finite-dimensional and vanishes for $j > d - 1$ (here $d - 1$ is the split semisimple rank of **J**). Since $H_{\mathrm{RZ}}^{i,j}[\rho] = \varinjlim_{K \subset K_0} H_{\mathrm{RZ}}^{i,j}[\rho]^K$, we obtain i) and ii) for this case.

Next we consider a general ρ. Let $\omega\colon \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ be the central character of ρ. Take $c \in \overline{\mathbb{Q}}_\ell^\times$ such that $c^2 = \omega(p)$, and $\chi\colon \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ the character given by $\chi(a) = c^{-v_p(a)}$. Lemma 3.5 tells us that $H_{\mathrm{RZ}}^{i,j}[\rho] = H_{\mathrm{RZ}}^{i,j}[\rho \otimes \chi_J] \otimes \chi_G^{-1}$. Since $\rho \otimes \chi_J$ is trivial on $p^{\mathbb{Z}}$, the right hand side is admissible and vanishes for $j > d - 1$. This concludes the proof. ∎

By the corollary above, we can take the alternating sum of $H_{\mathrm{RZ}}^{i,j}[\rho]$.

Definition 3.8 For an irreducible smooth representation ρ of J, we put $H_{\mathrm{RZ}}[\rho] = \sum_{i,j \geq 0} (-1)^{i+j} H_{\mathrm{RZ}}^{i,j}[\rho]$, where the sum is taken in the Grothendieck group of admissible representations of G.

The goal of this paper is to investigate $H_{\mathrm{RZ}}[\rho]$ by means of the Lefschetz trace formula.

In the sequel, we fix Haar measures on G and J. For each $g \in G^{\mathrm{reg}}$, we also fix a Haar measure on $Z(g)$. If g is elliptic, then we normalize the measure so that $\mathrm{vol}(Z(g)/p^{\mathbb{Z}}) = 1$, where $p^{\mathbb{Z}} \subset G$ is endowed with the counting measure. Note that

if $g_1, g_2 \in G^{\mathrm{ell}}$ are stably conjugate, then the isomorphism $\iota_{g_1, g_2} \colon Z(g_1) \xrightarrow{\cong} Z(g_2)$ preserves the measures. For $g \in G^{\mathrm{ell}}$ and a locally constant function f on G whose support is compact modulo Z_G, we have

$$O_g(f) = \int_{G/p^{\mathbb{Z}}} f(h^{-1}gh)dh, \quad SO_g(f) = \sum_{g' \in \{g\}_{\mathrm{st}}/\sim} \int_{G/p^{\mathbb{Z}}} f(h^{-1}g'h)dh.$$

Similarly we fix a Haar measure of the centralizer of each regular element of J. For $g \in G^{\mathrm{ell}}$ and $h \in J^{\mathrm{ell}}$ with $g \leftrightarrow h$, the isomorphism $\iota_{h,g} \colon Z(h) \xrightarrow{\cong} Z(g)$ preserves the measures.

3.2 Period space and period map

The goal of the rest of this section is to count fixed points under the action of $(g, h) \in G \times J$ on $M_K/p^{\mathbb{Z}}$. As in [Str08, §2.6], we use the period map introduced in [RZ96, Chapter 5].

Put $L_0 = \mathrm{Frac}\, W(\overline{\mathbb{F}}_p)$ and denote the Frobenius automorphism on L_0 by σ. Although L_0 is isomorphic to $\mathbb{Q}_{p^\infty}$, we distinguish them as in [RZ96]. An isocrystal over $\overline{\mathbb{F}}_p$ is a finite-dimensional L_0-vector space equipped with a bijective σ-linear endomorphism (*cf.* [RZ96, §1.1]).

Let $D(\mathbb{X})_{\mathbb{Q}} = (N, \Phi)$ be the rational Dieudonné module of $\mathbb{X}$, which is a d-dimensional isocrystal over $\overline{\mathbb{F}}_p$. The fixed polarization λ_0 on $\mathbb{X}$ gives the alternating pairing $\psi \colon N \times N \longrightarrow L_0$ satisfying $\psi(\Phi(x), \Phi(y)) = p\sigma(\psi(x, y))$ for every $x, y \in N$. We define the algebraic group $\mathbf{J}$ over $\mathbb{Q}_p$ as follows: for a $\mathbb{Q}_p$-algebra R, the group $\mathbf{J}(R)$ consists of elements $g \in \mathrm{Aut}_{R \otimes_{\mathbb{Q}_p} L_0}(R \otimes_{\mathbb{Q}_p} N)$ such that

- g commutes with Φ, i.e., $g \circ (\mathrm{id}_R \otimes \Phi) = (\mathrm{id}_R \otimes \Phi) \circ g$,
- and g preserves the pairing ψ up to scalar multiplication, i.e., there exists $c(g) \in (R \otimes_{\mathbb{Q}_p} L_0)^{\times}$ such that $\psi(gx, gy) = c(g)\psi(x, y)$ for every $x, y \in R \otimes_{\mathbb{Q}_p} N$.

Representability of $\mathbf{J}$ is shown in [RZ96, Proposition 1.12]. By the Dieudonné theory, we have $\mathbf{J}(\mathbb{Q}_p) = J$.

Since the isocrystal (N, Φ) is basic, $\mathbf{J}$ is known to be an inner form of GSp_{2d} ([RZ96, Corollary 1.14, Remark 1.15]). For later use, we will observe it directly. Put $N^{\circ} = N^{p^{-1}\Phi^2}$. It is a Φ-stable $\mathbb{Q}_{p^2}$-subspace of N satisfying $L_0 \otimes_{\mathbb{Q}_{p^2}} N^{\circ} = N$. For $x, y \in N^{\circ}$, we have $\sigma^2(\psi(x, y)) = p^{-2}\psi(\Phi^2(x), \Phi^2(y)) = p^{-2}\psi(px, py) = \psi(x, y)$, and thus $\psi(x, y) \in \mathbb{Q}_{p^2}$. Therefore ψ gives a perfect alternating bilinear pairing $\psi \colon N^{\circ} \times N^{\circ} \longrightarrow \mathbb{Q}_{p^2}$. Its base change from $\mathbb{Q}_{p^2}$ to L_0 coincides with the original ψ. By using N° and the restrictions of Φ and ψ on it, we can describe $\mathbf{J}$ as follows: for each $\mathbb{Q}_p$-algebra R,

$$\mathbf{J}(R) = \left\{ g \in \mathrm{Aut}_{R \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}}(R \otimes_{\mathbb{Q}_p} N^{\circ}) \,\middle|\, g \text{ satisfies the similar conditions as above} \right\}.$$

In the sequel, we always use this description of $\mathbf{J}$. Now we can prove the following:

Lemma 3.9 *We have a natural isomorphism $\xi\colon \mathbf{J}\otimes_{\mathbb{Q}_p}\mathbb{Q}_{p^2} \xrightarrow{\cong} \mathrm{GSp}(N^\circ,\psi)$ of algebraic groups over $\mathbb{Q}_{p^2}$.*

Proof. Take a $\mathbb{Q}_{p^2}$-algebra R. Then we have $R\otimes_{\mathbb{Q}_p}N^\circ \cong (R\otimes_{\mathbb{Q}_{p^2}}N^\circ)\oplus(R\otimes_{\mathbb{Q}_{p^2}}{}^\sigma N^\circ)$, where ${}^\sigma N^\circ$ is the scalar extension of N° by $\sigma\colon \mathbb{Q}_{p^2}\longrightarrow \mathbb{Q}_{p^2}$. Under this isomorphism, $\mathrm{id}_R\otimes\Phi\colon R\otimes_{\mathbb{Q}_p}N^\circ \longrightarrow R\otimes_{\mathbb{Q}_p}N^\circ$ is expressed by the matrix

$$\begin{pmatrix} 0 & \mathrm{id}_R\otimes\Phi_1 \\ \mathrm{id}_R\otimes\Phi_2 & 0 \end{pmatrix},$$

where Φ_1 (resp. Φ_2) denotes the $\mathbb{Q}_{p^2}$-homomorphism ${}^\sigma N^\circ \longrightarrow N^\circ$ (resp. $N^\circ \longrightarrow {}^\sigma N^\circ$) induced by Φ. Therefore, every element $g\in \mathrm{Aut}_{R\otimes_{\mathbb{Q}_p}\mathbb{Q}_{p^2}}(R\otimes_{\mathbb{Q}_p}N^\circ)$ can be written as $g'\oplus g''$ with $g'\in \mathrm{Aut}_R(R\otimes_{\mathbb{Q}_{p^2}}N^\circ)$ and $g''\in \mathrm{Aut}_R(R\otimes_{\mathbb{Q}_{p^2}}{}^\sigma N^\circ)$, and the condition $g\circ(\mathrm{id}_R\otimes\Phi)=(\mathrm{id}_R\otimes\Phi)\circ g$ is equivalent to $g'\circ(\mathrm{id}_R\otimes\Phi_1)=(\mathrm{id}_R\otimes\Phi_1)\circ g''$ and $g''\circ(\mathrm{id}_R\otimes\Phi_2)=(\mathrm{id}_R\otimes\Phi_2)\circ g'$. For $g'\in \mathrm{Aut}_{R\otimes_{\mathbb{Q}_p}\mathbb{Q}_{p^2}}(R\otimes_{\mathbb{Q}_p}N^\circ)$, put $g''=(\mathrm{id}_R\otimes\Phi_1)^{-1}\circ g'\circ(\mathrm{id}_R\otimes\Phi_1)$. Then the pair (g',g'') satisfies the conditions above (note that $\Phi_1\circ\Phi_2=p$). In other words, the group

$$\left\{ g\in \mathrm{Aut}_{R\otimes_{\mathbb{Q}_p}\mathbb{Q}_{p^2}}(R\otimes_{\mathbb{Q}_p}N^\circ) \,\middle|\, g\circ(\mathrm{id}_R\otimes\Phi)=(\mathrm{id}_R\otimes\Phi)\circ g\right\}$$

can be identified with the group $\mathrm{Aut}_R(R\otimes_{\mathbb{Q}_{p^2}}N^\circ)$. Now it is straightforward to see that (g',g'') preserves the pairing ψ on $R\otimes_{\mathbb{Q}_p}N^\circ$ up to scalar if and only if $g\in \mathrm{GSp}(N^\circ,\psi)(R)$. This concludes the proof. ∎

By the construction of the isomorphism ξ, we have the following:

Corollary 3.10 *A natural homomorphism $\mathbf{J}\longrightarrow \mathrm{Res}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}\mathrm{GSp}(N^\circ,\psi)$ corresponds to ξ by the adjointness between base change and the Weil restriction. In particular, the composite $\mathbf{J}(\mathbb{Q}_p)\hookrightarrow \mathbf{J}(\mathbb{Q}_{p^2})\xrightarrow[\cong]{\xi}\mathrm{GSp}(N^\circ,\psi)$ is nothing but the natural inclusion.*

Remark 3.11 Actually, we can describe $\mathbf{J}$ more explicitly as follows.

Let Σ_2 be a (unique) one-dimensional p-divisible group with slope $1/2$ over $\overline{\mathbb{F}}_p$. It is well-known that there exists a polarization λ_{Σ_2} on Σ_2; for example, a principal polarization on a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ induces such a polarization. Put $D=\mathrm{End}(\Sigma_2)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$. Then D is a quaternion division algebra over $\mathbb{Q}_p$ and λ_{Σ_2} induces an involution on it. By [IM10, Lemma 4.1], we know that $(\mathbb{X},\lambda_0)$ and $(\Sigma_2^{\oplus d},\lambda_{\Sigma_2}^{\oplus d})$ are isogenous. Therefore, we can prove without difficulty that the algebraic group $\mathbf{J}$ is isomorphic to the quaternionic unitary similitude group $\mathrm{GU}(d,D)$.

Next we introduce the period space for GSp_{2d}.

Definition 3.12   i) Let $\mathcal{F}$ be the Grassmannian over L_0 parameterizing d-dimensional subspaces $\mathrm{Fil}\subset N$ such that $\mathrm{Fil}^\perp=\mathrm{Fil}$.

ii) Let L be a finite extension of L_0. An element $\mathrm{Fil} \subset L \otimes_{L_0} N$ of $\mathcal{F}(L)$ is said to be weakly admissible if, for every subspace N' of N which is stable under Φ, the following inequality holds:

$$\dim_L\big((L \otimes_{L_0} N') \cap \mathrm{Fil}\big) \leq \frac{1}{2} \dim N'.$$

It is known that there exists a canonical open rigid subspace $\Omega \subset \mathcal{F}^{\mathrm{ad}}$ such that $\Omega(L) = \{\mathrm{Fil} \in \mathcal{F}(L) \mid \mathrm{Fil}$ is weakly admissible$\}$ for every finite extension L of L_0 ([RZ96, Proposition 1.36]). We call this Ω a period space for GSp_{2d}. The group $\mathrm{GSp}(N, \psi)$ naturally acts on $\mathcal{F}$ and the induced action of $J = \mathbf{J}(\mathbb{Q}_p) \subset \mathrm{GSp}(N, \psi)$ preserves $\Omega \subset \mathcal{F}$.

The following theorem is due to Rapoport-Zink:

Theorem 3.13   i) *There exists a J-equivariant étale morphism $\wp \colon M \longrightarrow \Omega$ over L_0 called the period morphism. For a finite extension L of L_0 and an L-valued point $x = (X, \rho)$ of M, $\wp(x)$ is given by the subspace $\rho_*^{-1}(\mathrm{Fil}_X)$ of $L \otimes_{L_0} N$, where $\rho_* \colon D(\mathbb{X})_{\mathbb{Q}} \xrightarrow{\cong} D(X)_{\mathbb{Q}}$ is the isomorphism between rational Dieudonné modules induced by ρ, and $\mathrm{Fil}_X \subset L \otimes_{L_0} D(X)_{\mathbb{Q}}$ is the Hodge filtration of X.*
ii) *The period map $\wp$ induces a surjection on classical points. Namely, for every finite extension L of L_0 and every L-valued point x of Ω, there exist a finite extension L' of L and an L'-valued point $\widetilde{x}$ of M such that $\wp(\widetilde{x}) = x$.*

Proof. The period map $\wp$ is constructed in [RZ96, 5.16]. Precisely speaking, our $\wp$ is the first factor $\breve{\pi}_1$ of the period map $\breve{\pi}$ defined by Rapoport-Zink.

ii) follows from [RZ96, Proposition 5.28]; note that Fontaine's conjecture assumed in the proposition has been solved by Kisin ([Kis06, Corollary 2.2.6]). ∎

The following proposition is the first step of our point counting:

Proposition 3.14 *Let h be a regular element of J. Then all fixed points of $\mathcal{F}$ under h are discrete with multiplicity one. If moreover h is elliptic, then every fixed point lies in Ω.*

The former part is well-known. In order to see the latter part, we will use the theory of Harder-Narasimhan filtrations. Let us fix a finite extension L of L_0 and an element $\mathrm{Fil} \in \mathcal{F}(L)$. For a non-zero subspace N' of N which is stable under Φ, we put

$$\mu(N') = \frac{\dim_L\big((L \otimes_{L_0} N') \cap \mathrm{Fil}\big) - 1/2 \dim_{L_0} N'}{\dim_{L_0} N'}.$$

We say that $N' \neq 0$ is semi-stable if every non-zero Φ-stable subspace $N'' \subset N'$ satisfies $\mu(N'') \leq \mu(N')$.

The following proposition is a part of [RZ96, Proposition 1.4]:

Proposition 3.15 *There exists a unique Φ-stable subspace $N_0 \subset N$ satisfying the following conditions:*

- *N_0 is semi-stable (in particular non-zero).*
- *For every Φ-stable N' with $N_0 \subsetneq N' \subset N$, we have $\mu(N_0) > \mu(N')$.*

Proof of Proposition 3.14. Let $h \in J$ be a regular element, L a finite extension of L_0 and Fil an element of $\mathcal{F}(L)$ which is fixed by h. We will assume Fil $\notin \Omega$ and prove that h is not elliptic. Since Fil $\notin \Omega$, $\mu(N') > 0 = \mu(N)$ for some Φ-stable subspace $N' \subset N$. Therefore N is not semi-stable and thus $N_0 \subsetneq N$, where N_0 is given in Proposition 3.15. Since Fil is fixed by h, we have $hN_0 = N_0$ by the uniqueness.

Let us prove that $N_0 \subset N_0^{\perp}$. The following argument is inspired by [RZ96, Proposition 1.43]. Put $W = N_0 \cap N_0^{\perp}$ and assume that $W \subsetneq N_0$. Then ψ induces a perfect alternating pairing $N_0/W \times N_0/W \longrightarrow L_0$. Denote the image of $(L \otimes_{L_0} N_0) \cap$ Fil under $L \otimes_{L_0} N_0 \longrightarrow (L \otimes_{L_0} N_0)/(L \otimes_{L_0} W)$ by Fil$'$. Since Fil$^{\perp} =$ Fil, we have Fil$' \subset$ Fil$'^{\perp}$. Therefore we have $\dim_L$ Fil$' \leq 1/2 \dim_{L_0}(N_0/W)$. On the other hand, by the definition of N_0, $N_0 \subsetneq N$ implies $\mu(N_0) > \mu(N) = 0$. Hence, if $W \neq 0$,

$$
\begin{aligned}
\mu(W) &= \frac{\dim_L\big((L \otimes_{L_0} W) \cap \text{Fil}\big)}{\dim_{L_0} W} - \frac{1}{2} = \frac{\dim_L\big((L \otimes_{L_0} N_0) \cap \text{Fil}\big) - \dim_L \text{Fil}'}{\dim_{L_0} W} - \frac{1}{2} \\
&\geq \frac{\dim_L\big((L \otimes_{L_0} N_0) \cap \text{Fil}\big) - 1/2 \dim_{L_0}(N_0/W)}{\dim_{L_0} W} - \frac{1}{2} = \frac{\dim_{L_0} N_0}{\dim_{L_0} W} \mu(N_0) \\
&> \mu(N_0),
\end{aligned}
$$

which contradicts to semi-stability of N_0. If $W = 0$,

$$
\dim_L\big((L \otimes_{L_0} N_0) \cap \text{Fil}\big) - \frac{1}{2} \dim_{L_0} N_0 = \dim_L \text{Fil}' - \frac{1}{2} \dim_{L_0}(N_0/W) \leq 0,
$$

which contradicts to $\mu(N_0) > 0$. Thus we get $N_0 \subset N_0^{\perp}$.

Put $N_0^{\circ} = (N_0)^{p^{-1}\Phi^2} \subset N^{\circ}$. Since (N_0, Φ) is isoclinic of slope $1/2$, we have $L_0 \otimes_{\mathbb{Q}_{p^2}} N_0^{\circ} = N_0$. In particular $0 \subsetneq N_0^{\circ} \subsetneq N^{\circ}$. Moreover we have $N_0^{\circ} \subset (N_0^{\circ})^{\perp}$, since $N_0 \subset N_0^{\perp}$. Consider the subgroup $\mathbf{P}$ of $\mathbf{J}$ given as follows (R denotes a $\mathbb{Q}_p$-algebra):

$$
\mathbf{P}(R) = \big\{ h' \in \mathbf{J}(R) \,\big|\, h'(R \otimes_{\mathbb{Q}_p} N_0^{\circ}) = R \otimes_{\mathbb{Q}_p} N_0^{\circ} \big\}.
$$

Here $\mathbf{J}(R)$ is regarded as a subgroup of $\text{Aut}_{R \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}}(R \otimes_{\mathbb{Q}_p} N^{\circ})$. In the similar way as in the proof of Lemma 3.9, we can see that $\mathbf{P} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}$ is isomorphic to the stabilizer subgroup $\text{Stab}_{\text{GSp}(N^{\circ}, \psi)}(N_0^{\circ})$ of N_0° in $\text{GSp}(N^{\circ}, \psi)$. Therefore $\mathbf{P} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}$ is a proper parabolic subgroup of $\mathbf{J} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}$, and thus $\mathbf{P}$ is a proper parabolic subgroup of $\mathbf{J}$. Since $h \in \mathbf{P}(\mathbb{Q}_p)$, the following lemma says that h is not elliptic. This completes the proof. ∎

Lemma 3.16 *Let F be a p-adic field, $\mathbf{G}$ a connected reductive group over F and g a regular elliptic element of $\mathbf{G}(F)$. Then, for every proper parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ defined over F, g does not lie in $\mathbf{P}(F)$.*

Proof. Assume that there exists a proper parabolic subgroup of $\mathbf{G}$ defined over F such that $g \in \mathbf{P}(F)$. Then, since $g \in \mathbf{G}(F)$ is semisimple, there exists a Levi subgroup $\mathbf{L}$ of $\mathbf{P}$ defined over F such that $g \in \mathbf{L}(F)$ (*cf.* [Spr98, 13.3.8 (i), 8.4.4, 16.1.4]). By the restricted root decomposition, it is easy to see that the split center of $\mathbf{L}$ is strictly bigger than that of $\mathbf{G}$. In other words, the center $\mathbf{Z_L}$ of $\mathbf{L}$ is not anisotropic modulo $\mathbf{Z_G}$. Therefore the centralizer of g, that contains $\mathbf{Z_L}$, is not anisotropic modulo $\mathbf{Z_G}$. This contradicts to the assumption that g is regular elliptic. ∎

Remark 3.17 By the Bruhat decomposition, we can easily calculate the number of fixed points in Proposition 3.14; the number is $d!2^{d-1}$.

3.3 Counting fixed points under the group action

Let (g, h) be an element of $G \times J$ and K a compact open subgroup of K_0 normalized by g. Let $\wp_K \colon M_K \longrightarrow \Omega$ and $\wp_{K,p} \colon M_K/p^{\mathbb{Z}} \longrightarrow \Omega$ be the étale morphisms induced from the period map $\wp$. In Proposition 3.14, we considered fixed points on the period space Ω. Thus, to count fixed points on $M_K/p^{\mathbb{Z}}$, it suffices to investigate the action of (g, h) on the fiber $\wp_{K,p}^{-1}(x)$ of each point x in Ω fixed by h.

Definition 3.18 Let L be a finite extension of L_0 and $x \in \Omega(L)$ a point fixed by h. We denote the subspace of $L \otimes_{L_0} N$ corresponding to x by Fil_x. Then, since $(N, \Phi, \mathrm{Fil}_x)$ is a weakly admissible filtered isocrystal, there exists a $2d$-dimensional p-adic Galois representation V_x of $\mathrm{Gal}(\overline{L}/L)$ such that $D_{\mathrm{crys}}(V_x) \cong (N, \Phi, \mathrm{Fil}_x)$ (*cf.* [CF00]). As the functor D_{crys} is fully faithful and compatible with tensor products, the alternating bilinear pairing $\psi \colon N \times N \longrightarrow L_0$ induces an alternating bilinear pairing $\psi_x \colon V_x \times V_x \longrightarrow \mathbb{Q}_p(1)$. Since $h \colon N \xrightarrow{\cong} N$ commutes with Φ, preserves Fil_x and preserves ψ up to $\mathbb{Q}_p^{\times}$-multiplication, it induces a $\mathrm{Gal}(\overline{L}/L)$-automorphism $g_{h,x}$ on V_x preserving ψ_x up to $\mathbb{Q}_p^{\times}$-multiplication. By choosing isomorphisms $\mathbb{Q}_p(1) \cong \mathbb{Q}_p$ and $(V_x, \psi_x) \cong (\mathbb{Q}_p^{2d}, \langle\ ,\ \rangle)$, $g_{h,x}$ can be regarded as an element of G. Obviously, the conjugacy class of $g_{h,x}$ is independent of the choice of the isomorphisms above.

Proposition 3.19 *The element $g_{h,x} \in G$ is a transfer of $h \in J$ with respect to ξ. Namely, if we fix an isomorphism $(N^{\circ}, \psi) \cong (\mathbb{Q}_{p^2}^{2d}, \langle\ ,\ \rangle)$, the image of h under the composite*

$$\mathbf{J}(\mathbb{Q}_p) \longhookrightarrow \mathbf{J}(\mathbb{Q}_{p^2}) \xrightarrow[\cong]{\xi} \mathrm{GSp}(N^{\circ}, \psi) \cong \mathrm{GSp}_{2d}(\mathbb{Q}_{p^2})$$

and $g_{h,x} \in G$ are conjugate in $\mathrm{GSp}_{2d}(\overline{\mathbb{Q}}_p)$.

In particular, $g_{h,x} \in G$ is regular (resp. regular elliptic) if and only if h is regular (resp. regular elliptic).

Proof. Since $D_{\mathrm{crys}}(V_x) \cong (N, \Phi, \mathrm{Fil}_x)$, we have an isomorphism

$$V_x \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong N \otimes_{L_0} B_{\mathrm{dR}} = N^{\circ} \otimes_{\mathbb{Q}_{p^2}} B_{\mathrm{dR}}.$$

Since we have fixed isomorphisms $(V_x, \psi_x) \cong (\mathbb{Q}_p^{2d}, \langle \ , \ \rangle)$ and $(N^\circ, \psi) \cong (\mathbb{Q}_{p^2}^{2d}, \langle \ , \ \rangle)$, the isomorphism above induces

$$B_{\mathrm{dR}}^{2d} \xrightarrow{\cong} V_x \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \xrightarrow{\cong} N^\circ \otimes_{\mathbb{Q}_{p^2}} B_{\mathrm{dR}} \xrightarrow{\cong} B_{\mathrm{dR}}^{2d},$$

which gives an element $\alpha \in \mathrm{GSp}_{2d}(B_{\mathrm{dR}})$. By the definition of $g_{h,x}$ and Corollary 3.10, we have $g_{h,x} = \alpha^{-1}\xi(h)\alpha$ in $\mathrm{GSp}_{2d}(B_{\mathrm{dR}})$. Therefore $g_{h,x}$ and $\xi(h)$ are conjugate in $\mathrm{GSp}_{2d}(B_{\mathrm{dR}})$, and thus conjugate in $\mathrm{GSp}_{2d}(\overline{\mathbb{Q}}_p)$. ∎

We will consider how the conjugacy class of $g_{h,x}$ changes when we vary h inside its stable conjugacy class. Assume that h is regular. Let $\mathbf{T}_h = \mathbf{Z}(h)$ be the centralizer of h in $\mathbf{J}$, which is a maximal torus of $\mathbf{J}$.

Lemma 3.20 *For every $\mathbb{Q}_p$-algebra R and an element $t \in \mathbf{T}_h(R)$, t stabilizes the subspace $R \otimes_{\mathbb{Q}_p} \mathrm{Fil}_x \subset R \otimes_{\mathbb{Q}_p} (L \otimes_{L_0} N) = L \otimes_{L_0} (R \otimes_{\mathbb{Q}_p} N)$.*

Proof. Let $\boldsymbol{\mathcal{F}}^\circ$ be the Grassmannian over $\mathbb{Q}_{p^2}$ parameterizing d-dimensional subspaces $\mathrm{Fil} \subset N^\circ$ such that $\mathrm{Fil}^\perp = \mathrm{Fil}$. Then, clearly $\boldsymbol{\mathcal{F}}^\circ \otimes_{\mathbb{Q}_{p^2}} L_0 = \boldsymbol{\mathcal{F}}$. Moreover, since h is regular, the fixed point x in $\boldsymbol{\mathcal{F}}$ comes from a closed point of $\boldsymbol{\mathcal{F}}^\circ$; namely, there exists a finite extension F of $\mathbb{Q}_{p^2}$ contained in L and $x' \in \boldsymbol{\mathcal{F}}^\circ(F)$ such that $\mathrm{Fil}_x = L \otimes_F \mathrm{Fil}_{x'}$. Therefore, we may consider the subspace $\mathrm{Fil}_{x'} \subset F \otimes_{\mathbb{Q}_{p^2}} N^\circ$ instead of $\mathrm{Fil}_x \subset L \otimes_{L_0} N$.

Put $\mathbf{G} = \mathrm{GSp}(N^\circ, \psi) \otimes_{\mathbb{Q}_{p^2}} F$. This algebraic group acts on $\boldsymbol{\mathcal{F}}_F^\circ = \boldsymbol{\mathcal{F}}^\circ \otimes_{\mathbb{Q}_{p^2}} F$. Let $\mathbf{P}$ be the stabilizer of $x' \in \boldsymbol{\mathcal{F}}_F^\circ(F)$ in $\mathbf{G}$. It is a parabolic subgroup of $\mathbf{G}$. We have homomorphisms $\mathbf{J} \longrightarrow \mathrm{Res}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p} \mathrm{GSp}(N^\circ, \psi) \longrightarrow \mathrm{Res}_{F/\mathbb{Q}_p} \mathbf{G}$, which induce an action of $\mathbf{J}$ on $\mathrm{Res}_{F/\mathbb{Q}_p} \boldsymbol{\mathcal{F}}_F^\circ$. What we would like to show is that the subgroup $\mathbf{T}_h$ of $\mathbf{J}$ stabilizes $x' \in (\mathrm{Res}_{F/\mathbb{Q}_p} \boldsymbol{\mathcal{F}}_F^\circ)(\mathbb{Q}_p)$; in other words, $\mathbf{T}_h \subset \mathrm{Res}_{F/\mathbb{Q}_p} \mathbf{P}$. Let us denote by h' the image of $h \in \mathbf{J}(\mathbb{Q}_p)$ in $(\mathrm{Res}_{F/\mathbb{Q}_p} \mathbf{G})(\mathbb{Q}_p) = \mathbf{G}(F)$. Note that $h' \in \mathbf{P}(F)$, for h stabilizes x'. Moreover h' is regular, since it is the image of $\xi(h) \in \mathrm{GSp}(N^\circ, \psi)$ under $\mathrm{GSp}(N^\circ, \psi) \longrightarrow \mathrm{GSp}(N^\circ \otimes_{\mathbb{Q}_{p^2}} F, \psi) = \mathbf{G}(F)$. It suffices to show that the centralizer $\mathbf{S}_{h'}$ of h' in $\mathbf{G}$ is contained in $\mathbf{P}$. We can pass to an algebraic closure $\overline{F}$ of F; we simply write $\mathbf{G}$ and $\mathbf{P}$ for $\mathbf{G} \otimes_F \overline{F}$ and $\mathbf{P} \otimes_F \overline{F}$, respectively. Take a maximal torus $\mathbf{T}'$ of $\mathbf{P}$ containing h'. As $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$, it contains a maximal torus $\mathbf{T}''$ of $\mathbf{G}$. Since $\mathbf{T}'$ and $\mathbf{T}''$ are conjugate in $\mathbf{P}$, $\mathbf{T}'$ is also a maximal torus of $\mathbf{G}$. This implies that $\mathbf{T}' = \mathbf{S}_{h'}$. In particular $\mathbf{S}_{h'}$ is contained in $\mathbf{P}$. ∎

Definition 3.21 Fix isomorphisms $\mathbb{Q}_p(1) \cong \mathbb{Q}_p$ and $(V_x, \psi_x) \cong (\mathbb{Q}_p^{2d}, \langle \ , \ \rangle)$ as in Definition 3.18. We define a homomorphism $\iota_{h,x} \colon \mathbf{T}_h \longrightarrow \mathrm{GSp}_{2d}$ of algebraic groups over $\mathbb{Q}_p$ as follows. For a $\mathbb{Q}_p$-algebra R, each element $t \in \mathbf{T}_h(R)$ gives an automorphism of the filtered isocrystal $(R \otimes_{\mathbb{Q}_p} N, \Phi, R \otimes_{\mathbb{Q}_p} \mathrm{Fil}_x)$ by the previous lemma. The induced automorphism on $(R \otimes_{\mathbb{Q}_p} N \otimes_{L_0} B_{\mathrm{crys}})^{\Phi, \mathrm{Fil}} = R \otimes_{\mathbb{Q}_p} V_x$ defines an element $\iota_{h,x}(t) \in \mathrm{GSp}(R \otimes_{\mathbb{Q}_p} V_x, \psi_x) \cong \mathrm{GSp}_{2d}(R)$.

By definition, we have $\iota_{h,x}(h) = g_{h,x}$. Clearly $\iota_{h,x}$ is independent of the choice of $\mathbb{Q}_p(1) \cong \mathbb{Q}_p$, and the $\mathrm{GSp}_{2d}(\mathbb{Q}_p)$-conjugacy class of $\iota_{h,x}$ is independent of the choice of $(V_x, \psi_x) \cong (\mathbb{Q}_p^{2d}, \langle \ , \ \rangle)$.
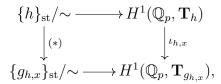
Proposition 3.22 *Let $\mathbf{T}_{g_{h,x}} = \mathbf{Z}(g_{h,x})$ be the centralizer of $g_{h,x}$ in GSp_{2d}. The homomorphism $\iota_{h,x}$ induces an isomorphism $\mathbf{T}_h \xrightarrow{\cong} \mathbf{T}_{g_{h,x}}$.*

Proof. The proof of Proposition 3.19 tells us that the base change of $\iota_{h,x}$ to B_{dR} can be described as the composite of

$$\mathbf{T}_h \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \hookrightarrow \mathbf{J} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \xrightarrow[\cong]{\xi} \mathrm{GSp}_{2d} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \xrightarrow[\cong]{\mathrm{Ad}(\alpha^{-1})} \mathrm{GSp}_{2d} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}},$$

where $\alpha \in \mathrm{GSp}_{2d}(B_{\mathrm{dR}})$ is the element defined in the proof of Proposition 3.19. Now it is clear that the homomorphism $\mathbf{T}_h \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \longrightarrow \mathbf{T}_{g_{h,x}} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ induced by $\iota_{h,x}$ is an isomorphism, and thus $\iota_{h,x} \colon \mathbf{T}_h \longrightarrow \mathbf{T}_{g_{h,x}}$ is also an isomorphism. ∎

Proposition 3.23   i) *For $h' \in \{h\}_{\mathrm{st}}$, take $\gamma \in \mathbf{J}(\overline{\mathbb{Q}}_p)$ such that $h' = \gamma^{-1}h\gamma$. Then, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $c_{h,h'}(\sigma) = \gamma\sigma(\gamma)^{-1}$ lies in $\mathbf{T}_h(\overline{\mathbb{Q}}_p)$, and $c_{h,h'}$ gives an element of $H^1(\mathbb{Q}_p, \mathbf{T}_h)$. The map $h' \longmapsto c_{h,h'}$ induces a bijection $\{h\}_{\mathrm{st}}/\!\sim \xrightarrow{\cong} H^1(\mathbb{Q}_p, \mathbf{T}_h)$. Similarly, we have a natural bijection $\{g_{h,x}\}_{\mathrm{st}}/\!\sim \xrightarrow{\cong} H^1(\mathbb{Q}_p, \mathbf{T}_{g_{h,x}})$.*

ii) *Assume that h is elliptic. Then we have a commutative diagram*

$$\begin{array}{ccc}
\{h\}_{\mathrm{st}}/\!\sim & \longrightarrow & H^1(\mathbb{Q}_p, \mathbf{T}_h) \\
\downarrow{\scriptstyle (*)} & & \downarrow{\scriptstyle \iota_{h,x}} \\
\{g_{h,x}\}_{\mathrm{st}}/\!\sim & \longrightarrow & H^1(\mathbb{Q}_p, \mathbf{T}_{g_{h,x}}),
\end{array}$$

where the map $(*)$ is given by $h' = \gamma^{-1}h\gamma \longmapsto g_{h',\xi(\gamma)^{-1}x}$. (Recall that ξ carries $\gamma \in \mathbf{J}(\overline{\mathbb{Q}}_p)$ to an element of $\mathrm{GSp}(N^\circ \otimes_{\mathbb{Q}_{p^2}} \overline{\mathbb{Q}}_p, \psi) \subset \mathrm{GSp}(N \otimes_{L_0} \overline{L}_0, \psi)$. Corollary 3.10 ensures that $\xi(\gamma)^{-1}x$ is fixed by h', and Proposition 3.14 tells us that $\xi(\gamma)^{-1}x$ lies in Ω.)

Proof. i) It is well-known that the map $h' \longmapsto c_{h,h'}$ induces a bijection

$$\{h\}_{\mathrm{st}}/\!\sim \xrightarrow{\cong} \mathrm{Ker}\big(H^1(\mathbb{Q}_p, \mathbf{T}_h) \longrightarrow H^1(\mathbb{Q}_p, \mathbf{J})\big).$$

Therefore it suffices to show $H^1(\mathbb{Q}_p, \mathbf{J}) = 1$. Since the derived group $\mathbf{J}_{\mathrm{der}}$ is simply connected, we have $H^1(\mathbb{Q}_p, \mathbf{J}_{\mathrm{der}}) = 1$ by Kneser's theorem. Now the exact sequence

$$1 \longrightarrow \mathbf{J}_{\mathrm{der}} \longrightarrow \mathbf{J} \longrightarrow \mathbb{G}_m \longrightarrow 1$$

gives the desired vanishing result. Similarly we can prove $H^1(\mathbb{Q}_p, \mathrm{GSp}_{2d}) = 1$.

ii) Take a finite extension F of $\mathbb{Q}_{p^2}$ and $x' \in \mathcal{F}^\circ(F)$ as in the proof of Lemma 3.20. Let E be a finite Galois extension of $\mathbb{Q}_{p^2}$ such that $\gamma \in \mathbf{J}(E)$. Extending F if necessary, we may assume that $E \subset F$ (although we can take $E = F$, it is better to distinguish them in order to avoid confusion). Put $y' = \xi(\gamma)^{-1}(x') \in \mathcal{F}^\circ(F)$.

First we prove that the subspace $E \otimes_{\mathbb{Q}_p} \mathrm{Fil}_{x'} \subset E \otimes_{\mathbb{Q}_p} (F \otimes_{\mathbb{Q}_{p^2}} N^\circ)$ is mapped to $E \otimes_{\mathbb{Q}_p} \mathrm{Fil}_{y'}$ by $\gamma^{-1} \in \mathbf{J}(E)$. Put $W = \gamma^{-1}(E \otimes_{\mathbb{Q}_p} \mathrm{Fil}_{x'}) \subset E \otimes_{\mathbb{Q}_p} (F \otimes_{\mathbb{Q}_{p^2}} N^\circ)$. As $h' = \gamma^{-1} h \gamma \in \mathbf{J}(\mathbb{Q}_p)$, we have $\gamma \sigma(\gamma)^{-1} \in \mathbf{T}_h(E)$ for every $\sigma \in \mathrm{Gal}(E/\mathbb{Q}_{p^2})$. By Lemma 3.20, we have $W = \sigma(\gamma)^{-1}(E \otimes_{\mathbb{Q}_p} \mathrm{Fil}_{x'})$. The commutative diagram

$$
\begin{array}{ccc}
E \otimes_{\mathbb{Q}_p} (F \otimes_{\mathbb{Q}_{p^2}} N^\circ) & \xrightarrow{\gamma^{-1}} & E \otimes_{\mathbb{Q}_p} (F \otimes_{\mathbb{Q}_{p^2}} N^\circ) \\
\downarrow{\scriptstyle\sigma \otimes \mathrm{id}} & & \downarrow{\scriptstyle\sigma \otimes \mathrm{id}} \\
E \otimes_{\mathbb{Q}_p} (F \otimes_{\mathbb{Q}_{p^2}} N^\circ) & \xrightarrow{\sigma(\gamma)^{-1}} & E \otimes_{\mathbb{Q}_p} (F \otimes_{\mathbb{Q}_{p^2}} N^\circ)
\end{array}
$$

tells us that $(\sigma \otimes \mathrm{id})(W) = W$. Therefore, by the Galois descent, there exists an F-subspace $W' \subset F \otimes_{\mathbb{Q}_{p^2}} N^\circ$ such that $W = E \otimes_{\mathbb{Q}_p} W'$. It suffices to show that $W' = \mathrm{Fil}_{y'}$. By definition, W' is the image of $\mathrm{Fil}_{x'}$ under the map $F \otimes_{\mathbb{Q}_{p^2}} N^\circ \xrightarrow{(*)} F \otimes_{\mathbb{Q}_{p^2}} N^\circ$ obtained as the base change of $\gamma^{-1} \colon E \otimes_{\mathbb{Q}_p} (F \otimes_{\mathbb{Q}_{p^2}} N^\circ) \longrightarrow E \otimes_{\mathbb{Q}_p} (F \otimes_{\mathbb{Q}_{p^2}} N^\circ)$ by $E \otimes_{\mathbb{Q}_p} F \longrightarrow F$. Corollary 3.10 tells us that the base change of $\gamma^{-1} \colon E \otimes_{\mathbb{Q}_p} N^\circ \longrightarrow E \otimes_{\mathbb{Q}_p} N^\circ$ by $E \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2} \longrightarrow E$ coincides with $\xi(\gamma)^{-1} \colon E \otimes_{\mathbb{Q}_{p^2}} N^\circ \longrightarrow E \otimes_{\mathbb{Q}_{p^2}} N^\circ$. Hence $(*)$ is equal to the base change of $\xi(\gamma)^{-1}$ by $E \longrightarrow F$, and we conclude that $W' = \xi(\gamma)^{-1}(\mathrm{Fil}_{x'}) = \mathrm{Fil}_{y'}$, as desired.

Now we know that $\gamma \colon E \otimes_{\mathbb{Q}_p} N^\circ \xrightarrow{\cong} E \otimes_{\mathbb{Q}_p} N^\circ$ gives an isomorphism of filtered isocrystals $(E \otimes_{\mathbb{Q}_p} N^\circ, \Phi, E \otimes_{\mathbb{Q}_p} \mathrm{Fil}_{y'}) \xrightarrow{\cong} (E \otimes_{\mathbb{Q}_p} N^\circ, \Phi, E \otimes_{\mathbb{Q}_p} \mathrm{Fil}_{x'})$. It induces an isomorphism of corresponding $\mathrm{Gal}(\overline{F}/F)$-representations $\widetilde{\gamma} \colon E \otimes_{\mathbb{Q}_p} V_{y'} \xrightarrow{\cong} E \otimes_{\mathbb{Q}_p} V_{x'}$. Since $h' = \sigma(\gamma)^{-1} h \sigma(\gamma)$ and $y' = \xi(\sigma(\gamma))^{-1}(x')$ for each $\sigma \in \mathrm{Gal}(E/\mathbb{Q}_p)$, we may also define $\widetilde{\sigma(\gamma)} \colon E \otimes_{\mathbb{Q}_p} V_{y'} \xrightarrow{\cong} E \otimes_{\mathbb{Q}_p} V_{x'}$, which clearly coincides with $\sigma(\widetilde{\gamma}) = (\sigma \otimes \mathrm{id}) \circ \widetilde{\gamma} \circ (\sigma \otimes \mathrm{id})^{-1}$ by functoriality. By the construction, we have $\widetilde{\gamma} \sigma(\widetilde{\gamma})^{-1} = \widetilde{\gamma} \widetilde{\sigma(\gamma)}^{-1} = \iota_{h,x}(\gamma \sigma(\gamma)^{-1})$. The relation $h' = \gamma^{-1} h \gamma$ is translated into the commutativity of the following diagram:

$$
\begin{array}{ccc}
E \otimes_{\mathbb{Q}_p} V_{y'} & \xrightarrow{g_{h',y'}} & E \otimes_{\mathbb{Q}_p} V_{y'} \\
\downarrow{\scriptstyle\widetilde{\gamma}} & & \downarrow{\scriptstyle\widetilde{\gamma}} \\
E \otimes_{\mathbb{Q}_p} V_{x'} & \xrightarrow{g_{h,x'}} & E \otimes_{\mathbb{Q}_p} V_{x'}.
\end{array}
$$

In other words, if we fix $(V_{x'}, \psi_{x'}) \cong (\mathbb{Q}_p^{2d}, \langle\,,\,\rangle) \cong (V_{y'}, \psi_{y'})$ and regard $\widetilde{\gamma}$ as an element of $\mathrm{GSp}_{2d}(E)$, then we have $g_{h',y'} = \widetilde{\gamma}^{-1} g_{h,x'} \widetilde{\gamma}$. Therefore, under the isomorphism $\{g_{h,x}\}/\sim \xrightarrow{\cong} H^1(\mathbb{Q}_p, \mathbf{T}_{g_{h,x}})$, the conjugacy class of $g_{h',y'}$ corresponds to the cohomology class of the cocycle $\sigma \longmapsto \widetilde{\gamma} \sigma(\widetilde{\gamma})^{-1}$. This concludes the proof, since we have $\widetilde{\gamma} \sigma(\widetilde{\gamma})^{-1} = \iota_{h,x}(\gamma \sigma(\gamma)^{-1}) = \iota_{h,x}(c_{h,h'}(\sigma))$ as mentioned above. $\blacksquare$

Corollary 3.24 *Assume that h is elliptic. Let $g_h \in G$ be an arbitrary element with $g_h \leftrightarrow h$. Then, each $[g] \in \{g_h\}_{\mathrm{st}}/\sim$ satisfies the following:*

$$\#\{([h'], x) \mid [h'] \in \{h\}_{\mathrm{st}}/\sim, x \in \mathrm{Fix}(h'; \Omega), [g_{h',x}] = [g]\} = d! 2^{d-1}.$$

Here $[-]$ denotes the conjugacy class.

Proof. For $h' \in \{h\}_{\mathrm{st}}$, take $\gamma \in \mathbf{J}(\overline{\mathbb{Q}}_p)$ such that $h' = \gamma^{-1} h \gamma$. Then $\xi(\gamma)^{-1}$ induces a bijection $\mathrm{Fix}(h; \Omega) \longrightarrow \mathrm{Fix}(h'; \Omega)$. Thus by Proposition 3.23 we have

$$\#\{x \in \mathrm{Fix}(h', \Omega) \mid [g_{h',x}] = [g]\} = \#\{x \in \mathrm{Fix}(h, \Omega) \mid [g_{h',\xi(\gamma)^{-1}x}] = [g]\}$$
$$= \#\{x \in \mathrm{Fix}(h, \Omega) \mid \iota_{h,x}(c_{h'}) = c_g\},$$

where $c_{h'} \in H^1(\mathbb{Q}_p, \mathbf{T}_h)$ (resp. $c_g \in H^1(\mathbb{Q}_p, \mathbf{T}_{g_{h,x}})$) is the element corresponding to $[h'] \in \{h\}_{\mathrm{st}}/\sim$ (resp. $[g] \in \{g_h\}_{\mathrm{st}}/\sim = \{g_{h,x}\}_{\mathrm{st}}/\sim$) under the bijection in Proposition 3.23 i). Varying $[h']$ (or equivalently $c_{h'}$), we obtain

$$\#\{([h'], x) \mid [h'] \in \{h\}_{\mathrm{st}}/\sim, x \in \mathrm{Fix}(h'; \Omega), [g_{h',x}] = [g]\}$$
$$= \#\{(c, x) \in H^1(\mathbb{Q}_p, \mathbf{T}_h) \times \mathrm{Fix}(h; \Omega) \mid \iota_{h,x}(c) = c_g\},$$

whose right hand side is clearly equal to $\# \mathrm{Fix}(h; \Omega) = d! 2^{d-1}$ (*cf.* Remark 3.17). ■

The following is an analogue of [Fal94, Theorem 1] and [Str08, Theorem 2.6.8]:

Proposition 3.25 *Let x be as in Definition 3.18. Assume that $g \in G$ is regular elliptic. Then, the number of fixed points in $\wp_{K,p}^{-1}(x)$ under the action of (g, h) is given by*
$$\#\{\gamma K p^{\mathbb{Z}} \in G/K p^{\mathbb{Z}} \mid \gamma K p^{\mathbb{Z}} = g_{h,x}^{-1} \gamma g K p^{\mathbb{Z}}\}.$$
If h is elliptic, this number is equal to the orbital integral

$$O_{g_{h,x}}\left(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\mathrm{vol}(K)}\right),$$

where $\mathbf{1}_{gKp^{\mathbb{Z}}}$ denotes the characteristic function of $gKp^{\mathbb{Z}} \subset G$ (for our normalization of the Haar measure, see the last paragraph of Section 3.1).

Proof. As in Theorem 3.13 ii), there exist a finite extension L' of L and $\widetilde{x} = (X, \rho, \overline{\eta}) \in M_K(L')$ such that $\wp_K(\widetilde{x}) = x$. Here, (X, ρ) is an $\mathcal{O}_{L'}$-valued point of $\breve{\mathscr{M}}$ and $\overline{\eta}$ is a K-level structure on X, namely, a $\mathrm{Gal}(\overline{L}/L')$-invariant K-orbit of isomorphisms $\mathbb{Q}_p^{2d} \xrightarrow{\cong} V_p(X)$ preserving symplectic pairings up to $\mathbb{Q}_p^\times$-multiplication. Fix $\eta \in \overline{\eta}$. Then, by [RZ96, Proposition 5.37], the fiber $\wp_K^{-1}(x)$ of $\wp_K \colon M_K \longrightarrow \Omega$ at x can be identified with G/K. The identification is given as follows. Let $\widetilde{x}' = (X', \rho', \overline{\eta}')$ be another point in the fiber. Then, there exists a unique quasi-isogeny $f \colon X' \longrightarrow X$ satisfying $\rho = (f \bmod p) \circ \rho'$. This f automatically preserves

polarizations on X and X' up to $\mathbb{Q}_p^{\times}$-multiplication. Choose a representative η' in the level structure $\overline{\eta}'$, and define $\gamma \in G$ by the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{Q}_p^{2d} & \xrightarrow[\cong]{\eta'} & V_p(X') \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle V_p(f)} \\
\mathbb{Q}_p^{2d} & \xrightarrow[\cong]{\eta} & V_p(X).
\end{array}
$$

The class of γ in G/K does not depend on the choice of η'.

Since (g, h) acts on $\wp_K^{-1}(x)$, it also acts on G/K by the identification above. We will show that this action is given by $\gamma K \longmapsto g_{h,x}^{-1} \gamma g K$, where the element $g_{h,x} \in G$ is given by the isomorphism $V_x \cong V_p(X) \xrightarrow[\cong]{\eta^{-1}} \mathbb{Q}_p^{2d}$ (cf. Definition 3.18). Take $(X', \rho', \overline{\eta}') \in \wp_K^{-1}(x)$ corresponding to $\gamma K \in G/K$. Then $(g, h)(X', \rho', \overline{\eta}') = (X', \rho' \circ h, \overline{\eta'} \circ g)$. Since $\rho_* \circ h \circ (\rho_*)^{-1} \colon D(X)_{\mathbb{Q}} \longrightarrow D(X)_{\mathbb{Q}}$ preserves the Hodge filtration, there exists a quasi-isogeny $\phi \colon X \longrightarrow X$ such that $\rho \circ h = (\phi \bmod p) \circ \rho$. For this ϕ, we have $\rho = ((\phi^{-1} \circ f) \bmod p) \circ (\rho' \circ h)$, where $f \colon X' \longrightarrow X$ is a quasi-isogeny as above. Now we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathbb{Q}_p^{2d} & \xrightarrow[\cong]{\eta' \circ g} & V_p(X') \\
\downarrow{\scriptstyle g} & & \| \\
\mathbb{Q}_p^{2d} & \xrightarrow[\cong]{\eta'} & V_p(X') \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle V_p(f)} \\
\mathbb{Q}_p^{2d} & \xrightarrow[\cong]{\eta} & V_p(X) \\
\downarrow{\scriptstyle g_{h,x}^{-1}} & & \downarrow{\scriptstyle V_p(\phi)^{-1}} \\
\mathbb{Q}_p^{2d} & \xrightarrow[\cong]{\eta} & V_p(X).
\end{array}
$$

This diagram tells us that $(g, h)(X', \rho', \overline{\eta}')$ corresponds to $g_{h,x}^{-1} \gamma g K \in G/K$, as desired.

Now we consider the fiber $\wp_{K,p}^{-1}(x) = \wp_K^{-1}(x)/p^{\mathbb{Z}}$. By [RZ96, Lemma 5.36], it is in bijection with $G/Kp^{\mathbb{Z}}$. The action of (g, h) is given by $\gamma Kp^{\mathbb{Z}} \longmapsto g_{h,x}^{-1} \gamma g Kp^{\mathbb{Z}}$. Therefore, the number of fixed points under this action is equal to

$$
\#\{\gamma Kp^{\mathbb{Z}} \in G/Kp^{\mathbb{Z}} \mid \gamma Kp^{\mathbb{Z}} = g_{h,x}^{-1} \gamma g Kp^{\mathbb{Z}}\}.
$$

If h is elliptic, then $g_{h,x}$ is regular elliptic by Proposition 3.19, and thus the number above is equal to

$$
\frac{1}{\mathrm{vol}(Kp^{\mathbb{Z}}/p^{\mathbb{Z}})} \int_{G/p^{\mathbb{Z}}} \mathbf{1}_{gKp^{\mathbb{Z}}}(\gamma^{-1} g_{h,x} \gamma) d\gamma = O_{g_{h,x}}\left(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\mathrm{vol}(K)}\right). \qquad \blacksquare
$$

For later use, we will state our result for the action of the inverse element (g^{-1}, h^{-1}).

Theorem 3.26 *Let $(g, h) \in G \times J$ be an element and K a compact open subgroup of K_0 normalized by g.*

i) *If h is regular elliptic, then we have*

$$\# \operatorname{Fix}\big((g^{-1}, h^{-1}); M_K/p^{\mathbb{Z}}\big) = \sum_{x \in \operatorname{Fix}(h; \Omega)} O_{g_{h,x}}\Big(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\operatorname{vol}(K)}\Big),$$

where the Haar measure on $Z(g_{h,x})$ is normalized as in the last paragraph of Section 3.1. The left hand side denotes the number of fixed points in the sense of [Mie10a, Definition 2.6].

ii) *Assume that h is regular non-elliptic, and gK consists of regular elliptic elements. Then there is no point on $M_K/p^{\mathbb{Z}}$ fixed by (g^{-1}, h^{-1}).*

Proof. i) Since the multiplicity at each fixed point in Ω is one and $\wp_{K,p}$ is étale, the multiplicity at each fixed point in $M_K/p^{\mathbb{Z}}$ is equal to one (*cf.* [Mie10a, Proposition 2.10, Proposition 2.11]). Therefore, we have only to count points in $M_K/p^{\mathbb{Z}}$ fixed by (g^{-1}, h^{-1}) (or equivalently, fixed by (g, h)) as sets. The desired equality immediately follows from Proposition 3.25.

ii) If there exists a fixed point, then we can find $\gamma \in G$ and an integer n such that $p^{-n}\gamma^{-1}g_{h,x}\gamma \in gK$. Since gK consists of regular elliptic elements, $g_{h,x}$ is also regular elliptic. By Proposition 3.19, this contradicts to the assumption that h is non-elliptic. ∎

Corollary 3.27 *In the setting of Theorem 3.26 i), we have*

$$\sum_{[h'] \in \{h\}_{\mathrm{st}}/\sim} \# \operatorname{Fix}\big((g^{-1}, h^{-1}); M_K/p^{\mathbb{Z}}\big) = d! 2^{d-1} SO_{g_h}\Big(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\operatorname{vol}(K)}\Big),$$

where $g_h \in G$ is an arbitrary element with $g_h \leftrightarrow h$.

Proof. By Theorem 3.26 i) and Corollary 3.24, we have

$$\sum_{[h'] \in \{h\}_{\mathrm{st}}/\sim} \# \operatorname{Fix}\big((g^{-1}, h^{-1}); M_K/p^{\mathbb{Z}}\big) = \sum_{[h'] \in \{h\}_{\mathrm{st}}/\sim} \sum_{x \in \operatorname{Fix}(h'; \Omega)} O_{g_{h,x}}\Big(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\operatorname{vol}(K)}\Big)$$

$$= d! 2^{d-1} \sum_{[g'] \in \{g_h\}_{\mathrm{st}}/\sim} O_{g'}\Big(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\operatorname{vol}(K)}\Big) = d! 2^{d-1} SO_{g_h}\Big(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\operatorname{vol}(K)}\Big),$$

as desired. ∎

4   Some formal models

In order to apply the Lefschetz trace formula [Mie10a, Theorem 4.5], we need to construct a formal model of M_K for K in a certain family of compact open subgroups of K_0.

4.1   Construction of formal models

First let us recall the definition of a chain of lattices of $\mathbb{Q}_p^{2d}$ considered in [RZ96, §3]:

Definition 4.1 The set $\mathscr{L}$ of $\mathbb{Z}_p$-lattices of $\mathbb{Q}_p^{2d}$ is said to be a chain of lattices if the following hold (*cf.* [RZ96, Definition 3.1]):

  – for $L, L' \in \mathscr{L}$, we have either $L \subset L'$ or $L \supset L'$,

  – and for $L \in \mathscr{L}$ and $a \in \mathbb{Q}_p^\times$, we have $aL \in \mathscr{L}$.

Moreover, $\mathscr{L}$ is said to be self-dual if $L \in \mathscr{L}$ implies $L^\vee \in \mathscr{L}$, where $L^\vee$ denotes the dual lattice with respect to the fixed symplectic form $\langle \ , \ \rangle$ on $\mathbb{Q}_p^{2d}$.

  For a self-dual chain of lattices $\mathscr{L}$, write $K_\mathscr{L}$ for the stabilizer of $\mathscr{L}$ in G. It is a parahoric subgroup of G. Moreover, for an integer $m \geq 0$, we put

$$K_{\mathscr{L},m} = \{ g \in K_\mathscr{L} \mid \text{for every } L \in \mathscr{L},\ g \text{ acts trivially on } L/p^m L \},$$

which is an open normal subgroup of $K_\mathscr{L}$.

  Let $N_\mathscr{L}$ be the set of $g \in G$ such that $g\mathscr{L} = \mathscr{L}$. It coincides with the normalizer of $K_\mathscr{L}$ in G. Furthermore, $N_\mathscr{L}$ also normalizes $K_{\mathscr{L},m}$ for every $m \geq 0$. It is known that $N_\mathscr{L}$ is a compact-mod-center subgroup of G.

  In this subsection, we will construct a formal model $\breve{\mathscr{M}}_{\mathscr{L},m}$ of $M_{K_{\mathscr{L},m}}$ on which $N_\mathscr{L}$ acts as isomorphisms.

  In the following, we fix a self-dual chain of lattices $\mathscr{L}$. First we recall the Rapoport-Zink space with parahoric level $\breve{\mathscr{M}}_\mathscr{L}$ introduced in [RZ96, Definition 3.21]:

Definition 4.2 Let $\breve{\mathscr{M}}_\mathscr{L}$ be the contravariant functor **Nilp** $\longrightarrow$ **Set** that associates S with the set of isomorphism classes of $\{(X_L, \rho_L)\}_{L \in \mathscr{L}}$ where

  – X_L is a d-dimensional p-divisible group X over S,

  – and $\rho_L \colon \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{S} \longrightarrow X_L \otimes_S \overline{S}$ is a quasi-isogeny (as in the definition of $\breve{\mathscr{M}}$, we put $\overline{S} = S \otimes_{\mathbb{Z}_{p^\infty}} \overline{\mathbb{F}}_p$),

satisfying the following conditions:

  – For $L, L' \in \mathscr{L}$ with $L \subset L'$, the quasi-isogeny $\rho_{L'} \circ \rho_L^{-1} \colon X_L \otimes_S \overline{S} \longrightarrow X_{L'} \otimes_S \overline{S}$ lifts to an isogeny $\widetilde{\rho}_{L',L} \colon X_L \longrightarrow X_{L'}$.

  – For L, L' as above, $\deg \widetilde{\rho}_{L',L} = \log_p \#(L'/L)$.

  – For $L \in \mathscr{L}$, the quasi-isogeny $\rho_{pL} \circ [p] \circ \rho_L^{-1} \colon X_L \otimes_S \overline{S} \longrightarrow X_{pL} \otimes_S \overline{S}$ lifts to an isomorphism $\theta_p \colon X_L \xrightarrow{\cong} X_{pL}$ (here $[p]$ denotes the multiplication by p).

- There exists a constant $a \in \mathbb{Q}_p^\times$ such that for every $L \in \mathscr{L}$, we can find an isomorphism $\lambda_L \colon X_L \longrightarrow (X_{L^\vee})^\vee$ which makes the following diagram commutative:

$$\begin{array}{ccc} \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{S} & \xrightarrow{\rho_L} & X_L \otimes_S \overline{S} \\ \downarrow{\scriptstyle a\lambda_0 \otimes \mathrm{id}} & & \downarrow{\scriptstyle \lambda_L \otimes \mathrm{id}} \\ \mathbb{X}^\vee \otimes_{\overline{\mathbb{F}}_p} \overline{S} & \xleftarrow{(\rho_{L^\vee})^\vee} & (X_{L^\vee})^\vee \otimes_S \overline{S} \end{array}$$

(such λ_L automatically satisfies $\lambda_L^\vee = -\lambda_{L^\vee}$).

The functor $\breve{\mathscr{M}}_{\mathscr{L}}$ is represented by a special formal scheme over $\mathrm{Spf}\,\mathbb{Z}_{p^\infty}$.

The group $N_{\mathscr{L}} \times J$ naturally acts on $\breve{\mathscr{M}}_{\mathscr{L}}$ on the right; for $(g, h) \in N_{\mathscr{L}} \times J$, the action $(g, h) \colon \breve{\mathscr{M}}_{\mathscr{L}}(S) \longrightarrow \breve{\mathscr{M}}_{\mathscr{L}}(S)$ is given by $\{(X_L, \rho_L)\} \longmapsto \{(X_{gL}, \rho_{gL} \circ h)\}$. It is easy to see that $(g, 1)$ with $g \in K_{\mathscr{L}}$ and (p, p) act trivially on $\breve{\mathscr{M}}_{\mathscr{L}}$.

Next we consider level structures on $\breve{\mathscr{M}}_{\mathscr{L}}$.

Definition 4.3 For an integer $m \geq 0$, let $\breve{\mathscr{M}}'_{\mathscr{L},m}$ be the contravariant functor **Nilp** $\longrightarrow$ **Set** that associates S with the set of isomorphism classes of $\{(X_L, \rho_L, \eta_{L,m})\}_{L \in \mathscr{L}}$ where

- $\{(X_L, \rho_L)\}_{L \in \mathscr{L}} \in \breve{\mathscr{M}}_{\mathscr{L}}(S)$, and
- $\eta_{L,m} \colon L/p^m L \longrightarrow X_L[p^m]$ is a homomorphism

satisfying the following conditions:

- $\eta_{L,m}$ is a Drinfeld m-level structure. Namely, the image of $\eta_{L,m}$ gives a full set of sections of $X_L[p^m]$ (*cf.* [KM85, §1.8]).
- For $L, L' \in \mathscr{L}$ with $L \subset L'$, the following diagrams are commutative:

$$\begin{array}{ccc} L/p^m L & \longrightarrow & L'/p^m L' \\ \downarrow{\scriptstyle \eta_{L,m}} & & \downarrow{\scriptstyle \eta_{L',m}} \\ X_L[p^m] & \xrightarrow{\widetilde{\rho}_{L',L}} & X_{L'}[p^m], \end{array} \qquad \begin{array}{ccc} L/p^m L & \xrightarrow[\cong]{\times p} & pL/p^{m+1}L \\ \downarrow{\scriptstyle \eta_{L,m}} & & \downarrow{\scriptstyle \eta_{pL,m}} \\ X_L[p^m] & \xrightarrow[\cong]{\theta_p} & X_{pL}[p^m]. \end{array}$$

- There exists a homomorphism $\mathbb{Z}/p^m\mathbb{Z} \longrightarrow \mu_{p^m}$ such that for every $L \in \mathscr{L}$ the diagram below is commutative up to constant which is independent of L:

$$\begin{array}{ccc} L/p^m L \times L^\vee/p^m L^\vee & \xrightarrow{\quad\langle\ ,\ \rangle\quad} & \mathbb{Z}/p^m\mathbb{Z} \\ \downarrow{\scriptstyle \eta_{L,m} \times \eta_{L^\vee,m}} & & \downarrow \\ X_L[p^m] \times X_{L^\vee}[p^m] \xrightarrow{\mathrm{id}\times\lambda_{L^\vee}} X_L[p^m] \times X_L^\vee[p^m] & \longrightarrow & \mu_{p^m}. \end{array}$$

It is easy to see that $\breve{\mathscr{M}}'_{\mathscr{L},m}$ is represented by a formal scheme which is finite over $\breve{\mathscr{M}}_{\mathscr{L}}$ (*cf.* [KM85, Proposition 1.9.1]). The group $N_{\mathscr{L}} \times J$ naturally acts on $\breve{\mathscr{M}}'_{\mathscr{L},m}$ on the right; by $(g, h) \in N_{\mathscr{L}} \times J$, $\{(X_L, \rho_L, \eta_{L,m})\}_{L \in \mathscr{L}}$ is mapped to $\{(X_{gL}, \rho_{gL} \circ$

$h, \eta_{gL,m} \circ g)\}_{L \in \mathscr{L}}$. It is easy to see that $(g, 1)$ with $g \in K_{\mathscr{L},m}$ and (p, p) act trivially on $\breve{\mathscr{M}}'_{\mathscr{L},m}$. By [Man04, Lemma 7.2], $\{\breve{\mathscr{M}}'_{\mathscr{L},m}\}_{m \geq 0}$ forms a projective system of formal schemes equipped with actions of $N_{\mathscr{L}} \times J$.

Let $\breve{\mathscr{M}}_{\mathscr{L},m}$ be the intersection of the scheme-theoretic images of $\breve{\mathscr{M}}'_{\mathscr{L},m'} \longrightarrow \breve{\mathscr{M}}'_{\mathscr{L},m}$ for all $m' \geq m$. Obviously $\{\breve{\mathscr{M}}_{\mathscr{L},m}\}_{m \geq 0}$ again forms a projective system of formal schemes.

Finally, let $\breve{\mathscr{M}}^\flat_{\mathscr{L},m}$ be the closed formal subscheme of $\breve{\mathscr{M}}_{\mathscr{L},m}$ defined by the quasi-coherent ideal of $\mathcal{O}_{\breve{\mathscr{M}}_{\mathscr{L},m}}$ consisting of elements killed by p^l for some integer $l \geq 0$. It is flat over $\mathrm{Spf}\, \mathbb{Z}_{p^\infty}$. We obtain a projective system of formal schemes $\{\breve{\mathscr{M}}^\flat_{\mathscr{L},m}\}_{m \geq 0}$ equipped with actions of $N_{\mathscr{L}} \times J$. Obviously we can identify $\breve{\mathscr{M}}^{\flat,\mathrm{rig}}_{\mathscr{L},m}$ with $\breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m}$.

Remark 4.4 By [RZ96, Proposition A.21], we can show that the natural map $\breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L}}$ is surjective for every $m \geq 0$. In particular, we have $\breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},0} = \breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L}}$.

The formal scheme $\breve{\mathscr{M}}^\flat_{\mathscr{L},m}$ gives a formal model of the rigid space $M_{K_{\mathscr{L},m}}$:

Proposition 4.5 *Assume that $K_{\mathscr{L},m} \subset K_0$. Then, we have a natural isomorphism $\breve{\mathscr{M}}^{\flat,\mathrm{rig}}_{\mathscr{L},m} \cong M_{K_{\mathscr{L},m}}$ of rigid spaces over $\mathbb{Q}_{p^\infty}$ which is compatible with actions of $N_{\mathscr{L}} \times J$ and change of m.*

Proof. First we construct a morphism $M_{K_{\mathscr{L},m}} \longrightarrow \breve{\mathscr{M}}'^{\mathrm{rig}}_m$. Let $\mathcal{S}$ be a formal scheme of finite type over $\mathrm{Spf}\, \mathbb{Z}_{p^\infty}$ such that $\mathcal{S}^{\mathrm{rig}}$ is connected, and $\mathcal{S}^{\mathrm{rig}} \longrightarrow M_{K_{\mathscr{L},m}}$ a morphism over $\mathbb{Q}_{p^\infty}$. It suffices to construct a morphism $\mathcal{S}^{\mathrm{rig}} \longrightarrow \breve{\mathscr{M}}'^{\mathrm{rig}}_m$.

By changing $\mathcal{S}$ by its admissible blow-up, we may assume that the composite $\mathcal{S}^{\mathrm{rig}} \longrightarrow M_{K_{\mathscr{L},m}} \longrightarrow M$ extends to $\mathcal{S} \longrightarrow \breve{\mathscr{M}}$. Therefore, we have the following data:

- a p-divisible group X on $\mathcal{S}$,
- a quasi-isogeny $\rho \colon \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{\mathcal{S}} \longrightarrow X \times_{\mathcal{S}} \overline{\mathcal{S}}$ satisfying the same condition as in the definition of $\breve{\mathscr{M}}$,
- and a $\pi_1(\mathcal{S}^{\mathrm{rig}}, \overline{x})$-stable $K_{\mathscr{L},m}$-orbit $\overline{\eta}$ of an isomorphism $\eta \colon \mathbb{Z}_p^{2d} \xrightarrow{\cong} T_p X^{\mathrm{rig}}_{\overline{x}}$ which preserves polarizations up to multiplication by $\mathbb{Z}_p^\times$.

Here we put $\overline{\mathcal{S}} = \mathcal{S} \otimes_{\mathbb{Z}_{p^\infty}} \overline{\mathbb{F}}_p$ and fix a geometric point $\overline{x}$ of $\mathcal{S}^{\mathrm{rig}}$.

Fix $\eta \in \overline{\eta}$. It corresponds to a homomorphism $\eta \colon \mathbb{Q}_p^{2d}/\mathbb{Z}_p^{2d} \longrightarrow X^{\mathrm{rig}}_{\overline{x}}$. Choose $L_0 \in \mathscr{L}$ with $\mathbb{Z}_p^{2d} \subset L_0$ and consider $L \in \mathscr{L}$ with $L_0 \subset L \subsetneq p^{-1} L_0$. The image of $L/\mathbb{Z}_p^{2d} \subset \mathbb{Q}_p^{2d}/\mathbb{Z}_p^{2d}$ under η corresponds to a finite étale subgroup scheme of X^{rig}. Since there are only finitely many such L, by the flattening theorem (*cf.* [BL93]), after replacing $\mathcal{S}$ by an admissible blow-up we may assume that for each L there exists a finite flat subgroup scheme Y_L of X whose rigid generic fiber Y_L^{rig} corresponds to the image $\eta(L/\mathbb{Z}_p^{2d})$. Put $X_L = X/Y_L$, and write $\varphi_L \colon X \longrightarrow X_L$ for the canonical isogeny. The homomorphism $\eta \colon \mathbb{Q}_p^{2d}/\mathbb{Z}_p^{2d} \longrightarrow X^{\mathrm{rig}}_{\overline{x}}$ induces $\eta_L \colon \mathbb{Q}_p^{2d}/L \longrightarrow X^{\mathrm{rig}}_{L,\overline{x}}$,

which corresponds to $\eta_L \colon L \longrightarrow T_p X_{L,\overline{x}}^{\mathrm{rig}}$. It is easy to see that this homomorphism fits into the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_p^{2d} & \longrightarrow & L & \longrightarrow & L/\mathbb{Z}_p^{2d} & \longrightarrow & 0 \\
& & \cong \downarrow \eta & & \downarrow \eta_L & & \cong \downarrow & & \\
0 & \longrightarrow & T_p X_{\overline{x}}^{\mathrm{rig}} & \longrightarrow & T_p X_{L,\overline{x}}^{\mathrm{rig}} & \longrightarrow & Y_L^{\mathrm{rig}} & \longrightarrow & 0.
\end{array}
$$

In particular, η_L is an isomorphism.

For $L, L' \in \mathscr{L}$ with $L_0 \subset L \subset L' \subsetneq p^{-1}L_0$, we have a natural closed immersion $Y_L^{\mathrm{rig}} \longrightarrow Y_{L'}^{\mathrm{rig}}$. Since Y_L and $Y_{L'}$ are finite flat closed subgroup schemes of X, this extends to a closed immersion $Y_L \longrightarrow Y_{L'}$. Therefore we have a natural isogeny $\widetilde{\rho}_{L',L} \colon X_L \longrightarrow X_{L'}$. Clearly the following diagrams are commutative:

$$
\begin{array}{ccccc}
\mathbb{Q}_p^{2d}/\mathbb{Z}_p^{2d} & \longrightarrow & \mathbb{Q}_p^{2d}/L & \longrightarrow & \mathbb{Q}_p^{2d}/L' \\
\downarrow \eta & & \downarrow \eta_L & & \downarrow \eta_{L'} \\
X_{\overline{x}}^{\mathrm{rig}} & \xrightarrow{\varphi_L} & X_{L,\overline{x}}^{\mathrm{rig}} & \xrightarrow{\widetilde{\rho}_{L',L}} & X_{L',\overline{x}}^{\mathrm{rig}},
\end{array}
\qquad
\begin{array}{ccccc}
\mathbb{Z}_p^{2d} & \longrightarrow & L & \longrightarrow & L' \\
\cong \downarrow \eta & & \cong \downarrow \eta_L & & \cong \downarrow \eta_{L'} \\
T_p X_{\overline{x}}^{\mathrm{rig}} & \xrightarrow{\varphi_L} & T_p X_{L,\overline{x}}^{\mathrm{rig}} & \xrightarrow{\widetilde{\rho}_{L',L}} & T_p X_{L',\overline{x}}^{\mathrm{rig}}.
\end{array}
$$

For $L \in \mathscr{L}$ with $L_0 \subset L \subsetneq p^{-1}L_0$, $X_{L_0} \xrightarrow{[p]} X_{L_0}$ factors as $X_{L_0} \xrightarrow{\widetilde{\rho}_{L,L_0}} X_L \xrightarrow{\theta_{L_0,L}} X_{L_0}$. By the construction, we have the following commutative diagrams:

$$
\begin{array}{ccc}
\mathbb{Q}_p^{2d}/L & \longrightarrow & \mathbb{Q}_p^{2d}/p^{-1}L_0 \\
\downarrow \eta_L & & \downarrow \eta_{L_0}\circ p \\
X_{L,\overline{x}}^{\mathrm{rig}} & \xrightarrow{\theta_{L_0,L}} & X_{L_0,\overline{x}}^{\mathrm{rig}},
\end{array}
\qquad
\begin{array}{ccc}
L & \longrightarrow & p^{-1}L_0 \\
\cong \downarrow \eta_L & & \cong \downarrow \eta_{L_0}\circ p \\
T_p X_{L,\overline{x}}^{\mathrm{rig}} & \xrightarrow{\theta_{L_0,L}} & T_p X_{L_0,\overline{x}}^{\mathrm{rig}}.
\end{array}
$$

Next consider the level structure. For $L \in \mathscr{L}$ with $L_0 \subset L \subsetneq p^{-1}L_0$, let $K_{L,m}$ be the kernel of $\mathrm{GL}(L) \longrightarrow \mathrm{GL}(L/p^m L)$. As the $K_{\mathscr{L},m}$-orbit of $\eta_L \colon L \longrightarrow T_p X_{L,\overline{x}}^{\mathrm{rig}}$ is $\pi_1(\mathcal{S}^{\mathrm{rig}}, \overline{x})$-invariant, so is the $K_{L,m}$-orbit of η_L. Therefore we obtain an isomorphism $L/p^m L \xrightarrow{\cong} X_L^{\mathrm{rig}}[p^m]$. Again by the flattening theorem, we may assume that this isomorphism extends to a homomorphism $\eta_{m,L} \colon L/p^m L \longrightarrow X_L[p^m]$ of group schemes. This is a Drinfeld m-level structure (*cf.* [KM85, Lemma 1.8.3, Proposition 1.9.1]). For $L, L' \in \mathscr{L}$ with $L_0 \subset L \subset L' \subsetneq p^{-1}L_0$, the following diagrams are clearly commutative:

$$
\begin{array}{ccc}
L/p^m L & \longrightarrow & L'/p^m L' \\
\downarrow \eta_{L,m} & & \downarrow \eta_{L',m} \\
X_L[p^m] & \xrightarrow{\widetilde{\rho}_{L',L}} & X_{L'}[p^m],
\end{array}
\qquad
\begin{array}{ccc}
L/p^m L & \longrightarrow & p^{-1}L_0/p^{m-1}L_0 \\
\downarrow \eta_{L,m} & & \downarrow \eta_{L_0,m}\circ p \\
X_L[p^m] & \xrightarrow{\theta_{L_0,L}} & X_{L_0}[p^m].
\end{array}
$$

Now we extend the construction above to all $L \in \mathscr{L}$; take an integer n such that $L_0 \subset p^n L \subsetneq p^{-1}L_0$, and put $X_L = X_{p^n L}$, $\varphi_L = p^{-n} \circ \varphi_{p^n L} \colon X \longrightarrow X_L$, $\eta_L =$

$\eta_{p^n L} \circ p^n \colon L \xrightarrow{\cong} T_p X_{L,\overline{x}}^{\mathrm{rig}}$ and $\eta_{L,m} = \eta_{p^n L,m} \circ p^n \colon L/p^m L \longrightarrow X_L[p^m]$. Then, for $L, L' \in \mathscr{L}$ with $L \subset L'$, we have a natural isogeny $\widetilde{\rho}_{L',L} \colon X_L \longrightarrow X_{L'}$ which makes the following diagrams commutative:

$$
\begin{array}{ccc}
L \longrightarrow L' & \qquad & L/p^m L \longrightarrow L'/p^m L' \\
{\scriptstyle \cong}\Big\downarrow{\scriptstyle \eta_L} \quad {\scriptstyle \cong}\Big\downarrow{\scriptstyle \eta_{L'}} & & \Big\downarrow{\scriptstyle \eta_{L,m}} \qquad \Big\downarrow{\scriptstyle \eta_{L',m}} \\
T_p X_{L,\overline{x}}^{\mathrm{rig}} \xrightarrow{\widetilde{\rho}_{L',L}} T_p X_{L',\overline{x}}^{\mathrm{rig}}, & & X_L[p^m] \xrightarrow{\widetilde{\rho}_{L',L}} X_{L'}[p^m].
\end{array}
$$

Indeed, take integers n, n' such that $L_0 \subset p^n L \subsetneq p^{-1} L_0$ and $L_0 \subset p^{n'} L' \subsetneq p^{-1} L_0$ (the assumption $L \subset L'$ implies $n \leq n'$). If $p^n L \subset p^{n'} L'$, we set

$$
\widetilde{\rho}_{L',L} \colon X_L = X_{p^n L} \xrightarrow{\widetilde{\rho}_{p^{n'} L', p^n L} \circ [p^{n'-n}]} X_{p^{n'} L'} = X_{L'}.
$$

If $p^n L \supset p^{n'} L'$ (in this case $n < n'$), we set

$$
\widetilde{\rho}_{L',L} \colon X_L = X_{p^n L} \xrightarrow{\theta_{L_0, p^n L} \circ [p^{n'-n-1}]} X_{L_0} \xrightarrow{\widetilde{\rho}_{p^{n'} L', L_0}} X_{p^{n'} L'} = X_{L'}.
$$

Put $\rho_L = \varphi_L \circ \rho \colon \mathbb{X} \otimes_{\mathbb{F}_p} \overline{\mathcal{S}} \longrightarrow X_L \times_{\mathcal{S}} \overline{\mathcal{S}}$. Then, it is straightforward to check that $\{(X_L, \rho_L, \eta_{L,m})\}_{L \in \mathscr{L}}$ satisfies the conditions in the definition of $\breve{\mathscr{M}}'_{\mathscr{L},m}$ (for the condition on polarizations, we can work on the generic fibers). Moreover, the isomorphism class of $\{(X_L, \rho_L, \eta_{L,m})\}_{L \in \mathscr{L}}$ is independent of the choice of $\overline{x}$, η and L_0. Hence we get a morphism $\mathcal{S} \longrightarrow \breve{\mathscr{M}}'_{\mathscr{L},m}$, and thus a morphism $M_{K_{\mathscr{L},m}} \longrightarrow \breve{\mathscr{M}}'^{\mathrm{rig}}_{\mathscr{L},m}$. By the construction, it is easy to verify that this morphism is compatible with the actions of $N_{\mathscr{L}} \times J$ and change of m.

For $m' \geq m$, the transition morphism $M_{K_{\mathscr{L},m'}} \longrightarrow M_{K_{\mathscr{L},m}}$ is surjective. Therefore, the morphism $M_{K_{\mathscr{L},m}} \longrightarrow \breve{\mathscr{M}}'^{\mathrm{rig}}_{\mathscr{L},m}$ factors through the image of $\breve{\mathscr{M}}'^{\mathrm{rig}}_{\mathscr{L},m'} \longrightarrow \breve{\mathscr{M}}'^{\mathrm{rig}}_{\mathscr{L},m}$ for every $m' \geq m$. Since $\breve{\mathscr{M}}'^{\mathrm{rig}}_{\mathscr{L},m'} \longrightarrow \breve{\mathscr{M}}'^{\mathrm{rig}}_{\mathscr{L},m}$ is finite étale, $\breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m}$ coincides with the intersection of the images of $\breve{\mathscr{M}}'^{\mathrm{rig}}_{\mathscr{L},m'} \longrightarrow \breve{\mathscr{M}}'^{\mathrm{rig}}_{\mathscr{L},m}$ for all $m' \geq m$. Hence we have a morphism $M_{K_{\mathscr{L},m}} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m}$ of rigid spaces over $\mathbb{Q}_{p^\infty}$.

Next we construct an inverse morphism $\breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m} \longrightarrow M_{K_{\mathscr{L},m}}$. Let $\mathcal{S}$ be a formal scheme of finite type over $\mathrm{Spf}\, \mathbb{Z}_{p^\infty}$ such that $\mathcal{S}^{\mathrm{rig}}$ is connected, and $\mathcal{S}^{\mathrm{rig}} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m}$ a morphism over $\mathbb{Q}_{p^\infty}$. It suffices to construct a morphism $\mathcal{S}^{\mathrm{rig}} \longrightarrow M_{K_{\mathscr{L},m}}$. By changing $\mathcal{S}$ by its admissible blow-up, we may assume that the morphism $\mathcal{S}^{\mathrm{rig}} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m}$ comes from a morphism $\mathcal{S} \longrightarrow \breve{\mathscr{M}}_{\mathscr{L},m}$ of formal schemes over $\mathbb{Z}_{p^\infty}$. Therefore, we have the data $\{(X_L, \rho_L, \eta_{L,m})\}$ where

- X_L is a p-divisible group on $\mathcal{S}$,
- $\rho_L \colon \mathbb{X} \otimes_{\mathbb{F}_p} \overline{\mathcal{S}} \longrightarrow X_L \times_{\mathcal{S}} \overline{\mathcal{S}}$ is a quasi-isogeny,
- and $\eta_{L,m} \colon L/p^m L \longrightarrow X_L[p^m]$

satisfying suitable conditions (*cf.* Definition 4.3).

Fix a geometric point $\overline{x}$ of $\mathcal{S}^{\mathrm{rig}}$. By the definition of $\breve{\mathcal{M}}_{\mathscr{L},m}$, we can find a family of isomorphisms $\{\eta_L \colon L \xrightarrow{\cong} T_p X_{L,\overline{x}}^{\mathrm{rig}}\}_{L \in \mathscr{L}}$ satisfying the following:

(a) $\eta_L \bmod p^m = \eta_{L,m,\overline{x}}$.

(b) For $L, L' \in \mathscr{L}$ with $L \subset L'$, the following diagrams are commutative:

$$
\begin{array}{ccc}
L & \longrightarrow & L' \\
\cong \downarrow \eta_L & & \cong \downarrow \eta_{L'} \\
T_p X_{L,\overline{x}}^{\mathrm{rig}} & \xrightarrow{\widetilde{\rho}_{L',L}} & T_p X_{L',\overline{x}}^{\mathrm{rig}},
\end{array}
\qquad
\begin{array}{ccc}
L & \xrightarrow[\cong]{\times p} & pL \\
\cong \downarrow \eta_L & & \cong \downarrow \eta_{pL} \\
T_p X_{L,\overline{x}}^{\mathrm{rig}} & \xrightarrow[\cong]{\theta_p} & T_p X_{pL,\overline{x}}^{\mathrm{rig}}.
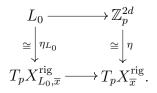\end{array}
$$

(c) Fix an isomorphism $\mathbb{Z}_p \xrightarrow{\cong} \mathbb{Z}_p(1)$. Then, for every $L \in \mathscr{L}$, the diagram below is commutative up to constant which is independent of L:

$$
\begin{array}{ccc}
L \times L^{\vee} & \xrightarrow{\langle\ ,\ \rangle} & \mathbb{Z}_p \\
\cong \downarrow \eta_L \times \eta_{L^{\vee}} & & \downarrow \cong \\
T_p X_{L,\overline{x}}^{\mathrm{rig}} \times T_p X_{L^{\vee},\overline{x}}^{\mathrm{rig}} \xrightarrow{\mathrm{id} \times \lambda_{L^{\vee}}} T_p X_{L,\overline{x}}^{\mathrm{rig}} \times T_p X_{L,\overline{x}}^{\vee\mathrm{rig}} & \longrightarrow & \mathbb{Z}_p(1).
\end{array}
$$

Fix $L_0 \in \mathscr{L}$ such that $L_0 \subset \mathbb{Z}_p^{2d}$. The homomorphism $\eta_{L_0} \colon L_0 \longrightarrow T_p X_{L_0,\overline{x}}^{\mathrm{rig}}$ can be identified with $\eta_{L_0} \colon \mathbb{Q}_p^{2d}/L_0 \longrightarrow X_{L_0,\overline{x}}^{\mathrm{rig}}$. We will see that the image $\eta_{L_0}(\mathbb{Z}_p^{2d}/L_0)$ of $\mathbb{Z}_p^{2d}/L_0$ under η_{L_0} is $\pi_1(\mathcal{S}^{\mathrm{rig}}, \overline{x})$-stable. Indeed, for each $\sigma \in \pi_1(\mathcal{S}^{\mathrm{rig}}, \overline{x})$ and $L \in \mathscr{L}$, we can find $g_{L,\sigma} \in \mathrm{GL}(L)$ such that $\sigma \circ \eta_L = \eta_L \circ g_{L,\sigma}$. By (b) above, if we regard $\mathrm{GL}(L)$ as a subset of $\mathrm{GL}_{2d}(\mathbb{Q}_p)$, then $g_{L,\sigma} = g_{L_0,\sigma}$ for every $L \in \mathscr{L}$. Moreover, (a) tells us that $g_{L_0,\sigma} = g_{L,\sigma}$ lies in the kernel of $\mathrm{GL}(L) \longrightarrow \mathrm{GL}(L/p^m L)$, and (c) tells us that $g_{L_0,\sigma} \in \mathrm{GSp}_{2d}(\mathbb{Q}_p)$. We conclude that $g_{L_0,\sigma} \in K_{\mathscr{L},m}$, and thus $g_{L_0,\sigma} \in K_0$ by the assumption. Hence $\sigma(\eta_{L_0}(\mathbb{Z}_p^{2d}/L_0)) = \eta_{L_0}(g_{L_0,\sigma}(\mathbb{Z}_p^{2d}/L_0)) = \eta_{L_0}(\mathbb{Z}_p^{2d}/L_0)$, as desired. Therefore, the subset $\eta_{L_0}(\mathbb{Z}_p^{2d}/L_0) \subset T_p X_{L_0,\overline{x}}^{\mathrm{rig}}$ corresponds to a finite étale closed subgroup scheme of $X_{L_0}^{\mathrm{rig}}$. By the flattening theorem, after replacing $\mathcal{S}$, we may assume that this subgroup scheme comes from a finite flat closed subgroup scheme Y of X_{L_0}. Put $X = X_{L_0}/Y$. Then, by the similar argument as above, we have an isomorphism $\eta \colon \mathbb{Z}_p^{2d} \xrightarrow{\cong} T_p X_{\overline{x}}^{\mathrm{rig}}$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
L_0 & \longrightarrow & \mathbb{Z}_p^{2d} \\
\cong \downarrow \eta_{L_0} & & \cong \downarrow \eta \\
T_p X_{L_0,\overline{x}}^{\mathrm{rig}} & \longrightarrow & T_p X_{\overline{x}}^{\mathrm{rig}}.
\end{array}
$$

Let ρ be the composite $\mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{\mathcal{S}} \xrightarrow{\rho_{L_0}} X_{L_0} \times_{\mathcal{S}} \overline{\mathcal{S}} \longrightarrow X \times_{\mathcal{S}} \overline{\mathcal{S}}$. We will show that the triple $(X, \rho, \overline{\eta})$ gives an $\mathcal{S}^{\mathrm{rig}}$-valued point of $M_{K_{\mathscr{L},m}}$. For existence of a polarization,

let $\lambda\colon X \longrightarrow X^\vee$ be the quasi-isogeny $X \longrightarrow X_L \xrightarrow{\lambda_L} (X_{L^\vee})^\vee \xrightarrow{(*)} X_L^\vee \longrightarrow X^\vee$, where $(*)$ is the quasi-isogeny $\widetilde{\rho}^\vee_{L^\vee, L}$ if $L \subset L^\vee$, and $(\widetilde{\rho}^\vee_{L, L^\vee})^{-1}$ if $L^\vee \subset L$. Then, the following diagram is commutative up to multiplication by $\mathbb{Z}_p^\times$:

$$
\begin{array}{ccc}
\mathbb{Q}_p^{2d} \times \mathbb{Q}_p^{2d} & \xrightarrow{\ \langle\ ,\ \rangle\ } & \mathbb{Q}_p \\
{\scriptstyle \cong}\downarrow{\scriptstyle \eta \times \eta} & & \downarrow{\scriptstyle \cong} \\
V_p X^{\mathrm{rig}}_{\overline{x}} \times V_p X^{\mathrm{rig}}_{\overline{x}} \xrightarrow{\mathrm{id} \times \lambda} V_p X^{\mathrm{rig}}_{\overline{x}} \times V_p X^{\vee\mathrm{rig}}_{\overline{x}} & \longrightarrow & \mathbb{Q}_p(1),
\end{array}
$$

where the isomorphism $\mathbb{Q}_p \longrightarrow \mathbb{Q}_p(1)$ is induced from the isomorphism $\mathbb{Z}_p \longrightarrow \mathbb{Z}_p(1)$ fixed in (c) above. In particular $\lambda(T_p X^{\mathrm{rig}}_{\overline{x}}) = T_p X^{\vee\mathrm{rig}}_{\overline{x}}$, and thus λ is an isomorphism. It obviously satisfies that $\rho^\vee \circ \lambda \circ \rho = a\lambda_0$ for some $a \in \mathbb{Q}_p^\times$. The diagram above also tells us that η preserves polarizations up to multiplication by $\mathbb{Z}_p^\times$. On the other hand, the $K_{\mathscr{L},m}$-orbit of η is invariant under $\pi_1(\mathcal{S}^{\mathrm{rig}}, \overline{x})$ simply because the $K_{\mathscr{L},m}$-orbit of η_{L_0} is invariant (recall that we have proved $g_{L_0,\sigma} \in K_{\mathscr{L},m}$).

Now we obtain an $\mathcal{S}^{\mathrm{rig}}$-valued point $(X, \rho, \overline{\eta})$ of $M_{K_{\mathscr{L},m}}$. It is easy to see that it is independent of the choice of $\overline{x}$ and L_0. Hence we have a canonical morphism $\mathcal{S}^{\mathrm{rig}} \longrightarrow M_{K_{\mathscr{L},m}}$, and thus $\breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m} \longrightarrow M_{K_{\mathscr{L},m}}$.

It is not difficult to show that the morphisms $M_{K_{\mathscr{L},m}} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m}$ and $\breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m} \longrightarrow M_{K_{\mathscr{L},m}}$ we have obtained are inverse to each other. As $\breve{\mathscr{M}}^{\mathrm{rig}}_{\mathscr{L},m} \cong \breve{\mathscr{M}}^{\flat,\mathrm{rig}}_{\mathscr{L},m}$, we have a desired isomorphism $M_{K_{\mathscr{L},m}} \cong \breve{\mathscr{M}}^{\flat,\mathrm{rig}}_{\mathscr{L},m}$. ∎

4.2 Boundary strata

Definition 4.6 For an integer h with $1 \leq h \leq d$, let $\mathcal{S}_{\infty,h}$ be the set of totally isotropic subspaces of dimension h of $\mathbb{Q}_p^{2d}$. Recall that a subspace V of $\mathbb{Q}_p^{2d}$ is said to be totally isotropic if $V \subset V^\perp$. For a lattice L of $\mathbb{Z}_p^{2d}$ and an integer $m \geq 0$, denote by $\mathcal{S}_{L,m,h}$ the set of direct summands of rank h of $L/p^m L$.

For a self-dual chain of lattices $\mathscr{L}$ of $\mathbb{Q}_p^{2d}$ and an integer $m \geq 0$, we have a natural map $\mathcal{S}_{\infty,h} \longrightarrow \prod_{L \in \mathscr{L}} \mathcal{S}_{L,m,h}$; $V \longmapsto V_{\mathscr{L},m} = (V \cap L \bmod p^m)_{L \in \mathscr{L}}$. We denote its image by $\mathcal{S}_{\mathscr{L},m,h}$. The group $N_{\mathscr{L}}$ acts naturally on $\mathcal{S}_{\infty,h}$ and $\prod_{L \in \mathscr{L}} \mathcal{S}_{L,m,h}$, and the map above is equivariant with respect to these actions. Therefore $N_{\mathscr{L}}$ also acts on $\mathcal{S}_{\mathscr{L},m,h}$.

Put $\mathcal{S}_\infty = \bigcup_{h=1}^d \mathcal{S}_{\infty,h}$ and $\mathcal{S}_{\mathscr{L},m} = \bigcup_{h=1}^d \mathcal{S}_{\mathscr{L},m,h}$. For two elements $\alpha = (\alpha_L)_L$, $\beta = (\beta_L)_L$ in $\mathcal{S}_{\mathscr{L},m}$, we write $\alpha \prec \beta$ if $\alpha_L \supset \beta_L$ for every $L \in \mathscr{L}$. This gives a partial order on $\mathcal{S}_{\mathscr{L},m}$. The action of $N_{\mathscr{L}}$ on $\mathcal{S}_{\mathscr{L},m}$ obviously preserves this partial order.

Lemma 4.7 For every $m \geq 0$, $\mathcal{S}_{\mathscr{L},m}$ is a finite set.

Proof. Since $K_{\mathscr{L},m}$ acts trivially on $\mathcal{S}_{\mathscr{L},m}$, we have a natural surjection $K_{\mathscr{L},m} \backslash \mathcal{S}_\infty \longrightarrow \mathcal{S}_{\mathscr{L},m}$. Therefore it suffices to show that $K_{\mathscr{L},m} \backslash \mathcal{S}_{\infty,h}$ is a finite set for each $1 \leq h \leq d$. For an integer $m' \geq 0$, let $K_{m'}$ be the kernel of $K_0 \longrightarrow \mathrm{GSp}_{2d}(\mathbb{Z}/p^{m'}\mathbb{Z})$. Then,

for a sufficiently large m', we have $K_{m'} \subset K_{\mathscr{L},m}$. Therefore it suffices to show that $K_{m'}\backslash\mathcal{S}_{\infty,h}$ is a finite set. Fix an element V of $\mathcal{S}_{\infty,h}$, and denote by P the stabilizer of V in G. It is well-known that G acts transitively on $\mathcal{S}_{\infty,h}$ and P is a parabolic subgroup of G. Therefore, by the Iwasawa decomposition $G = K_0 P$ for the hyperspecial subgroup K_0, we have

$$K_{m'}\backslash\mathcal{S}_{\infty,h} \cong K_{m'}\backslash G/P \cong K_{m'}\backslash K_0/P \cap K_0,$$

which is obviously a finite set. ∎

Put $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,s} = \breve{\mathscr{M}}^{\flat}_{\mathscr{L},m} \otimes_{\mathbb{Z}_{p^\infty}} \overline{\mathbb{F}}_p$. To $\alpha \in \mathcal{S}_{\mathscr{L},m}$, we will attach a closed formal subscheme $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha}$ of $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,s}$. A family $\{\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha}\}_{\alpha \in \mathcal{S}_{\mathscr{L},m}}$ of these formal subschemes plays a role of the "boundary" of $M_{K_{\mathscr{L},m}}$ inside $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m}$.

Definition 4.8 For $\alpha = (\alpha_L)_L \in \mathcal{S}_{\mathscr{L},m}$, we define the subfunctor $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha}$ of $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,s}$ as follows: for an $\overline{\mathbb{F}}_p$-scheme S, $\{(X_L, \rho_L, \eta_{L,m})\}_{L \in \mathscr{L}} \in \breve{\mathscr{M}}^{\flat}_{\mathscr{L},m}(S)$ lies in $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha}(S)$ if and only if $\eta_{L,m}|_{\alpha_L^\perp} = 0$ for every $L \in \mathscr{L}$.

Lemma 4.9 The subfunctor $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha}$ is represented by a closed formal subscheme of $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,s}$.

Proof. Obviously, for each $L \in \mathscr{L}$ the condition $\eta_{L,m}|_{\alpha_L^\perp} = 0$ gives a closed formal subscheme of $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,s}$. Therefore it suffices to show the equivalence of $\eta_{L,m}|_{\alpha_L^\perp} = 0$ and $\eta_{pL,m}|_{\alpha_{pL}^\perp} = 0$. Since $\alpha \in \mathcal{S}_{\mathscr{L},m}$, we have $\alpha_{pL} = p\alpha_L$. Therefore the commutative diagram

$$
\begin{array}{ccc}
L/p^m L & \xrightarrow[\cong]{\times p} & pL/p^{m+1}L \\
\downarrow{\scriptstyle \eta_{L,m}} & & \downarrow{\scriptstyle \eta_{pL,m}} \\
X_L[p^m] & \xrightarrow[\cong]{\theta_p} & X_{pL}[p^m]
\end{array}
$$

in the definition of $\mathscr{M}'_{\mathscr{L},m}$ gives the equivalence. ∎

The following lemma is clear from the definition:

Lemma 4.10 i) *The action of J on $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m}$ preserves the closed formal subscheme $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha}$ for each $\alpha \in \mathcal{S}_{\mathscr{L},m}$.*

  ii) *For $g \in N_{\mathscr{L}}$, the right action $g\colon \breve{\mathscr{M}}^{\flat}_{\mathscr{L},m} \longrightarrow \breve{\mathscr{M}}^{\flat}_{\mathscr{L},m}$ induces an isomorphism $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha} \longrightarrow \breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,g^{-1}\alpha}$ for each $\alpha \in \mathcal{S}_{\mathscr{L},m}$.*

For $\alpha \in \mathcal{S}_{\mathscr{L},m}$, put $M^{\flat}_{\mathscr{L},m,\alpha} = t(\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha})_a$. It is a closed analytic adic subspace of $M^{\flat}_{\mathscr{L},m,s} = t(\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,s})_a$. Moreover, set $M^{\flat}_{\mathscr{L},m,(\alpha)} = M^{\flat}_{\mathscr{L},m,\alpha} \setminus \bigcup_{\beta \succ \alpha} M^{\flat}_{\mathscr{L},m,\beta}$. It is a locally closed subset of $M^{\flat}_{\mathscr{L},m,s}$.

Proposition 4.11   i) *For $\alpha, \beta \in \mathcal{S}_{\mathscr{L},m}$ with $\alpha \neq \beta$, $M^{\flat}_{\mathscr{L},m,(\alpha)} \cap M^{\flat}_{\mathscr{L},m,(\beta)} = \varnothing$.*

ii) *Assume that $d = 2$. Then, we have $M^{\flat}_{\mathscr{L},m,s} = \bigcup_{\alpha \in \mathcal{S}_{\mathscr{L},m}} M^{\flat}_{\mathscr{L},m,(\alpha)}$.*

Proof. We use the p-adic uniformization theorem. Fix a compact open subgroup K^p of $\mathrm{GSp}_{2d}(\mathbb{A}_f^p)$. By [IM10, Lemma 4.1], we may assume that $(\mathbb{X}, \lambda_0)$ comes from a polarized d-dimensional abelian variety. Let $\mathrm{Sh}_{\mathscr{L},K^p}$ be the moduli space over $\mathbb{Z}_{p^{\infty}}$ of "$\mathscr{L}$-sets" of abelian varieties with principal polarizations and K^p-level structures introduced in [RZ96, Definition 6.9]. We shrink K^p so that $\mathrm{Sh}_{\mathscr{L},K^p}$ is represented by a quasi-projective scheme over $\mathbb{Z}_{p^{\infty}}$. For simplicity, we write $\mathrm{Sh}_{\mathscr{L}}$ for $\mathrm{Sh}_{\mathscr{L},K^p}$. By using Drinfeld level structures, we can construct towers $\{\mathrm{Sh}'_{\mathscr{L},m}\}_{m \geq 0}$, $\{\mathrm{Sh}_{\mathscr{L},m}\}_{m \geq 0}$ and $\{\mathrm{Sh}^{\flat}_{\mathscr{L},m}\}_{m \geq 0}$ over $\mathrm{Sh}_{\mathscr{L}}$ similarly as in Definition 4.3. These towers are endowed with right actions of $N_{\mathscr{L}}$. Let $Y^{\flat}_m$ be a closed subset of $\mathrm{Sh}^{\flat}_{\mathscr{L},m}$ consisting of supersingular points, and $(\mathrm{Sh}^{\flat}_{\mathscr{L},m})^{\wedge}_{/Y^{\flat}_m}$ the formal completion of $\mathrm{Sh}^{\flat}_{\mathscr{L},m}$ along $Y^{\flat}_m$. The p-adic uniformization theorem [RZ96, Theorem 6.30] tells us that we have a natural $N_{\mathscr{L}}$-equivariant étale surjection

$$\theta_m \colon \breve{\mathscr{M}}^{\flat}_{\mathscr{L},m} \longrightarrow (\mathrm{Sh}^{\flat}_{\mathscr{L},m})^{\wedge}_{/Y^{\flat}_m}$$

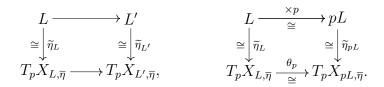(note that GSp_{2d} satisfies the Hasse principle; *cf.* [Kot92, §7]).

     Put $\mathrm{Sh}^{\flat}_{\mathscr{L},m,s} = \mathrm{Sh}^{\flat}_{\mathscr{L},m} \otimes_{\mathbb{Z}_{p^{\infty}}} \overline{\mathbb{F}}_p$. For $\alpha \in \mathcal{S}_{\mathscr{L},m}$, we can define a closed subscheme $\mathrm{Sh}^{\flat}_{\mathscr{L},m,\alpha}$ of $\mathrm{Sh}^{\flat}_{\mathscr{L},m,s}$ in the same way as $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha}$. By the construction of θ_m, $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,\alpha}$ is isomorphic to the fiber product $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m,s} \times_{\mathrm{Sh}^{\flat}_{\mathscr{L},m,s}} \mathrm{Sh}^{\flat}_{\mathscr{L},m,\alpha}$. Thus, as in [Mie10a, Example 4.2], we can reduce our problem to an analogue for $\{\mathrm{Sh}^{\flat}_{\mathscr{L},m,\alpha}\}$; namely, for $\mathrm{Sh}^{\flat}_{\mathscr{L},m,(\alpha)} = \mathrm{Sh}^{\flat}_{\mathscr{L},m,\alpha} \setminus \bigcup_{\beta \succ \alpha} \mathrm{Sh}^{\flat}_{\mathscr{L},m,\beta}$, it suffices to show the following:

(I)   For $\alpha, \beta \in \mathcal{S}_{\mathscr{L},m}$ with $\alpha \neq \beta$, $\mathrm{Sh}^{\flat}_{\mathscr{L},m,(\alpha)} \cap \mathrm{Sh}^{\flat}_{\mathscr{L},m,(\beta)} = \varnothing$.

(II)   Under the assumption $d = 2$, we have $\mathrm{Sh}^{\flat}_{\mathscr{L},m,s} \setminus Y^{\flat}_m = \bigcup_{\alpha \in \mathcal{S}_{\mathscr{L},m}} \mathrm{Sh}^{\flat}_{\mathscr{L},m,(\alpha)}$.

Let $x = \{(A_L, \eta_{L,m})\}_{L \in \mathscr{L}} \in \mathrm{Sh}^{\flat}_{\mathscr{L},m,s}(\overline{\mathbb{F}}_p)$. We shall prove that $((\mathrm{Ker}\,\eta_{L,m})^{\perp})_L$ lies in $\mathcal{S}_{\mathscr{L},m}$ provided that $\eta_{L,m} \neq 0$ for some L. Since $\mathrm{Sh}^{\flat}_{\mathscr{L},m}$ is flat over $\mathbb{Z}_{p^{\infty}}$, x can be lifted to a point on the generic fiber. Therefore, we can find

    – a finite extension F of $\mathbb{Q}_{p^{\infty}}$,

    – an "$\mathscr{L}$-set" of p-divisible groups $\{X_L\}_{L \in \mathscr{L}}$ with a principal polarization $\{\lambda_L\}_{L \in \mathscr{L}}$ (*cf.* [RZ96, Definition 6.5, Definition 6.6]) whose special fiber can be identified with $\{A_L[p^{\infty}]\}_{L \in \mathscr{L}}$ (with the implicit polarization),

    – and a family of isomorphisms $\{\widetilde{\eta}_L \colon L \xrightarrow{\cong} T_p X_{L,\overline{\eta}}\}_{L \in \mathscr{L}}$ (here $X_{L,\overline{\eta}}$ denotes the geometric generic fiber of X_L) satisfying the following conditions:

    (a)   Denote by η_L the composite $L \xrightarrow{\widetilde{\eta}_L} T_p X_{L,\overline{\eta}} \longrightarrow T_p X_{L,s}$, where $X_{L,s}$ denotes the special fiber of X_L. Then, $\eta_L \bmod p^m$ coincides with $\eta_{L,m}$ under the identification of $A_L[p^{\infty}]$ and $X_{L,s}$.

(b) For $L, L' \in \mathscr{L}$ with $L \subset L'$, the following diagrams are commutative:

$$
\begin{array}{ccc}
L & \longrightarrow & L' \\
{\scriptstyle\cong}\downarrow{\scriptstyle\widetilde{\eta}_L} & & {\scriptstyle\cong}\downarrow{\scriptstyle\widetilde{\eta}_{L'}} \\
T_p X_{L,\overline{\eta}} & \longrightarrow & T_p X_{L',\overline{\eta}},
\end{array}
\qquad
\begin{array}{ccc}
L & \xrightarrow{\ \times p\ }_{\cong} & pL \\
{\scriptstyle\cong}\downarrow{\scriptstyle\widetilde{\eta}_L} & & {\scriptstyle\cong}\downarrow{\scriptstyle\widetilde{\eta}_{pL}} \\
T_p X_{L,\overline{\eta}} & \xrightarrow[\cong]{\ \theta_p\ } & T_p X_{pL,\overline{\eta}}.
\end{array}
$$

(c) Fix an isomorphism $\mathbb{Z}_p \xrightarrow{\cong} \mathbb{Z}_p(1)$. Then, for every $L \in \mathscr{L}$, the diagram below is commutative up to constant which is independent of L:

$$
\begin{array}{ccc}
L \times L^{\vee} & \xrightarrow{\qquad\qquad \langle\ ,\ \rangle \qquad\qquad} & \mathbb{Z}_p \\
{\scriptstyle\cong}\downarrow{\scriptstyle\widetilde{\eta}_L \times \widetilde{\eta}_{L^\vee}} & & \downarrow{\scriptstyle\cong} \\
T_p X_{L,\overline{\eta}} \times T_p X_{L^\vee,\overline{\eta}} & \xrightarrow{\ \mathrm{id}\,\times\lambda_{L^\vee}\ } T_p X_{L,\overline{\eta}} \times T_p X_{L,\overline{\eta}}^{\vee} \longrightarrow & \mathbb{Z}_p(1).
\end{array}
$$

Let V be the kernel of $\eta_L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \colon \mathbb{Q}_p^{2d} = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow V_p X_{L,s}$. By the first diagram in (b) above, V is independent of the choice of L. Moreover, we have $\operatorname{Ker} \eta_L = V \cap L$, and thus $\operatorname{Ker} \eta_{L,m} = V \cap L \bmod p^m$ by (a). In particular $V \neq \mathbb{Q}_p^{2d}$, as we are assuming that $\eta_{L,m} \neq 0$ for some L. Therefore, it suffices to prove that $V \subset \mathbb{Q}_p^{2d}$ is coisotropic, that is to say, satisfies $V^{\perp} \subset V$. Denote by λ the quasi-isogeny $X_L \xrightarrow{(*)} X_{L^\vee} \xrightarrow{\lambda_{L^\vee}} X_L^{\vee}$, where $(*)$ comes from the $\mathscr{L}$-set structure of $\{X_L\}_{L \in \mathscr{L}}$. This induces an alternating pairing

$$
\langle\ ,\ \rangle_\lambda \colon V_p X_{L,\overline{\eta}} \times V_p X_{L,\overline{\eta}} \xrightarrow{\ \mathrm{id}\,\times\lambda\ } V_p X_{L,\overline{\eta}} \times V_p X_{L,\overline{\eta}}^{\vee} \longrightarrow \mathbb{Q}_p(1).
$$

By the commutative diagrams in (b) and (c) above, $\widetilde{\eta}_L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \colon \mathbb{Q}_p^{2d} = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\cong} V_p X_{L,\overline{\eta}}$ maps the pairing $\langle\ ,\ \rangle$ to a scalar multiple of $\langle\ ,\ \rangle_\lambda$ under an identification of $\mathbb{Q}_p \cong \mathbb{Q}_p(1)$. Thus, it suffices to show that the kernel of the specialization map $V_p X_{L,\overline{\eta}} \longrightarrow V_p X_{L,s}$ is a coisotropic subspace of $V_p X_{L,\overline{\eta}}$ with respect to $\langle\ ,\ \rangle_\lambda$. To prove it, we can argue in the same way as in the proof of [IM10, Lemma 5.8]. Let us recall the argument briefly. Take an exact sequence of p-divisible groups $0 \longrightarrow X_{L,s}^0 \longrightarrow X_{L,s} \longrightarrow X_{L,s}^{\text{ét}} \longrightarrow 0$, where $X_{L,s}^0$ is connected and $X_{L,s}^{\text{ét}}$ is étale. It is canonically lifted to an exact sequence $0 \longrightarrow X_L^0 \longrightarrow X_L \longrightarrow X_L^{\text{ét}} \longrightarrow 0$ over $\mathcal{O}_F$, where $X_L^{\text{ét}}$ is an étale p-divisible group (cf. [Mes72, p. 76]). It is easy to see that the kernel of $V_p X_{L,\overline{\eta}} \longrightarrow V_p X_{L,s}$ coincides with $V_p X_{L,\overline{\eta}}^0$. Therefore it suffices to prove that the composite of Galois-equivariant homomorphisms $(V_p X_{L,\overline{\eta}}^0)^{\perp} \hookrightarrow V_p X_{L,\overline{\eta}} \longrightarrow V_p X_{L,\overline{\eta}}^{\text{ét}}$ is zero. This follows from the p-adic Hodge theory, noting that $\lambda \colon V_p X_{L,\overline{\eta}} \xrightarrow{\cong} V_p X_{L,\overline{\eta}}^{\vee} = (V_p X_{L,\overline{\eta}})^{\vee}(1)$ induces a Galois-equivariant isomorphism $(V_p X_{L,\overline{\eta}}^0)^{\perp} \cong (V_p X_{L,\overline{\eta}}^{\text{ét}})^{\vee}(1)$.

Now we can show (I) and (II) above. For (I), assume that $x = \{(A_L, \eta_{L,m})\}_{L \in \mathscr{L}} \in \operatorname{Sh}_{\mathscr{L},m,s}^{\flat}(\overline{\mathbb{F}}_p)$ lies in $\operatorname{Sh}_{\mathscr{L},m,(\alpha)}^{\flat}$. Then $\alpha_L^{\perp} \subset \operatorname{Ker} \eta_{L,m}$ for every $L \in \mathscr{L}$, or equivalently,

$\alpha \prec ((\operatorname{Ker}\eta_{L,m})^{\perp})_L$. As x belongs to $\operatorname{Sh}^{\flat}_{\mathscr{L},m,((\operatorname{Ker}\eta_{L,m})^{\perp})_L}(\overline{\mathbb{F}}_p)$, we conclude that $\alpha = ((\operatorname{Ker}\eta_{L,m})^{\perp})_L$. In particular, for $x \in \operatorname{Sh}^{\flat}_{\mathscr{L},m,s}(\overline{\mathbb{F}}_p)$, there is at most one $\alpha \in \mathcal{S}_{\mathscr{L},m}$ with $x \in \operatorname{Sh}^{\flat}_{\mathscr{L},m,(\alpha)}(\overline{\mathbb{F}}_p)$. This concludes the proof of (I). For (II), assume that $\eta_{L,m} = 0$ for every $L \in \mathscr{L}$. By [HT01, Lemma II.2.1], $A_L[p^{\infty}]$ has no étale part. Since the rational Dieudonné module $D(A_L[p^{\infty}])_{\mathbb{Q}}$ is polarized, [IM10, Lemma 4.1] tells us that, under the assumption $d = 2$, A_L is supersingular, namely, $x \in Y^{\flat}_m(\overline{\mathbb{F}}_p)$. Hence we conclude that $\operatorname{Sh}^{\flat}_{\mathscr{L},m,s} \setminus Y^{\flat}_m = \bigcup_{\alpha \in \mathcal{S}_{\mathscr{L},m}} \operatorname{Sh}^{\flat}_{\mathscr{L},m,(\alpha)}$, as desired. ∎

Remark 4.12 For every quasi-compact open formal subscheme $\mathscr{U}$ of $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m}$, we can choose the level $K^p \subset \operatorname{GSp}_{2d}(\mathbb{A}^p_f)$ such that $\theta_m|_{\mathscr{U}}$ is an isomorphism from $\mathscr{U}$ onto the image (*cf.* [Far04, Corollaire 3.1.4]). In particular, the formal scheme $\breve{\mathscr{M}}^{\flat}_{\mathscr{L},m}$ is locally algebraizable in the sense of [Mie10b, Definition 3.18].

4.3 Group action on $\mathcal{S}_{\mathscr{L},m}$

In this subsection, we assume that $d = 2$. In this case, by the Bruhat-Tits theory, we can easily classify self-dual chains of lattices:

Lemma 4.13 *For every self-dual chain of lattices $\mathscr{L}$, we can find $g \in G$ so that $g\mathscr{L}$ is one of the following:*
(hyperspecial type) $\mathscr{L}_0 = \{p^m\mathbb{Z}^4_p \mid m \in \mathbb{Z}\}$.
(paramodular type)

$$\mathscr{L}_{\text{para}} = \{p^{m-1}\mathbb{Z}_p \oplus p^m\mathbb{Z}_p \oplus p^m\mathbb{Z}_p \oplus p^m\mathbb{Z}_p, p^m\mathbb{Z}_p \oplus p^m\mathbb{Z}_p \oplus p^m\mathbb{Z}_p \oplus p^{m+1}\mathbb{Z}_p \mid m \in \mathbb{Z}\}.$$

(Siegel parahoric type)

$$\mathscr{L}_{\text{Siegel}} = \mathscr{L}_0 \cup \{p^m\mathbb{Z}_p \oplus p^m\mathbb{Z}_p \oplus p^{m+1}\mathbb{Z}_p \oplus p^{m+1}\mathbb{Z}_p \mid m \in \mathbb{Z}\}.$$

(Klingen parahoric type) $\mathscr{L}_{\text{Klingen}} = \mathscr{L}_0 \cup \mathscr{L}_{\text{para}}$.
(Iwahori type) $\mathscr{L}_{\text{Iw}} = \mathscr{L}_{\text{Siegel}} \cup \mathscr{L}_{\text{para}}$.

Lemma 4.14 *Every maximal compact-mod-center subgroup of G is conjugate to one of $N_{\mathscr{L}_0}$, $N_{\mathscr{L}_{\text{para}}}$ or $N_{\mathscr{L}_{\text{Siegel}}}$.*

Proof. Since a compact-mod-center subgroup is contained in the normalizer of some parahoric subgroup, it suffices to show that $N_{\mathscr{L}_{\text{Klingen}}} = N_{\mathscr{L}_0} \cap N_{\mathscr{L}_{\text{para}}}$ and $N_{\mathscr{L}_{\text{Iw}}} = N_{\mathscr{L}_{\text{Siegel}}} \cap N_{\mathscr{L}_{\text{para}}}$.

For a self-dual chain of lattices $\mathscr{L}$, let $\mathscr{L}_+$ be the subset of $\mathscr{L}$ consisting of lattices L such that $L^{\vee} = aL$ for some $a \in \mathbb{Q}_p$, and put $\mathscr{L}_- = \mathscr{L} \setminus \mathscr{L}_+$. Then, clearly $g \in N_{\mathscr{L}}$ preserves $\mathscr{L}_+$ and $\mathscr{L}_-$. If $\mathscr{L} = \mathscr{L}_{\text{Klingen}}$, then $\mathscr{L}_+ = \mathscr{L}_0$ and $\mathscr{L}_- = \mathscr{L}_{\text{para}}$. This implies that $N_{\mathscr{L}_{\text{Klingen}}}$ is contained in $N_{\mathscr{L}_0} \cap N_{\mathscr{L}_{\text{para}}}$. The other inclusion is obvious. If $\mathscr{L} = \mathscr{L}_{\text{Iw}}$, then we have $\mathscr{L}_+ = \mathscr{L}_{\text{Siegel}}$ and $\mathscr{L}_- = \mathscr{L}_{\text{para}}$, which gives $N_{\mathscr{L}_{\text{Iw}}} = N_{\mathscr{L}_{\text{Siegel}}} \cap N_{\mathscr{L}_{\text{para}}}$. ∎

The following proposition is very important to control the group action on the "boundary strata" of the Rapoport-Zink spaces introduced in the previous subsection:

Proposition 4.15 *Let $\mathscr{L}$ be one of $\mathscr{L}_0$, $\mathscr{L}_{\mathrm{para}}$ or $\mathscr{L}_{\mathrm{Siegel}}$. Then, for $1 \leq h \leq 2$ and $m \geq 1$, the natural surjection $\mathcal{S}_{\infty,h} \longrightarrow \mathcal{S}_{\mathscr{L},m,h}$ induces a bijection $K_{\mathscr{L},m}\backslash\mathcal{S}_{\infty,h} \longrightarrow \mathcal{S}_{\mathscr{L},m,h}$.*

Before proving it, we give its immediate corollary:

Corollary 4.16 *Let $\mathscr{L}$ be as in Proposition 4.15. For $g \in N_{\mathscr{L}}$ and an integer $m \geq 1$, assume that $gK_{\mathscr{L},m} = K_{\mathscr{L},m}g$ consists of regular elliptic elements. Then the action of g on $\mathcal{S}_{\mathscr{L},m}$ has no fixed point.*

Proof. Assume that g has a fixed point $V_{\mathscr{L},m} \in \mathcal{S}_{\mathscr{L},m}$ with $V \in \mathcal{S}_\infty$. Then Proposition 4.15 says that $K_{\mathscr{L},m}g$ intersects the stabilizer P of V in G. Since P is a proper parabolic subgroup of G, it has no regular elliptic element (Lemma 3.16). This contradicts to the assumption. ∎

We will give a case-by-case proof of Proposition 4.15.

4.3.1 Hyperspecial case

Here we consider the case $\mathscr{L} = \mathscr{L}_0$. For $V, V' \in \mathcal{S}_{\infty,h}$, put $I = V \cap \mathbb{Z}_p^4$ and $I' = V' \cap \mathbb{Z}_p^4$. Under the assumption $I \equiv I' \pmod{p^m}$, we will prove that there exists $g \in K_{\mathscr{L}_0,m}$ with $I' = gI$.

First consider the case where $h = 2$. Fix a basis $x, y \in I$. Since the lattice $\mathbb{Z}_p^4$ is self-dual, we can find $z, w \in \mathbb{Z}_p^4$ such that $\langle x, z \rangle = \langle y, w \rangle = 1$, $\langle x, w \rangle = \langle y, z \rangle = 0$ (note that $\operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^4, \mathbb{Z}_p) \longrightarrow \operatorname{Hom}_{\mathbb{Z}_p}(I, \mathbb{Z}_p)$ is surjective, as $I \subset \mathbb{Z}_p^4$ is a direct summand). Replacing w by $w + \langle z, w \rangle x$, we may assume that $\langle z, w \rangle = 0$. It is easy to see that x, y, z, w spans a self-dual $\mathbb{Z}_p$-lattice of $\mathbb{Q}_p^4$ contained in $\mathbb{Z}_p^4$. Therefore x, y, z, w form a basis of $\mathbb{Z}_p^4$. Take $x', y' \in I'$ such that $x \equiv x', y \equiv y' \pmod{p^m}$. Since x', y', z, w form a basis of $\mathbb{Z}_p^4$, we can take $A, B, C, D \in \mathbb{Z}_p$ so that $z' = Az + Bw$ and $w' = Cz + Dw$ satisfy $\langle x', z' \rangle = \langle y', w' \rangle = 1$ and $\langle x', w' \rangle = \langle y', z' \rangle = 0$ (note that we have $\langle z', w' \rangle = 0$ automatically). These are equivalent to the following identity of matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \langle x', z \rangle & \langle y', z \rangle \\ \langle x', w \rangle & \langle y', w \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By using the fact $x' - x, y' - y \in p^m \mathbb{Z}_p^4$, it is immediate to observe that

$$\begin{pmatrix} \langle x', z \rangle & \langle y', z \rangle \\ \langle x', w \rangle & \langle y', w \rangle \end{pmatrix} \in 1 + p^m M_2(\mathbb{Z}_p).$$

Therefore we can conclude that $A, D \in 1 + p^m \mathbb{Z}_p$ and $B, C \in p^m \mathbb{Z}_p$. In other words, we have $z' \equiv z, w' \equiv w \pmod{p^m}$. In particular, x', y', z', w' form a basis of $\mathbb{Z}_p^4$. Let

g be the automorphism of $\mathbb{Z}_p^4$ that maps x to x', y to y', z to z' and w to w'. Clearly g is an element of $K_{\mathscr{L}_0,m}$ satisfying $gI = I'$.

Next consider the case where $h = 1$. Take a basis x of I and a basis x' of I' such that $x \equiv x' \pmod{p^m}$. Put $I^\perp = V^\perp \cap \mathbb{Z}_p^4$ and $I'^\perp = V'^\perp \cap \mathbb{Z}_p^4$, which are direct summands of rank 3 of $\mathbb{Z}_p^4$ satisfying $I^\perp \equiv I'^\perp \pmod{p^m}$. There exists $y \in I^\perp$ such that x and y span a totally isotropic direct summand I_1 of rank 2 of $\mathbb{Z}_p^4$. Take an arbitrary element $y' \in I'^\perp$ such that $y \equiv y' \pmod{p^m}$. Then x' and y' span a totally isotropic direct summand I'_1 of $\mathbb{Z}_p^4$. By the argument in the case $h = 2$, there exists $g \in K_{\mathscr{L}_0,m}$ satisfying $g(x) = x'$ and $g(y) = y'$. In particular, we have $gI = I'$.

4.3.2 Paramodular case

Here assume that $\mathscr{L} = \mathscr{L}_{\mathrm{para}}$. Put $L = p^{-1}\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$. Then we have $L^\vee = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus p\mathbb{Z}_p$ and $[L : L^\vee] = p^2$. In particular, $g \in G$ lies in $K_{\mathscr{L}}$ if and only if $gL = L$. For every $x, y \in L$, we have $\langle x, y \rangle \in p^{-1}\mathbb{Z}_p$. We introduce temporary terminology:

Definition 4.17 An element x of L is said to be primitive if $x \notin pL$. A primitive element x of L is said to be type (I) if $\langle x, y \rangle \in p^{-1}\mathbb{Z}_p \setminus \mathbb{Z}_p$ for some $y \in L$, and x is said to be type (II) if $\langle x, y \rangle \in \mathbb{Z}_p$ for every $y \in L$.

Remark 4.18 A primitive element of type (I) is of the form $(p^{-1}a, b, c, d)$ where $a, b, c, d \in \mathbb{Z}_p$ and either $a \notin p\mathbb{Z}_p$ or $d \notin p\mathbb{Z}_p$. A primitive element of type (II) is of the form (a, b, c, pd) where $a, b, c, d \in \mathbb{Z}_p$ and either $b \notin p\mathbb{Z}_p$ or $c \notin p\mathbb{Z}_p$.

Lemma 4.19 i) *Let $x \in L$ be a primitive element of type (I), and $y \in L$ an element satisfying $\langle x, y \rangle \in \mathbb{Z}_p$. Then there exists $\lambda \in \mathbb{Z}_p$ such that $y - \lambda x$ is not primitive of type (I).*

ii) *For every primitive element $x \in L$ of type (II), there exists a primitive element $y \in L$ of type (II) such that $\langle x, y \rangle = 1$.*

iii) *Let $m \geq 1$ be an integer. For $x, x' \in L$ with $x \equiv x' \pmod{p^m}$, x is primitive of type (I) (resp. (II)) if and only if x' is primitive of type (I) (resp. (II)).*

iv) *Let x and y be primitive elements of L. If x is type (I) and y is type (II), then the $\mathbb{Z}_p$-submodule of L generated by x and y is a direct summand of rank 2 of L.*

Proof. i) Write $x = (p^{-1}a, b, c, d)$ and $y = (p^{-1}a', b', c', d')$ with $a, b, c, d, a', b', c', d' \in \mathbb{Z}_p$. Since x is primitive of type (I), a or d is a unit. If a is a unit, by replacing y by $y - (a'/a)x$, we may assume that $a' = 0$. Since $\langle x, y \rangle = p^{-1}ad' + bc' - b'c \in \mathbb{Z}_p$, d' lies in $p\mathbb{Z}_p$. Therefore y is not primitive of type (I). If d is a unit, by replacing y by $y - (d'/d)x$, we may assume that $d' = 0$. A similar argument as above tells us that $a' \in p\mathbb{Z}_p$, namely, y is not primitive of type (I).

ii) Since x is primitive of type (II), we have $x = (a, b, c, pd)$, where $a, b, c, d \in \mathbb{Z}_p$ and either $b \notin p\mathbb{Z}_p$ or $c \notin p\mathbb{Z}_p$. We can find $b', c' \in \mathbb{Z}_p$ such that $bc' - cb' = 1$. For $y = (0, b', c', 0) \in L$, we have $\langle x, y \rangle = 1$, as desired.

iii) It is clear that x is primitive if and only if x' is primitive. Moreover, noting that $\langle x, y \rangle - \langle x', y \rangle = \langle x - x', y \rangle \in p^{m-1}\mathbb{Z}_p \subset \mathbb{Z}_p$, it is also immediate to see that x is type (I) (resp. (II)) if and only if x' is type (I) (resp. (II)).

iv) Write $x = (p^{-1}a, b, c, d)$ and $y = (a', b', c', pd')$ as in Remark 4.18. To show that they generates a direct summand of rank 2 of L, it suffices to see that there exists a 2×2-minor of the following matrix whose determinant is a unit in $\mathbb{Z}_p$:

$$\begin{pmatrix} a & pa' \\ b & b' \\ c & c' \\ d & pd' \end{pmatrix}.$$

It is clear from the assumptions that a or d is a unit, and b' or c' is a unit. ■

Lemma 4.20 *Let I be a totally isotropic direct summand of rank 2 of L. Then there exist a primitive element $x \in I$ of type (I) and a primitive element $y \in I$ of type (II). These elements x, y form a basis of I.*

Proof. Take a basis x, y of I. Obviously they are primitive. First prove that one of x and y is type (I). Suppose that both x and y are type (II), and write $x = (a, b, c, pd)$ and $y = (a', b', c', pd')$ as in Remark 4.18. Then, since x and y generates a direct summand of L, there exists a 2×2-minor of the following matrix whose determinant is a unit in $\mathbb{Z}_p$:

$$\begin{pmatrix} pa & pa' \\ b & b' \\ c & c' \\ pd & pd' \end{pmatrix}.$$

Namely, $bc' - b'c \in \mathbb{Z}_p^\times$. Therefore, we have $\langle x, y \rangle = p(ad' - a'd) + bc' - b'c \in \mathbb{Z}_p^\times$. This contradicts to $\langle x, y \rangle = 0$.

Next, assume that x is type (I), and find an element of I which is primitive of type (II). Since $\langle x, y \rangle = 0 \in \mathbb{Z}_p$, Lemma 4.19 i) tells us that there exists $\lambda \in \mathbb{Z}_p$ such that $y - \lambda x$ is not primitive of type (I). On the other hand, $y - \lambda x$ is primitive since $x, y - \lambda x$ form a basis of a direct summand I. Thus we can conclude that $y - \lambda x$ is primitive of type (II).

Finally, let x (resp. y) be an arbitrary primitive element of type (I) (resp. (II)) of I. Then, Lemma 4.19 iv) tells us that x and y form a basis of I. ■

Lemma 4.21 *Let I be a totally isotropic direct summand of rank 2 of L, $x \in I$ a primitive element of type (I) and $y \in I$ a primitive element of type (II). Then there exist $z, w \in L$ satisfying the following conditions:*

 − x, y, z, w form a basis of L.
 − $\langle x, z \rangle = p^{-1}$, $\langle y, w \rangle = 1$, $\langle x, w \rangle = \langle y, z \rangle = \langle z, w \rangle = 0$.

Moreover, if we are given an element $u \in L$ satisfying $\langle x, u \rangle \in p^{-1}\mathbb{Z}_p \setminus \mathbb{Z}_p$, then we can find z and w so that $\langle z, u \rangle = \langle w, u \rangle = 0$ holds.

Proof. It suffices to show the latter part, since we can always find such $u \in L$.

It is easy to see that the images $\overline{x}, \overline{y}, \overline{u} \in L/pL$ of x, y, u are linearly independent over $\mathbb{F}_p$. Therefore, we can find $v \in L$ such that $\overline{x}, \overline{y}, \overline{u}, \overline{v}$ form a basis of L/pL. Then x, y, u, v form a basis of L. Since $\langle u, v \rangle \in p^{-1}\mathbb{Z}_p$ and $\langle u, x \rangle \in p^{-1}\mathbb{Z}_p \setminus \mathbb{Z}_p$, there exists $a \in \mathbb{Z}_p$ satisfying $\langle u, v \rangle = a \langle u, x \rangle$. Therefore, by replacing v by $v - ax$, we may assume that $\langle u, v \rangle = 0$.

Denote by I' the $\mathbb{Z}_p$-submodule of L generated by u, v. It is a totally isotropic direct summand of rank 2 of L. The pairing $\langle \ , \ \rangle$ induces an isomorphism of $\mathbb{Q}_p$-vector spaces $I'_{\mathbb{Q}_p} \xrightarrow{\cong} (I_{\mathbb{Q}_p})^\vee$, where $(-)_{\mathbb{Q}_p} = (-) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Thus we obtain a basis $x^\vee, y^\vee$ of $I'_{\mathbb{Q}_p}$ satisfying $\langle x, x^\vee \rangle = \langle y, y^\vee \rangle = 1$ and $\langle x, y^\vee \rangle = \langle y, x^\vee \rangle = 0$. Since I' is totally isotropic, $x^\vee$ and $y^\vee$ belongs to $L^\vee$. As $L^\vee \subset L$, we have $x^\vee, y^\vee \in I'_{\mathbb{Q}_p} \cap L = I'$.

Let I'' be the $\mathbb{Z}_p$-submodule of I' generated by $x^\vee, y^\vee$, and L'' the $\mathbb{Z}_p$-lattice of $\mathbb{Q}_p^4$ generated by $x, y, x^\vee, y^\vee$. Then we have $L/L'' = I'/I''$. Since L'' is self-dual, we have $L^\vee \subsetneq (L'')^\vee = L'' \subsetneq L$. Therefore $[L : L''] = p$, and thus $I'/I'' = L/L'' \cong \mathbb{Z}/p\mathbb{Z}$. In particular, there exist $a, b \in \mathbb{Q}_p$ such that $ax^\vee + by^\vee \in I' \setminus I''$. Since y is primitive of type (II), $b = \langle y, ax^\vee + by^\vee \rangle \in \mathbb{Z}_p$. Therefore $a \notin \mathbb{Z}_p$, and thus $p^{-1}x^\vee \in L$. Now we can easily observe that $z = p^{-1}x^\vee$ and $w = y^\vee$ satisfy all conditions in the proposition; note that $\langle p^{-1}x^\vee, u \rangle = \langle y^\vee, u \rangle = 0$ since $u, p^{-1}x^\vee$ and $y^\vee$ belong to I', which is a totally isotropic direct summand. ∎

Now we can prove Proposition 4.15 for $h = 2$.

Proposition 4.22 *If $\mathscr{L} = \mathscr{L}_{\mathrm{para}}$ and $h = 2$, we have $K_{\mathscr{L},m} \backslash \mathcal{S}_{\infty,h} \cong \mathcal{S}_{\mathscr{L},m,h}$.*

Proof. Let V and V' be elements of $\mathcal{S}_{\infty,2}$ such that $V_{\mathscr{L},m} = V'_{\mathscr{L},m}$. Put $I = V \cap L$ and $I' = V' \cap L$. Then I and I' are totally isotropic and $I \equiv I' \pmod{p^m}$. It suffices to find $g \in K_{\mathscr{L},m}$ satisfying $gI = I'$.

By Lemma 4.20, there exist a primitive element $x \in I$ of type (I) and a primitive element $y \in I$ of type (II). Take $x', y' \in I'$ such that $x \equiv x', y \equiv y' \pmod{p^m}$. Then x' (resp. y') is primitive of type (I) (resp. type (II)) by Lemma 4.19 iii), and x', y' form a basis of I' by Lemma 4.20. As $p^{-m}(y' - y) \in L$, we have $\langle x, p^{-m}(y' - y) \rangle \in p^{-1}\mathbb{Z}_p$. Moreover, if $\langle x, p^{-m}(y' - y) \rangle \in \mathbb{Z}_p$, Lemma 4.19 i) enables us to find $\lambda \in \mathbb{Z}_p$ such that $p^{-m}(y' - y) - \lambda x$ is not primitive of type (I). Replacing y by $y + p^m \lambda x$, we may assume that $p^{-m}(y' - y)$ is not primitive of type (I).

By Lemma 4.21, there exist $z, w \in L$ such that

- x, y, z, w form a basis of L,
- $\langle x, z \rangle = p^{-1}$, $\langle y, w \rangle = 1$, $\langle x, w \rangle = \langle y, z \rangle = \langle z, w \rangle = 0$,
- and $\langle z, p^{-m}(y' - y) \rangle, \langle w, p^{-m}(y' - y) \rangle \in \mathbb{Z}_p$.

Indeed, if $\langle x, p^{-m}(y' - y) \rangle \notin \mathbb{Z}_p$, we can apply Lemma 4.21 to $u = p^{-m}(y' - y)$; otherwise the third condition is automatic since $p^{-m}(y' - y)$ is not primitive of type (I).

Since x', y', z, w form a basis of L, we can take $A, B, C, D \in \mathbb{Q}_p$ so that $z' = Az + Bw$ and $w' = Cz + Dw$ satisfy $\langle x', z' \rangle = p^{-1}$, $\langle y', w' \rangle = 1$ and $\langle x', w' \rangle = \langle y', z' \rangle = 0$

(note that we have $\langle z', w' \rangle = 0$ automatically). These are equivalent to the following identity of matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p\langle x', z \rangle & \langle y', z \rangle \\ p\langle x', w \rangle & \langle y', w \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By using the facts $x' - x \in p^m L$ and $\langle z, p^{-m}(y' - y) \rangle, \langle w, p^{-m}(y' - y) \rangle \in \mathbb{Z}_p$, it is immediate to observe that

$$\begin{pmatrix} p\langle x', z \rangle & \langle y', z \rangle \\ p\langle x', w \rangle & \langle y', w \rangle \end{pmatrix} \in 1 + p^m M_2(\mathbb{Z}_p).$$

Therefore we can conclude that $A, D \in 1 + p^m \mathbb{Z}_p$ and $B, C \in p^m \mathbb{Z}_p$. In other words, we have $z', w' \in L$ and $z' \equiv z, w' \equiv w \pmod{p^m}$. In particular, x', y', z', w' form a basis of L. Let g be the automorphism of L that maps x to x', y to y', z to z' and w to w'. It is clear that g is an element of $K_{\mathscr{L},m}$ satisfying $gI = I'$. This completes the proof. ∎

Finally we prove Proposition 4.15 for $h = 1$.

Proposition 4.23 If $\mathscr{L} = \mathscr{L}_{\mathrm{para}}$ and $h = 1$, we have $K_{\mathscr{L},m} \backslash \mathcal{S}_{\infty,h} \cong \mathcal{S}_{\mathscr{L},m,h}$.

Proof. Let V and V' be elements of $\mathcal{S}_{\infty,1}$ such that $V_{\mathscr{L},m} = V'_{\mathscr{L},m}$. Put $I = V \cap L$ and $I' = V' \cap L$. It suffices to find $g \in K_{\mathscr{L},m}$ satisfying $gI = I'$.

Take a basis x of I and a basis x' of I' satisfying $x \equiv x' \pmod{p^m}$. First consider the case where x is primitive of type (I). Then x' is also primitive of type (I) by Lemma 4.19 iii). Take $z, z' \in L$ such that $\langle x, z \rangle = \langle x', z' \rangle = p^{-1}$. Take an arbitrary primitive element $y \in L$ of type (II). By replacing it by $y - p\langle x, y \rangle z$, we may assume that $\langle x, y \rangle = 0$. Put $y' = y - p\langle x', y \rangle z'$. Then y' is a primitive element of type (II) satisfying $\langle x', y' \rangle = 0$ and $y \equiv y' \pmod{p^m}$. By the proof of Proposition 4.22, we can find $g \in K_{\mathscr{L},m}$ such that $gx = x'$.

Next consider the case where x is primitive of type (II). In this case x' is also primitive of type (II). By Lemma 4.19 ii), there exist primitive elements $z, z' \in L$ of type (II) such that $\langle x, z \rangle = \langle x', z' \rangle = 1$. Let us prove that we can find a primitive element $y \in L$ of type (I) satisfying $\langle x, y \rangle = 0$ and $\langle p^{-m}(x' - x), y \rangle \in \mathbb{Z}_p$. If $p^{-m}(x' - x)$ is primitive of type (I), put $y = p^{-m}(x' - x) - az$ where $a = \langle x, p^{-m}(x' - x) \rangle$. Then $a \in \mathbb{Z}_p$ as x is primitive of type (II), and $\langle p^{-m}(x' - x), y \rangle = a\langle z, y \rangle \in \mathbb{Z}_p$ since z is primitive of type (II). If $p^{-m}(x' - x)$ is not primitive of type (I), take an arbitrary primitive element $w \in L$ of type (I) and put $y = w - \langle x, w \rangle z$. In this case, the condition $\langle p^{-m}(x' - x), y \rangle \in \mathbb{Z}_p$ is automatic. Put $y' = y - \langle x', y \rangle z'$. Then we have $\langle x', y' \rangle = 0$. Furthermore we have $y \equiv y' \pmod{p^m}$, for $\langle x', y \rangle = p^m \langle p^{-m}(x' - x), y \rangle \in p^m \mathbb{Z}_p$. Hence, again by the proof of Proposition 4.22, we can find $g \in K_{\mathscr{L},m}$ such that $gx = x'$. ∎

Now a proof of Proposition 4.15 for the paramodular case is complete.

4.3.3 Siegel parahoric case

Here we assume that $\mathscr{L} = \mathscr{L}_{\mathrm{Siegel}}$. Put $L_0 = \mathbb{Z}_p^4$ and $L_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus p\mathbb{Z}_p \oplus p\mathbb{Z}_p$. We have $pL_0 \subset L_1 \subset L_0$ and $L_0/L_1 \cong L_1/pL_0 \cong \mathbb{F}_p^2$. An element $g \in G$ lies in $K_{\mathscr{L}}$ if and only if $gL_0 = L_0$ and $gL_1 = L_1$.

Lemma 4.24 *Let $x, y \in L_0$ and $z, w \in L_1$ be elements satisfying*

$$\langle x, y \rangle = 0, \quad \langle x, z \rangle = \langle y, w \rangle = 1, \quad \langle x, w \rangle = \langle y, z \rangle = 0, \quad \langle z, w \rangle = 0.$$

Then, x, y, z, w (resp. px, py, z, w) form a basis of L_0 (resp. L_1).

Proof. Let us prove that the images $\overline{x}, \overline{y}$ of x, y in L_0/L_1 form an $\mathbb{F}_p$-basis of L_0/L_1. Assume that $u = ax + by \in L_1$ for $a, b \in \mathbb{Z}_p$. Then, $a = \langle u, z \rangle \in p\mathbb{Z}_p$ and $b = \langle u, w \rangle \in p\mathbb{Z}_p$, since the pairing $\langle \, , \, \rangle$ on L_1 takes its value in $p\mathbb{Z}_p$. This implies that $\overline{x}, \overline{y} \in L_0/L_1$ are linearly independent over $\mathbb{F}_p$, and thus they form a basis. Similarly, we can prove that $\overline{z}, \overline{w} \in L_1/pL_0$ form a basis of L_1/pL_0. From this, it is immediate to conclude the lemma. $\blacksquare$

Proposition 4.25 *Let I_0 be a totally isotropic direct summand of rank 2 of L_0 and put $I_1 = I_0 \cap L_1$.*

i) *Assume that $I_0 = I_1$. For a basis $x, y \in I_0 = I_1$, there exist $z, w \in L_0$ such that*

$$\langle x, z \rangle = \langle y, w \rangle = 1, \quad \langle x, w \rangle = \langle y, z \rangle = 0, \quad \langle z, w \rangle = 0.$$

ii) *Assume that $[I_0 : I_1] = p$. For $x \in I_0 \setminus I_1$ and $y \in I_1 \setminus pI_0$, x, y form a basis of I_0. Moreover, there exist $z \in L_1$ and $w \in L_0$ such that*

$$\langle x, z \rangle = \langle y, w \rangle = 1, \quad \langle x, w \rangle = \langle y, z \rangle = 0, \quad \langle z, w \rangle = 0.$$

iii) *Assume that $[I_0 : I_1] = p^2$. For a basis $x, y \in I_0$, there exist $z, w \in L_1$ such that*

$$\langle x, z \rangle = \langle y, w \rangle = 1, \quad \langle x, w \rangle = \langle y, z \rangle = 0, \quad \langle z, w \rangle = 0.$$

Proof. i) Take $u, v \in L_0$ such that x, y, u, v form a basis of L_0. Since $y \notin pL_0$ and $\langle x, y \rangle = 0$, either $\langle u, y \rangle$ or $\langle v, y \rangle$ is a unit in $\mathbb{Z}_p$. Therefore we may assume that $\langle u, y \rangle \in \mathbb{Z}_p^\times$. Then there exists $a \in \mathbb{Z}_p$ such that $\langle u, v - ay \rangle = 0$. Replacing v by $v - ay$, we may assume that $\langle u, v \rangle = 0$. Let J be a direct summand of L_0 generated by u, v, which is totally isotropic. The pairing $\langle \, , \, \rangle$ induces an isomorphism $J \xrightarrow{\cong} I_0^\vee$. Therefore, there exists a basis z, w of J satisfying

$$\langle x, z \rangle = \langle y, w \rangle = 1, \quad \langle x, w \rangle = \langle y, z \rangle = 0.$$

This concludes i).

ii) It is immediate to see that x, y form a basis of L_0. We shall prove the existence of z, w. Take $u \in L_0$ (resp. $v \in L_1$) so that $\overline{x}, \overline{u}$ (resp. $\overline{y}, \overline{v}$) form a basis of L_0/L_1

(resp. L_1/pL_0). Since $y \notin pL_0$ and $\langle x, y \rangle = 0$, either $\langle u, y \rangle$ or $\langle v, y \rangle$ is a unit in $\mathbb{Z}_p$. As $\langle v, y \rangle \in p\mathbb{Z}_p$, we have $\langle u, y \rangle \in \mathbb{Z}_p^\times$. Replacing v by $v - ay$ for a suitable $a \in \mathbb{Z}_p$, we may assume that $\langle u, v \rangle = 0$. By the same argument as above, we can find $z, w \in \mathbb{Z}_p u + \mathbb{Z}_p v$ satisfying

$$\langle x, z \rangle = \langle y, w \rangle = 1, \quad \langle x, w \rangle = \langle y, z \rangle = 0.$$

We should prove that $z \in L_1$. Since px, y, pu, v form a basis of L_1, it is easy to see that $\langle u, z \rangle \in p\mathbb{Z}_p$ for every $u \in L_1$. Therefore $z \in pL_1^\vee = L_1$, as desired.

iii) Take $u, v \in L_1$ such that $\overline{u}, \overline{v} \in L_1/pL_0$ form a basis of L_1/pL_0. Since $L_0/L_1 \cong I_0/I_1$, x, y, u, v form a basis of L_0. As in i) and ii), we may assume that $\langle u, y \rangle \in \mathbb{Z}_p^\times$. For $a = \langle u, y \rangle^{-1} \langle u, v \rangle \in \mathbb{Z}_p$, we have $\langle u, v - ay \rangle = 0$. Since $\langle u, v \rangle \in p\mathbb{Z}_p$, the element ay belongs to L_1. Therefore, replacing v by $v - ay$, we may assume that $\langle u, v \rangle = 0$. By the same argument as above, we can find $z, w \in \mathbb{Z}_p u + \mathbb{Z}_p v \subset L_1$ satisfying

$$\langle x, z \rangle = \langle y, w \rangle = 1, \quad \langle x, w \rangle = \langle y, z \rangle = 0. \qquad \blacksquare$$

Proposition 4.26 *If $\mathscr{L} = \mathscr{L}_{\mathrm{Siegel}}$ and $h = 2$, we have $K_{\mathscr{L},m} \backslash \mathcal{S}_{\infty,h} \cong \mathcal{S}_{\mathscr{L},m,h}$.*

Proof. Let V and V' be elements of $\mathcal{S}_{\infty,2}$ such that $V_{\mathscr{L},m} = V'_{\mathscr{L},m}$. Put $I_i = V \cap L_i$ and $I'_i = V' \cap L_i$ for $i = 0, 1$. Then I_i and I'_i are totally isotropic and $I_i \equiv I'_i$ (mod p^m). It suffices to find $g \in K_{\mathscr{L},m}$ satisfying $gI_0 = I'_0$.

Let us take $x, y, z, w \in L_0$ as in Proposition 4.25. For $x', y' \in I'_0$ with $x' - x, y' - y \in p^m L_0$, we can take $A, B, C, D \in \mathbb{Q}_p$ so that $z' = Az + Bw$ and $w' = Cz + Dw$ satisfy $\langle x', z' \rangle = \langle y', w' \rangle = 1$ and $\langle x', w' \rangle = \langle y', z' \rangle = 0$. These are equivalent to the following identity of matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \langle x', z \rangle & \langle y', z \rangle \\ \langle x', w \rangle & \langle y', w \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By using the assumption $x' - x, y' - y \in p^m L_0$, it is immediate to observe that

$$\begin{pmatrix} \langle x', z \rangle & \langle y', z \rangle \\ \langle x', w \rangle & \langle y', w \rangle \end{pmatrix} \in 1 + p^m M_2(\mathbb{Z}_p).$$

Therefore we can conclude that $A, D \in 1 + p^m \mathbb{Z}_p$ and $B, C \in p^m \mathbb{Z}_p$. In other words, z', w' lie in $\mathbb{Z}_p z + \mathbb{Z}_p w$ and $z' - z, w' - w \in p^m \mathbb{Z}_p z + p^m \mathbb{Z}_p w$.

Now consider the three cases in Proposition 4.25 separately. First assume that $I_0 = I_1$. Then $I'_0 = I'_1$, as $I'_0 \subset I_0 + p^m L_0 \subset L_1 + pL_0 = L_1$. By the assumption $I_1 \equiv I'_1$ (mod p^m), we may take $x', y' \in I'_0$ above so that $x - x', y - y' \in p^m L_1$. Lemma 4.24 tells us that

- x, y, z, w and x', y', z', w' are bases of L_0,
- x, y, pz, pw and x', y', pz', pw' are bases of L_1, and
- x', y' form a basis of I'_0.

Let g be the automorphism of L_0 that maps x to x', y to y', z to z' and w to w'. It is clear that g is an element of $K_{\mathscr{L},m}$ satisfying $gI_0 = I_0'$.

Next consider the case where $[I_0 : I_1] = p$. In this case, we can choose y' so that y' belongs to I_1' and $y' - y \in p^m L_1$, as $I_1 \equiv I_1' \pmod{p^m}$. Then $z' \in L_1$ and $z' - z \in p^m L_1$; indeed, we have $\langle y', z \rangle = p^m \langle p^{-m}(y' - y), z \rangle \in p^{m+1}\mathbb{Z}_p$ (note that $z \in L_1$) and thus $B \in p^{m+1}\mathbb{Z}_p$. Therefore, Lemma 4.24 tells us that

- x, y, z, w and x', y', z', w' are bases of L_0,
- px, y, z, pw and px', y', z', pw' are bases of L_1, and
- x', y' form a basis of I_0'.

Hence the automorphism of L_0 defined as above gives an element $g \in K_{\mathscr{L},m}$ satisfying $gI_0 = I_0'$.

Finally consider the case where $[I_0 : I_1] = p^2$. Since $z, w \in L_1$, we have $z', w' \in L_1$ and $z' - z, w' - w \in p^m L_1$. Lemma 4.24 tells us that

- x, y, z, w and x', y', z', w' are bases of L_0,
- px, py, z, w and px', py', z', w' are bases of L_1, and
- x', y' form a basis of I_0'.

Hence the automorphism of L_0 defined as above gives an element $g \in K_{\mathscr{L},m}$ satisfying $gI_0 = I_0'$. ∎

Proposition 4.27 If $\mathscr{L} = \mathscr{L}_{\mathrm{Siegel}}$ and $h = 1$, we have $K_{\mathscr{L},m} \backslash \mathcal{S}_{\infty,h} \cong \mathcal{S}_{\mathscr{L},m,h}$.

Proof. Let V and V' be elements of $\mathcal{S}_{\infty,1}$ such that $V_{\mathscr{L},m} = V'_{\mathscr{L},m}$. Put $I_i = V \cap L_i$ and $I_i' = V' \cap L_i$ for $i = 0, 1$. Then we have $I_i \equiv I_i' \pmod{p^m}$. It suffices to find $g \in K_{\mathscr{L},m}$ satisfying $gI_0 = I_0'$.

Take a basis x of I_0 and an element $x' \in I_0'$ such that $x' - x \in p^m L_0$. Then x' is also a basis of I_0', for $x' \notin pL_0$. Moreover, since $x' - x \in pL_0 \subset L_1$, $x \in L_1$ if and only if $x' \in L_1$.

First consider the case where $I_0 = I_1$. In this case, $I_0' = I_1'$ as $x' \in I_1'$. By the assumption $I_1 \equiv I_1' \pmod{p^m}$, we may take x' above so that $x - x' \in p^m L_1$. Let us observe that there exists $y \in L_0 \setminus L_1$ such that $\langle x, y \rangle = 0$. Consider an $\mathbb{F}_p$-linear map $\langle x, - \rangle \colon L_0 / L_1 \longrightarrow \mathbb{F}_p$. Since $\dim_{\mathbb{F}_p} L_0 / L_1 = 2$, the kernel of this map is non-trivial. In other words, there exists $u \in L_0 \setminus L_1$ such that $\langle x, u \rangle \in p\mathbb{Z}_p$. Take $v \in L_0$ such that $\langle x, v \rangle = 1$ and put $y = u - \langle x, u \rangle v$. Then we have $\langle x, y \rangle = 0$ and $y \in L_0 \setminus L_1$ (note that $\langle x, u \rangle v \in pL_0 \subset L_1$). Take $z \in L_0$ satisfying $\langle x', z \rangle = 1$ and put $y' = y - \langle x', y \rangle z$. Since $\langle x', y \rangle = \langle x' - x, y \rangle \in p^m \mathbb{Z}_p$, we have $y' \in L_0 \setminus L_1$ and $y' - y \in p^m L_0$. Let $\widetilde{V}$ (resp. $\widetilde{V}'$) be a subspace of $\mathbb{Q}_p^4$ generated by x, y (resp. x', y'), and put $\widetilde{I}_i = \widetilde{V} \cap L_i$, $\widetilde{I}_i' = \widetilde{V}' \cap L_i$ for $i = 1, 2$. Then it is easy to see the following:

- $\widetilde{V}, \widetilde{V}' \in \mathcal{S}_{\infty,2}$.
- x, y (resp. x, py) form a basis of $\widetilde{I}_0$ (resp. $\widetilde{I}_1$).
- x', y' (resp. x', py') form a basis of $\widetilde{I}_0'$ (resp. $\widetilde{I}_1'$).
- $\widetilde{I}_i \equiv \widetilde{I}_i' \pmod{p^m}$.

Therefore, by the proof of Proposition 4.26, we can find $g \in K_{\mathscr{L},m}$ such that $gx = x'$.

Next consider the case where $I_0 \neq I_1$. In this case, $I_0' \neq I_1'$ since $x' \notin I_1'$. In the same way as above, we can find $y \in L_1 \setminus pL_0$ such that $\langle x, y \rangle = 0$. Since $x' \notin L_1 = pL_1^\vee$, there exists $z \in L_1$ such that $\langle x', z \rangle = 1$. Put $y' = y - \langle x', y \rangle z$. As $\langle x', y \rangle = \langle x' - x, y \rangle \in p^m \mathbb{Z}_p$, we have $y' \in L_1 \setminus pL_0$ and $y' - y \in p^m L_1$. Define $\widetilde{V}, \widetilde{V}'$, $\widetilde{I}_i$ and $\widetilde{I}_i'$ as above. Then it is easy to see the following:

- $\widetilde{V}, \widetilde{V}' \in \mathcal{S}_{\infty,2}$.
- x, y (resp. px, y) form a basis of $\widetilde{I}_0$ (resp. $\widetilde{I}_1$).
- x', y' (resp. px', y') form a basis of $\widetilde{I}_0'$ (resp. $\widetilde{I}_1'$).
- $\widetilde{I}_i \equiv \widetilde{I}_i' \pmod{p^m}$.

Therefore, by the proof of Proposition 4.26, we can find $g \in K_{\mathscr{L},m}$ such that $gx = x'$. ∎

Now a proof of Proposition 4.15 for the Siegel parahoric case is complete.

5 Open covering of the Rapoport-Zink space

From this section, we convert the right action of $G \times J$ on the Rapoport-Zink tower to the left action by taking inverse. Therefore, the action of $(g, h) \in G \times J$ on the cohomology H_{RZ}^i is given by $(g^{-1}, h^{-1})^*$.

5.1 Definition of open covering

Fix a chain of lattices $\mathscr{L}$ of $\mathbb{Q}_p^{2d}$. Then we can consider $\breve{\mathscr{M}}_{\mathscr{L}}^\flat$ (*cf.* Definition 4.3) and its rigid generic fiber $M_{\mathscr{L}}$. We write $\mathrm{Irr}(\mathscr{L})$ for the set of irreducible components of $\breve{\mathscr{M}}_{\mathscr{L}}^{\flat,\mathrm{red}}$. For each $\alpha \in \mathrm{Irr}(\mathscr{L})$, the set $\breve{\mathscr{M}}_{\mathscr{L}}^{\flat,\mathrm{red}} \setminus \bigcup_{\beta \in \mathrm{Irr}(\mathscr{L}), \alpha \cap \beta = \varnothing} \beta$ is open in $\breve{\mathscr{M}}_{\mathscr{L}}^{\flat,\mathrm{red}}$, as $\breve{\mathscr{M}}_{\mathscr{L}}^{\flat,\mathrm{red}}$ is locally of finite type over $\overline{\mathbb{F}}_p$. Let $\mathscr{U}_\alpha$ be the open formal subscheme of $\breve{\mathscr{M}}_{\mathscr{L}}^\flat$ satisfying $\mathscr{U}_\alpha^{\mathrm{red}} = \breve{\mathscr{M}}_{\mathscr{L}}^{\flat,\mathrm{red}} \setminus \bigcup_{\beta \in \mathrm{Irr}(\mathscr{L}), \alpha \cap \beta = \varnothing} \beta$, and U_α the rigid generic fiber of $\mathscr{U}_\alpha$. As $\alpha \subset \mathscr{U}_\alpha^{\mathrm{red}}$, $\{\mathscr{U}_\alpha\}_{\alpha \in \mathrm{Irr}(\mathscr{L})}$ (resp. $\{U_\alpha\}_{\alpha \in \mathrm{Irr}(\mathscr{L})}$) gives an open covering of $\breve{\mathscr{M}}_{\mathscr{L}}^\flat$ (resp. $M_{\mathscr{L}}$). For $\alpha, \beta \in \mathrm{Irr}(\mathscr{L})$, note that $U_\alpha \cap U_\beta \neq \varnothing$ if and only if $\mathscr{U}_\alpha^{\mathrm{red}} \cap \mathscr{U}_\beta^{\mathrm{red}} \neq \varnothing$, since $U_\alpha \cap U_\beta = (\mathscr{U}_\alpha \cap \mathscr{U}_\beta)^{\mathrm{rig}}$ and $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$ is flat over $\mathbb{Z}_{p^\infty}$.

Clearly J naturally acts on $\mathrm{Irr}(\mathscr{L})$, and we have $h\mathscr{U}_\alpha = \mathscr{U}_{h\alpha}$ for $h \in J$.

Lemma 5.1   i) *Put* $\mathrm{Irr}(\mathscr{L})_{\alpha,0} = \{\alpha\}$ *and for* $m \geq 1$ *define the set* $\mathrm{Irr}(\mathscr{L})_{\alpha,m}$ *inductively as follows;* $\mathrm{Irr}(\mathscr{L})_{\alpha,m}$ *consists of* $\beta \in \mathrm{Irr}(\mathscr{L})$ *which intersects some element in* $\mathrm{Irr}(\mathscr{L})_{\alpha,m-1}$. *Then* $\mathrm{Irr}(\mathscr{L})_{\alpha,m}$ *is a finite set.*

ii) *The action of* J *on* $\mathrm{Irr}(\mathscr{L})$ *has finite orbits.*

iii) *For every* $\alpha, \beta \in \mathrm{Irr}(\mathscr{L})$, *the subset* $\{h \in J \mid h\alpha \cap \beta \neq \varnothing\}$ *of* J *is contained in a compact subset of* J.

iv) *For each* $\alpha \in \mathrm{Irr}(\mathscr{L})$, *the subgroup* $J_\alpha = \{h \in J \mid h\alpha = \alpha\}$ *is open and compact.*

Proof. Since $\breve{\mathscr{M}}_{\mathscr{L}}^{\flat,\mathrm{red}}$ is locally of finite type and every irreducible component of $\breve{\mathscr{M}}_{\mathscr{L}}^{\flat,\mathrm{red}}$ is projective, $\mathrm{Irr}(\mathscr{L})_{\alpha,1}$ is a finite set. If $\mathrm{Irr}(\mathscr{L})_{\alpha,m-1}$ is a finite set, $\mathrm{Irr}(\mathscr{L})_{\alpha,m} = \bigcup_{\beta\in\mathrm{Irr}(\mathscr{L})_{\alpha,m-1}} \mathrm{Irr}(\mathscr{L})_{\beta,1}$ is also a finite set. This concludes the proof of i).

We assume that $\mathscr{L} = \mathscr{L}_0 = \{p^m \mathbb{Z}_p^{2d} \mid m \in \mathbb{Z}\}$ and prove ii) and iii). In this case, $\breve{\mathscr{M}}_{\mathscr{L}_0}^{\flat} = \breve{\mathscr{M}}$ and the action of J on $\mathrm{Irr}(\mathscr{L}_0)$ is transitive ([Vie08, Theorem 2]). Thus ii) is immediate. Let us prove iii). By the transitivity, we may assume that $\alpha = \beta$. Let $\breve{\mathscr{M}}_{\mathrm{GL}}$ be the Rapoport-Zink space for $\mathrm{GL}(2d)$ corresponding to the p-divisible group $\mathbb{X}$ and J_{GL} the group of self-quasi-isogenies of $\mathbb{X}$ which naturally acts on $\breve{\mathscr{M}}_{\mathrm{GL}}$. The group J is a closed subgroup of J_{GL}. By forgetting the condition on polarization, we have a natural closed immersion $\breve{\mathscr{M}} \hookrightarrow \breve{\mathscr{M}}_{\mathrm{GL}}$ and the actions of J and J_{GL} are compatible. Therefore, it suffices to show that $\{h \in J_{\mathrm{GL}_n} \mid h\alpha \cap \alpha \neq \varnothing\}$ is contained in a compact subset of J_{GL}, where α is regarded as a quasi-compact closed subset of $\breve{\mathscr{M}}_{\mathrm{GL}}^{\mathrm{red}}$. This is essentially proved in [RZ96, proof of Proposition 2.34].

Next we assume that $\mathscr{L} = \mathscr{L}_{\mathrm{Iw}} = \{p^m \mathbb{Z}_p^i \oplus p^{m+1} \mathbb{Z}_p^{2d-i} \mid m \in \mathbb{Z}, 0 \leq i \leq 2d\}$. Then we have a natural morphism $\pi\colon \breve{\mathscr{M}}_{\mathscr{L}_{\mathrm{Iw}}}^{\flat} \longrightarrow \breve{\mathscr{M}}_{\mathscr{L}_0}^{\flat}$, which is proper. Fix an arbitrary element $\beta \in \mathrm{Irr}(\mathscr{L}_0)$ and let $\alpha_1, \ldots, \alpha_m$ be the collection of elements of $\mathrm{Irr}(\mathscr{L}_{\mathrm{Iw}})$ which are contained in $\pi^{-1}(\beta)$. Since π is J-equivariant, it is easy to observe $\mathrm{Irr}(\mathscr{L}_{\mathrm{Iw}}) = \bigcup_{h\in J} \bigcup_{i=1}^m h\alpha_i$. This concludes the proof of ii) in this case. To prove iii), let α_0 (resp. β_0) be an element of $\mathrm{Irr}(\mathscr{L}_0)$ which contains $\pi(\alpha)$ (resp. $\pi(\beta)$). Then, $\{h \in J \mid h\alpha \cap \beta \neq \varnothing\}$ is contained in $\{h \in J \mid h\alpha_0 \cap \beta_0 \neq \varnothing\}$, and thus we are reduced to the case $\mathscr{L} = \mathscr{L}_0$.

Finally we consider an arbitrary $\mathscr{L}$. Changing $\mathscr{L}$ by $g\mathscr{L}$ with $g \in G$, we may assume that $\mathscr{L}$ is contained in $\mathscr{L}_{\mathrm{Iw}}$. Then we have a natural morphism $\pi\colon \breve{\mathscr{M}}_{\mathscr{L}_{\mathrm{Iw}}}^{\flat} \longrightarrow \breve{\mathscr{M}}_{\mathscr{L}}^{\flat}$, which is proper. By Remark 4.4 and Proposition 4.5, the induced morphism $\pi\colon M_{\mathscr{L}_{\mathrm{Iw}}} \longrightarrow M_{\mathscr{L}}$ is surjective, and therefore the morphism $\breve{\mathscr{M}}_{\mathscr{L}_{\mathrm{Iw}}}^{\flat,\mathrm{red}} \longrightarrow \breve{\mathscr{M}}_{\mathscr{L}}^{\flat,\mathrm{red}}$ is also surjective (note that $\breve{\mathscr{M}}_{\mathscr{L}}^{\flat}$ is flat over $\mathbb{Z}_{p^\infty}$). For ii), let $\beta_1, \ldots, \beta_m \in \mathrm{Irr}(\mathscr{L}_{\mathrm{Iw}})$ be elements such that $\mathrm{Irr}(\mathscr{L}_{\mathrm{Iw}}) = \bigcup_{h\in J} \bigcup_{i=1}^m h\beta_i$, and α_i an element of $\mathrm{Irr}(\mathscr{L})$ which contains $\pi(\beta_i)$. Then $\{h\alpha_i\}_{h\in J, 1\leq i\leq m}$ cover $\breve{\mathscr{M}}_{\mathscr{L}}^{\flat,\mathrm{red}}$, and we have $\mathrm{Irr}(\mathscr{L}) = \bigcup_{h\in J} \bigcup_{i=1}^m h\alpha_i$. To prove iii), take $\alpha_1', \ldots, \alpha_k', \beta_1', \ldots, \beta_l' \in \mathrm{Irr}(\mathscr{L}_{\mathrm{Iw}})$ so that $\pi^{-1}(\alpha) \subset \bigcup_{i=1}^k \alpha_i'$ and $\pi^{-1}(\beta) \subset \bigcup_{j=1}^l \beta_j'$. We have

$$\{h \in J \mid h\alpha \cap \beta \neq \varnothing\} = \{h \in J \mid \pi^{-1}(h\alpha) \cap \pi^{-1}(\beta) \neq \varnothing\}$$
$$\subset \bigcup_{i=1}^k \bigcup_{j=1}^l \{h \in J \mid h\alpha_i' \cap \beta_j' \neq \varnothing\}.$$

We already know that the latter set is contained in a compact subset of J. This completes the proof of ii) and iii).

Let us prove iv). By [Far04, Proposition 2.3.11], there exists an open subgroup of J whose elements stabilize α. This implies that J_α is an open subgroup of J. In particular, J_α is a closed subgroup of J. On the other hand, by iii), J_α is contained in a compact subset of J. Thus J_α is compact, as desired. $\blacksquare$

Corollary 5.2 *For each $\alpha \in \mathrm{Irr}(\mathscr{L})$, there exist only finitely many $\beta \in \mathrm{Irr}(\mathscr{L})$ with $U_\alpha \cap U_\beta \neq \varnothing$.*

Proof. By the construction, it is easy to see that $\mathscr{U}_\alpha^{\mathrm{red}} \cap \mathscr{U}_\beta^{\mathrm{red}} = \varnothing$ unless $\beta \in \mathrm{Irr}(\mathscr{L})_{\alpha,2}$. Thus the claim follows from Lemma 5.1 i). ∎

Consider the quotient $\breve{\mathscr{M}}_\mathscr{L}^\flat / p^\mathbb{Z}$ of $\breve{\mathscr{M}}_\mathscr{L}^\flat$ by the discrete subgroup $p^\mathbb{Z} \subset J$. For $\overline{\alpha} \in \mathrm{Irr}(\mathscr{L})/p^\mathbb{Z}$, we write $\mathscr{U}_{\overline{\alpha}}$ (resp. $U_{\overline{\alpha}}$) for the image of $\mathscr{U}_\alpha$ (resp. U_α). Recall that we have an action of $N_\mathscr{L}$ on $\breve{\mathscr{M}}_\mathscr{L}^\flat / p^\mathbb{Z}$. Therefore, $N_\mathscr{L}$ also acts on $\mathrm{Irr}(\mathscr{L})/p^\mathbb{Z}$. This action factors through the quotient $N_\mathscr{L}/p^\mathbb{Z} K_\mathscr{L}$, which is a finite group. Therefore, each $N_\mathscr{L}$-orbit in $\mathrm{Irr}(\mathscr{L})/p^\mathbb{Z}$ is a finite set. We denote by $I(\mathscr{L})$ the set of $N_\mathscr{L}$-orbits in $\mathrm{Irr}(\mathscr{L})/p^\mathbb{Z}$. For $\lambda \in I(\mathscr{L})$, we put $\mathscr{U}_\lambda = \bigcup_{\overline{\alpha} \in \lambda} \mathscr{U}_{\overline{\alpha}}$ and $U_\lambda = \bigcup_{\overline{\alpha} \in \lambda} U_{\overline{\alpha}}$. Then $\mathscr{U}_\lambda$ and U_λ are stable under the action of $N_\mathscr{L}$. Clearly $J/p^\mathbb{Z}$ naturally acts on $I(\mathscr{L})$, and we have $h\mathscr{U}_\lambda = \mathscr{U}_{h\lambda}$ and $hU_\lambda = U_{h\lambda}$ for $h \in J/p^\mathbb{Z}$ and $\lambda \in I(\mathscr{L})$.

Since U_λ is a finite union of $U_{\overline{\alpha}}$'s, the following corollary is an immediate consequence of Lemma 5.1 and Corollary 5.2:

Corollary 5.3 i) *For each $\lambda \in I(\mathscr{L})$, there exist only finitely many $\mu \in I(\mathscr{L})$ with $U_\lambda \cap U_\mu \neq \varnothing$.*

ii) *The action of J on $I(\mathscr{L})$ has finite orbits. In particular, there exist finitely many elements $\lambda_1, \ldots, \lambda_m$ of $I(\mathscr{L})$ such that $M_\mathscr{L}/p^\mathbb{Z} = \bigcup_{h \in J} \bigcup_{i=1}^m hU_{\lambda_i}$.*

iii) *For every $\lambda, \mu \in I(\mathscr{L})$, the subset $\{h \in J/p^\mathbb{Z} \mid hU_\lambda \cap U_\mu \neq \varnothing\}$ of $J/p^\mathbb{Z}$ is contained in a compact subset of $J/p^\mathbb{Z}$.*

iv) *For each $\lambda \in I(\mathscr{L})$, the subgroup $J_\lambda = \{h \in J \mid h\lambda = \lambda\}$ is open and compact-mod-center.*

Corollary 5.4 *For $\lambda, \mu \in I(\mathscr{L})$, the set $\{h \in J/p^\mathbb{Z} \mid hU_\lambda \cap U_\mu \neq \varnothing\}$ is in fact a compact open subset of $J/p^\mathbb{Z}$.*

Proof. Put $C_{\lambda,\mu} = \{h \in J/p^\mathbb{Z} \mid hU_\lambda \cap U_\mu \neq \varnothing\}$ and consider the closure $\overline{C_{\lambda,\mu}}$ in $J/p^\mathbb{Z}$. By Corollary 5.3 iii), it is compact. The group J_λ acts on $C_{\lambda,\mu}$ and $\overline{C_{\lambda,\mu}}$ on the right. Since $J_\lambda/p^\mathbb{Z}$ is an open subgroup of $J/p^\mathbb{Z}$, the quotient $\overline{C_{\lambda,\mu}}/J_\lambda$ is finite. Therefore $C_{\lambda,\mu}/J_\lambda$ is also a finite set. Since $J_\lambda/p^\mathbb{Z}$ is compact open, so is $C_{\lambda,\mu}$. ∎

For a finite subset $\underline{\lambda} = \{\lambda_1, \ldots, \lambda_m\}$ of $I(\mathscr{L})$, we put $\mathscr{U}_{\underline{\lambda}} = \bigcap_{i=1}^m \mathscr{U}_{\lambda_i}$, $U_{\underline{\lambda}} = \bigcap_{i=1}^m U_{\lambda_i}$ and $J_{\underline{\lambda}} = \{h \in J \mid h\underline{\lambda} = \underline{\lambda}\}$. Since the group $J_{\underline{\lambda}}/\bigcap_{i=1}^m J_{\lambda_i}$ acts faithfully on $\underline{\lambda}$, it is a finite group. Thus, Corollary 5.3 iv) tells us that $J_{\underline{\lambda}}$ is an open compact-mod-center subgroup of J.

For an integer $s \geq 1$, let $I(\mathscr{L})_s$ be the set of subsets $\underline{\lambda} \subset I(\mathscr{L})$ such that $\#\underline{\lambda} = s$ and $U_{\underline{\lambda}} \neq \varnothing$. The group J (or $J/p^\mathbb{Z}$) naturally acts on $I(\mathscr{L})_s$.

Lemma 5.5 *For an integer $s \geq 1$, the action of J on $I(\mathscr{L})_s$ has finite orbits.*

Proof. Let $I(\mathscr{L})_s^{\sim}$ be the subset of $I(\mathscr{L})^s$ consisting of elements $(\lambda_1, \ldots, \lambda_s)$ such that $\lambda_1, \ldots, \lambda_s$ are mutually disjoint and $U_{\{\lambda_1, \ldots, \lambda_s\}} \neq \varnothing$. It is stable under the diagonal action of J on $I(\mathscr{L})^s$, and the natural surjection $I(\mathscr{L})_s^{\sim} \longrightarrow I(\mathscr{L})_s$; $(\lambda_1, \ldots, \lambda_s) \longmapsto \{\lambda_1, \ldots, \lambda_s\}$ is obviously J-equivariant. Therefore it suffices to see that the action of J on $I(\mathscr{L})_s^{\sim}$ has finite orbits.

Consider the first projection $I(\mathscr{L})_s^{\sim} \longrightarrow I(\mathscr{L})$, which is J-equivariant. By Corollary 5.3 i), each fiber of this map is finite. Thus, Corollary 5.3 ii) tells us the finiteness of J-orbits in $I(\mathscr{L})_s^{\sim}$. ∎

For an integer $m \geq 0$ and $\lambda \in I(\mathscr{L})$, let $\mathscr{U}_{\lambda,m}$ (resp. $U_{\lambda,m}$) be the inverse image of $\mathscr{U}_\lambda$ (resp. U_λ) under $\breve{\mathscr{M}}_{\mathscr{L},m}^\flat / p^{\mathbb{Z}} \longrightarrow \breve{\mathscr{M}}_{\mathscr{L}}^\flat / p^{\mathbb{Z}}$ (resp. $M_{\mathscr{L},m}/p^{\mathbb{Z}} \longrightarrow M_{\mathscr{L}}/p^{\mathbb{Z}}$, where we put $M_{\mathscr{L},m} = \breve{\mathscr{M}}_{\mathscr{L},m}^{\flat,\mathrm{rig}}$). These are stable under the natural action of $N_{\mathscr{L}} \times J_\lambda / p^{\mathbb{Z}}$ on $\breve{\mathscr{M}}_{\mathscr{L},m}^\flat / p^{\mathbb{Z}}$ and $M_{\mathscr{L},m}/p^{\mathbb{Z}}$. Similarly we define $\mathscr{U}_{\underline{\lambda},m}$ and $U_{\underline{\lambda},m}$.

As $\breve{\mathscr{M}}_{\mathscr{L},m}^\flat$ is locally algebraizable (Remark 4.12), [Mie10a, Proposition 4.6] tells us that the compactly supported cohomology

$$H_c^i(U_{\underline{\lambda},m}) := H_c^i(U_{\underline{\lambda},m} \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$$

is finite-dimensional. Two groups $N_{\mathscr{L}}$ and $J_{\underline{\lambda}}/p^{\mathbb{Z}}$ naturally act on it. The action of $K_{\mathscr{L},m} \subset N_{\mathscr{L}}$ is trivial, and the action of $J_{\underline{\lambda}}/p^{\mathbb{Z}}$ is known to be smooth ([Ber94, Corollary 7.7], [Far04, Corollaire 4.4.7]). Therefore, for fixed $g \in N_{\mathscr{L}}$, the function

$$\eta_{\underline{\lambda},gK_{\mathscr{L},m}}^i : h \longmapsto \mathrm{Tr}\big((g^{-1}, h^{-1}); H_c^i(U_{\underline{\lambda},m})\big)$$

on $J_{\underline{\lambda}}/p^{\mathbb{Z}}$ only depends on the coset $gK_{\mathscr{L},m}$ and is locally constant. Extending it by 0, we regard it as a locally constant compactly supported function on $J/p^{\mathbb{Z}}$.

The following construction is very important in this work.

Definition 5.6 For each integer $s \geq 0$, take a system of representatives $\underline{\lambda}_{s,1}, \ldots, \underline{\lambda}_{s,k_s}$ of the quotient $J \backslash I(\mathscr{L})_{s+1}$. For $m \geq 0$ and $g \in N_{\mathscr{L}}$, we define a locally constant compactly supported function $\eta_{gK_{\mathscr{L},m}}$ on $J/p^{\mathbb{Z}}$ as follows:

$$\eta_{gK_{\mathscr{L},m}} = \sum_{s,t \geq 0} \sum_{i=1}^{k_s} \frac{(-1)^{s+t}}{\mathrm{vol}(J_{\underline{\lambda}_{s,i}}/p^{\mathbb{Z}})} \eta_{\underline{\lambda}_{s,i},gK_{\mathscr{L},m}}^t.$$

Proposition 5.7 *Let $m \geq 0$ be an integer such that $K_{\mathscr{L},m} \subset K_0$, and g an element of $N_{\mathscr{L}}$. For every admissible representation ρ of $J/p^{\mathbb{Z}}$, we have*

$$\mathrm{Tr}(\eta_{gK_{\mathscr{L},m}}; \rho) = \mathrm{Tr}(g; H_{\mathrm{RZ}}[\rho]^{K_{\mathscr{L},m}})$$

(for a definition of $H_{\mathrm{RZ}}[\rho]$, see Definition 3.3 and Definition 3.8). Moreover, the image of $\eta_{gK_{\mathscr{L},m}}$ in $\overline{\mathcal{H}}(J/p^{\mathbb{Z}})$ is characterized by this property. In particular, the image is independent of the choice of $\underline{\lambda}_{s,1}, \ldots, \underline{\lambda}_{s,k_s}$.

Proof. By Remark 3.4, Lemma 3.6 and Proposition 4.5, we obtain the equality

$$\operatorname{Tr}(g; H_{\mathrm{RZ}}[\rho]^{K_{\mathscr{L},m}}) = \sum_{i,j\geq 0}(-1)^{i+j}\operatorname{Tr}\big(g; \operatorname{Ext}^j_{J/p^{\mathbb{Z}}}(H^i_c(M_{\mathscr{L},m}/p^{\mathbb{Z}}), \rho)\big).$$

On the other hand, we have an $N_{\mathscr{L}} \times J/p^{\mathbb{Z}}$-equivariant Čech spectral sequence

$$E_1^{-s,t} = \bigoplus_{\underline{\lambda}\in I(\mathscr{L})_{s+1}} H^t_c(U_{\underline{\lambda},m}) \Longrightarrow H^{-s+t}_c(M_{\mathscr{L},m}/p^{\mathbb{Z}}).$$

It is easy to see that $E_1^{-s,t} \cong \bigoplus_{i=1}^{k_s} \operatorname{c-Ind}^J_{J_{\underline{\lambda}_{s,i}}} H^t_c(U_{\underline{\lambda}_{s,i},m})$ as $N_{\mathscr{L}} \times J/p^{\mathbb{Z}}$-representations. Since $\operatorname{c-Ind}^J_{J_{\underline{\lambda}_{s,i}}} H^t_c(U_{\underline{\lambda}_{s,i},m})$ is projective in the category of smooth $J/p^{\mathbb{Z}}$-modules, in the Grothendieck group of finite-dimensional representations of $N_{\mathscr{L}}$ we can compute as follows:

$$\sum_{i,j\geq 0}(-1)^{i+j}\operatorname{Ext}^j_{J/p^{\mathbb{Z}}}\big(H^i_c(M_{\mathscr{L},m}/p^{\mathbb{Z}}), \rho\big)$$

$$= \sum_{s,t,j\geq 0}\sum_{i=1}^{k_s}(-1)^{s+t+j}\operatorname{Ext}^j_{J/p^{\mathbb{Z}}}\big(\operatorname{c-Ind}^J_{J_{\underline{\lambda}_{s,i}}} H^t_c(U_{\underline{\lambda}_{s,i},m}), \rho\big)$$

$$= \sum_{s,t\geq 0}\sum_{i=1}^{k_s}(-1)^{s+t}\operatorname{Hom}_{J_{\underline{\lambda}_{s,i}}/p^{\mathbb{Z}}}\big(H^t_c(U_{\underline{\lambda}_{s,i},m}), \rho\big).$$

Therefore,

$$\operatorname{Tr}(g; H_{\mathrm{RZ}}[\rho]^{K_{\mathscr{L},m}}) = \sum_{s,t\geq 0}\sum_{i=1}^{k_s}(-1)^{s+t}\operatorname{Tr}\big(g; \operatorname{Hom}_{J_{\underline{\lambda}_{s,i}}/p^{\mathbb{Z}}}(H^t_c(U_{\underline{\lambda}_{s,i},m}), \rho)\big)$$

$$= \sum_{s,t\geq 0}\sum_{i=1}^{k_s}\frac{(-1)^{s+t}}{\operatorname{vol}(J_{\underline{\lambda}_{s,i}}/p^{\mathbb{Z}})}\int_{J_{\underline{\lambda}_{s,i}}/p^{\mathbb{Z}}}\eta^t_{\underline{\lambda}_{s,i},gK_{\mathscr{L},m}}(h)\operatorname{Tr}\rho(h)dh$$

$$= \int_{J_{\underline{\lambda}_{s,i}}/p^{\mathbb{Z}}}\eta_{gK_{\mathscr{L},m}}(h)\operatorname{Tr}\rho(h)dh$$

$$= \operatorname{Tr}(\eta_{gK_{\mathscr{L},m}}; \rho),$$

as desired.

The uniqueness of an element of $\overline{\mathcal{H}}(J/p^{\mathbb{Z}})$ with this property follows from [Kaz86, Theorem 0]. ∎

5.2   Action of regular element

Let $\mathscr{L}$ and $m \geq 0$ be as in the previous section. We fix a regular element $\gamma \in J$ and consider how γ permutes the open subsets $\{U_\lambda\}_{\lambda\in I(\mathscr{L})}$.

First recall the following well-known lemma:

Lemma 5.8 *Let* **H** *be a connected reductive group over* $\mathbb{Q}_p$, $\mathbf{Z_H}$ *its center and* $\gamma \in$ $\mathbf{H}(\mathbb{Q}_p)$ *a regular element. Put* $H = \mathbf{H}(\mathbb{Q}_p)$, $Z_H = \mathbf{Z_H}(\mathbb{Q}_p)$ *and* $Z(\gamma) = \mathbf{Z}(\gamma)(\mathbb{Q}_p)$. *Then the map* $Z(\gamma)\backslash H \longrightarrow H/Z_H;\ h \longmapsto h^{-1}\gamma h$ *is proper.*

Proof. We have only to prove that the inverse image of any compact subset of H/Z_H is compact. It follows from [HC70, Lemma 18]. ∎

Proposition 5.9 *Let* $I(\mathscr{L})_\gamma$ *be the subset* $\{\lambda \in I(\mathscr{L}) \mid \gamma U_\lambda \cap U_\lambda \neq \varnothing\}$ *of* $I(\mathscr{L})$. *Then the left action of* $Z(\gamma)$ *on* $I(\mathscr{L})$ *preserves* $I(\mathscr{L})_\gamma$ *and* $Z(\gamma)\backslash I(\mathscr{L})_\gamma$ *is finite. If* γ *is elliptic,* $I(\mathscr{L})_\gamma$ *is a finite set.*

Proof. Take a system of representatives $\lambda_1, \ldots, \lambda_k$ of $J\backslash I(\mathscr{L})$. Then, $I(\mathscr{L})$ can be identified with $\coprod_{i=1}^{k} J/J_{\lambda_i}$. Since $J_{\lambda_i}/p^{\mathbb{Z}}$ is a compact open subgroup of $J/p^{\mathbb{Z}}$ (Corollary 5.3 iv)), to show the finiteness of $Z(\gamma)\backslash I(\mathscr{L})_\gamma$, it suffices to show that the set $\{h \in Z(\gamma)\backslash J \mid \gamma U_{h\lambda_i} \cap U_{h\lambda_i} \neq \varnothing\}$ is compact. The condition $\gamma U_{h\lambda_i} \cap U_{h\lambda_i} \neq \varnothing$ is equivalent to $U_{h^{-1}\gamma h\lambda_i} \cap U_{\lambda_i} \neq \varnothing$. By Corollary 5.4, the set $\{h' \in J/p^{\mathbb{Z}} \mid U_{h'\lambda_i} \cap U_{\lambda_i} \neq \varnothing\}$ is a compact subset of $J/p^{\mathbb{Z}}$. Therefore, Lemma 5.8 tells us that the set $\{h \in Z(\gamma)\backslash J \mid U_{h^{-1}\gamma h\lambda_i} \cap U_{\lambda_i} \neq \varnothing\}$ is compact.

Assume that γ is elliptic, namely, $Z(\gamma)/p^{\mathbb{Z}}$ is compact. Then, for each $\lambda \in I(\mathscr{L})_\gamma$, the $Z(\gamma)$-orbit of λ is finite; indeed, the $Z(\gamma)$-orbit can be identified with $(Z(\gamma)/p^{\mathbb{Z}})/((Z(\gamma) \cap J_\lambda)/p^{\mathbb{Z}})$, and $(Z(\gamma) \cap J_\lambda)/p^{\mathbb{Z}}$ is an open subgroup of $Z(\gamma)/p^{\mathbb{Z}}$. In other words, each fiber of the natural projection $I(\mathscr{L})_\gamma \longrightarrow Z(\gamma)\backslash I(\mathscr{L})_\gamma$ is a finite set. Hence $I(\mathscr{L})_\gamma$ is a finite set. ∎

Definition 5.10 Put

$$\mathscr{U}_{\gamma,m} = \bigcup_{\lambda \in I(\mathscr{L})_\gamma} \mathscr{U}_{\lambda,m}, \quad U_{\gamma,m} = \bigcup_{\lambda \in I(\mathscr{L})_\gamma} U_{\lambda,m}.$$

These are stable under the actions of $N_{\mathscr{L}}$ and $Z(\gamma)$.

Corollary 5.11 *Let* $I(\mathscr{L})'_\gamma$ *be the subset of* $I(\mathscr{L})$ *consisting of* λ *such that* $U_{\lambda,m} \cap U_{\gamma,m} \neq \varnothing$. *Then the left action of* $Z(\gamma)$ *on* $I(\mathscr{L})$ *preserves* $I(\mathscr{L})'_\gamma$ *and* $Z(\gamma)\backslash I(\mathscr{L})'_\gamma$ *is finite. If* γ *is elliptic,* $I(\mathscr{L})'_\gamma$ *is a finite set.*

Proof. It is an easy consequence of Proposition 5.9 and Corollary 5.3 i). Indeed, for a system of representatives $\lambda_1, \ldots, \lambda_k$ for $Z(\gamma)\backslash I(\mathscr{L})_\gamma$, $Z(\gamma)\backslash I(\mathscr{L})'_\gamma$ is contained in the image of a finite set $\bigcup_{i=1}^{k}\{\lambda \in I(\mathscr{L}) \mid U_\lambda \cap U_{\lambda_i} \neq \varnothing\}$. ∎

Definition 5.12 Put

$$\mathscr{U}'_{\gamma,m} = \bigcup_{\lambda \in I(\mathscr{L})'_\gamma} \mathscr{U}_{\lambda,m}, \quad U'_{\gamma,m} = \bigcup_{\lambda \in I(\mathscr{L})'_\gamma} U_{\lambda,m}.$$

These are stable under the actions of $N_{\mathscr{L}}$ and $Z(\gamma)$.

Remark 5.13 The closure $\overline{U_{\gamma,m}}$ of $U_{\gamma,m}$ is contained in $U'_{\gamma,m}$. Indeed, let x be a point in $\overline{U_{\gamma,m}}$ and take $\lambda \in I(\mathscr{L})$ such that $x \in U_{\lambda,m}$. Then $U_{\lambda,m}$ should be intersect $U_{\gamma,m}$ and therefore λ lies in $I(\mathscr{L})'_\gamma$.

If γ is elliptic, then $\mathscr{U}_{\gamma,m}$ is quasi-compact, and thus the cohomology $H^i_c(U_{\gamma,m})$ is finite-dimensional. For $g \in N_{\mathscr{L}}$, the alternating sum of the traces of (g^{-1}, γ^{-1}) on $H^i_c(U_{\gamma,m})$ can be computed by the function $\eta_{gK_{\mathscr{L},m}}$ introduced in the previous subsection:

Proposition 5.14 *Assume that γ is elliptic. Then, for $g \in N_{\mathscr{L}}$ we have*

$$\sum_i (-1)^i \operatorname{Tr}\big((g^{-1}, \gamma^{-1}); H^i_c(U_{\gamma,m})\big) = O_\gamma(\eta_{gK_{\mathscr{L},m}}).$$

For our normalization of the Haar measure, see the last paragraph of Section 3.1.

Proof. For an integer $s \geq 1$, put $I(\mathscr{L})_{\gamma,s} = \{\underline{\lambda} \in I(\mathscr{L})_s \mid \underline{\lambda} \subset I(\mathscr{L})_\gamma\}$. Then we have the Čech spectral sequence

$$E_1^{-s,t} = \bigoplus_{\underline{\lambda} \in I(\mathscr{L})_{\gamma,s+1}} H^t_c(U_{\underline{\lambda},m}) \Longrightarrow H^{-s+t}_c(U_{\gamma,m}).$$

Therefore, we can compute

$$\sum_i (-1)^i \operatorname{Tr}\big((g^{-1}, \gamma^{-1}); H^i_c(U_{\gamma,m})\big) = \sum_{s,t \geq 0} \sum_{\underline{\lambda} \in I(\mathscr{L})_{\gamma,s+1}} (-1)^{s+t} \operatorname{Tr}\big((g^{-1}, \gamma^{-1}); H^t_c(U_{\underline{\lambda},m})\big)$$

$$= \sum_{s,t \geq 0} \sum_{i=1}^{k_s} (-1)^{s+t} \sum_{\substack{h \in J/J_{\underline{\lambda}_{s,i}} \\ h\underline{\lambda}_{s,i} \subset I(\mathscr{L})_\gamma}} \operatorname{Tr}\big((g^{-1}, \gamma^{-1}); H^t_c(U_{h\underline{\lambda}_{s,i},m})\big)$$

$$= \sum_{s,t \geq 0} \sum_{i=1}^{k_s} (-1)^{s+t} \sum_{\substack{h \in J/J_{\underline{\lambda}_{s,i}} \\ h\underline{\lambda}_{s,i} \subset I(\mathscr{L})_\gamma}} \eta^t_{\underline{\lambda}_{s,i},gK_{\mathscr{L},m}}(g, h^{-1}\gamma h)$$

$$\overset{(*)}{=} \sum_{s,t \geq 0} \sum_{i=1}^{k_s} \frac{(-1)^{s+t}}{\operatorname{vol}(J_{\underline{\lambda}_{s,i}}/p^{\mathbb{Z}})} \int_{J/p^{\mathbb{Z}}} \eta^t_{\underline{\lambda}_{s,i},gK_{\mathscr{L},m}}(g, h^{-1}\gamma h)dh = O_\gamma(\eta_{gK_{\mathscr{L},m}}).$$

For $(*)$, note that if $h\underline{\lambda}_{s,i} \not\subset I(\mathscr{L})_\gamma$, then $\gamma h\underline{\lambda}_{s,i} \neq h\underline{\lambda}_{s,i}$; indeed, if an element λ in $h\underline{\lambda}_{s,i} \setminus I(\mathscr{L})_\gamma$ satisfies $\gamma\lambda \in h\underline{\lambda}_{s,i}$, we have $U_{h\underline{\lambda}_{s,i}} \subset U_\lambda \cap U_{\gamma\lambda} = \varnothing$, which contradicts to the fact $h\underline{\lambda}_{s,i} \in I(\mathscr{L})_{s+1}$. Thus $h\underline{\lambda}_{s,i} \not\subset I(\mathscr{L})_\gamma$ implies $h^{-1}\gamma h \notin J_{\underline{\lambda}_{s,i}}$ and $\eta^t_{\underline{\lambda}_{s,i},gK_{\mathscr{L},m}}(g, h^{-1}\gamma h) = 0$. ∎

Next we consider the case where γ is not elliptic. In this case, the centralizer $\mathbf{Z}(\gamma)$ is a maximal torus which is not anisotropic modulo the center of $\mathbf{J}$. We can take a discrete torsion-free cocompact subgroup Γ of $Z(\gamma)/p^{\mathbb{Z}}$.

Take a system of representatives $\lambda'_1, \ldots, \lambda'_l$ for $Z(\gamma) \backslash I(\mathscr{L})'_\gamma$. For each i with $1 \leq i \leq l$, let C_i be the subset of $J/p^{\mathbb{Z}}$ consisting of $h \in J/p^{\mathbb{Z}}$ such that there exist $\mu_1, \mu_2, \mu_3 \in I(\mathscr{L})$ with $U_{\lambda'_i} \cap U_{\mu_1} \neq \varnothing$, $U_{\mu_1} \cap U_{\mu_2} \neq \varnothing$, $U_{\mu_2} \cap U_{\mu_3} \neq \varnothing$ and $U_{\mu_3} \cap U_{h\lambda'_i} \neq \varnothing$. By Corollary 5.3 i) and Corollary 5.4, C_i is compact. Put $C = \bigcup_{i=1}^l C_i$.

Since Γ is discrete, $C \cap \Gamma$ is finite. Therefore, by shrinking Γ if necessary, we may assume that $C \cap \Gamma = \{1\}$.

Lemma 5.15 *The quotient $\Gamma \backslash I(\mathscr{L})'_\gamma$ is a finite set.*

Proof. As in the proof of Proposition 5.9, it follows from compactness of $Z(\gamma)/\Gamma$ and Corollary 5.11. ∎

Lemma 5.16 *For $\lambda \in I(\mathscr{L})'_\gamma$ and $\gamma' \in \Gamma$, assume that there exist $\mu_1, \mu_2, \mu_3 \in I(\mathscr{L})$ such that $U_\lambda \cap U_{\mu_1} \neq \varnothing$, $U_{\mu_1} \cap U_{\mu_2} \neq \varnothing$, $U_{\mu_2} \cap U_{\mu_3} \neq \varnothing$ and $U_{\mu_3} \cap U_{\gamma'\lambda} \neq \varnothing$. Then $\gamma' = 1$.*

In particular, if $\gamma' \neq 1$ then we have $U_{\gamma'\lambda} \cap U_\lambda = \varnothing$ for every $\lambda \in I(\mathscr{L})'_\gamma$.

Proof. We may write $\lambda = h\lambda'_i$ with $h \in Z(\gamma)$ and $1 \leq i \leq l$. If there exist μ_1, μ_2, μ_3 as above, then $U_{\lambda'_i} \cap U_{h^{-1}\mu_1} \neq \varnothing$, $U_{h^{-1}\mu_1} \cap U_{h^{-1}\mu_2} \neq \varnothing$, $U_{h^{-1}\mu_2} \cap U_{h^{-1}\mu_3} \neq \varnothing$ and $U_{h^{-1}\mu_3} \cap U_{h^{-1}\gamma'h\lambda'_i} \neq \varnothing$, and thus $h^{-1}\gamma'h \in C_i$. Since $\gamma', h \in Z(\gamma)$ and $Z(\gamma)$ is abelian, we have $\gamma' = h^{-1}\gamma'h$. Therefore $\gamma' \in C \cap \Gamma = \{1\}$, as desired. ∎

By the lemma above, we can take quotient of $\mathscr{U}_{\gamma,m}$, $\mathscr{U}'_{\gamma,m}$, $U_{\gamma,m}$ and $U'_{\gamma,m}$ by Γ. Lemma 5.15 tells us that $\Gamma \backslash \mathscr{U}_{\gamma,m}$ is quasi-compact, and thus the cohomology $H^i_c(\Gamma \backslash U_{\gamma,m})$ is finite-dimensional. We write p_Γ for the natural maps $\mathscr{U}'_{\gamma,m} \longrightarrow \Gamma \backslash \mathscr{U}'_{\gamma,m}$ and $U'_{\gamma,m} \longrightarrow \Gamma \backslash U'_{\gamma,m}$.

Proposition 5.17   i) *Let g be an element of $N_\mathscr{L}$. Assume that a point $x \in \Gamma \backslash U_{\gamma,m}$ is fixed by (g, γ). Then, every point $y \in U_{\gamma,m}$ satisfying $p_\Gamma(y) = x$ is fixed by (g, γ).*

   ii) *For $\lambda \in I(\mathscr{L})'_\gamma \setminus I(\mathscr{L})_\gamma$, we have $p_\Gamma(U_{\gamma\lambda,m}) \cap p_\Gamma(U_{\lambda,m}) = \varnothing$.*

Proof. i) Take $\lambda \in I(\mathscr{L})_\gamma$ such that $y \in U_{\lambda,m}$. As x is fixed by (g, γ), there exists $\gamma' \in \Gamma$ such that $(g, \gamma'\gamma)y = y$. In particular $U_{\gamma'\gamma\lambda,m} \cap U_{\lambda,m} \neq \varnothing$. Since $\lambda \in I(\mathscr{L})_\gamma$, we have $U_{\gamma\lambda,m} \cap U_{\lambda,m} \neq \varnothing$, and thus $U_{\gamma'\gamma\lambda,m} \cap U_{\gamma'\lambda,m} \neq \varnothing$. Therefore, by Lemma 5.16, we can conclude that $\gamma' = 1$. Hence y is fixed by (g, γ).

ii) Assume that $p_\Gamma(U_{\gamma\lambda,m}) \cap p_\Gamma(U_{\lambda,m})$ is non-empty. Then there exists $\gamma' \in \Gamma$ such that $U_{\gamma\lambda,m} \cap U_{\gamma'\lambda,m} \neq \varnothing$. Since $\lambda \in I(\mathscr{L})'_\gamma$, we can find $\mu \in I(\mathscr{L})_\gamma$ such that $U_{\lambda,m} \cap U_{\mu,m} \neq \varnothing$. By the definition of $I(\mathscr{L})_\gamma$, we have $U_{\mu,m} \cap U_{\gamma\mu,m} \neq \varnothing$. Therefore, all of $U_{\lambda,m} \cap U_{\mu,m}$, $U_{\mu,m} \cap U_{\gamma\mu,m}$, $U_{\gamma\mu,m} \cap U_{\gamma\lambda,m}$ and $U_{\gamma\lambda,m} \cap U_{\gamma'\lambda,m}$ are non-empty. Lemma 5.16 tells us that $\gamma' = 1$. Hence $U_{\gamma\lambda,m} \cap U_{\lambda,m} \neq \varnothing$, which contradicts to the assumption $\lambda \notin I(\mathscr{L})_\gamma$. ∎

By the same argument as in the proof of Proposition 5.14, we can obtain the following:

Proposition 5.18 *For $g \in N_{\mathscr{L}}$, we have*

$$\sum_i (-1)^i \operatorname{Tr}\big((g^{-1}, \gamma^{-1}); H_c^i(\Gamma \backslash U_{\gamma,m})\big) = \operatorname{vol}\big(Z(\gamma)/\widetilde{\Gamma}\big) O_\gamma(\eta_{gK_{\mathscr{L},m}}),$$

where $\widetilde{\Gamma}$ denotes the inverse image of Γ under $J \longrightarrow J/p^{\mathbb{Z}}$.

6 Application of Lefschetz trace formula

From now on, we assume that $d = 2$. In this section, we will apply the Lefschetz trace formula to compute the left hand side of the identities in Proposition 5.14 and Proposition 5.18. Let $\mathscr{L}$ be one of $\mathscr{L}_0$, $\mathscr{L}_{\mathrm{para}}$ or $\mathscr{L}_{\mathrm{Siegel}}$ in Lemma 4.13, and $m \geq 1$ be an integer. In this case, $K_{\mathscr{L},m}$ is contained in K_0. We put $K = K_{\mathscr{L},m}$ for simplicity. The goal of this section is as follows:

Theorem 6.1 *For $g \in N_{\mathscr{L}}$, assume that gK consists of regular elliptic elements.*

i) *For every regular elliptic element $\gamma \in J$, we have*

$$O_\gamma(\eta_{gK}) = \# \operatorname{Fix}\big((g, \gamma); M_{\mathscr{L},m}/p^{\mathbb{Z}}\big) = \sum_{x \in \operatorname{Fix}(\gamma;\Omega)} O_{g_{\gamma,x}}\Big(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\operatorname{vol}(K)}\Big),$$

(The latter equality has been proved in Theorem 3.26 and .)

ii) *For every regular non-elliptic element $\gamma \in J$, we have $O_\gamma(\eta_{gK}) = 0$.*

Before proving the theorem, we record the following corollary, which is an immediate consequence of Theorem 6.1 i) and Corollary 3.27:

Corollary 6.2 *Let g be an element of $N_{\mathscr{L}}$ such that gK consists of regular elliptic elements. For $\gamma \in J^{\mathrm{ell}}$, let g_γ be an element of G with $g_\gamma \leftrightarrow \gamma$. Then we have*

$$SO_\gamma(\eta_{gK}) = 4SO_{g_\gamma}\Big(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\operatorname{vol}(K)}\Big).$$

6.1 Complement on [Mie10a]

We would like to apply [Mie10a, Theorem 4.5] to $\mathscr{U}_{\gamma,m}$ and $\Gamma \backslash \mathscr{U}_{\gamma,m}$, but the rigid generic fiber of these formal schemes are not partially proper over $\mathbb{Q}_{p^\infty}$. Here we will give a slightly stronger version of [Mie10a, Theorem 4.5] which is applicable to our cases. All techniques we need are included in [Mie10a, §4].

We use the same notation as in [Mie10a, §4]. Let R be a complete discrete valuation ring and k an algebraic closure of the fraction field F of R. Put $\mathcal{S} = \operatorname{Spf} R$ and $S = \operatorname{Spa}(k, k^+)$, where k^+ is the valuation ring of k. Let $\mathcal{X}$ be a quasi-compact

special formal scheme which is separated over $\mathcal{S}$. Then we can associate $\mathcal{X}$ with the adic spaces $X = t(\mathcal{X})_a$, $X_\eta = t(\mathcal{X})_\eta$ and $X_{\overline{\eta}} = t(\mathcal{X})_{\overline{\eta}}$. We denote the special fiber of $\mathcal{X}$ (resp. X) by $\mathcal{X}_s$ (resp. X_s).

Let $\mathcal{X}'$ be a quasi-compact special formal scheme separated over $\mathcal{S}$ which contains $\mathcal{X}$ as an open formal subscheme. We can define X', X'_η, $X'_{\overline{\eta}}$, $\mathcal{X}'_s$ and X'_s similarly. Let $\mathcal{T}$ be a finite set equipped with a partial order and $\{\mathcal{Y}_\alpha\}_{\alpha \in \mathcal{T}}$ a family of closed formal subschemes of $\mathcal{X}_s$ indexed by $\mathcal{T}$. We put $Y_\alpha = t(\mathcal{Y}_\alpha)_a = t(\mathcal{Y}_\alpha) \times_{t(\mathcal{X})} X$, and assume the same condition as in [Mie10a, Assumption 4.1];

Assumption 6.3 i) $X_s = \bigcup_{\alpha \in \mathcal{T}} Y_\alpha$.

ii) For $\alpha \in \mathcal{T}$, put $Y_{(\alpha)} = Y_\alpha \backslash \bigcup_{\beta > \alpha} Y_\beta$. Then, for $\alpha, \beta \in \mathcal{T}$ with $\alpha \neq \beta$, $Y_{(\alpha)} \cap Y_{(\beta)} = \varnothing$.

Let $f \colon \mathcal{X}' \longrightarrow \mathcal{X}'$ be an isomorphism preserving $\mathcal{X}$. We also denote the induced isomorphism $X' \xrightarrow{\cong} X'$ by the same symbol f. The induced isomorphisms $X'_\eta \xrightarrow{\cong} X'_\eta$ and $X'_{\overline{\eta}} \xrightarrow{\cong} X'_{\overline{\eta}}$ are denoted by f_η and $f_{\overline{\eta}}$, respectively. We will make the same assumption as in [Mie10a, Assumption 4.3];

Assumption 6.4 There exist an order-preserving bijection $f \colon \mathcal{T} \xrightarrow{\cong} \mathcal{T}$ and a system of closed constructible subsets $\{Y_\alpha(n)\}_{n \geq 1}$ of X for each $\alpha \in \mathcal{T}$ satisfying the following:

i) $Y_\alpha(n+1) \subset Y_\alpha(n)$ for every $n \geq 1$.

ii) $\bigcap_{n \geq 1} Y_\alpha(n) = Y_\alpha$.

iii) $f(Y_\alpha(n)) = Y_{f(\alpha)}(n)$ for every $\alpha \in \mathcal{T}$ and $n \geq 1$.

iv) $f(\alpha) \neq \alpha$ for every $\alpha \in \mathcal{T}$.

Remark 6.5 As proved in [Mie10a, Proposition 4.18], if f induces an isomorphism of formal schemes $\mathcal{Y}_\alpha \xrightarrow{\cong} \mathcal{Y}_{f(\alpha)}$ for each $\alpha \in \mathcal{T}$, then we can find a system of closed constructible subsets $\{Y_\alpha(n)\}_{n \geq 1}$ of X satisfying Assumption 6.4 i), ii), iii).

Theorem 6.6 *In addition to Assumption 6.3 and Assumption 6.4, assume the following:*

(a) *$\mathcal{X}'$ is locally algebraizable ([Mie10b, Definition 3.18]) and X_η is smooth over* $\mathrm{Spa}(F, R)$.

(b) *The closure $\overline{X_\eta}$ of X_η in X'_η is partially proper over* $\mathrm{Spa}(F, R)$.

(c) *For every $x \in X'_{\overline{\eta}} \setminus X_{\overline{\eta}}$, $f_{\overline{\eta}}(x) \neq x$.*

Let $\Lambda = \mathbb{Z}/\ell^n \mathbb{Z}$ for a prime ℓ which is invertible in k^+ and some integer $n \geq 1$. Then, $\mathrm{Fix}(f_{\overline{\eta}}|_{X_{\overline{\eta}}})$ (cf. [Mie10a, Example 2.9]) is proper over S and we have

$$\mathrm{Tr}\big(f_{\overline{\eta}}^*; R\Gamma_c(X_{\overline{\eta}}, \Lambda)\big) = \# \mathrm{Fix}(f_{\overline{\eta}}|_{X_{\overline{\eta}}}).$$

For the definition of the right hand side, see [Mie10a, Definition 2.6].

Proof. First of all, [Mie10a, Proposition 4.6] tells us that $R\Gamma_c(X_{\overline{\eta}}, \Lambda)$ is a perfect Λ-complex, and therefore we can consider the trace $\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(X_{\overline{\eta}}, \Lambda))$.

Note that the proof of [Mie10a, Lemma 4.8] does not require partially properness of X_η; thus we can find an integer $n_\alpha \geq 1$ for each $\alpha \in \mathcal{T}$ satisfying the following conditions:

- For every $\alpha \in \mathcal{T}$, $n_\alpha = n_{f(\alpha)}$.
- For $\alpha \in \mathcal{T}$, put $U_\alpha = Y_\alpha(n_\alpha) \setminus \bigcup_{\beta > \alpha} Y_\beta(n_\beta)$. Then we have $U_\alpha \cap U_\beta = \varnothing$ for every $\alpha, \beta \in \mathcal{T}$ with $\alpha \neq \beta$.

Put $W = \bigcup_{\alpha \in \mathcal{T}} Y_\alpha(n_\alpha)$ and $X_0 = X \setminus W$. As in the proof of [Mie10a, Lemma 4.12 i)], we can show that X_0 is a quasi-compact open adic subspace of X_η. Moreover, by exactly the same method as in the proof of [Mie10a, Proposition 4.10], we can obtain the equality $\mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(X_{\overline{\eta}}, \Lambda)) = \mathrm{Tr}(f_{\overline{\eta}}^*; R\Gamma_c(X_{0,\overline{\eta}}, \Lambda))$.

Consider the closure $\overline{X_{0,\overline{\eta}}}$ of $X_{0,\overline{\eta}}$ in $X_{\overline{\eta}}'$. Since $X_{\overline{\eta}}'$ is taut (*cf.* [Mie10b, Lemma 4.14]), $\overline{X_{0,\overline{\eta}}}$ is quasi-compact. On the other hand, by the assumption (b), $\overline{X_{0,\overline{\eta}}}$ is partially proper over S. Thus $\overline{X_{0,\overline{\eta}}}$ is proper over S. Let us observe that for $x \in X_{\overline{\eta}}' \setminus X_{0,\overline{\eta}}$ we have $f_{\overline{\eta}}(x) \neq x$. Indeed, if $x \notin X_{\overline{\eta}}$, by the assumption (c) we have $f_{\overline{\eta}}(x) \neq x$. If $x \in X_{\overline{\eta}} \setminus X_{0,\overline{\eta}}$, then $x \in W_{\overline{\eta}} = \coprod_{\alpha \in \mathcal{T}} U_{\alpha,\overline{\eta}}$, and thus we can find $\alpha \in \mathcal{T}$ such that $x \in U_{\alpha,\overline{\eta}}$. Since $U_{\alpha,\overline{\eta}} \cap f(U_{\alpha,\overline{\eta}}) = U_{\alpha,\overline{\eta}} \cap U_{f(\alpha),\overline{\eta}} = \varnothing$, x and $f_{\overline{\eta}}(x)$ are distinct. Note that this implies that $\mathrm{Fix}(f_{\overline{\eta}}|_{X_{0,\overline{\eta}}}) = \mathrm{Fix}(f_{\overline{\eta}}|_{\overline{X_{0,\overline{\eta}}}}) = \mathrm{Fix}(f_{\overline{\eta}}|_{X_{\overline{\eta}}})$. In particular $\mathrm{Fix}(f_{\overline{\eta}}|_{X_{\overline{\eta}}})$ is proper over S.

Now we can apply the Lefschetz trace formula [Mie10a, Theorem 3.13] to $X_{0,\overline{\eta}} \hookrightarrow \overline{X_{0,\overline{\eta}}}$, and obtain

$$\mathrm{Tr}\big(f_{\overline{\eta}}^*; R\Gamma_c(X_{\overline{\eta}}, \Lambda)\big) = \mathrm{Tr}\big(f_{\overline{\eta}}^*; R\Gamma_c(X_{0,\overline{\eta}}, \Lambda)\big) = \# \mathrm{Fix}(f_{\overline{\eta}}|_{X_{0,\overline{\eta}}}) = \# \mathrm{Fix}(f_{\overline{\eta}}|_{X_{\overline{\eta}}}).$$

For the final equality, we use [Mie10a, Proposition 2.10]. ∎

Remark 6.7 At least when the characteristic of k is 0, we can deduce from Theorem 6.6 the analogous result for ℓ-adic coefficient simply by taking projective limit (*cf.* [Mie10b, proof of Corollary 4.40]).

6.2 Proof of Theorem 6.1

Here we give a proof of Theorem 6.1.

First we consider the case where $\gamma \in J$ is elliptic. We will apply Theorem 6.6 to $\mathscr{U}_{\gamma,m} \subset \mathscr{U}_{\gamma,m}'$ and $(g, \gamma)\colon \mathscr{U}_{\mathscr{L},m}' \longrightarrow \mathscr{U}_{\mathscr{L},m}'$. By Remark 4.12, $\mathscr{U}_{\mathscr{L},m}'$ is locally algebraizable (actually it is algebraizable). We know that the generic fiber $U_{\gamma,m}$ is smooth over $\mathbb{Q}_{p^\infty}$. Moreover, Remark 5.13 tells us that the closure $\overline{U_{\gamma,m}}$ of $U_{\gamma,m}$ inside $M_{\mathscr{L},m}/p^{\mathbb{Z}}$ is the same as that inside $U_{\gamma,m}'$. Since $M_{\mathscr{L},m}/p^{\mathbb{Z}}$ is partially proper over $\mathbb{Q}_{p^\infty}$, so is $\overline{U_{\gamma,m}}$. Let $x \in U_{\gamma,m}' \setminus U_{\gamma,m}$. Then we can find $\lambda \in I(\mathscr{L})_\gamma' \setminus I(\mathscr{L})_\gamma$ such that $x \in U_{\lambda,m}$. Since $U_{\lambda,m} \cap (g, \gamma) U_{\lambda,m} = U_{\lambda,m} \cap U_{\gamma\lambda,m} = \varnothing$, we have $x \neq (g, \gamma)x$. Thus the assumptions (a), (b), (c) in Theorem 6.6 are satisfied.

Recall that for each $\alpha \in \mathcal{S}_{\mathscr{L},m}$, a closed formal subscheme $\breve{\mathscr{M}}^\flat_{\mathscr{L},m,\alpha}$ is attached (Definition 4.8, Lemma 4.9). We denote by $\mathcal{Y}_\alpha$ the restriction of $\breve{\mathscr{M}}^\flat_{\mathscr{L},m,\alpha}/p^{\mathbb{Z}}$ to $\mathscr{U}_{\gamma,m}$. In Proposition 4.11, we have checked that $\{\mathcal{Y}_\alpha\}_{\alpha \in \mathcal{S}_{\mathscr{L},m}}$ satisfies Assumption 6.3. By Lemma 4.10, the isomorphism $(g,\gamma)\colon \mathscr{U}_{\gamma,m} \longrightarrow \mathscr{U}_{\gamma,m}$ induces an isomorphism $\mathcal{Y}_\alpha \longrightarrow \mathcal{Y}_{g\alpha}$. Therefore, Assumption 6.4 is also satisfied; note that Assumption 6.4 iv) is nothing but Corollary 4.16.

Therefore all the conditions of Theorem 6.6 are verified, and we have

$$\sum_i (-1)^i \operatorname{Tr}\big((g^{-1},h^{-1}); H^i_c(U_{\gamma,m})\big) = \#\operatorname{Fix}\big((g,\gamma); U_{\gamma,m}\big).$$

By Proposition 5.14, the left hand side is equal to $O_\gamma(\eta_{gK})$. As there is no fixed point under (g,γ) in $(M_{\mathscr{L},m}/p^{\mathbb{Z}})\backslash U_{\gamma,m}$, the right hand side is equal to $\#\operatorname{Fix}((g,\gamma); M_{\mathscr{L},m}/p^{\mathbb{Z}})$ (*cf.* [Mie10a, Proposition 2.10]). This completes the proof of Theorem 6.1 i).

The case where γ is non-elliptic is similar. Let $\Gamma \subset Z(\gamma)/p^{\mathbb{Z}}$ be as in Section 5.2, and apply Theorem 6.6 to $\Gamma\backslash\mathscr{U}_{\gamma,m} \subset \Gamma\backslash\mathscr{U}'_{\gamma,m}$ and $(g,\gamma)\colon \Gamma\backslash\mathscr{U}'_{\gamma,m} \longrightarrow \Gamma\backslash\mathscr{U}'_{\gamma,m}$. The conditions (a), (b), (c) in Theorem 6.6 can be proved in the same manner. (a) is clear. For (b), note that $\Gamma\backslash\overline{U_{\gamma,m}}$ is a closed subset of $\Gamma\backslash U'_{\gamma,m}$ which is partially proper over $\mathbb{Q}_{p^\infty}$. Thus the closure of $\Gamma\backslash U_{\gamma,m}$ in $\Gamma\backslash U'_{\gamma,m}$, being a closed subset of $\Gamma\backslash\overline{U_{\gamma,m}}$, is also partially proper over $\mathbb{Q}_{p^\infty}$. (c) can be proved in the same way as above by using Proposition 5.17 ii).

It is easy to show that $\mathcal{Y}_\alpha$ for $\alpha \in \mathcal{S}_{\mathscr{L},m}$ induces a closed formal subscheme $\Gamma\backslash\mathcal{Y}_\alpha$ of $\Gamma\backslash\mathscr{U}_{\gamma,m}$, and a family $\{\Gamma\backslash\mathcal{Y}_\alpha\}_{\alpha \in \mathcal{S}_{\mathscr{L},m}}$ satisfies Assumption 6.3 and Assumption 6.4.

Therefore we can apply Theorem 6.6, and we have

$$\sum_i (-1)^i \operatorname{Tr}\big((g^{-1},h^{-1}); H^i_c(\Gamma\backslash U_{\gamma,m})\big) = \#\operatorname{Fix}\big((g,\gamma); \Gamma\backslash U_{\gamma,m}\big).$$

By Proposition 5.18, the left hand side is equal to $\operatorname{vol}(Z(\gamma)/\widetilde{\Gamma})O_\gamma(\eta_{gK})$. By Proposition 5.17 i), the right hand side is equal to $\#\operatorname{Fix}((g,\gamma); U_{\gamma,m})$, which is zero by Theorem 3.26 ii). This completes the proof of Theorem 6.1 ii).

7 Computation of the character

7.1 Local Langlands correspondence for G and J

A candidate of the local Langlands correspondence for G and J has been constructed by Gan-Takeda [GT11a] and Gan-Tantono [GT]. Here we review their results briefly. In this subsection, let F be a p-adic field and put $G = \operatorname{GSp}_4(F)$, $J = \operatorname{GU}(2,D)$ where D is a quaternion division algebra over F.

First we recall basic definitions on L-parameters.

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Definition 7.1   i) An L-parameter for G is a homomorphism $\phi\colon W_F\times\mathrm{SL}_2(\mathbb{C})\longrightarrow$ $\mathrm{GSp}_4(\mathbb{C})$ satisfying the following conditions:

- The restriction to the first factor $\phi|_{W_F}\colon W_F\longrightarrow\mathrm{GSp}_4(\mathbb{C})$ is continuous, where $\mathrm{GSp}_4(\mathbb{C})$ is endowed with the discrete topology. The restriction to the second factor $\phi|_{\mathrm{SL}_2(\mathbb{C})}\colon\mathrm{SL}_2(\mathbb{C})\longrightarrow\mathrm{GSp}_4(\mathbb{C})$ comes from a homomorphism of algebraic groups.
- For $w\in W_F$, $\phi(w,1)\in\mathrm{GSp}_4(\mathbb{C})$ is semisimple.

We denote by $\Phi(G)$ the set of $\mathrm{GSp}_4(\mathbb{C})$-conjugacy classes of L-parameters for G.

ii) An L-parameter $\phi\colon W_F\times\mathrm{SL}_2(\mathbb{C})\longrightarrow\mathrm{GSp}_4(\mathbb{C})$ is said to be discrete if $\mathrm{Im}\,\phi$ is not contained in any proper parabolic subgroup of $\mathrm{GSp}_4(\mathbb{C})$.

iii) Let P be a proper parabolic subgroup of J, which is unique up to conjugacy. It determines a conjugacy class $\widehat{P}$ of parabolic subgroups of $\mathrm{GSp}_4(\mathbb{C})=\widehat{\mathrm{GSp}}_4$ (in fact, it is the conjugacy class containing the Siegel parabolic subgroup). A proper parabolic subgroup of $\mathrm{GSp}_4(\mathbb{C})$ belonging to $\widehat{P}$ is said to be relevant for J. An L-parameter $\phi\colon W_F\times\mathrm{SL}_2(\mathbb{C})\longrightarrow\mathrm{GSp}_4(\mathbb{C})$ is said to be relevant for J if $\mathrm{Im}\,\phi$ is not contained in any proper parabolic subgroup of $\mathrm{GSp}_4(\mathbb{C})$ which is not relevant for J. We denote by $\Phi(J)$ the set of $\mathrm{GSp}_4(\mathbb{C})$-conjugacy classes of L-parameters for G which are relevant for J.

The main theorem of [GT11a] can be summarized as follows:

Theorem 7.2 (Gan-Takeda [GT11a]) *Let* $\mathrm{Irr}(G)$ *denote the set of isomorphic classes of irreducible admissible representations of* G.

i) *There exists a natural surjection* $\mathrm{Irr}(G)\longrightarrow\Phi(G)$ *with finite fibers. For* $\phi\in\Phi(G)$, *we denote the fiber at* ϕ *by* Π_ϕ^G *and call it the* L-packet *corresponding to* ϕ.

ii) *An irreducible representation* $\pi\in\mathrm{Irr}(G)$ *is a (essentially) discrete series if and only if* $\pi\in\Pi_\phi^G$ *for a discrete* L-parameter $\phi\in\Phi(G)$.

iii) *For* $\phi\in\Phi(G)$, *we put* $A_\phi=\pi_0(Z_{\mathrm{GSp}_4}(\mathrm{Im}\,\phi))$ *and write* $\widehat{A}_\phi$ *the set of irreducible characters of* A_ϕ. *Then, there exists a natural bijection between* Π_ϕ^G *and* $\widehat{A}_\phi$.

iv) *For* $\phi\in\Phi(G)$ *and* $\pi\in\Pi_\phi^G$, *the central character of* π *is equal to* $(\mathrm{sim}\circ\phi)\circ\mathrm{Art}$, *where* $\mathrm{sim}\colon\mathrm{GSp}_4(\mathbb{C})\longrightarrow\mathbb{C}^\times$ *denotes the similitude character and* $\mathrm{Art}\colon F^\times\longrightarrow W_F^{\mathrm{ab}}$ *denotes the isomorphism of local class field theory (normalized so that a uniformizer is mapped to a lift of the geometric Frobenius element).*

v) *For* $\phi\in\Phi(G)$ *and a smooth character* χ *of* $F^\times$, *let* $\phi\otimes\chi$ *be the* L-parameter *given by* $(\phi\otimes\chi)(w,v)\longmapsto\chi(\mathrm{Art}^{-1}(w))\phi(w,v)$ *for* $(w,v)\in W_F\times\mathrm{SL}_2(\mathbb{C})$. *Then we have* $\Pi_{\phi\otimes\chi}^G=\{\pi\otimes\chi_G\mid\pi\in\Pi_\phi^G\}$, *where* $\chi_G=\chi\circ\mathrm{sim}$ *as in Lemma 3.5.*

vi) *There is a way to characterize the map* $\mathrm{Irr}(G)\longrightarrow\Phi(G)$ *by means of local factors and Plancherel measures.*

Moreover, from the construction in [GT11a], we have the following:

Theorem 7.3 *If $\phi \in \Phi(G)$ is discrete and trivial on $\mathrm{SL}_2(\mathbb{C})$, then Π_ϕ^G consists of supercuspidal representations.*

Proof. We freely use the notation in [GT11a] and [GT11b]. First assume that ϕ is not irreducible as a 4-dimensional representation of W_F. Then $\phi = \phi_1 \oplus \phi_2$ where ϕ_1 and ϕ_2 are two-dimensional irreducible representations of W_F with $\det \phi_1 = \det \phi_2$ and $\phi_1 \not\cong \phi_2$ (*cf.* [GT11a, Lemma 6.2 (ii)]). Let τ_i be the irreducible representation of $\mathrm{GL}_2(F)$ corresponding to ϕ_i by the local Langlands correspondence. By the assumption, τ_1 and τ_2 are supercuspidal and $\tau_1 \not\cong \tau_2$. By the construction of L-packets (*cf.* [GT11a, §7]), we have $\Pi_\phi^G = \{\theta_{(2,2)}(\tau_1 \boxtimes \tau_2), \theta_{(4,0)}(\tau_1^D \boxtimes \tau_2^D)\}$, where $\theta_{(2,2)}$ (resp. $\theta_{(4,0)}$) denotes the theta correspondence between GSp_4 and $\mathrm{GSO}_{2,2}$ (resp. $\mathrm{GSO}_{4,0}$) and τ_i^D denotes the representation of $D^\times$ corresponding to τ_i by the local Jacquet-Langlands correspondence. [GT11b, Theorem 8.2 (ii)] tells us that $\theta_{(2,2)}(\tau_1 \boxtimes \tau_2)$ is supercuspidal. [GT11b, Theorem 8.1 (iii)] tells us that $\theta_{(4,0)}(\tau_1^D \boxtimes \tau_2^D)$ is supercuspidal. Therefore Π_ϕ^G consists of two supercuspidal representations.

Next assume that ϕ is irreducible as a 4-dimensional representation. Then, Π_ϕ^G consists of a single representation $\pi \in \mathrm{Irr}(G)$ such that $\theta_{(3,3)}(\pi) = \Pi \boxtimes \mu$, where $\theta_{(3,3)}$ denotes the theta correspondence between GSp_4 and $\mathrm{GSO}_{3,3}$, Π is the irreducible representation of $\mathrm{GL}_4(F)$ corresponding to the 4-dimensional representation ϕ, and $\mu = (\mathrm{sim} \circ \phi) \circ \mathrm{Art}$ is the character of $F^\times$ corresponding to $\mathrm{sim} \circ \phi$. By the assumption on ϕ, Π is supercuspidal, and thus $\theta_{(3,3)}(\pi)$ is supercuspidal. Therefore, [GT11b, Section 14, Table 1] tells us that π is supercuspidal, as desired. ∎

The main theorem of [GT] is as follows:

Theorem 7.4 (Gan-Tantono [GT]) *Let $\mathrm{Irr}(J)$ denote the set of isomorphic classes of irreducible admissible representations of J.*

i) *There exists a natural surjection $\mathrm{Irr}(J) \longrightarrow \Phi(J)$ with finite fibers. For $\phi \in \Phi(J)$, we denote the fiber at ϕ by Π_ϕ^J and call it the L-packet corresponding to ϕ.*

ii) *An irreducible representation $\rho \in \mathrm{Irr}(G)$ is a (essentially) discrete series if and only if $\rho \in \Pi_\phi^J$ for a discrete L-parameter $\phi \in \Phi(J)$.*

iii) *For $\phi \in \Phi(J)$, we put $B_\phi = \pi_0(Z_{\mathrm{Sp}_4}(\mathrm{Im}\,\phi))$ and write $\widehat{B}_\phi$ the set of irreducible characters of B_ϕ. The natural map $B_\phi \longrightarrow A_\phi$ is surjective, and thus $\widehat{A}_\phi$ can be regarded as a subgroup of $\widehat{B}_\phi$. There exists a natural bijection between Π_ϕ^J and $\widehat{B}_\phi \setminus \widehat{A}_\phi$.*

iv) *For $\phi \in \Phi(J)$ and $\rho \in \Pi_\phi^J$, the central character of ρ is equal to $(\mathrm{sim} \circ \phi) \circ \mathrm{Art}$.*

v) *For $\phi \in \Phi(J)$, we have $\Pi_{\phi \otimes \chi}^J = \{\rho \otimes \chi_J \mid \rho \in \Pi_\phi^J\}$, where $\chi_J = \chi \circ \mathrm{sim}$ as in Lemma 3.5.*

vi) *There is a way to characterize the map $\mathrm{Irr}(J) \longrightarrow \Phi(J)$ by means of local factors and Plancherel measures.*

We also have an analogous result as Theorem 7.3:

Theorem 7.5 *If $\phi \in \Phi(J)$ is discrete and trivial on $\mathrm{SL}_2(\mathbb{C})$, then Π_ϕ^J consists of supercuspidal representations.*

Proof. We freely use the notation in [GT]. First assume that ϕ is not irreducible as a 4-dimensional representation of W_F. Then $\phi = \phi_1 \oplus \phi_2$ as in the proof of Theorem 7.3. Define τ_1 and τ_2 similarly. They are supercuspidal and not isomorphic to each other. By the construction of L-packets (*cf.* [GT, §7]), we have $\Pi_\phi^J = \{\theta_{(1,1)}(\tau_1^D \boxtimes \tau_2), \theta_{(1,1)}(\tau_2^D \boxtimes \tau_1)\}$, where $\theta_{(1,1)}$ denotes the theta correspondence between $J = \mathrm{GSp}(1,1)$ and $\mathrm{GO}^*(1,1)$. Therefore, [GT, Proposition 5.4 (iv)] tells us that Π_ϕ^J consists of two supercuspidal representations.

Next assume that ϕ is irreducible as a 4-dimensional representation. Then, Π_ϕ^J consists of a single representation $\rho \in \mathrm{Irr}(J)$ such that $\theta_{(3,0)}(\rho) = \Pi \boxtimes \mu$, where $\theta_{(3,0)}$ denotes the theta correspondence between $J = \mathrm{GSp}(1,1)$ and $\mathrm{GO}^*(3,0)$, Π is the irreducible representation of $D_4^\times$ corresponding to the 4-dimensional representation ϕ (here D_4 is the central division algebra over $\mathbb{Q}_p$ with invariant $1/4$), and $\mu = (\mathrm{sim} \circ \phi) \circ \mathrm{Art}$ is the character of $F^\times$ corresponding to $\mathrm{sim} \circ \phi$. By the assumption on ϕ, Π is not a character. Therefore, [GT, Proposition 5.7] tells us that ρ is supercuspidal; indeed, if ρ is neither supercuspidal nor a twist of the Steinberg representation, then $\theta_{(3,0)}(\rho) = 0$, and if ρ is a twist of the Steinberg representation, then $\theta_{(3,0)}(\rho)$ is a character. $\blacksquare$

For an irreducible admissible representation π of G, we denote the character of π by θ_π. It is locally constant on G^{reg}; namely, θ_π is a unique locally constant function on G^{reg} such that $\mathrm{Tr}(f; \pi) = \int_G f(g)\theta_\pi(g)dg$ for every $f \in \mathcal{H}(G)$ with $\mathrm{supp}\, f \subset G^{\mathrm{reg}}$. For $\phi \in \Phi(G)$, put $\theta_{\Pi_\phi^G} = \sum_{\pi \in \Pi_\phi^G} \theta_\pi$.

Similarly, we define θ_ρ and $\theta_{\Pi_\phi^J}$ for $\rho \in \mathrm{Irr}(J)$ and $\phi \in \Phi(J)$.

Definition 7.6   i) For $\phi \in \Phi(G)$, the corresponding L-packet Π_ϕ^G is said to be stable if $\theta_{\Pi_\phi^G}$ is a stable function on G^{reg}, that is, $\theta_{\Pi_\phi^G}(g) = \theta_{\Pi_\phi^G}(g')$ for every $g, g' \in G^{\mathrm{reg}}$ which are stably conjugate. Similarly we can define the stability of the L-packet Π_ϕ^J for $\phi \in \Phi(J)$.

   ii) For $\phi \in \Phi(J)$, we say that Π_ϕ^G and Π_ϕ^J satisfy the character relation if $\theta_{\Pi_\phi^G}(g) = -\theta_{\Pi_\phi^J}(h)$ for every $g \in G^{\mathrm{ell}}$, $h \in J^{\mathrm{ell}}$ with $g \leftrightarrow h$ (*cf.* Section 2).

Remark 7.7 For every discrete (or more generally, tempered) L-parameter ϕ, it is expected that Π_ϕ^G and Π_ϕ^J are stable and satisfy the character relation. It is plausible that one can deduce these properties from the stable trace formula.

If ϕ is a TRSELP in the sense of [DR09], then the stability and the character relation for Π_ϕ^G and Π_ϕ^J are already known due to [DR09], [Kal10] and [Lus11].

7.2   Computation of the character

Theorem 7.8 *Let $\phi \in \Phi(J)$ be an L-parameter such that Π_ϕ^G, Π_ϕ^J are stable and satisfy the character relation. Then, for every $f \in \mathcal{H}(G)$ with $\mathrm{supp}\, f \subset G^{\mathrm{ell}}$, we*

have

$$\sum_{\rho \in \Pi_\phi^J} \mathrm{Tr}(f; H_{\mathrm{RZ}}[\rho]) = -4 \sum_{\pi \in \Pi_\phi^G} \mathrm{Tr}(f; \pi).$$

Proof. By Theorem 7.2 iv) and Theorem 7.4 iv), all representations in Π_ϕ^G and Π_ϕ^J share the same central character $\omega = (\mathrm{sim} \circ \phi) \circ \mathrm{Art} \colon \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$. First we will reduce the theorem to the case where $\omega|_{p^{\mathbb{Z}}}$ is trivial. The method is similar to the proof of Corollary 3.7. Take $c \in \overline{\mathbb{Q}}_\ell^\times$ such that $c^2 = \omega(p)$, and $\chi \colon \mathbb{Q}_p^\times \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ the character given by $\chi(a) = c^{-v_p(a)}$. Consider the L-packets

$$\Pi_{\phi \otimes \chi}^G = \{\pi \otimes \chi_G \mid \pi \in \Pi_\phi^G\}, \quad \Pi_{\phi \otimes \chi}^J = \{\rho \otimes \chi_J \mid \rho \in \Pi_\phi^J\}$$

corresponding to $\phi \otimes \chi$ (*cf.* Theorem 7.2 v), Theorem 7.4 v)). It is clear that these L-packets are stable and satisfy the character relation. Moreover, every representation belonging to these L-packets has the central character trivial on $p^{\mathbb{Z}}$.

We have $H_{\mathrm{RZ}}[\rho \otimes \chi_J] = H_{\mathrm{RZ}}[\rho] \otimes \chi_G$ by Lemma 3.5. Therefore, if we have the theorem for the L-parameter $\phi \otimes \chi \in \Phi(J)$, then

$$\sum_{\rho \in \Pi_\phi^J} \mathrm{Tr}(f; H_{\mathrm{RZ}}[\rho]) = \sum_{\rho \otimes \chi_J \in \Pi_{\phi \otimes \chi}^J} \mathrm{Tr}(f; H_{\mathrm{RZ}}[\rho \otimes \chi_J] \otimes \chi_G^{-1})$$

$$= \sum_{\rho \otimes \chi_J \in \Pi_{\phi \otimes \chi}^J} \mathrm{Tr}(f \cdot \chi_G^{-1}; H_{\mathrm{RZ}}[\rho \otimes \chi_J])$$

$$= -4 \sum_{\pi \otimes \chi_G \in \Pi_{\phi \otimes \chi}^G} \mathrm{Tr}(f \cdot \chi_G^{-1}; \pi \otimes \chi_G) = -4 \sum_{\pi \in \Pi_\phi^G} \mathrm{Tr}(f; \pi),$$

and thus the theorem also holds for ϕ.

In the following, we assume that the central character ω is trivial on $p^{\mathbb{Z}}$.

By the similar way as in [Mie12, Lemma 3.5], we can prove that G^{ell} is contained in the union of all open compact-mod-center subgroups of G. Therefore, we may assume that $\mathrm{supp}\, f$ is contained in an open compact-mod-center subgroup of G. Recall that a maximal open compact-mod-center subgroup is conjugate to $N_{\mathscr{L}}$ where $\mathscr{L}$ is one of $\mathscr{L}_0$, $\mathscr{L}_{\mathrm{para}}$ or $\mathscr{L}_{\mathrm{Siegel}}$ (Lemma 4.14). Note that both sides of the identity in the theorem do not change if we replace f by its conjugate. Therefore, we may assume that $\mathrm{supp}\, f$ is contained in $N_{\mathscr{L}}$ where $\mathscr{L}$ is one of $\mathscr{L}_0$, $\mathscr{L}_{\mathrm{para}}$ or $\mathscr{L}_{\mathrm{Siegel}}$. Since $\{K_{\mathscr{L},m}\}_{m \geq 1}$ form a fundamental system of neighborhoods of $1 \in N_{\mathscr{L}}$ consisting of normal subgroups of $N_{\mathscr{L}}$, f can be written as a linear combination of $\mathbf{1}_{gK_{\mathscr{L},m}}$ with $g \in N_{\mathscr{L}}$. Hence we are reduced to the case where $f = \mathrm{vol}(K_{\mathscr{L},m})^{-1} \mathbf{1}_{gK_{\mathscr{L},m}}$. For simplicity, put $K = K_{\mathscr{L},m}$. Note that $gK = \mathrm{supp}\, f \subset G^{\mathrm{ell}}$.

By Proposition 5.7, Theorem 6.1, Corollary 6.2 and the stable version of Weyl's

integration formula, we can compute as follows:

$$
\sum_{\rho \in \Pi_\phi^J} \operatorname{Tr}\Big(\frac{\mathbf{1}_{gK}}{\operatorname{vol}(K)}; H_{\mathrm{RZ}}[\rho]\Big) = \sum_{\rho \in \Pi_\phi^J} \operatorname{Tr}(g; H_{\mathrm{RZ}}[\rho]^K) \overset{(1)}{=} \sum_{\rho \in \Pi_\phi^J} \operatorname{Tr}(\eta_{gK}; \rho)
$$

$$
= \sum_{\rho \in \Pi_\phi^J} \int_{J/p^{\mathbb{Z}}} \eta_{gK}(h)\theta_\rho(h)dh = \int_{J/p^{\mathbb{Z}}} \eta_{gK}(h)\theta_{\Pi_\phi^J}(h)dh
$$

$$
\overset{(2)}{=} \sum_{\{\mathbf{T}'\}_{\mathrm{st}} \in \mathcal{T}_{\mathbf{J},\mathrm{st}}^{\mathrm{ell}}} \frac{1}{\#W_{\mathbf{T}'}(\mathbb{Q}_p)} \int_{T'^{\mathrm{reg}}/p^{\mathbb{Z}}} D_{\mathbf{J}}(t')\theta_{\Pi_\phi^J}(t')SO_{t'}(\eta_{gK})dt'
$$

$$
\overset{(3)}{=} 4 \sum_{\{\mathbf{T}'\}_{\mathrm{st}} \in \mathcal{T}_{\mathbf{J},\mathrm{st}}^{\mathrm{ell}}} \frac{1}{\#W_{\mathbf{T}'}(\mathbb{Q}_p)} \int_{T'^{\mathrm{reg}}/p^{\mathbb{Z}}} D_{\mathbf{J}}(t')\theta_{\Pi_\phi^J}(t')SO_{g_{t'}}\Big(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\operatorname{vol}(K)}\Big)dt',
$$

where $g_{t'} \in G^{\mathrm{ell}}$ is an arbitrary element with $g_{t'} \leftrightarrow t'$, and $D_{\mathbf{J}}(t')$ is the Weyl denominator (*cf.* [Rog83, p. 185]). Here (1) follows from Proposition 5.7 and (3) from Corollary 6.2. (2) is a consequence of Theorem 6.1 ii), stability of $\theta_{\Pi_\phi^J}$ and the stable version of Weyl's integration formula for J. Other equalities are obvious.

Recall that we have a natural bijection $\mathcal{T}_{\mathbf{J},\mathrm{st}}^{\mathrm{ell}} \overset{\cong}{\longrightarrow} \mathcal{T}_{\mathbf{G},\mathrm{st}}^{\mathrm{ell}}$. For an elliptic maximal torus $\mathbf{T}'$ of $\mathbf{J}$ and an elliptic maximal torus $\mathbf{T}$ of $\mathbf{G}$ corresponding to $\mathbf{T}$, choose $t_0' \in T'^{\mathrm{reg}}$, $t_0 \in T^{\mathrm{reg}}$ with $t_0 \leftrightarrow t_0'$ and consider the isomorphism $\iota = \iota_{t_0',t_0}\colon \mathbf{T}' \overset{\cong}{\longrightarrow} \mathbf{T}$ (*cf.* Section 2). Then we can take $g_{t'}$ as $t = \iota(t')$. Obviously we have $D_{\mathbf{J}}(t') = D_{\mathbf{G}}(t)$. Since Π_ϕ^G and Π_ϕ^J satisfy the character relation, we have $\theta_{\Pi_\phi^J}(t') = -\theta_{\Pi_\phi^G}(t)$. Moreover, $\iota\colon T'/p^{\mathbb{Z}} \overset{\cong}{\longrightarrow} T/p^{\mathbb{Z}}$ is compatible with the fixed measures (*cf.* the last paragraph of Section 3.1). Together with Lemma 2.1, we can convert the sum with respect to $\mathcal{T}_{\mathbf{J},\mathrm{st}}^{\mathrm{ell}}$ to that with respect to $\mathcal{T}_{\mathbf{G},\mathrm{st}}^{\mathrm{ell}}$, and obtain

$$
\sum_{\rho \in \Pi_\phi^J} \operatorname{Tr}\Big(\frac{\mathbf{1}_{gK}}{\operatorname{vol}(K)}; H_{\mathrm{RZ}}[\rho]\Big)
$$

$$
= -4 \sum_{\{\mathbf{T}\}_{\mathrm{st}} \in \mathcal{T}_{\mathbf{G},\mathrm{st}}^{\mathrm{ell}}} \frac{1}{\#W_{\mathbf{T}}(\mathbb{Q}_p)} \int_{T^{\mathrm{reg}}/p^{\mathbb{Z}}} D_{\mathbf{G}}(t)\theta_{\Pi_\phi^G}(t)SO_t\Big(\frac{\mathbf{1}_{gKp^{\mathbb{Z}}}}{\operatorname{vol}(K)}\Big)dt
$$

$$
= \frac{-4}{\operatorname{vol}(K)} \int_{G/p^{\mathbb{Z}}} \mathbf{1}_{gKp^{\mathbb{Z}}}(x)\theta_{\Pi_\phi^G}(x)dx = \frac{-4}{\operatorname{vol}(K)} \int_G \mathbf{1}_{gK}(x)\theta_{\Pi_\phi^G}(x)dx
$$

$$
= -4 \sum_{\pi \in \Pi_\phi^G} \operatorname{Tr}\Big(\frac{\mathbf{1}_{gK}}{\operatorname{vol}(K)}; \pi\Big)
$$

(in the second equality, we use the stable version of Weyl's integration formula for $\mathbf{G}$; again we use the fact that $\mathbf{1}_{gKp^{\mathbb{Z}}}$ is supported on G^{ell}). This completes the proof. ∎

Corollary 7.9 *Let ϕ be an L-parameter as in Theorem 7.8. Assume that for every $\rho \in \Pi_\phi^J$ and integers $i, j \geq 0$, $H_{\mathrm{RZ}}^{i,j}[\rho]$ is a G-module of finite length. Then, for each*

$\rho \in \Pi_\phi^J$, we can consider the character $\theta_{H_{\mathrm{RZ}}[\rho]}$ of the virtual representation $H_{\mathrm{RZ}}[\rho]$. This character satisfies the following for every $g \in G^{\mathrm{ell}}$:

$$\sum_{\rho \in \Pi_\phi^J} \theta_{H_{\mathrm{RZ}}[\rho]}(g) = -4 \sum_{\pi \in \Pi_\phi^G} \theta_\pi(g).$$

Proof. This is an immediate consequence of Theorem 7.8. ∎

Remark 7.10 i) Since the G-representation $H_{\mathrm{RZ}}^{i,j}[\rho]$ is admissible, it has finite length if and only if it is finitely generated as a G-module.

ii) At least if $\rho \in \Pi_\phi^J$ is supercuspidal, the assumption in Corollary 7.9 will be proved once we establish an analogue of Faltings' isomorphism for the Rapoport-Zink tower $\{M_K\}_K$ (*cf.* [Mie12, Lemma 5.2]). In a recent preprint of Scholze and Weinstein [SW12], Faltings' isomorphism for a Rapoport-Zink tower of EL type is obtained. It is plausible that a similar method is applicable to our case.

Corollary 7.11 *Under the setting in Corollary 7.9, assume moreover that ϕ is discrete and trivial on $\mathrm{SL}_2(\mathbb{C})$. Then, for each $\pi \in \Pi_\phi^G$, the representation $\pi^\vee$ of G appears in H_{RZ}^3.*

Proof. By the similar method as in the proof of Theorem 7.8, we may reduce to the case where π is trivial on $p^{\mathbb{Z}}$. By Theorem 7.3 and Theorem 7.5, Π_ϕ^G and Π_ϕ^J consist of supercuspidal representations. Therefore $\mathrm{Ext}_J^j(H_{\mathrm{RZ}}^i, \rho) = 0$ if $j \geq 1$, and thus $H_{\mathrm{RZ}}[\rho] = \sum_i (-1)^i \mathrm{Hom}_J(H_{\mathrm{RZ}}^i, \rho)^{\mathrm{sm}}$. We denote the supercuspidal part of $H_{\mathrm{RZ}}[\rho]$ by $H_{\mathrm{RZ}}[\rho]_{\mathrm{cusp}}$. Write $\sum_{\rho \in \Pi_\phi^J} H_{\mathrm{RZ}}[\rho]_{\mathrm{cusp}} = \sum_{\pi'} a_{\pi'} \pi'$, where π' runs through irreducible supercuspidal representations of G. Assume that $\pi^\vee$ does not appear in H_{RZ}^3. Then, by [IM10, Theorem 1.1], $\pi^\vee$ can appear only in H_{RZ}^2 and H_{RZ}^4. Hence π can appear in $\mathrm{Hom}_J(H_{\mathrm{RZ}}^i, \rho)^{\mathrm{sm}}$ only if $i = 2, 4$, and thus $a_\pi \geq 0$. On the other hand, by Corollary 7.9 and the orthogonality relation of characters, we have

$$a_\pi = \Big\langle \theta_\pi, \sum_{\rho \in \Pi_\phi^J} \theta_{H_{\mathrm{RZ}}[\rho]} \Big\rangle_{\mathrm{ell}} = -4 \Big\langle \theta_\pi, \sum_{\pi' \in \Pi_\phi^G} \theta_{\pi'} \Big\rangle_{\mathrm{ell}} = -4$$

(for the definition of $\langle \, , \, \rangle_{\mathrm{ell}}$, see [Mie11, §5]). This is a contradiction. ∎

Remark 7.12 In a forthcoming paper with Tetsushi Ito, the author will give more precise description of the cuspidal part $H_{\mathrm{RZ}}^i[\rho]_{\mathrm{cusp}}$ of the individual cohomology $H_{\mathrm{RZ}}^i[\rho]$ via global method. We can also obtain information on the action of Weil group on $H_{\mathrm{RZ}}^i[\rho]_{\mathrm{cusp}}$; we find the local Langlands correspondence for G and J in $H_{\mathrm{RZ}}^3[\rho]_{\mathrm{cusp}}$, as expected.

Remark 7.13 In the end of this paper, we will give a remark on a related work of Xu Shen [She12a]. He applied the author's Lefschetz trace formula [Mie10a, Theorem 3.13] to a large quasi-compact open subset of the Rapoport-Zink space

M_K for GU$(1, n - 1)$ with an arbitrary level K. However, the author does not know how to deduce a result on the character of H_{RZ} from his computation [She12a, Theorem 11.3, Corollary 11.5]. He stated such a result [She12a, Corollary 11.6] and sketched a proof, but the first equality "$tr_{H(\pi)}(g) = \cdots$" in his proof is far from trivial, since the left hand side is the value of the distribution character of an infinite-dimensional representation of a p-adic reductive group, while the right hand side is the usual trace of an endomorphism on a finite-dimensional vector space. By a trivial reason, the trace on the cohomology of a large open subset of M_K is not necessarily equal to the character of the cohomology of the whole space. Therefore, to justify his argument, one at least should prove some geometric property of the quasi-compact open subset he chose; even in the Lubin-Tate case, one needed such a result [She12b, Proposition 3.5].

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