# Cuspidal representations in the $\ell$-adic cohomology of the Rapoport-Zink space for GSp(4)

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Abstract. In this paper, we study the $\ell$-adic cohomology of the Rapoport-Zink tower for GSp(4). We prove that the smooth representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ obtained as the $i$th compactly supported $\ell$-adic cohomology of the Rapoport-Zink tower has no quasi-cuspidal subquotient unless $i = 2, 3, 4$. Our proof is purely local and does not require global automorphic methods.

## 1 Introduction

In [RZ96], M. Rapoport and Th. Zink introduced certain moduli spaces of quasi-isogenies of $p$-divisible groups with additional structures called the *Rapoport-Zink spaces*. They constructed systems of rigid analytic coverings of them which we call the *Rapoport-Zink towers*, and established the $p$-adic uniformization theory of Shimura varieties generalizing classical Čerednik-Drinfeld uniformization. These spaces uniformize the rigid spaces associated with the formal completion of certain Shimura varieties along Newton strata.

Using the $\ell$-adic cohomology of the Rapoport-Zink tower, we can construct a representation of the product $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, where $G$ is the reductive group over $\mathbb{Q}_p$ corresponding to the Shimura datum, $J$ is an inner form of it, and $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the Weil group of the $p$-adic field $\mathbb{Q}_p$. It is widely believed that this realizes the local Langlands and Jacquet-Langlands correspondences (*cf.* [Rap95]). Classical examples of the Rapoport-Zink spaces are the Lubin-Tate space and the Drinfeld upper half space; these spaces were extensively studied by many people and many important results were obtained (*cf.* [Dri76], [Car90], [Har97], [HT01], [Dat07], [Boy09] and references therein). However, very little was known about the $\ell$-adic cohomology of other Rapoport-Zink spaces.

The aim of this paper is to study cuspidal representations in the $\ell$-adic cohomology of the Rapoport-Zink tower for $\mathrm{GSp}_4(\mathbb{Q}_p)$. Let us denote the Rapoport-Zink space for $\mathrm{GSp}_4(\mathbb{Q}_p)$ by $\breve{\mathscr{M}}$. It is a special formal scheme over $\mathbb{Z}_{p^\infty} = W(\overline{\mathbb{F}}_p)$ in the sense of Berkovich [Ber96]. Let $\breve{\mathscr{M}}^{\mathrm{rig}}$ be the Raynaud generic fiber of $\breve{\mathscr{M}}$, that is, the generic fiber of the adic space $t(\breve{\mathscr{M}})$ associated with $\breve{\mathscr{M}}$. Using level structures

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at $p$, we can construct the Rapoport-Zink tower

$$\cdots \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{m+1} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{m} \longrightarrow \cdots \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{2} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{1} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{0} = \breve{\mathscr{M}}^{\mathrm{rig}},$$

where $\breve{\mathscr{M}}^{\mathrm{rig}}_{m} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}$ is an étale Galois covering of rigid spaces with Galois group $\mathrm{GSp}_4(\mathbb{Z}/p^m\mathbb{Z})$. We take the compactly supported $\ell$-adic cohomology (in the sense of [Hub98]) and take the inductive limit of them. Then, on

$$H^i_{\mathrm{RZ}} := \varinjlim_{m} H^i_c(\breve{\mathscr{M}}^{\mathrm{rig}}_{m} \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell)$$

(here $\mathbb{Q}_{p^\infty} = \operatorname{Frac} \mathbb{Z}_{p^\infty}$), we have an action of a product

$$\mathrm{GSp}_4(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p),$$

where $J$ is an inner form of $\mathrm{GSp}_4$.

The main theorem of this paper is as follows:

**Theorem 1.1 (Theorem 3.2)** *The* $\mathrm{GSp}_4(\mathbb{Q}_p)$*-representation* $H^i_{\mathrm{RZ}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ *has no quasi-cuspidal subquotient unless* $i = 2, 3, 4$.

For the definition of quasi-cuspidal representations, see [Ber84, 1.20]. Note that since $\breve{\mathscr{M}}^{\mathrm{rig}}_{m}$ is 3-dimensional for every $m \geq 0$, $H^i_{\mathrm{RZ}} = 0$ unless $0 \leq i \leq 6$.

Our proof of this theorem is purely local. We do not use global automorphic methods. The main strategy of the proof is similar to that of [Mie10a], in which the analogous result for the Lubin-Tate tower is given; we construct the formal model $\breve{\mathscr{M}}_m$ of $\breve{\mathscr{M}}^{\mathrm{rig}}_{m}$ by using Drinfeld level structures and consider the geometry of its special fiber. However, our situation is much more difficult than the case of the Lubin-Tate tower. In the Lubin-Tate case, the tower consists of affine formal schemes $\{\operatorname{Spf} A_m\}_{m\geq 0}$, and we can associate it with the tower of affine schemes $\{\operatorname{Spec} A_m\}_{m\geq 0}$. In [Mie10a], the second author defined the stratification on the special fiber of $\operatorname{Spec} A_m$ by using the kernel of the universal Drinfeld level structure, and considered the local cohomology of the nearby cycle complex $R\psi\Lambda$ along the strata. On the other hand, our tower $\{\breve{\mathscr{M}}_m\}_{m\geq 0}$ does not consist of affine formal schemes and there is no canonical way to associate it with a tower of schemes. To overcome this problem, we take a sheaf-theoretic approach. For each direct summand $I$ of $(\mathbb{Z}/p^m\mathbb{Z})^4$, we will define the complex of sheaves $\mathcal{F}_{m,I}$ on $(\breve{\mathscr{M}}_m)_{\mathrm{red}}$ so that the cohomology $H^i((\breve{\mathscr{M}}_m)_{\mathrm{red}}, \mathcal{F}_{m,I})$ substitutes for the local cohomology of $R\psi\Lambda$ along the strata defined by $I$ in the Lubin-Tate case. For the definition of $\mathcal{F}_{m,I}$, we use the $p$-adic uniformization theorem by Rapoport and Zink.

There is another difficulty; since a connected component of $\breve{\mathscr{M}}$ is not quasi-compact, the representation $H^i_{\mathrm{RZ}}$ of $\mathrm{GSp}_4(\mathbb{Q}_p)$ is far from admissible. Therefore it is important to consider the action of $J(\mathbb{Q}_p)$ on $H^i_{\mathrm{RZ}}$, though it does not appear in our main theorem. However, the cohomology $H^i((\breve{\mathscr{M}}_m)_{\mathrm{red}}, \mathcal{F}_{m,I})$ has no apparent action of $J(\mathbb{Q}_p)$, since $J(\mathbb{Q}_p)$ does not act on the Shimura variety uniformized by

$\breve{\mathscr{M}}$. We use the variants of formal nearby cycle introduced by the second author in [Mie10b] to endow it with an action of $J(\mathbb{Q}_p)$. Furthermore, to ensure the smoothness of this action, we use a property of finitely generated pro-$p$ groups (Section 2). In fact, extensive use of the formalism developed in [Mie10b] make us possible to work mainly on the Rapoport-Zink tower itself and avoid the theory of $p$-adic uniformization except for proving that $\breve{\mathscr{M}}_m$ is locally algebraizable. However, for the reader's convenience, the authors decided to make this article as independent of [Mie10b] as possible.

The authors expect that the converse of Theorem 1.1 also holds. Namely, we expect that $H^i_{\mathrm{RZ}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ has a quasi-cuspidal subquotient if $i = 2, 3, 4$. We hope to investigate it in a future work.

The outline of this paper is as follows. In Section 2, we prepare a criterion for the smoothness of representations over $\mathbb{Q}_\ell$. It is elementary but very powerful for our purpose. In Section 3, we give some basic definitions concerning with the Rapoport-Zink space for GSp(4) and state the main theorem. Section 4 is devoted to introduce certain Shimura varieties related to our Rapoport-Zink tower and recall the theory of $p$-adic uniformization. The proof of the main theorem is accomplished in Section 5. The final Section 6 is an appendix on cohomological correspondences. The results in the section are used to define actions of $\mathrm{GSp}_4(\mathbb{Q}_p)$ on various cohomology groups.

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**Notation** Let $p$ be a prime number and take another prime $\ell$ with $\ell \neq p$. We denote the completion of the maximal unramified extension of $\mathbb{Z}_p$ by $\mathbb{Z}_{p^\infty}$ and its fraction field by $\mathbb{Q}_{p^\infty}$. Let $\mathbf{Nilp} = \mathbf{Nilp}_{\mathbb{Z}_{p^\infty}}$ be the category of $\mathbb{Z}_{p^\infty}$-schemes on which $p$ is locally nilpotent. For an object $S$ of $\mathbf{Nilp}$, we put $\overline{S} = S \otimes_{\mathbb{Z}_{p^\infty}} \overline{\mathbb{F}}_p$.

In this paper, we use the theory of adic spaces ([Hub94], [Hub96]) as a framework of rigid geometry. A rigid space over $\mathbb{Q}_{p^\infty}$ is understood as an adic space locally of finite type over $\mathrm{Spa}(\mathbb{Q}_{p^\infty}, \mathbb{Z}_{p^\infty})$.

Every sheaf and cohomology are considered in the étale topology. Every smooth representation is considered over $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$. For a $\mathbb{Q}_\ell$-vector space $V$, we put $V_{\overline{\mathbb{Q}}_\ell} = V \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$.

# 2 Preliminaries: smoothness of representations of profinite groups

Let $\mathbf{G}$ be a linear algebraic group over a $p$-adic field $F$. In this section, we give a convenient criterion for the smoothness of a $\mathbf{G}(F)$-representation over $\mathbb{Q}_\ell$. The following theorem is essential:

**Theorem 2.1** *Let $K$ be a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ and $(\pi, V)$ a finite-dimensional representation over $\mathbb{Q}_\ell$ of $K$ as an abstract group. Assume that there exists a $K$-stable $\mathbb{Z}_\ell$-lattice $\Lambda$ of $V$. Then this representation is automatically smooth.*

In order to prove this theorem, we require several facts on pro-$p$ groups. Put $K_1 = K \cap (1 + p\mathrm{M}_n(\mathbb{Z}_p))$, which is a pro-$p$ open subgroup of $K$.

**Lemma 2.2** *The pro-$p$ group $K_1$ is (topologically) finitely generated.*

*Proof.* By [DdSMS99, §5.1], the profinite group $\mathrm{GL}_n(\mathbb{Z}_p)$ has finite rank. In particular, $K_1$, a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$, has finite topological generators. ∎

**Lemma 2.3** *Every subgroup of finite index of $K_1$ is open.*

*Proof.* In fact, this is true for every finitely generated pro-$p$ group; this is due to Serre [Ser94, 4.2, Exercices 6)]. See also [DdSMS99, Theorem 1.17], which gives a complete proof. ∎

**Remark 2.4** More generally, every subgroup of finite index of a finitely generated profinite group is open ([NS03], [NS07a], [NS07b]). It is a very deep theorem.

**Lemma 2.5** *Let $G$ be a pro-$\ell$ group. Then every homomorphism $f\colon K_1 \longrightarrow G$ is trivial.*

*Proof.* Let $H$ be an open normal subgroup of $G$ and denote the composite $K_1 \xrightarrow{f} G \longrightarrow G/H$ by $f_H$. By Lemma 2.3, $\operatorname{Ker} f_H$ is an open normal subgroup of $K_1$. Thus $K_1/\operatorname{Ker} f_H$ is a finite $p$-group. On the other hand, $G/H$ is a finite $\ell$-group. Since we have an injection $K_1/\operatorname{Ker} f_H \hookrightarrow G/H$, we have $K_1/\operatorname{Ker} f_H = 1$, in other words, $f_H = 1$. Therefore the composite $K_1 \xrightarrow{f} G \xrightarrow{\cong} \varprojlim_H G/H$ is trivial. Hence we have $f = 1$, as desired. ∎

*Proof of Theorem 2.1.* Since $K_1$ is an open subgroup of $K$, we may replace $K$ by $K_1$. Take a $K_1$-stable $\mathbb{Z}_\ell$-lattice $\Lambda$ of $V$. Then, $\Lambda/\ell\Lambda$ is a finite abelian group. Therefore, by Lemma 2.3, there exists an open subgroup $U$ of $K_1$ which acts trivially on $\Lambda/\ell\Lambda$. In other words, the homomorphism $\pi\colon K_1 \longrightarrow \mathrm{GL}(\Lambda) \subset \mathrm{GL}(V)$ maps $U$ into the subgroup $1+\ell\operatorname{End}(\Lambda)$. Since $U$ is a closed subgroup of $1+p\mathrm{M}_n(\mathbb{Z}_p)$ and $1+\ell\operatorname{End}(\Lambda)$ is a pro-$\ell$ group, by Lemma 2.5, the homomorphism $\pi|_U\colon U \longrightarrow 1+\ell\operatorname{End}(\Lambda)$ is trivial. Namely, $\pi|_U$ is a trivial representation. ∎

**Lemma 2.6** *Let $F$ be a $p$-adic field and $\mathbf{G}$ a linear algebraic group over $F$. Then every compact subgroup $K$ of $\mathbf{G}(F)$ can be realized as a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ for some $n$.*

*Proof.* Take an embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_m$ defined over $F$. Since $\mathbf{G}(F)$ is a closed subgroup of $\mathrm{GL}_m(F)$, $K$ is also a closed subgroup of $\mathrm{GL}_m(F)$. Therefore we have a faithful continuous action of $K$ on $F^m$. By taking a $\mathbb{Q}_p$-basis of $F$, we have a faithful continuous action of $K$ on $\mathbb{Q}_p^n$ for some $n$. Since $K$ is compact, it is well-known that there is a $K$-stable $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^n$. Hence we have a continuous injection $K \hookrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$. Since $K$ is compact, it is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$. ∎

**Corollary 2.7** *Let $F$ and $\mathbf{G}$ be as in the previous proposition. Let $I$ be a filtered ordered set and $\{K_i\}_{i\in I}$ be a system of compact open subgroups of $\mathbf{G}(F)$ indexed by $I$.*

*Let $(\pi, V)$ be a (not necessarily finite-dimensional) $\mathbb{Q}_\ell$-representation of $\mathbf{G}(F)$ as an abstract group. Assume that there exists an inductive system $\{V_i\}_{i\in I}$ of finite-dimensional $\mathbb{Q}_\ell$-vector spaces satisfying the following:*

- *For every $i \in I$, $V_i$ is endowed with an action of $K_i$ as an abstract group.*
- *For every $i \in I$, $V_i$ has a $K_i$-stable $\mathbb{Z}_\ell$-lattice.*
- *There exists an isomorphism $\varinjlim_{i\in I} V_i \xrightarrow{\cong} V$ as $\mathbb{Q}_\ell$-vector spaces such that the composite $V_i \longrightarrow \varinjlim_{i\in I} V_i \xrightarrow{\cong} V$ is $K_i$-equivariant for every $i \in I$.*

*Then $(\pi, V)$ is a smooth representation of $\mathbf{G}(F)$.*

*Proof.* Let us take $x \in V$ and show that $\mathrm{Stab}_{\mathbf{G}(F)}(x)$, the stabilizer of $x$ in $\mathbf{G}(F)$, is open. There exists an element $i \in I$ such that $x$ lies in the image of $V_i \longrightarrow V$. Take $y \in V_i$ which is mapped to $x$. By Theorem 2.1 and Lemma 2.6, $V_i$ is a smooth representation of $K_i$. Therefore $\mathrm{Stab}_{K_i}(y)$ is open in $K_i$, hence is open in $\mathbf{G}(F)$. Since $V_i \longrightarrow V$ is $K_i$-equivariant, we have $\mathrm{Stab}_{K_i}(y) \subset \mathrm{Stab}_{K_i}(x) \subset \mathrm{Stab}_{\mathbf{G}(F)}(x)$. Thus $\mathrm{Stab}_{\mathbf{G}(F)}(x)$ is open in $\mathbf{G}(F)$, as desired. ∎

**Remark 2.8** Although we need the corollary above only for the case $F = \mathbb{Q}_p$, we proved it for a general $p$-adic field $F$ for the completeness.

# 3 Rapoport-Zink space for GSp(4)

## 3.1 The Rapoport-Zink space for GSp(4) and its rigid analytic coverings

In this subsection, we recall basic definitions concerning with Rapoport-Zink spaces. General definitions are given in [RZ96], but here we restrict them to our special case.

Let $\mathbb{X}$ be a 2-dimensional isoclinic $p$-divisible group over $\overline{\mathbb{F}}_p$ with slope 1/2, and $\lambda_0\colon \mathbb{X} \xrightarrow{\cong} \mathbb{X}^\vee$ a (principal) polarization of $\mathbb{X}$, namely, an isomorphism satisfying $\lambda_0^\vee = -\lambda_0$. Consider the contravariant functor $\breve{\mathscr{M}}\colon \mathbf{Nilp} \longrightarrow \mathbf{Set}$ that associates $S$ with the set of isomorphism classes of pairs $(X, \rho)$ consisting of

- a 2-dimensional $p$-divisible group $X$ over $S$,
- and a quasi-isogeny (*cf.* [RZ96, Definition 2.8]) $\rho\colon \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{S} \longrightarrow X \otimes_S \overline{S}$,

such that there exists an isomorphism $\lambda\colon X \longrightarrow X^\vee$ which makes the following diagram commutative up to multiplication by $\mathbb{Q}_p^\times$:

$$\begin{tikzcd} \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} \overline{S} \arrow[r, "\rho"] \arrow[d, "\lambda_0 \otimes \mathrm{id}"] & X \otimes_S \overline{S} \arrow[d, "\lambda \otimes \mathrm{id}"] \\ \mathbb{X}^\vee \otimes_{\overline{\mathbb{F}}_p} \overline{S} & X^\vee \otimes_S \overline{S}. \arrow[l, "\rho^\vee"'] \end{tikzcd}$$

Note that such $\lambda$ is uniquely determined by $(X, \rho)$ up to multiplication by $\mathbb{Z}_p^\times$ and gives a polarization of $X$. It is proved by Rapoport-Zink that $\breve{\mathscr{M}}$ is represented by a special formal scheme (*cf.* [Ber96]) over $\operatorname{Spf} \mathbb{Z}_{p^\infty}$. Moreover, $\breve{\mathscr{M}}$ is separated over $\operatorname{Spf} \mathbb{Z}_{p^\infty}$ [Far04, Lemme 2.3.23]. However, $\breve{\mathscr{M}}$ is neither quasi-compact nor $p$-adic. We put $\bar{\mathscr{M}} = \breve{\mathscr{M}}_{\mathrm{red}}$, which is a scheme locally of finite type and separated over $\overline{\mathbb{F}}_p$. It is known that $\bar{\mathscr{M}}$ is 1-dimensional (for example, see [Vie08]) and every irreducible component of $\bar{\mathscr{M}}$ is projective over $\overline{\mathbb{F}}_p$ [RZ96, Proposition 2.32]. In particular, $\bar{\mathscr{M}}$ has a locally finite quasi-compact open covering.

Let $D(\mathbb{X})_{\mathbb{Q}} = (N, \Phi)$ be the rational Dieudonné module of $\mathbb{X}$, which is a 4-dimensional isocrystal over $\mathbb{Q}_{p^\infty}$. The fixed polarization $\lambda_0$ gives the alternating pairing $\langle\ ,\ \rangle_{\lambda_0} \colon N \times N \longrightarrow \mathbb{Q}_{p^\infty}(1)$. We define the algebraic group $J$ over $\mathbb{Q}_p$ as follows: for a $\mathbb{Q}_p$-algebra $R$, the group $J(R)$ consists of elements $g \in \mathrm{GL}(R \otimes_{\mathbb{Q}_p} N)$ such that

- $g$ commutes with $\Phi$,
- and $g$ preserves the pairing $\langle\ ,\ \rangle_{\lambda_0}$ up to scalar multiplication, i.e., there exists $c(g) \in R^\times$ such that $\langle gx, gy\rangle_{\lambda_0} = c(g)\langle x, y\rangle_{\lambda_0}$ for every $x, y \in R \otimes_{\mathbb{Q}_p} N$.

It is an inner form of $\mathrm{GSp}(4)$, since $D(\mathbb{X})_{\mathbb{Q}}$ is the isocrystal associated with a basic Frobenius conjugacy class of $\mathrm{GSp}(4)$.

In the sequel, we also denote $J(\mathbb{Q}_p)$ by $J$. Every element $g \in J$ naturally induces a quasi-isogeny $g \colon \mathbb{X} \longrightarrow \mathbb{X}$ and the following diagram is commutative up to $\mathbb{Q}_p^\times$-multiplication:

$$\begin{tikzcd} \mathbb{X} \arrow[r, "g"] \arrow[d, "\lambda_0"] & \mathbb{X} \arrow[d, "\lambda_0"] \\ \mathbb{X}^\vee & \mathbb{X}^\vee. \arrow[l, "g^\vee"'] \end{tikzcd}$$

Therefore, we can define the left action of $J$ on $\breve{\mathscr{M}}$ by $g \colon \breve{\mathscr{M}}(S) \longrightarrow \breve{\mathscr{M}}(S)$; $(X, \rho) \longmapsto (X, \rho \circ g^{-1})$.

We denote the Raynaud generic fiber of $\breve{\mathscr{M}}$ by $\breve{\mathscr{M}}^{\mathrm{rig}}$. It is defined as $t(\breve{\mathscr{M}}) \backslash V(p)$, where $t(\breve{\mathscr{M}})$ is the adic space associated with $\breve{\mathscr{M}}$ (*cf.* [Hub94, Proposition 4.1]). As $\breve{\mathscr{M}}$ is separated and special over $\mathbb{Z}_{p^\infty}$, $\breve{\mathscr{M}}^{\mathrm{rig}}$ is separated and locally of finite type over $\mathrm{Spa}(\mathbb{Q}_{p^\infty}, \mathbb{Z}_{p^\infty})$. Since $\bar{\mathscr{M}}$ has a locally finite quasi-compact open covering, $\breve{\mathscr{M}}^{\mathrm{rig}}$ is taut by [Mie10b, Lemma 4.14]. Moreover, by using the period morphism [RZ96, Chapter 5], we can see that $\breve{\mathscr{M}}^{\mathrm{rig}}$ is 3-dimensional and smooth over $\mathrm{Spa}(\mathbb{Q}_{p^\infty}, \mathbb{Z}_{p^\infty})$ (*cf.* [RZ96, Proposition 5.17]).

Next we will consider level structures. Let $\widetilde{X}$ be the universal $p$-divisible group over $\breve{\mathscr{M}}$ and $\widetilde{X}^{\mathrm{rig}}$ be the associated $p$-divisible group over $\breve{\mathscr{M}}^{\mathrm{rig}}$. Note that $\widetilde{X}^{\mathrm{rig}}$ is an étale $p$-divisible group. Let us fix a polarization $\widetilde{\lambda}\colon \widetilde{X} \longrightarrow \widetilde{X}^{\vee}$ which is compatible with $\lambda_0$, i.e., satisfies the condition in the definition of $\breve{\mathscr{M}}$. Let $\mathsf{S}$ be a connected rigid space over $\mathbb{Q}_{p^\infty}$ (i.e., a connected adic space locally of finite type over $\mathrm{Spa}(\mathbb{Q}_{p^\infty}, \mathbb{Z}_{p^\infty})$), $\mathsf{S} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}$ a morphism over $\mathbb{Q}_{p^\infty}$ and $\widetilde{X}^{\mathrm{rig}}_{\mathsf{S}}$ the pull-back of $\widetilde{X}^{\mathrm{rig}}$. Fix a geometric point $\overline{x}$ of $\mathsf{S}$ and an isomorphism $T_p(\mu_{p^\infty,\mathsf{S}})_{\overline{x}} = \mathbb{Z}_p(1) \cong \mathbb{Z}_p$. Then $\widetilde{\lambda}$ induces an alternating bilinear form $\psi_{\widetilde{\lambda}}$ on the $\pi_1(\mathsf{S},\overline{x})$-module $(T_p\widetilde{X}^{\mathrm{rig}})_{\overline{x}}$;

$$\psi_{\widetilde{\lambda}}\colon (T_p\widetilde{X}^{\mathrm{rig}})_{\overline{x}} \times (T_p\widetilde{X}^{\mathrm{rig}})_{\overline{x}} \longrightarrow T_p(\mu_{p^\infty,\mathsf{S}})_{\overline{x}} \cong \mathbb{Z}_p.$$

Fix a free $\mathbb{Z}_p$-module $L$ of rank 4 and a perfect alternating bilinear form $\psi_0\colon L\times L \longrightarrow \mathbb{Z}_p$. Put $K_0 = \mathrm{GSp}(L,\psi_0)$, $V = L\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $G = \mathrm{GSp}(V,\psi_0)$. Let $T(\mathsf{S},\overline{x})$ be the set consisting of isomorphisms $\eta\colon L \xrightarrow{\cong} (T_p\widetilde{X}^{\mathrm{rig}})_{\overline{x}}$ which map $\psi_0$ to $\mathbb{Z}_p^\times$-multiples of $\psi_{\widetilde{\lambda}}$. It is independent of the choice of $\widetilde{\lambda}$ and $T_p(\mu_{p^\infty,\mathsf{S}})_{\overline{x}} \cong \mathbb{Z}_p$, since they are unique up to $\mathbb{Z}_p^\times$-multiplication. Obviously, the groups $K_0$ and $\pi_1(\mathsf{S},\overline{x})$ naturally act on $T(\mathsf{S},\overline{x})$.

For an open subgroup $K$ of $K_0$, a $K$-level structure of $\widetilde{X}^{\mathrm{rig}}_{\mathsf{S}}$ means an element of $(T(\mathsf{S},\overline{x})/K)^{\pi_1(\mathsf{S},\overline{x})}$. Note that, if we change a geometric point $\overline{x}$ to $\overline{x}'$, the sets $(T(\mathsf{S},\overline{x})/K)^{\pi_1(\mathsf{S},\overline{x})}$ and $(T(\mathsf{S},\overline{x}')/K)^{\pi_1(\mathsf{S},\overline{x}')}$ are naturally isomorphic. Thus the notion of $K$-level structures is independent of the choice of $\overline{x}$. The functor that associates $\mathsf{S}$ with the set of $K$-level structures of $\widetilde{X}^{\mathrm{rig}}_{\mathsf{S}}$ is represented by a finite Galois étale covering $\breve{\mathscr{M}}^{\mathrm{rig}}_K \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}$, whose Galois group is $K_0/K$. Since $T(\mathsf{S},\overline{x})$ is a $K_0$-torsor, $\breve{\mathscr{M}}^{\mathrm{rig}}_{K_0}$ coincides with $\breve{\mathscr{M}}^{\mathrm{rig}}$. If $K'$ is an open subgroup of $K$, we have a natural morphism $p_{KK'}\colon \breve{\mathscr{M}}^{\mathrm{rig}}_{K'} \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_K$. Therefore, we get the projective system of rigid spaces $\{\breve{\mathscr{M}}^{\mathrm{rig}}_K\}_K$ indexed by the filtered ordered set of open subgroups of $K_0$, which is called the *Rapoport-Zink tower*. Obviously, the group $J$ acts on the projective system $\{\breve{\mathscr{M}}^{\mathrm{rig}}_K\}_K$.

Let $g$ be an element of $G$ and $K$ an open subgroup of $K_0$ which is enough small so that $g^{-1}Kg \subset K_0$. Then we have a natural morphism $\breve{\mathscr{M}}^{\mathrm{rig}}_K \longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}_{g^{-1}Kg}$ over $\mathbb{Q}_{p^\infty}$. If $g \in K_0$, then it is given by $\eta \longmapsto \eta\circ g$; for other $g$, it is more complicated [RZ96, 5.34]. In any case, we get a right action of $G$ on the pro-object "$\varprojlim$" $\breve{\mathscr{M}}^{\mathrm{rig}}_K$.

**Definition 3.1** We put $H^i_{\mathrm{RZ}} = \varinjlim_K H^i_c(\breve{\mathscr{M}}^{\mathrm{rig}}_K \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell)$.

Here $H^i_c(\breve{\mathscr{M}}^{\mathrm{rig}}_K \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell)$ is the compactly supported $\ell$-adic cohomology of $\breve{\mathscr{M}}^{\mathrm{rig}}_K \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}$ defined in [Hub98]; note that $\breve{\mathscr{M}}^{\mathrm{rig}}_K$ is separated and taut. By the constructions above, $G\times J$ acts on $H^i_{\mathrm{RZ}}$ on the left (the action of $j\in J$ is given by $(j^{-1})^*$). Obviously the action of $G$ on $H^i_{\mathrm{RZ}}$ is smooth. On the other hand, it is known that the action of $J$ on $H^i_{\mathrm{RZ}}$ is also smooth. This is due to Berkovich (see [Far04, Corollaire 4.4.7]); see also Remark 5.12, where we give another proof of the smoothness. Hence we get the smooth representation $H^i_{\mathrm{RZ}}$ of $G\times J$.

Our main theorem is the following:

**Theorem 3.2 (Non-cuspidality)** *The smooth representation $H^i_{\mathrm{RZ},\overline{\mathbb{Q}}_\ell}$ of $G$ has no quasi-cuspidal subquotient unless $i=2,3,4$.*

For the definition of quasi-cuspidal representations, see [Ber84, 1.20].

Theorem 3.2 is proved in Section 5.

## 3.2 An integral model $\breve{\mathscr{M}}_m$ of $\breve{\mathscr{M}}^{\mathrm{rig}}_{K_m}$

For an integer $m\geq 1$, let $K_m$ be the kernel of $\mathrm{GSp}(L,\psi_0)\longrightarrow \mathrm{GSp}(L/p^mL,\psi_0)$. It is an open subgroup of $K_0$. We can describe the definition of $K_m$-level structures more concretely. As in the previous subsection, we fix a polarization $\widetilde{\lambda}$ of $\widetilde{X}^{\mathrm{rig}}$ which is compatible with $\lambda_0$. It induces the alternating bilinear morphism between finite étale group schemes $\psi_{\widetilde{\lambda}}\colon \widetilde{X}^{\mathrm{rig}}[p^m]\times \widetilde{X}^{\mathrm{rig}}[p^m]\longrightarrow \mu_{p^m}$. Let $\mathsf{S}\longrightarrow \breve{\mathscr{M}}^{\mathrm{rig}}$ be as in the previous subsection. Then a $K_m$-level structure of $\widetilde{X}^{\mathrm{rig}}_{\mathsf{S}}$ naturally corresponds bijectively to an isomorphism $\eta\colon L/p^mL\xrightarrow{\cong}\widetilde{X}^{\mathrm{rig}}_{\mathsf{S}}[p^m]$ between finite étale group schemes such that there exists an isomorphism $\mathbb{Z}/p^m\mathbb{Z}\xrightarrow{\cong}\mu_{p^m,\mathsf{S}}$ which makes the following diagram commutative:

$$\begin{CD} L/p^mL\times L/p^mL @>{\psi_0}>> \mathbb{Z}/p^m\mathbb{Z}\\ @V{\eta\times\eta}V{\cong}V @V{\cong}VV\\ \widetilde{X}^{\mathrm{rig}}_{\mathsf{S}}[p^m]\times \widetilde{X}^{\mathrm{rig}}_{\mathsf{S}}[p^m] @>{\psi_{\widetilde{\lambda}}}>> \mu_{p^m,\mathsf{S}}. \end{CD}$$

For simplicity, we write $\breve{\mathscr{M}}^{\mathrm{rig}}_m$ for $\breve{\mathscr{M}}^{\mathrm{rig}}_{K_m}$ and $p_{mn}$ for $p_{K_mK_n}$. In this subsection, we construct a formal model $\breve{\mathscr{M}}_m$ of $\breve{\mathscr{M}}^{\mathrm{rig}}_m$ by following [Man05, §6]. Let $\mathcal{S}$ be a formal scheme of finite type over $\breve{\mathscr{M}}^{\mathrm{rig}}$ and denote by $\widetilde{X}_{\mathcal{S}}$ the pull-back of $\widetilde{X}$ to $\mathcal{S}$. A Drinfeld $m$-level structure of $\widetilde{X}_{\mathcal{S}}$ is a morphism $\eta\colon L/p^mL\longrightarrow \widetilde{X}_{\mathcal{S}}[p^m]$ satisfying the following conditions:

- the image of $\eta$ gives a full set of sections of $\widetilde{X}_{\mathcal{S}}[p^m]$,
- and there exists a morphism $\mathbb{Z}/p^m\mathbb{Z}\longrightarrow\mu_{p^m,\mathcal{S}}$ which makes the following diagram commutative:

$$\begin{CD} L/p^mL\times L/p^mL @>{\psi_0}>> \mathbb{Z}/p^m\mathbb{Z}\\ @V{\eta\times\eta}VV @VVV\\ \widetilde{X}_{\mathcal{S}}[p^m]\times \widetilde{X}_{\mathcal{S}}[p^m] @>{\psi_{\widetilde{\lambda}}}>> \mu_{p^m,\mathcal{S}}. \end{CD}$$

It is known that the functor that associates $\mathcal{S}$ with the set of Drinfeld $m$-level structures of $\widetilde{X}_{\mathcal{S}}$ is represented by the formal scheme $\breve{\mathscr{M}}_m$ which is finite over $\breve{\mathscr{M}}$ (*cf.* [Man05, Proposition 15]). Note that, unlike the case of Lubin-Tate tower, $\breve{\mathscr{M}}_m$ is not necessarily flat over $\breve{\mathscr{M}}$. It is easy to show that $\breve{\mathscr{M}}_m$ gives a formal model of

$\breve{\mathscr{M}}_m^{\mathrm{rig}}$, namely, the Raynaud generic fiber of $\breve{\mathscr{M}}_m$ coincides with $\breve{\mathscr{M}}_m^{\mathrm{rig}}$. We denote $(\breve{\mathscr{M}}_m)_{\mathrm{red}}$ by $\bar{\mathscr{M}}_m$, which is a 1-dimensional scheme over $\overline{\mathbb{F}}_p$.

There is a natural left action of $J$ on $\breve{\mathscr{M}}_m$ which is compatible with that on $\breve{\mathscr{M}}_m^{\mathrm{rig}}$. On the other hand, the natural action $K_0$ on $L/p^mL$ induces a right action of $K_0$ on $\breve{\mathscr{M}}_m$, which is compatible with that on $\breve{\mathscr{M}}_{K_m}^{\mathrm{rig}}$.

We can also describe $\breve{\mathscr{M}}_m$ as a functor from **Nilp** to **Set**; for an object $S$ of **Nilp**, the set $\breve{\mathscr{M}}_m(S)$ consists of isomorphism classes of triples $(X,\rho,\eta)$, where $(X,\rho)\in\breve{\mathscr{M}}_m(S)$ and $\eta\colon L/p^mL\longrightarrow X[p^m]$ is a Drinfeld $m$-level structure of $X$. By this description, the action of $j\in J$ on $\breve{\mathscr{M}}_m$ is given by $(X,\rho,\eta)\longmapsto(X,\rho\circ j^{-1},\eta)$. On the other hand, the action of $g\in K_0$ on $\breve{\mathscr{M}}_m$ is given by $(X,\rho,\eta)\longmapsto(X,\rho,\eta\circ g)$.

By [Man04, Lemma 7.2], $\{\breve{\mathscr{M}}_m\}_{m\geq 0}$ forms a projective system of formal schemes equipped with the commuting action of $J$ and $K_0$.

## 3.3 Compactly supported cohomology of $\bar{\mathscr{M}}_m$

For $m\geq 0$, we denote the set of quasi-compact open subsets of $\bar{\mathscr{M}}_m$ by $\mathscr{Q}_m$. It has a natural filtered order by inclusion.

**Definition 3.3** For an object $\mathcal{F}$ of $D^b(\bar{\mathscr{M}}_m,\mathbb{Z}_\ell)$ or $D^b(\bar{\mathscr{M}}_m,\mathbb{Q}_\ell)$, we put

$$H^i_c(\bar{\mathscr{M}}_m,\mathcal{F})=\varinjlim_{U\in\mathscr{Q}_m}H^i_c(U,\mathcal{F}|_U).$$

Assume that $\mathcal{F}$ has a $J$-equivariant structure, namely, for every $g\in J$ an isomorphism $\varphi_g\colon g^*\mathcal{F}\xrightarrow{\cong}\mathcal{F}$ is given such that $\varphi_{gg'}=\varphi_{g'}\circ g'^*\varphi_g$ for every $g,g'\in J$. Then $J$ naturally acts on $H^i_c(\bar{\mathscr{M}}_m,\mathcal{F})$ on the right. Therefore we get a left action of $J$ on $H^i_c(\bar{\mathscr{M}}_m,\mathcal{F})$ by taking the inverse $J\longrightarrow J$; $g\longmapsto g^{-1}$.

**Theorem 3.4** *Let $\mathcal{F}^\circ$ be an object of $D^b_c(\bar{\mathscr{M}}_m,\mathbb{Z}_\ell)$ and $\mathcal{F}$ the object of $D^b_c(\bar{\mathscr{M}}_m,\mathbb{Q}_\ell)$ associated with $\mathcal{F}^\circ$. Assume that we are given a $J$-equivariant structure of $\mathcal{F}^\circ$ (thus $\mathcal{F}$ also has a $J$-equivariant structure). Then $H^i_c(\bar{\mathscr{M}}_m,\mathcal{F})$ is a finitely generated smooth $J$-representation.*

*Proof.* Let $U$ be an element of $\mathscr{Q}_m$. By [Far04, Proposition 2.3.11], there exists a compact open subgroup $K_U$ of $J$ which stabilizes $U$. Then $H^i_c(U,\mathcal{F}|_U)$ is a finite-dimensional $\mathbb{Q}_\ell$-vector space endowed with the action of $K_U$ and has the $K_U$-stable $\mathbb{Z}_\ell$-lattice $\mathrm{Im}(H^i_c(U,\mathcal{F}^\circ|_U)\longrightarrow H^i_c(U,\mathcal{F}|_U))$. Therefore $H^i_c(\bar{\mathscr{M}}_m,\mathcal{F})$ is a smooth $J$-representation by Corollary 2.7.

To prove that $H^i_c(\bar{\mathscr{M}}_m,\mathcal{F})$ is finitely generated, we may assume $m=0$, for $H^i_c(\bar{\mathscr{M}}_m,\mathcal{F})=H^i_c(\bar{\mathscr{M}}_0,p_{0m*}\mathcal{F})$. In this case, we can use the similar method as in [Far04, Proposition 4.4.13]. Let us explain the argument briefly. By [Far04, Théorème 2.4.13], there exists $W\in\mathscr{Q}_0$ such that $\bigcup_{g\in J}gW=\bar{\mathscr{M}}_0$. We put $K=\{g\in J\mid gW=W\}$ and $\Omega=\{g\in J\mid gW\cap W\neq\varnothing\}$. As in the proof of [Far04, Proposition 4.4.13], $K$ is a compact open subgroup of $J$ and $\Omega$ is a compact subset of $J$. For $\alpha = ([g_1], \ldots, [g_n]) \in (J/K)^n$, we put $W_\alpha = g_1 W \cap \cdots \cap g_n W$ and $K_\alpha = \bigcap_{j=1}^n g_j K g_j^{-1}$. For an open covering $\{gW\}_{g \in J/K}$, we can associate the Čech spectral sequence

$$E_1^{r,s} = \bigoplus_{\alpha \in (J/K)^{-r+1}} H_c^s(W_\alpha, \mathcal{F}|_{W_\alpha}) \Longrightarrow H_c^{r+s}(\bar{\mathscr{M}}_0, \mathcal{F}).$$

Consider the diagonal action of $J$ on $(J/K)^{-r+1}$. The coset

$$J \backslash \{\alpha \in (J/K)^{-r+1} \mid W_\alpha \neq \varnothing\}$$

is finite; indeed, if $W_\alpha \neq \varnothing$ for $\alpha = ([g_1], \ldots, [g_{-r+1}]) \in (J/K)^{-r+1}$, then $g_1^{-1}\alpha \in \{1\} \times \Omega/K \times \cdots \times \Omega/K$, which is a finite set.

Take a system of representatives $\alpha_1, \ldots, \alpha_n$ of the coset above. Then there is a natural isomorphism $\bigoplus_{\alpha \in J\alpha_j} H_c^s(W_\alpha, \mathcal{F}|_{W_\alpha}) \cong \text{c-Ind}_{K_{\alpha_j}}^J H_c^s(W_{\alpha_j}, \mathcal{F}|_{W_{\alpha_j}})$. Hence $E_1^{r,s} \cong \bigoplus_{j=1}^n \text{c-Ind}_{K_{\alpha_j}}^J H_c^s(W_{\alpha_j}, \mathcal{F}|_{W_{\alpha_j}})$ is a finitely generated $J$-module, since the cohomology $H_c^s(W_{\alpha_j}, \mathcal{F}|_{W_{\alpha_j}})$ is finite-dimensional for each $j$. By this and the fact that a finitely generated smooth $J$-module is noetherian [Ber84, Remarque 3.12], we conclude that $H_c^i(\bar{\mathscr{M}}_0, \mathcal{F})$ is finitely generated. ∎

**Lemma 3.5** *Let $\mathcal{F}$ be an object of $D_c^b(\bar{\mathscr{M}}_m, \mathbb{Q}_\ell)$ with a $K_0/K_m$-equivariant structure. Let $n$ be an integer with $0 \leq n \leq m$ and put $\mathcal{G} = (p_{nm*}\mathcal{F})^{K_n/K_m}$. Then we have $H_c^i(\bar{\mathscr{M}}_m, \mathcal{F})^{K_n/K_m} = H_c^i(\bar{\mathscr{M}}_n, \mathcal{G})$.*

*Proof.* Since the cardinality of $K_n/K_m$ is prime to $\ell$, $(-)^{K_n/K_m}$ commutes with $H_c^i$. Therefore, we have

$$\begin{aligned}
H_c^i(\bar{\mathscr{M}}_m, \mathcal{F})^{K_n/K_m} &= \varinjlim_{U \in \mathscr{Q}_m} H_c^i(U, \mathcal{F}|_U)^{K_n/K_m} = \varinjlim_{V \in \mathscr{Q}_n} H_c^i\bigl(p_{nm}^{-1}(V), \mathcal{F}|_{p_{nm}^{-1}(V)}\bigr)^{K_n/K_m} \\
&= \varinjlim_{V \in \mathscr{Q}_n} H_c^i\bigl(V, p_{nm*}(\mathcal{F}|_{p_{nm}^{-1}(V)})\bigr)^{K_n/K_m} = \varinjlim_{V \in \mathscr{Q}_n} H_c^i\Bigl(V, \bigl(p_{nm*}(\mathcal{F}|_{p_{nm}^{-1}(V)})\bigr)^{K_n/K_m}\Bigr) \\
&= \varinjlim_{V \in \mathscr{Q}_n} H_c^i(V, \mathcal{G}|_V) = H_c^i(\bar{\mathscr{M}}_n, \mathcal{G}).
\end{aligned}$$

∎

**Definition 3.6** A *system of coefficients* over the tower $\{\bar{\mathscr{M}}_m\}_{m \geq 0}$ is the data $\mathcal{F} = \{\mathcal{F}_m\}_{m \geq 0}$ where $\mathcal{F}_m$ is an object of $D_c^b(\bar{\mathscr{M}}_m, \mathbb{Q}_\ell)$ with a $K_0/K_m$-equivariant structure such that $(p_{nm*}\mathcal{F})^{K_n/K_m} = \mathcal{F}_n$ for every integers $m$, $n$ with $0 \leq n \leq m$. Then, by Lemma 3.5, we have $H_c^i(\bar{\mathscr{M}}_m, \mathcal{F}_m)^{K_n/K_m} = H_c^i(\bar{\mathscr{M}}_n, \mathcal{F}_n)$. We put $H_c^i(\bar{\mathscr{M}}_\infty, \mathcal{F}) = \varinjlim_m H_c^i(\bar{\mathscr{M}}_m, \mathcal{F}_m)$.

If each $\mathcal{F}_m$ is endowed with a $J$-equivariant structure which commutes with the given $K_0/K_m$-equivariant structure, and for every $0 \leq n \leq m$ the $J$-equivariant structures on $\mathcal{F}_m$ and $\mathcal{F}_n$ are compatible under the identification $(p_{nm*}\mathcal{F}_m)^{K_n/K_m} = \mathcal{F}_n$, then we say that we have a $J$-equivariant structure on $\mathcal{F}$. Such a structure naturally induces the action of $J$ on $H_c^i(\bar{\mathscr{M}}_\infty, \mathcal{F})$.

By replacing "$D_c^b(\bar{\mathscr{M}}_m, \mathbb{Q}_\ell)$" with "$D_c^b(\bar{\mathscr{M}}_m, \mathbb{Z}_\ell)$", we may also define a *system of integral coefficients* $\mathcal{F}^\circ$ over $\{\bar{\mathscr{M}}_m\}_{m \geq 0}$, the cohomology $H_c^i(\bar{\mathscr{M}}_\infty, \mathcal{F}^\circ)$ and a $J$-equivariant structure on $\mathcal{F}^\circ$.

**Corollary 3.7** *Let $\mathcal{F}^\circ$ be a system of integral coefficients over $\{\bar{\mathscr{M}}_m\}_{m\geq 0}$ with a $J$-equivariant structure and $\mathcal{F}$ the system of coefficients associated with $\mathcal{F}^\circ$. Then $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F})$ is a smooth $K_0 \times J$-representation and $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F})^{K_m}$ is a finitely generated smooth $J$-representation for every integer $m \geq 0$.*

*Proof.* The smoothness is clear from Theorem 3.4 and the definition of $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F})$. Since $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F})^{K_m} = H^i_c(\bar{\mathscr{M}}_m, \mathcal{F}_m)$, the second assertion also follows from Theorem 3.4. ∎

# 4 Shimura variety and $p$-adic uniformization

In this section, we introduce certain Shimura varieties (Siegel threefolds) related to our Rapoport-Zink tower. Let us fix a 4-dimensional $\mathbb{Q}$-vector space $V'$ and an alternating perfect pairing $\psi' \colon V' \times V' \longrightarrow \mathbb{Q}$. For an integer $m \geq 0$ and a compact open subgroup $K^p \subset \mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}}) = \mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}}, \psi'_{\mathbb{A}^{\infty,p}})$, consider the functor $\mathrm{Sh}_{m,K^p}$ from the category of locally noetherian $\mathbb{Z}_{p^\infty}$-schemes to the category of sets that associates $S$ with the set of isomorphism classes of quadruples $(A, \lambda, \eta^p, \eta_p)$ where

- $A$ is a projective abelian surface over $S$ up to prime-to-$p$ isogeny,
- $\lambda \colon A \longrightarrow A^\vee$ is a prime-to-$p$ polarization,
- $\eta^p$ is a $K^p$-level structure of $A$,
- and $\eta_p \colon L/p^m L \longrightarrow A[p^m]$ is a Drinfeld $m$-level structure

(for the detail, see [Kot92, §5]). Two quadruples $(A, \lambda, \eta^p, \eta_p)$ and $(A', \lambda', \eta'^p, \eta'_p)$ are said to be isomorphic if there exists a prime-to-$p$ isogeny from $A$ to $A'$ which carries $\lambda$ to a $\mathbb{Z}_{(p)}^\times$-multiple of $\lambda'$, $\eta^p$ to $\eta'^p$ and $\eta_p$ to $\eta'_p$. We put $\mathrm{Sh}_{K^p} = \mathrm{Sh}_{0,K^p}$. It is known that if $K^p$ is sufficiently small, $\mathrm{Sh}_{m,K^p}$ is represented by a quasi-projective scheme over $\mathbb{Z}_{p^\infty}$ with smooth generic fiber. In the sequel, we always assume that $K^p$ is enough small so that $\mathrm{Sh}_{m,K^p}$ is representable. We denote the special fiber of $\mathrm{Sh}_{m,K^p}$ (resp. $\mathrm{Sh}_{K^p}$) by $\overline{\mathrm{Sh}}_{m,K^p}$ (resp. $\overline{\mathrm{Sh}}_{K^p}$).

For a compact open subgroup $K'^p$ contained in $K^p$ and an integer $m' \geq m$, we have the natural morphism $\mathrm{Sh}_{m',K'^p} \longrightarrow \mathrm{Sh}_{m,K^p}$. This is a finite morphism and is moreover étale if $m' = m$.

Next we recall the $p$-adic uniformization theorem, which gives a relation between $\breve{\mathscr{M}}$ and $\mathrm{Sh}_{K^p}$. Let us fix a polarized abelian surface $(A_0, \lambda_{A_0})$ over $\overline{\mathbb{F}}_p$ such that $A_0[p^\infty]$ is an isoclinic $p$-divisible group with slope $1/2$. Note that such $(A_0, \lambda_{A_0})$ exists; for example, we can take $(A_0, \lambda_{A_0}) = (E^2, \lambda_E^2)$, where $E$ is a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ and $\lambda_E$ is a polarization of $E$. By definition, the rational Dieudonné module $D(A_0[p^\infty])_{\mathbb{Q}}$ is isomorphic to $D(\mathbb{X})_{\mathbb{Q}}$. Thus, by the subsequent lemma, there is an isomorphism of isocrystals $D(A_0[p^\infty])_{\mathbb{Q}} \cong D(\mathbb{X})_{\mathbb{Q}}$ which preserves the natural polarizations.

**Lemma 4.1** *We use the notation in [RR96, §1]. Let $d \geq 1$ be an integer.*

i) *Let $b$ be an element of $B(\mathrm{GSp}_{2d})$ and $b'$ the image of $b$ under the natural map $B(\mathrm{GSp}_{2d}) \longrightarrow B(\mathrm{GL}_{2d})$. Then $b$ is basic if and only if $b'$ is basic.*

ii) *The map* $B(\mathrm{GSp}_{2d})_{\mathrm{basic}} \longrightarrow B(\mathrm{GL}_{2d})_{\mathrm{basic}}$ *induced from i) is an injection.*

*Proof.* Note that the center of $\mathrm{GSp}_{2d}$ coincides with that of $\mathrm{GL}_{2d}$. Thus i) is clear, since $b$ (resp. $b'$) is basic if and only if the slope morphism $\nu_b\colon \mathbb{D} \longrightarrow \mathrm{GSp}_{2d}$ (resp. $\nu_{b'}\colon \mathbb{D} \xrightarrow{\nu_b} \mathrm{GSp}_{2d} \hookrightarrow \mathrm{GL}_{2d}$) factors through the center of $\mathrm{GSp}_{2d}$ (resp. $\mathrm{GL}_{2d}$).

We prove ii). By [RR96, Theorem 1.15], it suffices to show that the natural map $\pi_1(\mathrm{GSp}_{2d}) \longrightarrow \pi_1(\mathrm{GL}_{2d})$ is injective. Take a maximal torus $T$ (resp. $T'$) of $\mathrm{GSp}_{2d}$ (resp. $\mathrm{GL}_{2d}$) such that $T \subset T'$. Then, since $\mathrm{Sp}_{2d}$ (resp. $\mathrm{SL}_{2d}$) is simply connected, $\pi_1(\mathrm{GSp}_{2d})$ (resp. $\pi_1(\mathrm{GL}_{2d})$) can be identified with the quotient of $X_*(T)$ (resp. $X_*(T')$) induced by $c\colon T \twoheadrightarrow \mathbb{G}_m$ (resp. $\det\colon T' \twoheadrightarrow \mathbb{G}_m$), where $c$ denotes the similitude character of $\mathrm{GSp}_{2d}$. In particular, both $\pi_1(\mathrm{GSp}_{2d})$ and $\pi_1(\mathrm{GL}_{2d})$ are isomorphic to $\mathbb{Z}$.

The commutative diagram

$$\begin{tikzcd} \mathrm{GSp}_{2d} \arrow[r, two heads, "c"] \arrow[d] & \mathbb{G}_m \arrow[d, "z\mapsto z^d"] \\ \mathrm{GL}_{2d} \arrow[r, two heads, "\det"] & \mathbb{G}_m \end{tikzcd}$$

induces the commutative diagram

$$\begin{tikzcd} X_*(T) \arrow[r, two heads] \arrow[d] & X_*(\mathbb{G}_m) \arrow[r, equal] \arrow[d, "\times d"] & \pi_1(\mathrm{GSp}_{2d}) \\ X_*(T') \arrow[r, two heads] & X_*(\mathbb{G}_m) \arrow[r, equal] & \pi_1(\mathrm{GL}_{2d}). \end{tikzcd}$$

In particular, the natural map $\pi_1(\mathrm{GSp}_{2d}) \longrightarrow \pi_1(\mathrm{GL}_{2d})$ is injective. ∎

Therefore, there is a quasi-isogeny $\mathbb{X} \longrightarrow A[p^\infty]$ preserving polarizations. If we replace $(\mathbb{X}, \lambda_0)$ by the polarized $p$-divisible group $(A_0[p^\infty], \lambda_{A_0})$ associated with $(A_0, \lambda_{A_0})$, the $G$-representation $H^i_{\mathrm{RZ}}$ remains unchanged. Thus, in order to prove Theorem 3.2, we may assume that $(\mathbb{X}, \lambda_0) = (A_0[p^\infty], \lambda_{A_0})$. In the remaining part of this article, we always assume it. Moreover, we fix an isomorphism $H_1(A_0, \mathbb{A}^{\infty,p}) \cong V'_{\mathbb{A}^{\infty,p}}$ preserving alternating pairings.

Denote the isogeny class of $(A_0, \lambda_{A_0})$ by $\phi$ and put $I^\phi = \mathrm{Aut}(\phi)$. We have natural group homomorphisms $I^\phi \hookrightarrow J$ and $I^\phi \hookrightarrow \mathrm{Aut}(H_1(A_0, \mathbb{A}^{\infty,p})) = \mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})$. These are injective.

Let $Y_{K^p}$ be the reduced closed subscheme of $\overline{\mathrm{Sh}}_{K^p}$ such that $Y_{K^p}(\overline{\mathbb{F}}_p)$ consists of triples $(A, \lambda, \eta^p)$ where the $p$-divisible group associated with $(A, \lambda)$ is isogenous to $(\mathbb{X}, \lambda_0)$. It is the basic (or supersingular) stratum in the Newton stratification of $\overline{\mathrm{Sh}}_{K^p}$. Note that $(A, \lambda, \eta^p) \in \overline{\mathrm{Sh}}_{K^p}(\overline{\mathbb{F}}_p)$ belongs to $Y_{K^p}(\overline{\mathbb{F}}_p)$ if and only if $(A, \lambda) \in \phi$ ([Far04, Proposition 3.1.8], [Kot92, §7]). We denote the formal completion of $\mathrm{Sh}_{K^p}$ along $Y_{K^p}$ by $(\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}}$.

Now we can state the $p$-adic uniformization theorem:

**Theorem 4.2 ([RZ96, Theorem 6.30])** *There exists a natural isomorphism of formal schemes:*

$$\theta_{K^p}\colon I^\phi\backslash(\breve{\mathscr{M}} \times \mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})/K^p) \xrightarrow{\cong} (\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}}.$$

*In the left hand side, $I^\phi$ acts on $\breve{\mathscr{M}}$ through $I^\phi \hookrightarrow J$ and acts on $\mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})/K^p$ through $I^\phi \hookrightarrow \mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})$.*

*The isomorphisms $\{\theta_{K^p}\}_{K^p}$ are compatible with change of $K^p$. (It is also compatible with the Hecke action of $\mathrm{GSp}_4(V'_{\mathbb{A}^{\infty,p}})$, but we do not use it.)*

Let us briefly recall the construction of the isomorphism $\theta_{K^p}$. Take a lift $(\widetilde{\mathbb{X}}, \widetilde{\lambda}_0)$ of $(\mathbb{X}, \lambda_0)$ over $\mathbb{Z}_{p^\infty}$ (such a lift is unique up to isomorphism). Then, by the Serre-Tate theorem, the lift $(\widetilde{A}_0, \widetilde{\lambda}_{A_0})$ of $(A_0, \lambda_{A_0})$ is canonically determined. Let $S$ be an object of **Nilp**, $(X, \rho) \in \breve{\mathscr{M}}(S)$ and $[g] \in \mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})/K^p$. Then $\rho$ extends uniquely to the quasi-isogeny $\widetilde{\rho}\colon \widetilde{\mathbb{X}} \times_{\mathbb{Z}_{p^\infty}} S \longrightarrow X$. We can see that there exist a polarized abelian variety $(A, \lambda)$ and a $p$-quasi-isogeny $\widetilde{A}_0 \times_{\mathbb{Z}_{p^\infty}} S \longrightarrow A$ preserving polarizations, such that the associated quasi-isogeny $\widetilde{A}_0[p^\infty] \times_{\mathbb{Z}_{p^\infty}} S \longrightarrow A[p^\infty]$ coincides with $\widetilde{\rho}$. The fixed isomorphism $H_1(A_0, \mathbb{A}^{\infty,p}) \cong V'_{\mathbb{A}^{\infty,p}}$ naturally induces a $K^p$-level structure $\eta$ of $A$. The morphism $\theta_{K^p}$ is given by $\theta_{K^p}((X,\rho),[g]) = (A, \lambda, \eta \circ g)$.

By composing the morphism $\breve{\mathscr{M}} \longrightarrow \breve{\mathscr{M}} \times \mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})/K^p$; $x \longmapsto (x, [\mathrm{id}])$, we get a morphism $\breve{\mathscr{M}} \longrightarrow (\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}}$, which is also denoted by $\theta_{K^p}$. For $U \in \mathcal{Q}_0$, we denote the image of $U$ under $\theta_{K^p}$ by $Y_{K^p}(U)$. It is an open subset of $Y_{K^p}$.

**Proposition 4.3** *Let $U$ be an element of $\mathcal{Q}_0$. Then for a sufficiently small compact open subgroup $K^p$ of $\mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})$, $\theta_{K^p}$ induces an isomorphism $U \xrightarrow{\cong} Y_{K^p}(U)$. Moreover, if we denote the open formal subscheme of $\breve{\mathscr{M}}$ (resp. $(\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}}$) whose underlying topological space is $U$ (resp. $Y_{K^p}(U)$) by $\breve{\mathscr{M}}_{/U}$ (resp. $(\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}(U)}$), then $\theta_{K^p}$ induces an isomorphism $\theta_{K^p}\colon \breve{\mathscr{M}}_{/U} \xrightarrow{\cong} (\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}(U)}$.*

*Proof.* The proof is similar to [Far04, Corollaire 3.1.4]. Put $\Gamma_{K^p} = I^\phi \cap K^p$, where the intersection is taken in $\mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})$. It is known that $\Gamma_{K^p}$ is discrete and torsion-free [RZ96]. By Theorem 4.2, $\theta_{K^p}$ gives an isomorphism from $\Gamma_{K^p}\backslash\breve{\mathscr{M}}$ to an open and closed formal subscheme of $(\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}}$. By the same method as in [Far04, Lemme 3.1.2, Proposition 3.1.3], we can see that every element $\gamma \in \Gamma_{K^p}$ other than 1 satisfies $\gamma \cdot U \cap U = \varnothing$ if $K^p$ is sufficiently small. For such $K^p$, the natural morphism $\breve{\mathscr{M}}_{/U} \longrightarrow \Gamma_{K^p}\backslash\breve{\mathscr{M}}$ is an open immersion. Thus we have an open immersion $\breve{\mathscr{M}}_{/U} \hookrightarrow \Gamma_{K^p}\backslash\breve{\mathscr{M}} \xrightarrow[\cong]{\theta_{K^p}} (\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}}$, whose image is $(\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}(U)}$. ∎

Next we consider the case with Drinfeld level structures at $p$. Let $Y_{m,K^p}$ be the closed subscheme of $\overline{\mathrm{Sh}}_{m,K^p}$ obtained as the inverse image of $Y_{K^p}$ under $\overline{\mathrm{Sh}}_{m,K^p} \longrightarrow \overline{\mathrm{Sh}}_{K^p}$. By the construction of $\theta_{K^p}$ described above, we have the following result:

**Corollary 4.4** *Let $m \geq 0$ be an integer. We can construct naturally a morphism $\theta_{m,K^p} \colon \breve{\mathscr{M}}_m \longrightarrow (\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}}$ which makes the following diagram cartesian:*

$$\begin{array}{ccc} \breve{\mathscr{M}}_m & \xrightarrow{\theta_{m,K^p}} & (\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}} \\ \big\downarrow{\scriptstyle p_{m0}} & & \big\downarrow \\ \breve{\mathscr{M}} & \xrightarrow{\theta_{K^p}} & (\mathrm{Sh}_{K^p})^\wedge_{/Y_{K^p}}. \end{array}$$

*In particular, the similar result as Proposition 4.3 holds for $\theta_{m,K^p}$; that is, for $U \in \mathscr{Q}_m$, $\theta_{m,K^p}$ induces $(\breve{\mathscr{M}}_m)_{/U} \xrightarrow{\cong} (\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}(U)}$ if $K^p$ is sufficiently small.*

# 5 Proof of the non-cuspidality result

## 5.1 The system of coefficients $\mathcal{F}^{[h]}$, $\mathcal{F}^{(h)}$

**Definition 5.1** Let $m \geq 1$ and $0 \leq h \leq 2$ be integers. We denote by $\mathcal{S}_{m,h}$ the set of direct summands of $L/p^mL$ of rank $4-h$, and by $\mathcal{S}^{\mathrm{coi}}_{m,h}$ the subset of $\mathcal{S}_{m,h}$ consisting of coisotropic direct summands (recall that $I \in \mathcal{S}_{m,h}$ is said to be coisotropic if $I^\perp \subset I$). Put $\mathcal{S}_m = \bigcup_{h=0}^2 \mathcal{S}_{m,h}$ and $\mathcal{S}^{\mathrm{coi}}_m = \bigcup_{h=0}^2 \mathcal{S}^{\mathrm{coi}}_{m,h}$.

For $I \in \mathcal{S}_{m,h}$, let $\overline{\mathrm{Sh}}_{m,K^p,[I]}$ be the $\overline{\mathbb{F}}_p$-scheme defined by

$$\overline{\mathrm{Sh}}_{m,K^p,[I]}(S) = \bigl\{(A,\lambda,\eta^p,\eta_p) \in \overline{\mathrm{Sh}}_{m,K^p,[I]}(S) \bigm| I \subset \operatorname{Ker}\eta_p\bigr\}.$$

Clearly it is a closed subscheme of $\overline{\mathrm{Sh}}_{m,K^p}$. Similarly, we can define the closed formal subscheme $\breve{\mathscr{M}}_{m,[I]}$ of $\breve{\mathscr{M}}_m \otimes_{\mathbb{Z}_{p^\infty}} \overline{\mathbb{F}}_p$. Obviously, $\breve{\mathscr{M}}_{m,[I]}$ is stable under the action of $J$ on $\breve{\mathscr{M}}_m$.

We denote by $Y_{m,K^p,[I]}$ the closed subscheme of $\overline{\mathrm{Sh}}_{m,K^p,[I]}$ obtained as the inverse image of $Y_{m,K^p}$. As Corollary 4.4, we have the following cartesian diagram of formal schemes:

$$\begin{array}{ccc} \breve{\mathscr{M}}_{m,[I]} & \longrightarrow & (\overline{\mathrm{Sh}}_{m,K^p,[I]})^\wedge_{/Y_{m,K^p,[I]}} \\ \big\downarrow & & \big\downarrow \\ \breve{\mathscr{M}}_m & \xrightarrow{\theta_{m,K^p}} & (\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}}. \end{array}$$

**Definition 5.2** For $I \in \mathcal{S}_m$, we put

$$\overline{\mathrm{Sh}}_{m,K^p,(I)} = \overline{\mathrm{Sh}}_{m,K^p,[I]} \setminus \bigcup_{I' \in \mathcal{S}_m, I \subsetneq I'} \overline{\mathrm{Sh}}_{m,K^p,[I']},$$

which is an open subscheme of $\overline{\mathrm{Sh}}_{m,K^p,[I]}$, and thus is a subscheme of $\overline{\mathrm{Sh}}_{m,K^p}$. Moreover, for an integer $h$ with $0 \leq h \leq 2$, we put $\overline{\mathrm{Sh}}^{[h]}_{m,K^p} = \bigcup_{I \in \mathcal{S}_{m,h}} \overline{\mathrm{Sh}}_{m,K^p,[I]}$ and

$\overline{\mathrm{Sh}}^{(h)}_{m,K^p}=\bigcup_{I\in\mathcal{S}_{m,h}}\overline{\mathrm{Sh}}_{m,K^p,(I)}$. The former is a closed subscheme of $\overline{\mathrm{Sh}}_{m,K^p}$, which is the scheme theoretic image of $\coprod_{I\in\mathcal{S}_{m,h}}\overline{\mathrm{Sh}}_{m,K^p,[I]}\longrightarrow\overline{\mathrm{Sh}}_{m,K^p}$. The latter is an open subscheme of $\overline{\mathrm{Sh}}^{[h]}_{m,K^p}$, since $\overline{\mathrm{Sh}}^{(h)}_{m,K^p}=\overline{\mathrm{Sh}}^{[h]}_{m,K^p}\setminus\overline{\mathrm{Sh}}^{[h-1]}_{m,K^p}$ (if $h=0$, we put $\overline{\mathrm{Sh}}^{[-1]}_{m,K^p}=\varnothing$).

**Lemma 5.3** i) *Let $x=(A,\lambda,\eta^p,\eta_p)$ be an element of $\overline{\mathrm{Sh}}_{m,K^p}(\overline{\mathbb{F}}_p)$. Then, for $I\in\mathcal{S}_m$, $x\in\overline{\mathrm{Sh}}_{m,K^p,(I)}(\overline{\mathbb{F}}_p)$ if and only if $I=\operatorname{Ker}\eta_p$. For an integer $h$ with $0\le h\le 2$, $x\in\overline{\mathrm{Sh}}^{[h]}_{m,K^p}(\overline{\mathbb{F}}_p)$ (resp. $x\in\overline{\mathrm{Sh}}^{(h)}_{m,K^p}(\overline{\mathbb{F}}_p)$) if and only if $\operatorname{rank}_{\mathbb{F}_p}A[p]\le h$ (resp. $\operatorname{rank}_{\mathbb{F}_p}A[p]=h$).*

ii) *For every integer $h$ with $0\le h\le 2$, we have $\overline{\mathrm{Sh}}^{(h)}_{m,K^p}=\coprod_{I\in\mathcal{S}_{m,h}}\overline{\mathrm{Sh}}_{m,K^p,(I)}$ as schemes.*

iii) *We have $(\overline{\mathrm{Sh}}^{[2]}_{m,K^p})_{\mathrm{red}}=(\overline{\mathrm{Sh}}_{m,K^p})_{\mathrm{red}}$ and $(\overline{\mathrm{Sh}}^{[0]}_{m,K^p})_{\mathrm{red}}=(Y_{m,K^p})_{\mathrm{red}}$.*

*Proof.* Let us prove i). Put $X=A[p^\infty]$. Then there is an exact sequence $0\longrightarrow X_0\longrightarrow X\longrightarrow X_{\text{ét}}\longrightarrow 0$, where $X_0$ is a connected $p$-divisible group and $X_{\text{ét}}$ is an étale $p$-divisible group. By [HT01, Lemma II.2.1], $\operatorname{Ker}\eta_p$ is a direct summand of $L/p^mL$ and $(L/p^mL)/\operatorname{Ker}\eta_p\longrightarrow X_{\text{ét}}[p^m]$ is an isomorphism. Thus $\operatorname{Ker}\eta_p\in\mathcal{S}_{m,r}$, where $r=\operatorname{rank}_{\mathbb{Z}/p^m\mathbb{Z}}X_{\text{ét}}[p^m]=\operatorname{rank}_{\mathbb{F}_p}A[p]\le 2$. By this, all the claims in i) are immediate.

By i), $\overline{\mathrm{Sh}}^{(h)}_{m,K^p}$ coincides with $\coprod_{I\in\mathcal{S}_{m,h}}\overline{\mathrm{Sh}}_{m,K^p,(I)}$ as a set; thus to prove ii) it suffices to show that $\overline{\mathrm{Sh}}_{m,K^p,(I)}$ is closed (hence open) in $\overline{\mathrm{Sh}}^{(h)}_{m,K^p}$ for every $I\in\mathcal{S}_{m,h}$. It is clear from $\overline{\mathrm{Sh}}_{m,K^p,(I)}=\overline{\mathrm{Sh}}_{m,K^p,[I]}\cap\overline{\mathrm{Sh}}^{(h)}_{m,K^p}$.

The former equality in iii) follows immediately from i). We will prove the latter. For $x=(A,\lambda,\eta^p,\eta_p)\in\overline{\mathrm{Sh}}^{[0]}_{m,K^p}(\overline{\mathbb{F}}_p)$, $X=A[p^\infty]$ has no étale part by i). Since $X^\vee\cong X$, $X$ has no multiplicative part. Therefore $X$ is isoclinic of slope $1/2$; indeed, if a Newton polygon with the terminal point $(4,2)$ has neither slope 0 part nor slope 1 part, then it is a line of slope $1/2$. Thus, by Lemma 4.1, there is a quasi-isogeny $\mathbb{X}\longrightarrow X$ preserving polarizations; namely, $x\in Y_{m,K^p}(\overline{\mathbb{F}}_p)$. The opposite inclusion is clear. ∎

**Remark 5.4** The latter part of iii) in Lemma 5.3 is the only place where the same argument does not work in the case $\mathrm{GSp}(2d)$ with $d\ge 3$.

**Definition 5.5** Let $m\ge 1$ be an integer. Fix a compact open subgroup $K^p$ of $\mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})$. For $I\in\mathcal{S}_m$, denote the natural immersion $\overline{\mathrm{Sh}}_{m,K^p,(I)}\hookrightarrow\overline{\mathrm{Sh}}_{m,K^p}$ by $j_{m,I}$. For an integer $h$ with $0\le h\le 2$, denote the natural immersions $\overline{\mathrm{Sh}}^{[h]}_{m,K^p}\hookrightarrow\overline{\mathrm{Sh}}_{m,K^p}$ and $\overline{\mathrm{Sh}}^{(h)}_{m,K^p}\hookrightarrow\overline{\mathrm{Sh}}_{m,K^p}$ by $j^{[h]}_m$ and $j^{(h)}_m$, respectively.

We define $\mathcal{F}^\circ_{m,I}$, $\mathcal{F}_{m,I}$, $\mathcal{F}^{\circ[h]}_m$, $\mathcal{F}^{[h]}_m$, $\mathcal{F}^{\circ(h)}_m$ and $\mathcal{F}^{(h)}_m$ as follows:

$$\begin{aligned}
\mathcal{F}^\circ_{m,I}&=\theta_m^*(Rj_{m,I*}Rj^!_{m,I}R\psi\mathbb{Z}_\ell)|_{Y_{m,K^p}}, & \mathcal{F}_{m,I}&=\theta_m^*(Rj_{m,I*}Rj^!_{m,I}R\psi\mathbb{Q}_\ell)|_{Y_{m,K^p}},\\
\mathcal{F}^{\circ[h]}_m&=\theta_m^*(Rj^{[h]}_{m*}Rj^{[h]!}_mR\psi\mathbb{Z}_\ell)|_{Y_{m,K^p}}, & \mathcal{F}^{[h]}_m&=\theta_m^*(Rj^{[h]}_{m*}Rj^{[h]!}_mR\psi\mathbb{Q}_\ell)|_{Y_{m,K^p}},\\
\mathcal{F}^{\circ(h)}_m&=\theta_m^*(Rj^{(h)}_{m*}Rj^{(h)!}_mR\psi\mathbb{Z}_\ell)|_{Y_{m,K^p}}, & \mathcal{F}^{(h)}_m&=\theta_m^*(Rj^{(h)}_{m*}Rj^{(h)!}_mR\psi\mathbb{Q}_\ell)|_{Y_{m,K^p}}.
\end{aligned}$$

Here $\theta_m\colon \bar{\mathscr{M}}_m \longrightarrow Y_{m,K^p}$ is the morphism induced from $\theta_{m,K^p}$ in Corollary 4.4.

These are independent of the choice of $K^p$; indeed, for another compact open subgroup $K'^p$ contained in $K^p$, the natural map $\mathrm{Sh}_{m,K'^p} \longrightarrow \mathrm{Sh}_{m,K^p}$ is étale.

**Proposition 5.6** *Let $h$ be an integer with $1 \le h \le 2$.*

i) *We have the following distinguished triangle:*

$$\mathcal{F}_m^{[h-1]} \longrightarrow \mathcal{F}_m^{[h]} \longrightarrow \mathcal{F}_m^{(h)} \longrightarrow \mathcal{F}_m^{[h-1]}[1].$$

ii) *We have $\mathcal{F}_m^{(h)} = \bigoplus_{I\in\mathcal{S}_{m,h}} \mathcal{F}_{m,I}$.*

*Proof.* By the definition, i) is clear. ii) is also clear from Lemma 5.3 ii). ∎

**Proposition 5.7** *For $I \in \mathcal{S}_{m,h} \setminus \mathcal{S}_{m,h}^{\mathrm{coi}}$, we have $\mathcal{F}^{\circ}_{m,I} = \mathcal{F}_{m,I} = 0$.*

*Proof.* We will prove that $Rj^!_{m,I}R\psi\mathbb{Z}_\ell = 0$. Since the dual of $Rj^!_{m,I}R\psi\mathbb{Z}_\ell$ is isomorphic to $j^*_{m,I}R\psi\mathbb{Z}_\ell(3)[6]$, it suffices to show that, for every $x \in \overline{\mathrm{Sh}}_{m,K^p,(I)}(\overline{\mathbb{F}}_p)$, no point on the generic fiber of $\mathrm{Sh}_{m,K^p}$ specializes to $x$. In other words, for every complete discrete valuation ring $R$ with residue field $\overline{\mathbb{F}}_p$ which is a flat $\mathbb{Z}_{p^\infty}$-algebra, and every $\mathbb{Z}_{p^\infty}$-morphism $\widetilde{x}\colon \operatorname{Spec} R \longrightarrow \mathrm{Sh}_{m,K^p}$, the image of the closed point of $\operatorname{Spec} R$ under $\widetilde{x}$ does not lie in $\overline{\mathrm{Sh}}_{m,K^p,(I)}$. This is a consequence of the following lemma. ∎

**Lemma 5.8** *Let $R$ be a complete discrete valuation ring with perfect residue field $k$ and with mixed characteristic $(0,p)$, and $(X,\lambda)$ a polarized $p$-divisible group over $R$. We denote the generic (resp. special) fiber of $X$ by $X_\eta$ (resp. $X_s$). Then, for every $m \ge 1$, the kernel of the specialization map $X_\eta[p^m] \longrightarrow X_s[p^m]$ is a coisotropic direct summand of $X_\eta[p^m]$.*

*Proof.* We shall prove that the kernel of the specialization map $T_pX_\eta \longrightarrow T_pX_s$ is a coisotropic direct summand of $T_pX_\eta$. Consider the exact sequence $0 \longrightarrow X_{s,0} \longrightarrow X_s \longrightarrow X_{s,\text{ét}} \longrightarrow 0$ over $k$. It is canonically lifted to the exact sequence $0 \longrightarrow X_0 \longrightarrow X \longrightarrow X_{\text{ét}} \longrightarrow 0$ over $R$, where $X_{\text{ét}}$ is an étale $p$-divisible group (*cf.* [Mes72, p. 76]). Thus we have the following commutative diagram, whose rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & T_pX_{0,\eta} \arrow[r] \arrow[d] & T_pX_\eta \arrow[r] \arrow[d] & T_pX_{\text{ét},\eta} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & T_pX_s \arrow[r] & T_pX_{s,\text{ét}} \arrow[r] & 0.
\end{tikzcd}$$

Hence the kernel of $T_pX_\eta \longrightarrow T_pX_s$ coincides with $T_pX_{0,\eta}$. Therefore it suffices to show that the composite $(T_pX_{0,\eta})^\perp \longrightarrow T_pX_\eta \longrightarrow T_pX_{\text{ét},\eta}$ is 0.

On the other hand, by the polarization $T_pX_\eta \xrightarrow{\cong} (T_pX_\eta)^\vee(1)$, $(T_pX_{0,\eta})^\perp$ corresponds to $(T_pX_{\text{ét},\eta})^\vee(1) \cong T_pX^\vee_{\text{ét},\eta}$. Thus it suffices to prove that every Galois-equivariant homomorphism $T_pX^\vee_{\text{ét},\eta} \longrightarrow T_pX_{\text{ét},\eta}$ is 0. For this, we may replace the

Tate modules $T_pX^\vee_{\text{ét},\eta}$ and $T_pX_{\text{ét},\eta}$ by the rational Tate modules $V_pX^\vee_{\text{ét},\eta}$ and $V_pX_{\text{ét},\eta}$. These are crystalline representations and the corresponding filtered $\varphi$-modules are the rational Dieudonné modules $D(X^\vee_{s,\text{ét}})_{\mathbb{Q}}$ and $D(X_{s,\text{ét}})_{\mathbb{Q}}$, respectively. Since the slope of the former is 1 and that of the latter is 0, there is no $\varphi$-homomorphism other than 0 from $D(X^\vee_{s,\text{ét}})_{\mathbb{Q}}$ to $D(X_{s,\text{ét}})_{\mathbb{Q}}$. This completes the proof. ∎

The following corollary is immediate from Proposition 5.6 ii) and Proposition 5.7.

**Corollary 5.9** *For $h$ with $1\le h\le 2$, we have $\mathcal{F}^{(h)}_m=\bigoplus_{I\in\mathcal{S}^{\mathrm{coi}}_{m,h}}\mathcal{F}_{m,I}$.*

Let us consider the action of $K_0$. Since $K_0/K_m$ naturally acts on $\mathrm{Sh}_{m,K^p}$ and the action of $g\in K_0/K_m$ maps $\overline{\mathrm{Sh}}_{m,K^p,[I]}$ onto $\overline{\mathrm{Sh}}_{m,K^p,[g^{-1}I]}$, the subschemes $\overline{\mathrm{Sh}}^{[h]}_{m,K^p}$ and $\overline{\mathrm{Sh}}^{(h)}_{m,K^p}$ are preserved by the action of $K_0/K_m$. Therefore $\mathcal{F}^{\circ[h]}_m$, $\mathcal{F}^{[h]}_m$, $\mathcal{F}^{\circ(h)}_m$ and $\mathcal{F}^{(h)}_m$ have natural $K_0/K_m$-equivariant structures. Moreover, in the same way as in [Mie10a, Proposition 2.5], we can observe that $\mathcal{F}^{[h]}=\{\mathcal{F}^{[h]}_m\}_{m\ge1}$ and $\mathcal{F}^{(h)}=\{\mathcal{F}^{(h)}_m\}_{m\ge1}$ (resp. $\mathcal{F}^{\circ[h]}=\{\mathcal{F}^{\circ[h]}_m\}_{m\ge1}$ and $\mathcal{F}^{\circ(h)}=\{\mathcal{F}^{\circ(h)}_m\}_{m\ge1}$) form systems of coefficients (resp. integral coefficients) over $\{\bar{\mathscr{M}}_m\}_{m\ge1}$.

Thanks to [Mie10b], we can define $J$-equivariant structures on the systems of coefficients introduced above.

**Proposition 5.10** *The complexes $\mathcal{F}^\circ_{m,I}$, $\mathcal{F}^{\circ[h]}_m$, $\mathcal{F}^{\circ(h)}_m$, $\mathcal{F}_{m,I}$, $\mathcal{F}^{[h]}_m$ and $\mathcal{F}^{(h)}_m$ have natural $J$-equivariant structures. These structures are compatible with the distinguished triangles and the direct sum decompositions in Proposition 5.6.*

*Proof.* We will prove the proposition for $\mathcal{F}^{(h)}_m$; other cases are similar. Put

$$\overline{\mathrm{Sh}}^{[h]\wedge}_{m,K^p}=(\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}}\times_{\mathrm{Sh}_{m,K^p}}\overline{\mathrm{Sh}}^{[h]}_{m,K^p},\qquad \overline{\mathrm{Sh}}^{(h)\wedge}_{m,K^p}=\bigl(\overline{\mathrm{Sh}}^{[h]\wedge}_{m,K^p},\overline{\mathrm{Sh}}^{[h-1]\wedge}_{m,K^p}\bigr),$$
$$\breve{\mathscr{M}}^{[h]}_m=\breve{\mathscr{M}}_m\times_{(\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}}}\overline{\mathrm{Sh}}^{[h]\wedge}_{m,K^p},\qquad \breve{\mathscr{M}}^{(h)}_m=(\breve{\mathscr{M}}^{[h]}_m,\breve{\mathscr{M}}^{[h-1]}_m).$$

Then, by [Mie10b, Proposition 3.11], we have the canonical isomorphism

$$(Rj^{(h)}_{m*}Rj^{(h)!}_mR\psi\mathbb{Q}_\ell)|_{Y_{m,K^p}}\cong R\Psi_{(\mathrm{Sh}_{m,K^p})^\wedge_{/Y_{m,K^p}},\overline{\mathrm{Sh}}^{(h)\wedge}_{m,K^p}}\mathbb{Q}_\ell.$$

Moreover, since $\theta_{m,K^p}$ is étale (*cf.* Corollary 4.4), by [Mie10b, Proposition 3.14], we have the canonical isomorphism

$$\mathcal{F}^{(h)}_m\cong R\Psi_{\breve{\mathscr{M}}_m,\breve{\mathscr{M}}^{(h)}_m}\mathbb{Q}_\ell.$$

Since the action of $J$ on $\breve{\mathscr{M}}_m$ preserves the closed formal subscheme $\breve{\mathscr{M}}_{m,[I]}$ for every $I\in\mathcal{S}_m$, it also preserves the closed formal subscheme $\breve{\mathscr{M}}^{[h]}_m$ for every $h$. Thus, by the functoriality [Mie10b, Proposition 3.7], $R\Psi_{\breve{\mathscr{M}}_m,\breve{\mathscr{M}}^{(h)}_m}\mathbb{Q}_\ell$ has a natural $J$-equivariant structure. We may import the structure into $\mathcal{F}^{(h)}_m$ by the isomorphism above.

The compatibilities with the exact sequence and the direct sum decomposition are clear from the construction (*cf.* [Mie10b, Remark 3.8]). ∎

It is easy to see that the actions defined in the previous proposition give $J$-equivariant structures on the systems of (integral) coefficients $\mathcal{F}^{\circ[h]}$, $\mathcal{F}^{\circ(h)}$, $\mathcal{F}^{\circ[h]}$ and $\mathcal{F}^{\circ(h)}$. Thus we get the smooth representations $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[h]})$ and $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)})$ of $K_0 \times J$ (*cf.* Corollary 3.7).

**Proposition 5.11** *There exists an isomorphism* $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[0]}) \cong H^i_{\mathrm{RZ}}$*, which is compatible with the action of* $K_0 \times J$.

*Proof.* Let $m \geq 1$ be an integer and $U \in \mathcal{Q}_m$. Then, by [Mie10b, Corollary 4.40] and Proposition 4.3, we have the $J$-equivariant isomorphism

$$H^i_c(U, \mathcal{F}^{[0]}_m|_U) \cong H^i_c\big((\breve{\mathscr{M}}_m)^{\mathrm{rig}}_{/U} \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell\big).$$

Since this isomorphism is functorial, we have $K_0 \times J$-equivariant isomorphisms

$$H^i_c(\bar{\mathscr{M}}_m, \mathcal{F}^{[0]}_m) \cong \varinjlim_{U \in \mathcal{Q}_m} H^i_c\big((\breve{\mathscr{M}}_m)^{\mathrm{rig}}_{/U} \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell\big) \overset{(*)}{\cong} H^i_c(\breve{\mathscr{M}}^{\mathrm{rig}}_m \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell),$$
$$H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[0]}) \cong \varinjlim_{m} H^i_c(\breve{\mathscr{M}}^{\mathrm{rig}}_m \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_\ell) = H^i_{\mathrm{RZ}}.$$

For the isomorphy of $(*)$, we need [Hub98, Proposition 2.1 (iv)] and [Mie10b, Lemma 4.14]. ∎

**Remark 5.12** We can deduce from Proposition 5.11 and Corollary 3.7 that the action of $K_0 \times J$ on $H^i_{\mathrm{RZ}}$ is smooth.

## 5.2 $G$-action on $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[h]})$, $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)})$

In this subsection, we define actions of $G$ on $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[h]})$ and $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)})$ by using the method in [Man05, §6]. Put $G^+ = \{g \in G \mid g^{-1}L \subset L\}$, which is a submonoid of $G$. For $g \in G^+$, let $e(g)$ be the minimal non-negative integer such that $\mathrm{Ker}(g^{-1}\colon V/L \longrightarrow V/L)$ is contained in $p^{-e(g)}L/L$. Since $\mathrm{Ker}\, g^{-1} = (gL+L)/L$, we have $gL \subset p^{-e(g)}L$.

In the sequel, we fix a compact open subgroup $K^p$ of $\mathrm{GSp}(V'_{\mathbb{A}^{\infty,p}})$ and denote $\mathrm{Sh}_{m,K^p}$, $\overline{\mathrm{Sh}}_{m,K^p}$, $\overline{\mathrm{Sh}}_{m,K^p,[I]}$, … by $\mathrm{Sh}_m$, $\overline{\mathrm{Sh}}_m$, $\overline{\mathrm{Sh}}_{m,[I]}$, …, respectively. Moreover, we fix $g \in G^+$ and denote $e(g)$ by $e$ for simplicity.

Assume that $m \geq e$. Let us consider the $\mathbb{Z}_{p^\infty}$-scheme $\mathrm{Sh}_{m,g}$ such that for a $\mathbb{Z}_{p^\infty}$-scheme $S$, the set $\mathrm{Sh}_{m,g}(S)$ consists of isomorphism classes of quintuples $(A, \lambda, \eta^p, \eta_p, \mathcal{E})$ satisfying the following.

- The quadruple $(A, \lambda, \eta^p, \eta_p)$ gives an element of $\mathrm{Sh}_m(S)$.
- $\mathcal{E} \subset X[p^e]$ is a finite flat subgroup scheme of order $p^{v_p(\det g^{-1})}$, where we put $X = A[p^\infty]$. It is self-dual with respect to $\lambda$, and satisfies $\eta'_p(\mathrm{Ker}\, g^{-1}) \subset \mathcal{E}(S)$, where $\eta'_p$ denotes the composite $p^{-m}L/L \xrightarrow[\cong]{\times p^m} L/p^mL \xrightarrow{\eta_p} X[p^m]$.

– For $\mathcal{E}$ as above, we have the following commutative diagram:

$$\begin{tikzcd} & p^{-m}L/L \arrow[r, "\eta'_p"] \arrow[d, two heads, "g^{-1}"'] & X[p^m] \arrow[r, hook] \arrow[d] & X \arrow[d] \\ & p^{-m}g^{-1}L/L \arrow[r] & X[p^m]/\mathcal{E} \arrow[r, hook] & X/\mathcal{E} \\ L/p^{m-e}L \arrow[r, "\cong"] & p^{-m+e}L/L \arrow[u, hook] \arrow[rr] & & (X/\mathcal{E})[p^{m-e}]. \arrow[u, hook] \end{tikzcd}$$

We denote the composite of the lowest row by $\eta_p \circ g$ and assume that it gives a Drinfeld $(m-e)$-level structure.

We have the two natural morphisms

$$\mathrm{pr}\colon \mathrm{Sh}_{m,g} \longrightarrow \mathrm{Sh}_m; (A, \lambda, \eta^p, \eta_p, \mathcal{E}) \longmapsto (A, \lambda, \eta^p, \eta_p),$$
$$[g]\colon \mathrm{Sh}_{m,g} \longrightarrow \mathrm{Sh}_{m-e}; (A, \lambda, \eta^p, \eta_p, \mathcal{E}) \longmapsto (A/\mathcal{E}, \lambda, \eta^p, \eta_p \circ g).$$

It is known that these are proper morphisms, pr induces an isomorphism on the generic fibers, and $[g]$ induces the action of $g$ on the generic fibers [Man05, Proposition 16, Proposition 17].

We can easily see that $\{\mathrm{Sh}_{m,g}\}_{m\geq e}$ form a projective system whose transition maps are finite. Obviously, pr and $[g]$ are compatible with change of $m$.

Similarly we can define the formal scheme $\breve{\mathscr{M}}_{m,g}$ and the morphisms $\mathrm{pr}\colon \breve{\mathscr{M}}_{m,g} \longrightarrow \breve{\mathscr{M}}_m$ and $[g]\colon \breve{\mathscr{M}}_{m,g} \longrightarrow \breve{\mathscr{M}}_{m-e}$. The former morphism induces an isomorphism on the Raynaud generic fibers and the composite $[g]^{\mathrm{rig}} \circ (\mathrm{pr}^{\mathrm{rig}})^{-1}$ coincides with the action of $g$. The group $J$ naturally acts on $\breve{\mathscr{M}}_{m,g}$ and two morphisms pr and $[g]$ are compatible with the action of $J$. Moreover, if we denote by $Y_{m,g}$ the inverse image of $Y_m \subset \mathrm{Sh}_m$ under $\mathrm{pr}\colon \mathrm{Sh}_{m,g} \longrightarrow \mathrm{Sh}_m$, then we can construct a morphism $\theta_{m,g}\colon \breve{\mathscr{M}}_{m,g} \longrightarrow (\mathrm{Sh}_{m,g})^{\wedge}_{Y_{m,g}}$ which makes the following diagrams cartesian:

$$\begin{tikzcd} \breve{\mathscr{M}}_{m,g} \arrow[r, "\theta_{m,g}"] \arrow[d, "\mathrm{pr}"] & (\mathrm{Sh}_{m,g})^{\wedge}_{/Y_{m,g}} \arrow[d, "\mathrm{pr}"] \\ \breve{\mathscr{M}}_m \arrow[r, "\theta_m"] & (\mathrm{Sh}_m)^{\wedge}_{/Y_m}, \end{tikzcd} \qquad \begin{tikzcd} \breve{\mathscr{M}}_{m,g} \arrow[r, "\theta_{m,g}"] \arrow[d, "{[g]}"] & (\mathrm{Sh}_{m,g})^{\wedge}_{/Y_{m,g}} \arrow[d, "{[g]}"] \\ \breve{\mathscr{M}}_{m-e} \arrow[r, "\theta_{m-e}"] & (\mathrm{Sh}_{m-e})^{\wedge}_{/Y_{m-e}}. \end{tikzcd}$$

Now let $h$ be an integer with $1 \leq h \leq 2$ and $I \in \mathcal{S}_{m,h}$. Then we can define the subschemes $\overline{\mathrm{Sh}}_{m,g,[I]}$, $\overline{\mathrm{Sh}}_{m,g,(I)}$, $\overline{\mathrm{Sh}}^{[h]}_{m,g}$ and $\overline{\mathrm{Sh}}^{(h)}_{m,g}$ of $\mathrm{Sh}_{m,g}$ in the same way as $\overline{\mathrm{Sh}}_{m,[I]}$, $\overline{\mathrm{Sh}}_{m,(I)}$, $\overline{\mathrm{Sh}}^{[h]}_m$ and $\overline{\mathrm{Sh}}^{(h)}_m$. The following proposition is obvious:

**Proposition 5.13** *We have the commutative diagrams below:*

$$\begin{tikzcd} \overline{\mathrm{Sh}}_{m,g,(I)} \arrow[r] \arrow[d] & \overline{\mathrm{Sh}}_{m,g,[I]} \arrow[r] \arrow[d] & \overline{\mathrm{Sh}}_{m,g} \arrow[d, "\mathrm{pr}"] \\ \overline{\mathrm{Sh}}_{m,(I)} \arrow[r] & \overline{\mathrm{Sh}}_{m,[I]} \arrow[r] & \overline{\mathrm{Sh}}_m, \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathrm{Sh}}^{(h)}_{m,g} \arrow[r] \arrow[d] & \overline{\mathrm{Sh}}^{[h]}_{m,g} \arrow[r] \arrow[d] & \overline{\mathrm{Sh}}_{m,g} \arrow[d, "\mathrm{pr}"] \\ \overline{\mathrm{Sh}}^{(h)}_m \arrow[r] & \overline{\mathrm{Sh}}^{[h]}_m \arrow[r] & \overline{\mathrm{Sh}}_m. \end{tikzcd}$$

*The rectangles in the left diagram is cartesian. The rectangles in the right diagram is cartesian up to nilpotent elements (namely,* $\overline{\mathrm{Sh}}^{[h]}_{m,g} \longrightarrow \overline{\mathrm{Sh}}^{[h]}_{m} \times_{\overline{\mathrm{Sh}}_m} \overline{\mathrm{Sh}}_{m,g}$ *induces a homeomorphism on the underlying topological spaces, and so on).*

Let us consider how $\overline{\mathrm{Sh}}_{m,g,[I]}$ are mapped by $[g]\colon \mathrm{Sh}_{m,g} \longrightarrow \mathrm{Sh}_{m-e}$. For this purpose, let us introduce some notation.

**Definition 5.14** We denote by $\mathcal{S}_{\infty,h}$ the set of direct summands of $L$ of rank $4-h$ and by $\mathcal{S}^{\mathrm{coi}}_{\infty,h}$ the subset of $\mathcal{S}_{\infty,h}$ consisting of coisotropic direct summands. We can identify $\mathcal{S}_{\infty,h}$ with the set of direct summands of $V$ of rank $4-h$; thus $G$ naturally acts on $\mathcal{S}_{\infty,h}$ and $\mathcal{S}^{\mathrm{coi}}_{\infty,h}$. Let $g^{-1}\colon \mathcal{S}_{m,h} \longrightarrow \mathcal{S}_{m-e,h}$ be the unique map which makes the following diagram commutative:

$$\begin{tikzcd}
\mathcal{S}_{\infty,h} \arrow[r, two heads] \arrow[d, "g^{-1}"] & \mathcal{S}_{m,h} \arrow[d, "g^{-1}"] \\
\mathcal{S}_{\infty,h} \arrow[r, two heads] & \mathcal{S}_{m-e,h}.
\end{tikzcd}$$

The existence of such $g^{-1}$ follows from $p^mL \subset p^eL \subset g^{-1}L \subset L$. Indeed, for direct summands $I$, $I'$ of $V$, we have

$$\begin{aligned}
I\cap L + p^mL = I'\cap L + p^mL &\implies g^{-1}I\cap g^{-1}L + p^mL = g^{-1}I'\cap g^{-1}L + p^mL\\
&\implies g^{-1}I\cap g^{-1}L\cap p^eL + p^mL = g^{-1}I'\cap g^{-1}L\cap p^eL + p^mL\\
&\iff g^{-1}I\cap p^eL + p^mL = g^{-1}I'\cap p^eL + p^mL\\
&\iff g^{-1}I\cap L + p^{m-e}L = g^{-1}I'\cap L + p^{m-e}L.
\end{aligned}$$

Obviously $g^{-1}\colon \mathcal{S}_{m,h} \longrightarrow \mathcal{S}_{m-e,h}$ induces a map from $\mathcal{S}^{\mathrm{coi}}_{m,h}$ to $\mathcal{S}^{\mathrm{coi}}_{m-e,h}$.

**Proposition 5.15** i) *For* $h \in \{1,2\}$ *and* $I \in \mathcal{S}_{m,h}$, $[g]$ *induces morphisms*

$$\mathrm{Sh}_{m,g,[I]} \longrightarrow \mathrm{Sh}_{m-e,[g^{-1}I]}, \qquad \mathrm{Sh}_{m,g,(I)} \longrightarrow \mathrm{Sh}_{m-e,(g^{-1}I)},$$
$$\mathrm{Sh}^{[h]}_{m,g} \longrightarrow \mathrm{Sh}^{[h]}_{m-e}, \qquad \mathrm{Sh}^{(h)}_{m,g} \longrightarrow \mathrm{Sh}^{(h)}_{m-e}.$$

ii) *The rectangles of the following commutative diagram is cartesian up to nilpotent elements:*

$$\begin{tikzcd}
\overline{\mathrm{Sh}}^{(h)}_{m,g} \arrow[r] \arrow[d] & \overline{\mathrm{Sh}}^{[h]}_{m,g} \arrow[r] \arrow[d] & \overline{\mathrm{Sh}}_{m,g} \arrow[d, "{[g]}"] \\
\overline{\mathrm{Sh}}^{(h)}_{m-e} \arrow[r] & \overline{\mathrm{Sh}}^{[h]}_{m-e} \arrow[r] & \overline{\mathrm{Sh}}_{m-e}.
\end{tikzcd}$$

*Proof.* By the definition of $[g]$, it is clear that $[g]$ induces a morphism $\mathrm{Sh}_{m,g,[I]} \longrightarrow \mathrm{Sh}_{m-e,[g^{-1}I]}$ for $I \in \mathcal{S}_{m,h}$, and thus induces a morphism $\mathrm{Sh}^{[h]}_{m,g} \longrightarrow \mathrm{Sh}^{[h]}_{m-e}$. On the other hand, note that, for every $(A,\lambda,\eta^p,\eta_p,\mathcal{E}) \in \mathrm{Sh}_{m,g}(\overline{\mathbb{F}}_p)$, the $p$-divisible groups $A[p^\infty]$ and $A[p^\infty]/\mathcal{E}$ are isogenous, and thus have the same étale heights.

Therefore, by Lemma 5.3 i), the inverse image of $\overline{\mathrm{Sh}}^{[h]}_{m-e}$ (resp. $\overline{\mathrm{Sh}}^{(h)}_{m-e}$) under $[g]$ coincides with $\overline{\mathrm{Sh}}^{[h]}_{m,g}$ (resp. $\overline{\mathrm{Sh}}^{(h)}_{m,g}$) as sets. Therefore a morphism $\mathrm{Sh}^{(h)}_{m,g} \longrightarrow \mathrm{Sh}^{(h)}_{m-e}$ is naturally induced and the rectangles in the diagram above are cartesian up to nilpotent elements. Finally, since $\overline{\mathrm{Sh}}_{m,g,(I)} = \overline{\mathrm{Sh}}_{m,g,[I]} \cap \overline{\mathrm{Sh}}^{(h)}_{m,g}$ and $\overline{\mathrm{Sh}}_{m-e,(g^{-1}I)} = \overline{\mathrm{Sh}}_{m-e,[g^{-1}I]} \cap \overline{\mathrm{Sh}}^{(h)}_{m-e}$, $[g]$ induces a morphism $\overline{\mathrm{Sh}}_{m,g,(I)} \longrightarrow \overline{\mathrm{Sh}}_{m-e,(g^{-1}I)}$. ∎

By Proposition 5.13 and Proposition 5.15, we have the natural cohomological correspondence $\gamma_g$ from $\mathcal{F}^{[h]}_{m-e}$ (resp. $\mathcal{F}^{(h)}_{m-e}$) to $\mathcal{F}^{[h]}_{m}$ (resp. $\mathcal{F}^{(h)}_{m}$); see §6. This cohomological correspondence induces a homomorphism $\gamma_g$ from $H^i_c(\bar{\mathscr{M}}_{m-e}, \mathcal{F}^{[h]}_{m-e})$ (resp. $H^i_c(\bar{\mathscr{M}}_{m-e}, \mathcal{F}^{(h)}_{m-e})$) to $H^i_c(\bar{\mathscr{M}}_{m}, \mathcal{F}^{[h]}_{m})$ (resp. $H^i_c(\bar{\mathscr{M}}_{m}, \mathcal{F}^{(h)}_{m})$). Indeed, for $U \in \mathcal{Q}_{m-e}$, we can take $U' \in \mathcal{Q}_m$ which contains $\mathrm{pr}([g]^{-1}(U))$. Then $\gamma_g$ induces $H^i_c(U, \mathcal{F}^{[h]}_{m-e}|_U) \longrightarrow H^i_c(U', \mathcal{F}^{[h]}_{m}|_{U'})$, and therefore induces $H^i_c(\bar{\mathscr{M}}_{m-e}, \mathcal{F}^{[h]}_{m-e}) \longrightarrow H^i_c(\bar{\mathscr{M}}_{m}, \mathcal{F}^{[h]}_{m})$. It is easy to see that this homomorphism is compatible with change of $m$; hence we get the endomorphism $\gamma_g$ on $H^i_c(\bar{\mathscr{M}}_{\infty}, \mathcal{F}^{[h]})$ and $H^i_c(\bar{\mathscr{M}}_{\infty}, \mathcal{F}^{(h)})$.

**Lemma 5.16** *The endomorphism $\gamma_g$ commutes with the action of $J$ on $H^i_c(\bar{\mathscr{M}}_{\infty}, \mathcal{F}^{[h]})$ and $H^i_c(\bar{\mathscr{M}}_{\infty}, \mathcal{F}^{(h)})$.*

*Proof.* We will only consider $\gamma_g$ on $H^i_c(\bar{\mathscr{M}}_{\infty}, \mathcal{F}^{[h]})$, since the other case is similar. Let $U \in \mathcal{Q}_{m-e}$ and $U' \in \mathcal{Q}_m$ be as above and put $W = [g]^{-1}(U)$, $W' = \mathrm{pr}^{-1}(U')$. It suffices to show the commutativity of the following diagram for $j \in J$:

$$\begin{array}{ccccccc}
H^i_c(jU, \mathcal{F}^{[h]}_{m-e}|_{jU}) & \xrightarrow{[g]^*} & H^i_c(jW, \mathcal{F}^{[h]}_{m-e}|_{jW}) & \longrightarrow & H^i_c(jW', \mathcal{F}^{[h]}_{m-e}|_{jW'}) & \xrightarrow{\mathrm{pr}_*} & H^i_c(jU', \mathcal{F}^{[h]}_{m-e}|_{jU'}) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\
H^i_c(U, \mathcal{F}^{[h]}_{m-e}|_{U}) & \xrightarrow{[g]^*} & H^i_c(W, \mathcal{F}^{[h]}_{m-e}|_{W}) & \longrightarrow & H^i_c(W', \mathcal{F}^{[h]}_{m-e}|_{W'}) & \xrightarrow{\mathrm{pr}_*} & H^i_c(U', \mathcal{F}^{[h]}_{m-e}|_{U'}).
\end{array}$$

By the construction of the $J$-actions, the left and the middle rectangles are commutative. On the other hand, since pr is proper and induces an isomorphism on the generic fiber, $\mathrm{pr}_*$ is an isomorphism and its inverse is $\mathrm{pr}^*$. As $\mathrm{pr}^*$ commutes with the $J$-action, the right rectangle above is also commutative. This concludes the proof. ∎

**Lemma 5.17** i) *For $g, g' \in G^+$, $\gamma_{gg'} = \gamma_g \circ \gamma_{g'}$.*

ii) *For $g \in K_0$, $\gamma_g$ coincides with the action of $K_0$ on $H^i_c(\bar{\mathscr{M}}_{\infty}, \mathcal{F}^{[h]})$ or $H^i_c(\bar{\mathscr{M}}_{\infty}, \mathcal{F}^{(h)})$, which we already introduced.*

iii) *The endomorphism $\gamma_{p^{-1}\cdot\mathrm{id}}$ an isomorphism (in fact, it coincides with the action of $p^{-1} \cdot \mathrm{id} \in J$).*

*Proof.* i) follows from Corollary 6.3. ii) and iii) are consequences of [Man05, Proposition 16, Proposition 17] and the analogous properties for the Rapoport-Zink spaces (*cf.* [Man04, Proposition 7.4 (4), (5)]). ∎

Note that $G$ is generated by $G^+$ and $p\cdot\mathrm{id}$ as a monoid. Therefore, by the lemma above, we can extend the actions of $K_0$ on $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}^{[h]})$ and $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}^{(h)})$ to whole $G$. Together with Lemma 5.16, we have a smooth $G\times J$-module structures on $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}^{[h]})$ and $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}^{(h)})$. We can observe without difficulty that the isomorphism in Proposition 5.11 is in fact compatible with the action of $G$:

**Proposition 5.18** *The isomorphism $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}^{[0]})\cong H^i_{\mathrm{RZ}}$ in Proposition 5.11 is an isomorphism of $G\times J$-modules.*

Next we investigate the $G$-module structure of $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}^{(h)})$ for $h\in\{1,2\}$. Let us fix an element $\widetilde{I}(h)$ of $\mathcal{S}^{\mathrm{coi}}_{\infty,h}$ and denote its image under the natural map $\mathcal{S}^{\mathrm{coi}}_{\infty,h}\longrightarrow\mathcal{S}^{\mathrm{coi}}_{m,h}$ by $\widetilde{I}(h)_m$. Put $P_h=\mathrm{Stab}_G(\widetilde{I}(h))$, which is a maximal parabolic subgroup of $G$. Then we can identify $\mathcal{S}_{\infty,h}$ with $G/P_h=K_0/(P_h\cap K_0)$ and $\mathcal{S}_{m,h}$ with $K_m\backslash G/P_h=K_m\backslash K_0/(P_h\cap K_0)$. For $g\in G^+$ and an integer $m$ with $m\geq e:=e(g)$, $g^{-1}\colon\mathcal{S}_{m,h}\longrightarrow\mathcal{S}_{m-e,h}$ is identified with the map $K_m\backslash G/P_h\longrightarrow K_{m-e}\backslash G/P_h$; $K_mxP_h\longmapsto K_{m-e}g^{-1}xP_h$.

**Definition 5.19** We put $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}_{\widetilde{I}(h)})=\varinjlim_m H^i_c(\bar{\mathscr{M}}_m,\mathcal{F}_{m,\widetilde{I}(h)_m})$. Here the transition maps are given as follows: for integers $1\leq m\leq m'$,

$$\begin{aligned}H^i_c(\bar{\mathscr{M}}_m,\mathcal{F}_{m,\widetilde{I}(h)_m})\longrightarrow H^i_c(\bar{\mathscr{M}}_{m'},p^*_{mm'}\mathcal{F}_{m,\widetilde{I}(h)_m})\longrightarrow\bigoplus_{\substack{I'\in\mathcal{S}^{\mathrm{coi}}_{m',h}\\ I'/p^mI'=\widetilde{I}(h)_m}}H^i_c(\bar{\mathscr{M}}_{m'},\mathcal{F}_{m',I'})\\ \twoheadrightarrow H^i_c(\bar{\mathscr{M}}_{m'},\mathcal{F}_{m',\widetilde{I}(h)_{m'}}).\end{aligned}$$

It is easy to see that $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}_{\widetilde{I}(h)})$ has a structure of a smooth $P_h\times J$-module (use Theorem 3.4 and Proposition 5.15 i)). For each $m\geq 1$ we have the homomorphism

$$H^i_c(\bar{\mathscr{M}}_m,\mathcal{F}^{(h)}_m)=\bigoplus_{I\in\mathcal{S}^{\mathrm{coi}}_{m,h}}H^i_c(\bar{\mathscr{M}}_m,\mathcal{F}_{m,I})\twoheadrightarrow H^i_c(\bar{\mathscr{M}}_m,\mathcal{F}_{m,\widetilde{I}(h)_m}),$$

which induces the homomorphism $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}^{(h)})\longrightarrow H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}_{\widetilde{I}(h)})$. By Proposition 5.15 i), we can prove that this is a homomorphism of $P_h\times J$-modules.

**Proposition 5.20** *We have an isomorphism $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}^{(h)})\cong\mathrm{Ind}^G_{P_h}H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}_{\widetilde{I}(h)})$ of $G\times J$-modules.*

*Proof.* By the Frobenius reciprocity, we have a $G$-homomorphism $H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}^{(h)})\longrightarrow\mathrm{Ind}^G_{P_h}H^i_c(\bar{\mathscr{M}}_\infty,\mathcal{F}_{\widetilde{I}(h)})$. We shall observe that this is bijective. For an integer $m\geq 1$, we have

$$\begin{aligned}H^i_c(\bar{\mathscr{M}}_m,\mathcal{F}^{(h)}_m)&=\bigoplus_{I\in\mathcal{S}^{\mathrm{coi}}_{m,h}}H^i_c(\bar{\mathscr{M}}_m,\mathcal{F}_{m,I})=\bigoplus_{g\in K_m\backslash K_0/(P_h\cap K_0)}H^i_c(\bar{\mathscr{M}}_m,\mathcal{F}_{m,g^{-1}\widetilde{I}(h)_m})\\ &\cong\mathrm{Ind}^{K_0/K_m}_{(P_h\cap K_0)/(P_h\cap K_m)}H^i_c(\bar{\mathscr{M}}_m,\mathcal{F}_{m,\widetilde{I}(h)_m}),\end{aligned}$$

where the last isomorphism, due to [Boy99, Lemme 13.2], is an isomorphism as $K_0$-modules. By taking the inductive limit, we have isomorphisms

$$H_c^i(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)}) \xrightarrow{\cong} \mathrm{Ind}_{P_h\cap K_0}^{K_0} H_c^i(\bar{\mathscr{M}}_\infty, \mathcal{F}_{\widetilde{I}(h)_m}) \xleftarrow{\cong} \mathrm{Ind}_{P_h}^{G} H_c^i(\bar{\mathscr{M}}_\infty, \mathcal{F}_{\widetilde{I}(h)_m})$$

(the second isomorphy follows from the Iwasawa decomposition $G = P_hK_0$). By the proof of [Boy99, Lemme 13.2], it is easy to see that the first isomorphism above is nothing but the $K_0$-homomorphism obtained by the Frobenius reciprocity for $P_h \cap K_0 \subset K_0$. Therefore the composite of the two isomorphisms above coincides with the $G$-homomorphism introduced at the beginning of this proof. Thus we conclude the proof. ∎

## 5.3 Proof of the main theorem

We begin with the following result on non-cuspidality:

**Theorem 5.21** *For every $i \in \mathbb{Z}$ and $h \in \{1, 2\}$, the $G$-module $H_c^i(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell}$ has no quasi-cuspidal subquotient.*

By Proposition 5.20 and [Ber84, 2.4], it suffices to show the following proposition:

**Proposition 5.22** *Let $h \in \{1, 2\}$. The unipotent radical $U_h$ of $P_h$ acts trivially on $H_c^i(\bar{\mathscr{M}}_\infty, \mathcal{F}_{\widetilde{I}(h)})_{\overline{\mathbb{Q}}_\ell}$.*

To prove Proposition 5.22, we need some preparations. In the sequel, let $\mathbf{G}$ and $\mathbf{H}$ be connected reductive groups over $\mathbb{Q}_p$, $\mathbf{P}$ a parabolic subgroup of $\mathbf{G}$ and $\mathbf{U}$ the unipotent radical of $\mathbf{P}$. We put $P = \mathbf{P}(\mathbb{Q}_p)$, $H = \mathbf{H}(\mathbb{Q}_p)$ and $U = \mathbf{U}(\mathbb{Q}_p)$.

**Lemma 5.23** *Let $A$ be a noetherian $\mathbb{Q}$-algebra and $V$ an $A$-module with a smooth $P$-action. Assume that $V$ is $A$-admissible in the sense of [Ber84, 1.16]. Then $U$ acts on $V$ trivially.*

*Proof.* First assume that $A$ is Artinian. Then we can prove the lemma in the same way as [Boy99, Lemme 13.2.3] (we use length in place of dimension).

For the general case, we use noetherian induction. Assume that the lemma holds for every proper quotient of $A$. Take a minimal prime ideal $\mathfrak{p}$ of $A$. Then $A_\mathfrak{p}$ is Artinian and $V_\mathfrak{p}$ is an $A_\mathfrak{p}$-admissible representation of $P$ (note that $(V_\mathfrak{p})^K = (V^K)_\mathfrak{p}$ for every compact open subgroup $K$ of $P$). Therefore $U$ acts on $V_\mathfrak{p}$ trivially. Let $V'$ (resp. $V''$) be the kernel (resp. image) of $V \longrightarrow V_\mathfrak{p}$. Note that $V'$ and $V''$ are $A$-admissible representations of $P$, for $A$ is noetherian.

Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & V' \arrow[r] \arrow[d, "(1)"] & V \arrow[r] \arrow[d, "(2)"] & V'' \arrow[r] \arrow[d, "(3)"] & 0 \\
0 \arrow[r] & V'_U \arrow[r] & V_U \arrow[r] & (V_\mathfrak{p})_U .
\end{tikzcd}$$

It is well-known that the functor taking $U$-coinvariant $V \longmapsto V_U$ is an exact functor; thus the lower row in the diagram above is exact. On the other hand, the arrow labeled (3) is injective, since it is the composite of $V'' \hookrightarrow V_{\mathfrak{p}} \xrightarrow{\cong} (V_{\mathfrak{p}})_U$. Therefore, by the snake lemma, the injectivity of (2) is equivalent to that of (1). In other words, we have only to prove that the action of $U$ on $V'$ is trivial.

On the other hand, by the definition, $V'$ is the union of $V_s := \{x \in V \mid sx = 0\}$ for $s \in A \setminus \mathfrak{p}$. Since $V_s$ can be regarded as an admissible $A/(s)$-representation, $U$ acts on $V_s$ trivially by the induction hypothesis. Hence $U$ acts on $V'$ trivially. ∎

**Proposition 5.24** *Let $V$ be a smooth representation of $P \times H$ over $\overline{\mathbb{Q}}_\ell$ and assume that for every compact open subgroup $K$ of $P$, $V^K$ is a finitely generated $H$-module. Then $U$ acts on $V$ trivially.*

*Proof.* Since $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ are isomorphic as fields, we may replace $\overline{\mathbb{Q}}_\ell$ in the statement by $\mathbb{C}$. Let $\mathfrak{Z}$ be the Bernstein center of $H$ [Ber84]. It is decomposed as $\mathfrak{Z} = \prod_{\theta\in\Theta} \mathfrak{Z}_\theta$, where $\Theta$ denotes the set of connected components of the Bernstein variety of $H$. For $\theta \in \Theta$, we denote the $\theta$-part of $V$ by $V_\theta$. Then we have the canonical decomposition $V = \bigoplus_{\theta\in\Theta} V_\theta$, which is compatible with the action of $P \times H$. Therefore, by replacing $V$ with $V_\theta$, we may assume that the action of $\mathfrak{Z}$ on $V$ factors through $\mathfrak{Z}_\theta$ for some $\theta \in \Theta$.

By the assumption and [Ber84, Proposition 3.3], for every compact open subgroup $K$ of $P$, $V^K$ is a $\mathfrak{Z}_\theta$-admissible $H$-module. Namely, for every compact open subgroup $K$ (resp. $K'$) of $P$ (resp. $H$), $V^{K\times K'}$ is a finitely generated $\mathfrak{Z}_\theta$-module. In other words, for every compact open subgroup $K'$ of $H$, $V^{K'}$ is a $\mathfrak{Z}_\theta$-admissible $P$-module. Since $\mathfrak{Z}_\theta$ is a finitely generated $\mathbb{C}$-algebra, $U$ acts trivially on $V^{K'}$ by Lemma 5.23. Therefore $U$ acts trivially on $V$ also. ∎

*Proof of Proposition 5.22.* By Proposition 5.24, we have only to prove that, for every $m \geq 1$, $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}_{\widetilde{I}(h)})^{P_h\cap K_m}$ is a finitely generated $J$-module (recall that a finitely generated $J$-module is noetherian [Ber84, Remarque 3.12]). As a $J$-module, it is a direct summand of $(\mathrm{Ind}_{P_h}^G H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}_{\widetilde{I}(h)}))^{K_m} \cong H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)})^{K_m}$. On the other hand, by Corollary 3.7, $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)})^{K_m}$ is a finitely generated $J$-module. Thus $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}_{\widetilde{I}(h)})^{P_h\cap K_m}$ is also finitely generated. ∎

**Proposition 5.25** *Let $i$ be an integer. If $i \geq 5$, then $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[2]}) = 0$. On the other hand, if $i \leq 1$, then $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[0]}) = 0$.*

*Proof.* By the definition, it suffices to show that for every $m \geq 1$ and every $U \in \mathcal{Q}_m$ we have $H^i_c(U, \mathcal{F}^{[2]}_m|_U) = 0$ for $i \geq 5$ and $H^i_c(U, \mathcal{F}^{[0]}_m|_U) = 0$ for $i \leq 1$. Thus the claim is reduced to the following lemma. ∎

**Lemma 5.26** *Let $S$ be the spectrum of a strict henselian discrete valuation ring and $X$ a separated $S$-scheme of finite type. We denote its special (resp. generic)* *fiber by $X_s$ (resp. $X_\eta$). Let $Z$ be a closed subscheme of $X_s$ and denote the natural closed immersion $Z \hookrightarrow X$ by $i$. Assume that $X_\eta$ is smooth of pure dimension $d$ and $Z$ is purely $d'$-dimensional.*

*Then we have $H^n(Z, i^*R\psi_X\mathbb{Q}_\ell) = H^n_c(Z, i^*R\psi_X\mathbb{Q}_\ell) = 0$ for $n > d + d'$ and $H^n_c(Z, Ri^!R\psi_X\mathbb{Q}_\ell) = 0$ for $n < d - d'$.*

*Proof.* First note that $H^n(Z, i^*R^k\psi_X\mathbb{Q}_\ell) = H^n_c(Z, i^*R^k\psi_X\mathbb{Q}_\ell) = 0$ if $n > 2\dim Z$ or $n > 2\dim(\operatorname{supp} R^k\psi_X\mathbb{Q}_\ell)$. By [BBD82, Proposition 4.4.2], for each $k \geq 0$ we have $\dim(\operatorname{supp} R^k\psi_X\mathbb{Q}_\ell) \leq d - k$; therefore if $n + k > d + d'$ then we have

$$\begin{aligned} n > d' + (d-k) &\geq \dim Z + \dim(\operatorname{supp} R^k\psi_X\mathbb{Q}_\ell) \\ &\geq 2\min\{\dim Z, \dim(\operatorname{supp} R^k\psi_X\mathbb{Q}_\ell)\} \end{aligned}$$

and thus $H^n(Z, i^*R^k\psi_X\mathbb{Q}_\ell) = H^n_c(Z, i^*R^k\psi_X\mathbb{Q}_\ell) = 0$. By the spectral sequence, we have $H^n(Z, i^*R\psi_X\mathbb{Q}_\ell) = H^n_c(Z, i^*R\psi_X\mathbb{Q}_\ell) = 0$ for $n > d + d'$.

On the other hand, by the Poincaré duality, we have

$$\begin{aligned} H^n_c(Z, Ri^!R\psi_X\mathbb{Q}_\ell) &= H^{-n}(Z, D_Z(Ri^!R\psi_X\mathbb{Q}_\ell))^\vee = H^{-n}(Z, i^*R\psi_X D_{X_\eta}\mathbb{Q}_\ell)^\vee \\ &= H^{-n}(Z, i^*R\psi_X\mathbb{Q}_\ell(d)[2d])^\vee = H^{2d-n}(Z, i^*R\psi_X\mathbb{Q}_\ell)^\vee(-d), \end{aligned}$$

where $D_Z$ (resp. $D_{X_\eta}$) denotes the dualizing functor with respect to $Z$ (resp. $X_\eta$). Therefore it vanishes if $2d - n > d + d'$, namely, $n < d - d'$. ∎

Now we can prove our main theorem.

*Proof of Theorem 3.2.* By Proposition 5.11 and Proposition 5.25, we have $H^i_{\mathrm{RZ}} = 0$ for $i \leq 1$. Therefore we may assume that $i \geq 5$.

By Proposition 5.6 i), we have the exact sequence of smooth $G$-modules

$$H^{i-1}_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell} \longrightarrow H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[h-1]})_{\overline{\mathbb{Q}}_\ell} \longrightarrow H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[h]})_{\overline{\mathbb{Q}}_\ell}$$

for every $h$ with $1 \leq h \leq 2$. Moreover, $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}}_\ell}$ has no quasi-cuspidal subquotient by Theorem 5.21. Thus, starting from $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[2]})_{\overline{\mathbb{Q}}_\ell} = 0$ (Proposition 5.25), we can inductively prove that $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[h]})_{\overline{\mathbb{Q}}_\ell}$ has no quasi-cuspidal subquotient; indeed, the property that a representation has no quasi-cuspidal subquotient is stable under sub, quotient and extension (use the canonical decomposition in [Ber84, 2.3.1]). In particular, $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[0]})_{\overline{\mathbb{Q}}_\ell} \cong H^i_{\mathrm{RZ}, \overline{\mathbb{Q}}_\ell}$ (*cf.* Proposition 5.18) has no quasi-cuspidal subquotient. This completes the proof. ∎

# 6 Appendix: Complements on cohomological correspondences

In this section, we recall the notion of cohomological correspondences (*cf.* [SGA5, Exposé III], [Fuj97]) and give some results on them. These are used to define the action of $G$ on $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{[h]})$ and $H^i_c(\bar{\mathscr{M}}_\infty, \mathcal{F}^{(h)})$.

In this section, we change our notation. Let $k$ be a field and $\ell$ a prime number which is invertible in $k$. We denote one of $\mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Q}_\ell$ by $\Lambda$. Let $X_1$ and $X_2$ be schemes which are separated of finite type over $k$, and $L_i$ an object of $D_c^b(X_i,\Lambda)$ for $i=1,2$ respectively. A *cohomological correspondence* from $L_1$ to $L_2$ is a pair $(\gamma,c)$ consisting of a separated $k$-morphism of finite type $\gamma\colon\Gamma\longrightarrow X_1\times X_2$ and a morphism $c\colon\gamma_1^*L_1\longrightarrow R\gamma_2^!L_2$ in the category $D_c^b(\Gamma,\Lambda)$, where we denote $\mathrm{pr}_i\circ\gamma$ by $\gamma_i$. For simplicity, we also write $c$ for $(\gamma,c)$, if there is no risk of confusion. If we are given a cohomological correspondence $(\gamma,c)$ where $\gamma_1$ is proper, then we have the associated morphism $R\Gamma_c(c)\colon R\Gamma_c(X_1,L_1)\longrightarrow R\Gamma_c(X_2,L_2)$ by composing

$$\begin{aligned}R\Gamma_c(X_1,L_1)&\xrightarrow{\gamma_1^*}R\Gamma_c(\Gamma,\gamma_1^*L_1)\xrightarrow{R\Gamma_c(c)}R\Gamma_c(\Gamma,R\gamma_2^!L_1)=R\Gamma_c(X_2,R\gamma_{2!}R\gamma_2^!L_2)\\&\xrightarrow{\mathrm{adj}}R\Gamma_c(X_2,L_2).\end{aligned}$$

We can compose two cohomological correspondences. Let $X_3$ be another scheme which is separated of finite type over $k$ and $L_3\in D_c^b(X_3,\Lambda)$. Let $(\gamma',c')$ be a cohomological correspondence from $L_2$ to $L_3$. Consider the following diagram

$$\begin{tikzcd}\Gamma\times_{X_2}\Gamma' \arrow[r,"\mathrm{pr}_2"] \arrow[d,"\mathrm{pr}_1"] & \Gamma' \arrow[r,"\gamma_2'"] \arrow[d,"\gamma_1'"] & X_3\\ \Gamma \arrow[r,"\gamma_2"] \arrow[d,"\gamma_1"] & X_2 & \\ X_1. & & \end{tikzcd}$$

Let $\gamma''$ be the natural morphism $\Gamma\times_{X_2}\Gamma'\longrightarrow X_1\times X_3$ and $c''\colon\gamma_1''^*L_1\longrightarrow R\gamma_2''^!L_3$ the composite of

$$\gamma_1''^*L_1=\mathrm{pr}_1^*\gamma_1^*L_1\xrightarrow{\mathrm{pr}_1^*(c)}\mathrm{pr}_1^*R\gamma_2^!L_2\xrightarrow{\text{b.c.}}R\,\mathrm{pr}_2^!\gamma_1'^*L_2\xrightarrow{R\,\mathrm{pr}_2^!(c')}R\,\mathrm{pr}_2^!R\gamma_2'^!L_3=R\gamma_2''^!L_3,$$

where b.c. denotes the base change morphism. We call the cohomological correspondence $(\gamma'',c'')$ the composite of $(\gamma,c)$ and $(\gamma',c')$, and denote it by $c'\circ c$. It is not difficult to see that if $\gamma_1$ and $\gamma_1'$ are proper, then $\gamma_1''$ is also proper and $R\Gamma_c(c'\circ c)=R\Gamma_c(c')\circ R\Gamma_c(c)$.

Let us recall some operations for cohomological correspondences. Let $X_1$, $X_2$, $X_1'$ and $X_2'$ be schemes which are separated of finite type over $k$, and $\gamma\colon\Gamma\longrightarrow X_1\times X_2$ and $\gamma'\colon\Gamma'\longrightarrow X_1'\times X_2'$ separated $k$-morphisms of finite type. Assume that the following commutative diagram is given:

$$\begin{tikzcd}X_1' \arrow[d,"a_1"] & \Gamma' \arrow[l,"\gamma_1'"'] \arrow[r,"\gamma_2'"] \arrow[d,"a"] & X_2' \arrow[d,"a_2"]\\ X_1 & \Gamma \arrow[l,"\gamma_1"'] \arrow[r,"\gamma_2"] & X_2.\end{tikzcd}$$

First assume that every vertical morphism is proper. Let $L_i'$ be an object of $D_c^b(X_i', \Lambda)$ for each $i = 1, 2$ and $(\gamma', c')$ a cohomological correspondence from $L_1'$ to $L_2'$. Then we can define the cohomological correspondence $(\gamma, a_*c')$ from $Ra_{1*}L_1'$ to $Ra_{2*}L_2'$ by

$$\begin{aligned}\gamma_1^*Ra_{1*}L_1' &\xrightarrow{\text{b.c.}} Ra_*\gamma_1'^*L_1' \xrightarrow{Ra_*(c')} Ra_*R\gamma_2'^!L_2' = Ra_!R\gamma_2'^!L_2' \\ &\xrightarrow{\text{b.c.}} R\gamma_2^!Ra_{2!}L_2' = R\gamma_2^!Ra_{2*}L_2'.\end{aligned}$$

The cohomological correspondence $(\gamma, a_*c')$ is called the push-forward of $(\gamma', c')$ by $a$. It is easy to see that push-forward is compatible with composition. Moreover, we have the following lemma whose proof is also immediate:

**Lemma 6.1** *In the above diagram, assume that $X_1 = X_1'$, $X_2 = X_2'$, $a_1 = a_2 = \mathrm{id}$ and $a$ is proper. Let $L_i$ be an object of $D_c^b(X_i, \Lambda)$ for each $i = 1, 2$ and $(\gamma', c')$ a cohomological correspondence from $L_1$ to $L_2$. Then we have $R\Gamma_c(a_*c') = R\Gamma_c(c')$.*

Next we assume that the right rectangle in the diagram above is cartesian. Let $L_i'$ and $(\gamma', c')$ be as above. Then we have the cohomological correspondence $(\gamma, a_*c')$ from $Ra_{1*}L_1'$ to $Ra_{2*}L_2'$ by

$$\gamma_1^*Ra_{1*}L_1' \xrightarrow{\text{b.c.}} Ra_*\gamma_1'^*L_1' \xrightarrow{Ra_*(c')} Ra_*R\gamma_2'^!L_2' \xrightarrow{\text{b.c.}} R\gamma_2^!Ra_{2*}L_2'.$$

On the other hand, let $L_i$ be an object of $D_c^b(X_i, \Lambda)$ for each $i = 1, 2$ and $(\gamma, c)$ a cohomological correspondence from $L_1$ to $L_2$. Then we have the cohomological correspondence $(\gamma, a^*c)$ from $a_1^*L_1$ to $a_2^*L_2$ by

$$\gamma_1'^*a_1^*L_1 = a^*\gamma_1^*L_1 \xrightarrow{a^*(c)} a^*R\gamma_2^!L_2 \xrightarrow{\text{b.c.}} R\gamma_2'^!a_2^*L_2.$$

Finally assume that the left rectangle in the diagram above is cartesian. Let $L_i$ and $(\gamma, c)$ be as above. Then we have the cohomological correspondence $(\gamma, a^!c)$ from $Ra_1^!L_1$ to $Ra_2^!L_2$ by

$$\gamma_1'^*Ra_1^!L_1 \xrightarrow{\text{b.c.}} Ra^!\gamma_1^*L_1 \xrightarrow{Ra^!(c)} Ra^!R\gamma_2^!L_2 = R\gamma_2'^!Ra_2^!L_2.$$

These constructions are also compatible with composition.

Next we recall the specialization of cohomological correspondences. Let $S$ be the spectrum of a strict henselian discrete valuation ring on which $\ell$ is invertible. For an $S$-scheme $X$, we denote its special (resp. generic) fiber by $X_s$ (resp. $X_\eta$).

Let $X_1$, $X_2$ be schemes which are separated of finite type over $S$ and $\gamma\colon \Gamma \longrightarrow X_1 \times_S X_2$ a separated $S$-morphism of finite type. Let $L_i$ be an object of $D_c^b(X_{i,\eta}, \Lambda)$ for each $i = 1, 2$ and $(\gamma_\eta, c)$ a cohomological correspondence from $L_1$ to $L_2$. Then we have the cohomological correspondence $(\gamma_s, R\psi(c))$ from $R\psi L_1$ to $R\psi L_2$ by

$$\gamma_{1,s}^*R\psi L_1 \longrightarrow R\psi\gamma_{1,\eta}^*L_1 \xrightarrow{R\psi(c)} R\psi R\gamma_{2,\eta}^!L_2 \longrightarrow R\gamma_{2,s}^!R\psi L_2.$$

It is easy to see that this construction is compatible with composition and proper push-forward (*cf.* [Fuj97, Proposition 1.6.1]).

Now we will give the main result in this section. Let $X_i$, $\gamma$, $L_i$ be as above and $Y_i$ (resp. $Z_i$) a closed (resp. locally closed) subscheme of $X_{i,s}$. Assume that $\gamma_{1,s}^{-1}(Y_1) = \gamma_{2,s}^{-1}(Y_2)$ and $\gamma_{1,s}^{-1}(Z_1) = \gamma_{2,s}^{-1}(Z_2)$ as subschemes of $\Gamma_s$, and denote the former by $\Gamma_Y$ and the latter by $\Gamma_Z$. Then we have the following diagrams whose rectangles are cartesian:

$$\begin{tikzcd} Y_1 \arrow[d, "i_1"] & \Gamma_Y \arrow[l, "\gamma_{Y,1}"'] \arrow[r, "\gamma_{Y,2}"] \arrow[d, "i"] & Y_2 \arrow[d, "i_2"] \\ X_{1,s} & \Gamma_s \arrow[l, "\gamma_{1,s}"'] \arrow[r, "\gamma_{2,s}"] & X_{2,s}, \end{tikzcd} \qquad \begin{tikzcd} Z_1 \arrow[d, "j_1"] & \Gamma_Z \arrow[l, "\gamma_{Z,1}"'] \arrow[r, "\gamma_{Z,2}"] \arrow[d, "j"] & Z_2 \arrow[d, "j_2"] \\ X_{1,s} & \Gamma_s \arrow[l, "\gamma_{1,s}"'] \arrow[r, "\gamma_{2,s}"] & X_{2,s}. \end{tikzcd}$$

Therefore, for a cohomological correspondence $(\gamma_\eta, c)$ from $L_1$ to $L_2$, the cohomological correspondence $i^*j_*j^!R\psi(c)$ from $i_1^*Rj_{1*}Rj_1^!R\psi L_1$ to $i_2^*Rj_{2*}Rj_2^!R\psi L_2$ is induced. If moreover we assume that $\gamma_1$ is proper, then we have

$$R\Gamma_c(i^*j_*j^!R\psi(c))\colon R\Gamma_c(X_{1,s}, i_1^*Rj_{1*}Rj_1^!R\psi L_1) \longrightarrow R\Gamma_c(X_{2,s}, i_2^*Rj_{2*}Rj_2^!R\psi L_2).$$

**Proposition 6.2** *The morphism $R\Gamma_c(i_*j_*j^!R\psi(c))$ depends only on the cohomological correspondence $(\gamma_\eta, c)$. More precisely, if another $S$-morphism $\gamma'\colon \Gamma' \longrightarrow X_1 \times_S X_2$ has the same generic fiber as $\gamma$ and satisfies the conditions that $\gamma'^{-1}_{1,s}(Y_1) = \gamma'^{-1}_{2,s}(Y_2)$, $\gamma'^{-1}_{1,s}(Z_1) = \gamma'^{-1}_{2,s}(Z_2)$ and $\gamma'_1$ is proper, then the morphism $R\Gamma_c(i'^*j'_*j'^!R\psi(c))$ induced from $\gamma'$ is equal to $R\Gamma_c(i^*j_*j^!R\psi(c))$ (here $i'$ and $j'$ are defined in the same way as $i$ and $j$).*

*Proof.* Since $\Gamma$ and $\Gamma'$ have the same generic fiber, there is the "diagonal" in the generic fiber of $\Gamma \times_{X_1 \times_S X_2} \Gamma'$. Let $\Gamma''$ be the closure of it in $\Gamma \times_{X_1 \times_S X_2} \Gamma'$. Then $\Gamma''$ has the same generic fiber as $\Gamma$. We have natural morphisms $\Gamma'' \longrightarrow \Gamma$ and $\Gamma'' \longrightarrow \Gamma'$, which are proper since $\gamma$ and $\gamma'$ are proper. Therefore $\gamma''\colon \Gamma'' \longrightarrow X_1 \times_S X_2$ also satisfies the same conditions as $\gamma$ and $\gamma'$. By replacing $\gamma'$ by $\gamma''$, we may assume that there exists a proper morphism $a\colon \Gamma' \longrightarrow \Gamma$ such that $\gamma \circ a = \gamma'$.

Then, it is easy to see that the push-forward of the cohomological correspondence $(\gamma'_s, i'^*j'_*j'^!R\psi(c))$ by $a_s$ coincides with $(\gamma_s, i^*j_*j^!R\psi(c))$. Therefore the proposition follows from Lemma 6.1. ∎

**Corollary 6.3** *Let $X_1$, $X_2$ and $X_3$ be schemes which are separated of finite type over $S$, $Y_i$ (resp. $Z_i$) a closed (resp. locally closed) subscheme of $X_i$, and $L_i$ an object of $D_c^b(X_{i,\eta}, \Lambda)$ for each $i = 1, 2, 3$. Let $\gamma\colon \Gamma \longrightarrow X_1 \times_S X_2$ (resp. $\gamma'\colon \Gamma' \longrightarrow X_2 \times_S X_3$, resp. $\gamma''\colon \Gamma'' \longrightarrow X_1 \times_S X_3$) be an $S$-morphism such that $\gamma_1$ (resp. $\gamma'_1$, resp. $\gamma''_1$) is proper, and $(\gamma_\eta, c)$ (resp. $(\gamma'_\eta, c')$, resp. $(\gamma''_\eta, c'')$) a cohomological correspondence from $L_1$ to $L_2$ (resp. from $L_2$ to $L_3$, resp. from $L_1$ to $L_3$). Moreover we assume that $\gamma_{1,s}^{-1}(Y_1) = \gamma_{2,s}^{-1}(Y_2)$, $\gamma_{1,s}^{-1}(Z_1) = \gamma_{2,s}^{-1}(Z_2)$, $\gamma'^{-1}_{1,s}(Y_2) = \gamma'^{-1}_{2,s}(Y_3)$, $\gamma'^{-1}_{1,s}(Z_2) = \gamma'^{-1}_{2,s}(Z_3)$,*

$\gamma''^{-1}_{1,s}(Y_1) = \gamma''^{-1}_{2,s}(Y_3)$ *and* $\gamma''^{-1}_{1,s}(Z_1) = \gamma''^{-1}_{2,s}(Z_3)$. *Then, as above, the morphisms*

$$R\Gamma_c(i^*j_*j^!R\psi(c))\colon R\Gamma_c(X_{1,s}, i_1^*Rj_{1*}Rj_1^!R\psi L_1) \longrightarrow R\Gamma_c(X_{2,s}, i_2^*Rj_{2*}Rj_2^!R\psi L_2),$$
$$R\Gamma_c(i^*j_*j^!R\psi(c'))\colon R\Gamma_c(X_{2,s}, i_2^*Rj_{2*}Rj_2^!R\psi L_2) \longrightarrow R\Gamma_c(X_{3,s}, i_3^*Rj_{3*}Rj_3^!R\psi L_3),$$
$$R\Gamma_c(i^*j_*j^!R\psi(c''))\colon R\Gamma_c(X_{1,s}, i_1^*Rj_{1*}Rj_1^!R\psi L_1) \longrightarrow R\Gamma_c(X_{3,s}, i_3^*Rj_{3*}Rj_3^!R\psi L_3)$$

*are induced. Assume that the composite of* $(\gamma_\eta, c)$ *and* $(\gamma'_\eta, c')$ *coincides with* $(\gamma''_\eta, c'')$. *Then we have* $R\Gamma_c(i^*j_*j^!R\psi(c')) \circ R\Gamma_c(i^*j_*j^!R\psi(c)) = R\Gamma_c(i^*j_*j^!R\psi(c''))$.

*Proof.* By Proposition 6.2, we may replace $\gamma''$ by $\Gamma \times_{X_2} \Gamma' \longrightarrow X_1 \times_S X_3$. Then the equality is clear, since all the operations for cohomological correspondences are compatible with composition. ∎

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