Cuspidal representations in the $\ell$-adic cohomology of the Rapoport-Zink space for GSp(4)

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Abstract. In this paper, we study the $\ell$-adic cohomology of the Rapoport-Zink tower for GSp(4). We prove that the smooth representation of $\text{GSp}_4(\mathbb{Q}_p)$ obtained as the $i$th compactly supported $\ell$-adic cohomology of the Rapoport-Zink tower has no quasi-cuspidal subquotient unless $i = 2, 3, 4$. Our proof is purely local and does not require global automorphic methods.

1 Introduction

In [RZ96], M. Rapoport and Th. Zink introduced certain moduli spaces of quasi-isogenies of $p$-divisible groups with additional structures called the Rapoport-Zink spaces. They constructed systems of rigid analytic coverings of them which we call the Rapoport-Zink towers, and established the $p$-adic uniformization theory of Shimura varieties generalizing classical Čerednik-Drinfeld uniformization. These spaces uniformize the rigid spaces associated with the formal completion of certain Shimura varieties along Newton strata.

Using the $\ell$-adic cohomology of the Rapoport-Zink tower, we can construct a representation of the product $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, where $G$ is the reductive group over $\mathbb{Q}_p$ corresponding to the Shimura datum, $J$ is an inner form of it, and $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the Weil group of the $p$-adic field $\mathbb{Q}_p$. It is widely believed that this realizes the local Langlands and Jacquet-Langlands correspondences (cf. [Rap95]). Classical examples of the Rapoport-Zink spaces are the Lubin-Tate space and the Drinfeld upper half space; these spaces were extensively studied by many people and many important results were obtained (cf. [Dri76], [Car90], [Har97], [HT01], [Dat07], [Boy09] and references therein). However, very little was known about the $\ell$-adic cohomology of other Rapoport-Zink spaces.

The aim of this paper is to study cuspidal representations in the $\ell$-adic cohomology of the Rapoport-Zink tower for $\text{GSp}_4(\mathbb{Q}_p)$. Let us denote the Rapoport-Zink space for $\text{GSp}_4(\mathbb{Q}_p)$ by $\mathcal{M}$. It is a special formal scheme over $\mathbb{Z}_{p^\infty} = W(\overline{\mathbb{F}}_p)$ in the sense of Berkovich [Ber96]. Let $\mathcal{M}^{\text{rig}}$ be the Raynaud generic fiber of $\mathcal{M}$, that is, the generic fiber of the adic space $t(\mathcal{M})$ associated with $\mathcal{M}$. Using level structures

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Tetsushi Ito and Yoichi Mieda

at $p$, we can construct the Rapoport-Zink tower

$$
\cdots \to \mathcal{M}_{m+1} \to \mathcal{M}_m \to \cdots \to \mathcal{M}_2 \to \mathcal{M}_1 \to \mathcal{M}_0 = \mathcal{M}_{\text{rig}},
$$

where $\mathcal{M}_{m} \to \mathcal{M}_{m+1}$ is an étale Galois covering of rigid spaces with Galois group $\text{GSp}_4(\mathbb{Z}/p^m\mathbb{Z})$. We take the compactly supported $\ell$-adic cohomology (in the sense of [Hub98]) and take the inductive limit of them. Then, on

$$
H^i_{\text{RZ}} := \lim_{\to} H^i_{\text{c}}(\mathcal{M}_m \otimes_{\mathbb{Q}_{p^\infty}} \overline{\mathbb{Q}}_{p^\infty}, \mathbb{Q}_{\ell})
$$

(here $\mathbb{Q}_{p^\infty} = \text{Frac} \, \mathbb{Z}_{p^\infty}$), we have an action of a product

$$
\text{GSp}_4(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p),
$$

where $J$ is an inner form of $\text{GSp}_4$.

The main theorem of this paper is as follows:

**Theorem 1.1 (Theorem 3.2)** The $\text{GSp}_4(\mathbb{Q}_p)$-representation $H^i_{\text{RZ}} \otimes_{\mathbb{Q}_r} \overline{\mathbb{Q}}_{\ell}$ has no quasi-cuspidal subquotient unless $i = 2, 3, 4$.

For the definition of quasi-cuspidal representations, see [Ber84, 1.20]. Note that since $\mathcal{M}_m$ is 3-dimensional for every $m \geq 0$, $H^0_{\text{RZ}} = 0$ unless $0 \leq i \leq 6$.

Our proof of this theorem is purely local. We do not use global automorphic methods. The main strategy of the proof is similar to that of [Mie10a], in which the analogous result for the Lubin-Tate tower is given; we construct the formal model $\mathcal{M}_m$ of $\mathcal{M}_{\text{rig}}$ by using Drinfeld level structures and consider the geometry of its special fiber. However, our situation is much more difficult than the case of the Lubin-Tate tower. In the Lubin-Tate case, the tower consists of affine formal schemes $\{\text{Spf} \, A_m\}_{m \geq 0}$, and we can associate it with the tower of affine schemes $\{\text{Spec} \, A_m\}_{m \geq 0}$. In [Mie10a], the second author defined the stratification on the special fiber of $\text{Spec} \, A_m$ by using the kernel of the universal Drinfeld level structure, and considered the local cohomology of the nearby cycle complex $R\psi \Lambda$ along the strata. On the other hand, our tower $\{\mathcal{M}_m\}_{m \geq 0}$ does not consist of affine formal schemes and there is no canonical way to associate it with a tower of schemes. To overcome this problem, we take a sheaf-theoretic approach. For each direct summand $I$ of $(\mathbb{Z}/p^n\mathbb{Z})^4$, we will define the complex of sheaves $F_{m,I}$ on $\mathcal{M}_{\text{red}}$ so that the cohomology $H^i((\mathcal{M}_m)_{\text{red}}, F_{m,I})$ substitutes for the local cohomology of $R\psi \Lambda$ along the strata defined by $I$ in the Lubin-Tate case. For the definition of $F_{m,I}$, we use the $p$-adic uniformization theorem by Rapoport and Zink.

There is another difficulty; since a connected component of $\mathcal{M}$ is not quasi-compact, the representation $H^0_{\text{RZ}}$ of $\text{GSp}_4(\mathbb{Q}_p)$ is far from admissible. Therefore it is important to consider the action of $J(\mathbb{Q}_p)$ on $H^0_{\text{RZ}}$, though it does not appear in our main theorem. However, the cohomology $H^i((\mathcal{M}_m)_{\text{red}}, F_{m,I})$ has no apparent action of $J(\mathbb{Q}_p)$, since $J(\mathbb{Q}_p)$ does not act on the Shimura variety uniformized by
ℓ-adic cohomology of the Rapoport-Zink space for $\text{GSp}(4)$

We use the variants of formal nearby cycle introduced by the second author in [Mie10b] to endow it with an action of $J(\mathbb{Q}_p)$. Furthermore, to ensure the smoothness of this action, we use a property of finitely generated pro-$p$ groups (Section 2). In fact, extensive use of the formalism developed in [Mie10b] make us possible to work mainly on the Rapoport-Zink tower itself and avoid the theory of $p$-adic uniformization except for proving that $\mathcal{M}_m$ is locally algebraizable. However, for the reader’s convenience, the authors decided to make this article as independent of [Mie10b] as possible.

The authors expect that the converse of Theorem 1.1 also holds. Namely, we expect that $H^i_{\text{RZ}} \otimes \mathbb{Q}_\ell$ has a quasi-cuspidal subquotient if $i = 2, 3, 4$. We hope to investigate it in a future work.

The outline of this paper is as follows. In Section 2, we prepare a criterion for the smoothness of representations over $\mathbb{Q}_\ell$. It is elementary but very powerful for our purpose. In Section 3, we give some basic definitions concerning with the Rapoport-Zink space for $\text{GSp}(4)$ and state the main theorem. Section 4 is devoted to introduce certain Shimura varieties related to our Rapoport-Zink tower and recall the theory of $p$-adic uniformization. The proof of the main theorem is accomplished in Section 5. The final Section 6 is an appendix on cohomological correspondences. The results in the section are used to define actions of $\text{GSp}_4(\mathbb{Q}_p)$ on various cohomology groups.

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Notation Let $p$ be a prime number and take another prime $\ell$ with $\ell \neq p$. We denote the completion of the maximal unramified extension of $\mathbb{Z}_p$ by $\mathbb{Z}_p^{\infty}$ and its fraction field by $\mathbb{Q}_p^{\infty}$. Let $\text{Nilp} = \text{Nilp}_{\mathbb{Z}_p^{\infty}}$ be the category of $\mathbb{Z}_p^{\infty}$-schemes on which $p$ is locally nilpotent. For an object $S$ of $\text{Nilp}$, we put $\overline{S} = S \otimes_{\mathbb{Z}_p^{\infty}} \overline{\mathbb{F}_p}$.

In this paper, we use the theory of adic spaces ([Hub94], [Hub96]) as a framework of rigid geometry. A rigid space over $\mathbb{Q}_p^{\infty}$ is understood as an adic space locally of finite type over $\text{Spa}(\mathbb{Q}_p^{\infty}, \mathbb{Z}_p^{\infty})$.

Every sheaf and cohomology are considered in the étale topology. Every smooth representation is considered over $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}_\ell}$. For a $\mathbb{Q}_\ell$-vector space $V$, we put $V_{\overline{\mathbb{Q}_\ell}} = V \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$.

2 Preliminaries: smoothness of representations of profinite groups

Let $G$ be a linear algebraic group over a $p$-adic field $F$. In this section, we give a convenient criterion for the smoothness of a $G(F)$-representation over $\mathbb{Q}_\ell$. The following theorem is essential:

**Theorem 2.1** Let $K$ be a closed subgroup of $\text{GL}_n(\mathbb{Z}_p)$ and $(\pi, V)$ a finite-dimensional representation over $\mathbb{Q}_\ell$ of $K$ as an abstract group. Assume that there exists a $K$-stable $\mathbb{Z}_\ell$-lattice $\Lambda$ of $V$. Then this representation is automatically smooth.
In order to prove this theorem, we require several facts on pro-

Lemma 2.2 The pro-

Lemma 2.3 Every subgroup of finite index of $K_1$ is open.

Remark 2.4 More generally, every subgroup of finite index of a finitely generated

Lemma 2.5 Let $G$ be a pro-\ell group. Then every homomorphism $f: K_1 \rightarrow G$ is trivial.

Proof. Let $H$ be an open normal subgroup of $G$ and denote the composite $K_1 \xrightarrow{f} G \twoheadrightarrow G/H$ by $f_H$. By Lemma 2.3, $\ker f_H$ is an open normal subgroup of $K_1$. Thus $K_1/\ker f_H$ is a finite $p$-group. On the other hand, $G/H$ is a finite \ell-group. Since we have an injection $K_1/\ker f_H \hookrightarrow G/H$, we have $K_1/\ker f_H = 1$, in other words, $f_H = 1$. Therefore the composite $K_1 \xrightarrow{f} G \cong \varprojlim_H G/H$ is trivial. Hence we have $f = 1$, as desired.

Proof of Theorem 2.1. Since $K_1$ is an open subgroup of $K$, we may replace $K$ by $K_1$. Take a $K_1$-stable $\mathbb{Z}_p$-lattice $\Lambda$ of $V$. Then, $\Lambda/\ell \Lambda$ is a finite abelian group. Therefore, by Lemma 2.3, there exists an open subgroup $U$ of $K_1$ which acts trivially on $\Lambda/\ell \Lambda$. In other words, the homomorphism $\pi: K_1 \twoheadrightarrow \text{GL}(\Lambda) \subset \text{GL}(V)$ maps $U$ into the subgroup $1 + \ell \text{End}(\Lambda)$. Since $U$ is a closed subgroup of $1 + pM_n(\mathbb{Z}_p)$ and $1 + \ell \text{End}(\Lambda)$ is a pro-\ell group, by Lemma 2.5, the homomorphism $\pi|_U: U \twoheadrightarrow 1 + \ell \text{End}(\Lambda)$ is trivial. Namely, $\pi|_U$ is a trivial representation.

Lemma 2.6 Let $F$ be a $p$-adic field and $G$ a linear algebraic group over $F$. Then every compact subgroup $K$ of $G(F)$ can be realized as a closed subgroup of $\text{GL}_n(\mathbb{Z}_p)$ for some $n$.

Proof. Take an embedding $G \hookrightarrow \text{GL}_m$ defined over $F$. Since $G(F)$ is a closed subgroup of $\text{GL}_m(F)$, $K$ is also a closed subgroup of $\text{GL}_m(F)$. Therefore we have a faithful continuous action of $K$ on $F^m$. By taking a $\mathbb{Q}_p$-basis of $F$, we have a faithful continuous action of $K$ on $\mathbb{Q}_p^n$ for some $n$. Since $K$ is compact, it is well-known that there is a $K$-stable $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^n$. Hence we have a continuous injection $K \hookrightarrow \text{GL}_n(\mathbb{Z}_p)$. Since $K$ is compact, it is isomorphic to a closed subgroup of $\text{GL}_n(\mathbb{Z}_p)$.
Corollary 2.7 Let $F$ and $G$ be as in the previous proposition. Let $I$ be a filtered ordered set and $\{K_i\}_{i \in I}$ be a system of compact open subgroups of $G(F)$ indexed by $I$.

Let $(\pi, V)$ be a (not necessarily finite-dimensional) $\mathbb{Q}_\ell$-representation of $G(F)$ as an abstract group. Assume that there exists an inductive system $\{V_i\}_{i \in I}$ of finite-dimensional $\mathbb{Q}_\ell$-vector spaces satisfying the following:

1. For every $i \in I$, $V_i$ is endowed with an action of $K_i$ as an abstract group.
2. For every $i \in I$, $V_i$ has a $K_i$-stable $\mathbb{Z}_\ell$-lattice.
3. There exists an isomorphism $\lim_{\longrightarrow i \in I} V_i \cong V$ as $\mathbb{Q}_\ell$-vector spaces such that the composite $V_i \longrightarrow \lim_{\longrightarrow i \in I} V_i \longrightarrow V$ is $K_i$-equivariant for every $i \in I$.

Then $(\pi, V)$ is a smooth representation of $G(F)$.

Proof. Let us take $x \in V$ and show that $\text{Stab}_{G(F)}(x)$, the stabilizer of $x$ in $G(F)$, is open. There exists an element $i \in I$ such that $x$ lies in the image of $V_i \longrightarrow V$. Take $y \in V_i$ which is mapped to $x$. By Theorem 2.1 and Lemma 2.6, $V_i$ is a smooth representation of $K_i$. Therefore $\text{Stab}_{K_i}(y)$ is open in $K_i$, hence is open in $G(F)$. Since $V_i \longrightarrow V$ is $K_i$-equivariant, we have $\text{Stab}_{K_i}(y) \subset \text{Stab}_{K_i}(x) \subset \text{Stab}_{G(F)}(x)$. Thus $\text{Stab}_{G(F)}(x)$ is open in $G(F)$, as desired. 

Remark 2.8 Although we need the corollary above only for the case $F = \mathbb{Q}_p$, we proved it for a general $p$-adic field $F$ for the completeness.

3 Rapoport-Zink space for $GSp(4)$

3.1 The Rapoport-Zink space for $GSp(4)$ and its rigid analytic coverings

In this subsection, we recall basic definitions concerning with Rapoport-Zink spaces. General definitions are given in [RZ96], but here we restrict them to our special case.

Let $X$ be a 2-dimensional isoclinic $p$-divisible group over $\overline{\mathbb{F}}_p$ with slope $1/2$, and $\lambda_0 : X \cong X^\vee$ a (principal) polarization of $X$, namely, an isomorphism satisfying $\lambda_0^\vee = -\lambda_0$. Consider the contravariant functor $\mathcal{M} : \text{Nilp} \longrightarrow \text{Set}$ that associates $S$ with the set of isomorphism classes of pairs $(X, \rho)$ consisting of

1. a 2-dimensional $p$-divisible group $X$ over $S$,
2. and a quasi-isogeny (cf. [RZ96, Definition 2.8]) $\rho : X \otimes_{\mathbb{F}_p} \overline{S} \longrightarrow X \otimes_S \overline{S}$,

such that there exists an isomorphism $\lambda : X \longrightarrow X^\vee$ which makes the following
Tetsushi Ito and Yoichi Mieda

Diagram commutative up to multiplication by $\mathbb{Q}_p^\times$:

$$
\begin{array}{ccc}
X \otimes_{\mathbf{T}_p} \mathcal{S} & \xrightarrow{\rho} & X \otimes_{\mathcal{S}} \mathcal{S} \\
\downarrow_{\lambda_0 \otimes \text{id}} & & \downarrow_{\lambda \otimes \text{id}} \\
X^\vee \otimes_{\mathbf{T}_p} \mathcal{S} & \xrightarrow{\rho^\vee} & X^\vee \otimes_{\mathcal{S}} \mathcal{S}.
\end{array}
$$

Note that such $\lambda$ is uniquely determined by $(X, \rho)$ up to multiplication by $\mathbb{Z}_p^\times$ and gives a polarization of $X$. It is proved by Rapoport-Zink that $\mathcal{M}$ is represented by a special formal scheme (cf. [Ber96]) over $\text{Spf} \mathbb{Z}_p^\infty$. Moreover, $\mathcal{M}$ is separated over $\text{Spf} \mathbb{Z}_p^\infty$ [Far04, Lemme 2.3.23]. However, $\mathcal{M}$ is neither quasi-compact nor $p$-adic.

We put $\mathcal{M} = \mathcal{M}_{\text{red}}$, which is a scheme locally of finite type and separated over $\mathbb{F}_p$. It is known that $\mathcal{M}$ is 1-dimensional (for example, see [Vie08]) and every irreducible component of $\mathcal{M}$ is projective over $\mathbb{F}_p$ [RZ96, Proposition 2.32]. In particular, $\mathcal{M}$ has a locally finite quasi-compact open covering.

Let $D(X)_\mathbb{Q} = (N, \Phi)$ be the rational Dieudonné module of $X$, which is a 4-dimensional isocrystal over $\mathbb{Q}_p^\infty$. The fixed polarization $\lambda_0$ gives the alternating pairing $\langle \ , \ \rangle_{\lambda_0} : N \times N \longrightarrow \mathbb{Q}_p^\infty(1)$. We define the algebraic group $J$ over $\mathbb{Q}_p$ as follows: for a $\mathbb{Q}_p$-algebra $R$, the group $J(R)$ consists of elements $g \in \text{GL}(R \otimes_{\mathbb{Q}_p} N)$ such that

- $g$ commutes with $\Phi$,
- and $g$ preserves the pairing $\langle \ , \ \rangle_{\lambda_0}$ up to scalar multiplication, i.e., there exists $c(g) \in R^\times$ such that $(gx, gy)_{\lambda_0} = c(g)(x, y)_{\lambda_0}$ for every $x, y \in R \otimes_{\mathbb{Q}_p} N$.

It is an inner form of $\text{GSp}(4)$, since $D(X)_\mathbb{Q}$ is the isocrystal associated with a basic Frobenius conjugacy class of $\text{GSp}(4)$.

In the sequel, we also denote $J(\mathbb{Q}_p)$ by $J$. Every element $g \in J$ naturally induces a quasi-isogeny $g : X \longrightarrow X$ and the following diagram is commutative up to $\mathbb{Q}_p^\times$-multiplication:

$$
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow_{\lambda_0} & & \downarrow_{\lambda_0} \\
X^\vee & \xleftarrow{g^\vee} & X^\vee.
\end{array}
$$

Therefore, we can define the left action of $J$ on $\mathcal{M}$ by $g : \mathcal{M}(S) \longrightarrow \mathcal{M}(S)$; $(X, \rho) \longmapsto (X, \rho \circ g^{-1})$.

We denote the Raynaud generic fiber of $\mathcal{M}$ by $\mathcal{M}^{\text{rig}}$. It is defined as $t(\mathcal{M}) \setminus V(p)$, where $t(\mathcal{M})$ is the adic space associated with $\mathcal{M}$ (cf. [Hub94, Proposition 4.1]). As $\mathcal{M}$ is separated and special over $\mathbb{Z}_p^\infty$, $\mathcal{M}^{\text{rig}}$ is separated and locally of finite type over $\text{Spa}(\mathbb{Q}_p^\infty, \mathbb{Z}_p^\infty)$. Since $\mathcal{M}$ has a locally finite quasi-compact open covering, $\mathcal{M}^{\text{rig}}$ is taut by [Mie10b, Lemma 4.14]. Moreover, by using the period morphism [RZ96, Chapter 5], we can see that $\mathcal{M}^{\text{rig}}$ is 3-dimensional and smooth over $\text{Spa}(\mathbb{Q}_p^\infty, \mathbb{Z}_p^\infty)$ (cf. [RZ96, Proposition 5.17]).
Next we will consider level structures. Let $\tilde{X}$ be the universal $p$-divisible group over $\mathcal{M}$ and $\tilde{X}^{\text{rig}}$ be the associated $p$-divisible group over $\mathcal{M}^{\text{rig}}$. Note that $X^{\text{rig}}$ is an étale $p$-divisible group. Let us fix a polarization $\tilde{\lambda}: \tilde{X} \to \tilde{X}^\vee$ which is compatible with $\lambda_0$, i.e., satisfies the condition in the definition of $\mathcal{M}$. Let $S$ be a connected rigid space over $\mathbb{Q}_p$ (i.e., a connected adic space locally of finite type over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$), $\mathcal{M} \to \mathcal{M}^{\text{rig}}$ a morphism over $\mathbb{Q}_p$ and $\tilde{X}_S^{\text{rig}}$ the pull-back of $\tilde{X}^{\text{rig}}$. Fix a geometric point $\overline{x}$ of $S$ and an isomorphism $T_p(\mu_{p^\infty}, x) = \mathbb{Z}_p(1) \cong \mathbb{Z}_p$. Then $\tilde{\lambda}$ induces an alternating bilinear form $\psi_\overline{x}$ on the $\pi_1(S, \overline{x})$-module $(T_p(\tilde{X})^{\text{rig}})_{\overline{x}}$.

$$\psi_\overline{x}: (T_p(\tilde{X})^{\text{rig}})_{\overline{x}} \times (T_p(\tilde{X})^{\text{rig}})_{\overline{x}} \to T_p(\mu_{p^\infty}, x) \cong \mathbb{Z}_p.$$  

Fix a free $\mathbb{Z}_p$-module $L$ of rank 4 and a perfect alternating bilinear form $\psi_0: L \times L \to \mathbb{Z}_p$. Put $K_0 = \text{GSp}(L, \psi_0)$, $V = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $G = \text{GSp}(V, \psi_0)$. Let $T(S, \overline{x})$ be the set consisting of isomorphisms $\eta: L \to (T_p(\tilde{X})^{\text{rig}})_{\overline{x}}$ which map $\psi_0$ to $\mathbb{Z}_p^\times$-multiples of $\psi_\overline{x}$. It is independent of the choice of $\tilde{\lambda}$ and $T_p(\mu_{p^\infty}, x) \cong \mathbb{Z}_p$, since they are unique up to $\mathbb{Z}_p^\times$-multiplication. Obviously, the groups $K_0$ and $\pi_1(S, \overline{x})$ naturally act on $T(S, \overline{x})$.

For an open subgroup $K$ of $K_0$, a $K$-level structure of $\tilde{X}_S^{\text{rig}}$ means an element of $(T(S, \overline{x})/K)_{\pi_1(S, \overline{x})}$. Note that, if we change a geometric point $\overline{x}$ to $\overline{x}'$, the sets $(T(S, \overline{x})/K)_{\pi_1(S, \overline{x})}$ and $(T(S, \overline{x}')/K)_{\pi_1(S, \overline{x}')}^\text{rig}$ are naturally isomorphic. Thus the notion of $K$-level structures is independent of the choice of $\overline{x}$. The functor that associates $S$ with the set of $K$-level structures of $\tilde{X}_S^{\text{rig}}$ is represented by a finite Galois étale covering $\mathcal{M}_K^{\text{rig}} \to \mathcal{M}^{\text{rig}}$, whose Galois group is $K_0/K$. Since $T(S, \overline{x})$ is a $K_0$-torsor, $\mathcal{M}_K^{\text{rig}}$ coincides with $\mathcal{M}^{\text{rig}}$. If $K'$ is an open subgroup of $K$, we have a natural morphism $p_{KK'}: \mathcal{M}_K^{\text{rig}} \to \mathcal{M}_{K'}^{\text{rig}}$. Therefore, we get the projective system of rigid spaces $\{\mathcal{M}_K^{\text{rig}}\}_K$ indexed by the filtered ordered set of open subgroups of $K_0$, which is called the Rapoport-Zink tower. Obviously, the group $J$ acts on the projective system $\{\mathcal{M}_K^{\text{rig}}\}_K$.

Let $g$ be an element of $G$ and $K$ an open subgroup of $K_0$ which is enough small so that $g^{-1}Kg \subset K_0$. Then we have a natural morphism $\mathcal{M}_K^{\text{rig}} \to \mathcal{M}_{g^{-1}Kg}^{\text{rig}}$ over $\mathbb{Q}_p$. If $g \in K_0$, then it is given by $\eta \mapsto \eta \circ g$; for other $g$, it is more complicated [RZ96, 5.34]. In any case, we get a right action of $G$ on the pro-object $\varprojlim \mathcal{M}_K^{\text{rig}}$.

**Definition 3.1** We put $H^\ell_{\text{RZ}} = \lim_K H^\ell_{\mathcal{M}}(\mathcal{M}_K^{\text{rig}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p, \mathbb{Q}_\ell)$.

Here $H^\ell_{\mathcal{M}}(\mathcal{M}_K^{\text{rig}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p, \mathbb{Q}_\ell)$ is the compactly supported $\ell$-adic cohomology of $\mathcal{M}_K^{\text{rig}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p$ defined in [Hüb98]; note that $\mathcal{M}_K^{\text{rig}}$ is separated and taut. By the constructions above, $G \times J$ acts on $H^\ell_{\text{RZ}}$ on the left (the action of $j \in J$ is given by $(j^{-1})^*$). Obviously the action of $G$ on $H^\ell_{\text{RZ}}$ is smooth. On the other hand, it is known that the action of $J$ on $H^\ell_{\text{RZ}}$ is also smooth. This is due to Berkovich (see [Far04, Corollaire 4.4.7]); see also Remark 5.12, where we give another proof of the smoothness. Hence we get the smooth representation $H^\ell_{\text{RZ}}$ of $G \times J$.

Our main theorem is the following:
Theorem 3.2 (Non-cuspidality) The smooth representation $H^i_{RZ, \overline{Q}_l}$ of $G$ has no quasi-cuspidal subquotient unless $i = 2, 3, 4$.

For the definition of quasi-cuspidal representations, see [Ber84, 1.20]. Theorem 3.2 is proved in Section 5.

3.2 An integral model $\mathcal{M}_m$ of $\mathcal{M}^{\text{rig}}_{K_m}$

For an integer $m \geq 1$, let $K_m$ be the kernel of $GSp(L, \psi_0) \to GSp(L/p^m L, \psi_0)$. It is an open subgroup of $K_0$. We can describe the definition of $K_m$-level structures more concretely. As in the previous subsection, we fix a polarization $\lambda$ of $X^{\text{rig}}$ which is compatible with $\lambda_0$. It induces the alternating bilinear morphism between finite étale group schemes $\psi_0^{\lambda} : X^{\text{rig}}_S[p^m] \times X^{\text{rig}}_S[p^m] \to \mu_{p^m}$. Let $S \to \mathcal{M}^{\text{rig}}$ be as in the previous subsection. Then a $K_m$-level structure of $X^{\text{rig}}_S$ naturally corresponds bijectively to an isomorphism $\eta : L/p^m L \cong X^{\text{rig}}_S[p^m]$ between finite étale group schemes such that there exists an isomorphism $\mathbb{Z}/p^m \mathbb{Z} \cong \mu_{p^m,S}$ which makes the following diagram commutative:

\[
\begin{array}{ccc}
L/p^m L \times L/p^m L & \xrightarrow{\psi_0} & \mathbb{Z}/p^m \mathbb{Z} \\
\eta \times \eta \downarrow \cong & & \downarrow \cong \\
X^{\text{rig}}_S[p^m] \times X^{\text{rig}}_S[p^m] & \xrightarrow{\psi_0^{\lambda}} & \mu_{p^m,S}.
\end{array}
\]

For simplicity, we write $\mathcal{M}^{\text{rig}}_m$ for $\mathcal{M}^{\text{rig}}_{K_m}$ and $p_m$ for $p_{K_m K_n}$. In this subsection, we construct a formal model $\mathcal{M}_m$ of $\mathcal{M}^{\text{rig}}_{K_m}$ by following [Man05, §6]. Let $S$ be a formal scheme of finite type over $\mathcal{M}^{\text{rig}}$ and denote by $\tilde{X}_S$ the pull-back of $\tilde{X}$ to $S$. A Drinfeld $m$-level structure of $\tilde{X}_S$ is a morphism $\eta : L/p^m L \to \tilde{X}_S[p^m]$ satisfying the following conditions:

- the image of $\eta$ gives a full set of sections of $\tilde{X}_S[p^m]$, 
- and there exists a morphism $\mathbb{Z}/p^m \mathbb{Z} \to \mu_{p^m,S}$ which makes the following diagram commutative:

\[
\begin{array}{ccc}
L/p^m L \times L/p^m L & \xrightarrow{\psi_0} & \mathbb{Z}/p^m \mathbb{Z} \\
\eta \times \eta \downarrow & & \downarrow \\
\tilde{X}_S[p^m] \times \tilde{X}_S[p^m] & \xrightarrow{\psi_0^{\lambda}} & \mu_{p^m,S}.
\end{array}
\]

It is known that the functor that associates $S$ with the set of Drinfeld $m$-level structures of $\tilde{X}_S$ is represented by the formal scheme $\mathcal{M}_m$ which is finite over $\mathcal{M}$ (cf. [Man05, Proposition 15]). Note that, unlike the case of Lubin-Tate tower, $\mathcal{M}_m$ is not necessarily flat over $\mathcal{M}$. It is easy to show that $\mathcal{M}_m$ gives a formal model of
3.3 Compactly supported cohomology of \( \mathcal{M}_m \)

For \( m \geq 0 \), we denote the set of quasi-compact open subsets of \( \mathcal{M}_m \) by \( \mathcal{Q}_m \). It has a natural filtered order by inclusion.

**Definition 3.3** For an object \( \mathcal{F} \) of \( D^b(\mathcal{M}_m, \mathbb{Z}_\ell) \) or \( D^b(\mathcal{M}_m, \mathbb{Q}_\ell) \), we put

\[
H^i_c(\mathcal{M}_m, \mathcal{F}) = \lim_{U \in \mathcal{Q}_m} H^i_c(U, \mathcal{F}|_U).
\]

Assume that \( \mathcal{F} \) has a \( J \)-equivariant structure, namely, for every \( g \in J \) an isomorphism \( \varphi_g \colon g^* \mathcal{F} \xrightarrow{\sim} \mathcal{F} \) is given such that \( \varphi_{gg'} = \varphi_{g'} \circ g^* \varphi_g \) for every \( g, g' \in J \). Then \( J \) naturally acts on \( H^i_c(\mathcal{M}_m, \mathcal{F}) \) on the right. Therefore we get a left action of \( J \) on \( H^i_c(\mathcal{M}_m, \mathcal{F}) \) by taking the inverse \( J \to J^\prime = J^{-1} \).

**Theorem 3.4** Let \( \mathcal{F}^0 \) be an object of \( D^b_c(\mathcal{M}_m, \mathbb{Z}_\ell) \) and \( \mathcal{F} \) the object of \( D^b_c(\mathcal{M}_m, \mathbb{Q}_\ell) \) associated with \( \mathcal{F}^0 \). Assume that we are given a \( J \)-equivariant structure of \( \mathcal{F}^0 \) (thus \( \mathcal{F} \) also has a \( J \)-equivariant structure). Then \( H^i_c(\mathcal{M}_m, \mathcal{F}) \) is a finitely generated smooth \( J \)-representation.

**Proof**. Let \( U \) be an element of \( \mathcal{Q}_m \). By [Far04, Proposition 2.3.11], there exists a compact open subgroup \( K_U \) of \( J \) which stabilizes \( U \). Then \( H^i_c(U, \mathcal{F}|_U) \) is a finite-dimensional \( \mathbb{Q}_\ell \)-vector space endowed with the action of \( K_U \) and has the \( K_U \)-stable \( \mathbb{Z}_\ell \)-lattice \( \text{Im}(H^i_c(U, \mathcal{F}|_U) \to H^i_c(U, \mathcal{F}|_U)) \). Therefore \( H^i_c(\mathcal{M}_m, \mathcal{F}) \) is a smooth \( J \)-representation by Corollary 2.7.

To prove that \( H^i_c(\mathcal{M}_m, \mathcal{F}) \) is finitely generated, we may assume \( m = 0 \), for \( H^i_c(\mathcal{M}_m, \mathcal{F}) = H^i_c(\mathcal{M}_0, p_{0m*} \mathcal{F}) \). In this case, we can use the similar method as in [Far04, Proposition 4.4.13]. Let us explain the argument briefly. By [Far04, Théorème 2.4.13], there exists \( W \in \mathcal{Q}_0 \) such that \( \bigcup_{g \in J} gW = \mathcal{M}_0 \). We put \( K = \{ g \in J \mid gW = W \} \) and \( \Omega = \{ g \in J \mid gW \cap W \neq \emptyset \} \). As in the proof of [Far04, Proposition 4.4.13], \( K \) is a compact open subgroup of \( J \) and \( \Omega \) is a compact
subset of $J$. For $\alpha = ([g_1], \ldots, [g_n]) \in (J/K)^n$, we put $W_\alpha = g_1W \cap \cdots \cap g_nW$ and $K_\alpha = \bigcap_{j=1}^n g_jKg_j^{-1}$. For an open covering $\{gW\}_{g \in J/K}$, we can associate the Čech spectral sequence

$$E_1^{r,s} = \bigoplus_{\alpha \in (J/K)^{-r+1}} H_c^s(W_\alpha, \mathcal{F}|_{W_\alpha}) \implies H_c^{r+s}(\mathcal{M}_0, \mathcal{F}).$$

Consider the diagonal action of $J$ on $(J/K)^{-r+1}$. The coset

$$J \setminus \{\alpha \in (J/K)^{-r+1} \mid W_\alpha \neq \emptyset\}$$

is finite; indeed, if $W_\alpha \neq \emptyset$ for $\alpha = ([g_1], \ldots, [g_{r-1}]) \in (J/K)^{-r+1}$, then $g_1^{-1} \alpha \in \{1\} \times \Omega/K \times \cdots \times \Omega/K$, which is a finite set.

Take a system of representatives $\alpha_1, \ldots, \alpha_n$ of the coset above. Then there is a natural isomorphism $\bigoplus_{\alpha \in J_\alpha} H_c^s(W_\alpha, \mathcal{F}|_{W_\alpha}) \cong \text{c-Ind}_{K_\alpha}^{J_\alpha} H_c^s(W_{\alpha_j}, \mathcal{F}|_{W_{\alpha_j}})$. Hence $E_1^{r,s} \cong \bigoplus_{j=1}^n \text{c-Ind}_{K_\alpha}^{J_\alpha} H_c^s(W_{\alpha_j}, \mathcal{F}|_{W_{\alpha_j}})$ is a finitely generated $J$-module, since the cohomology $H_c^s(W_{\alpha_j}, \mathcal{F}|_{W_{\alpha_j}})$ is finite-dimensional for each $j$. By this and the fact that a finitely generated smooth $J$-module is noetherian [Ber84, Remarque 3.12], we conclude that $H_c^i(\mathcal{M}_0, \mathcal{F})$ is finitely generated.

**Lemma 3.5** Let $\mathcal{F}$ be an object of $D_c^b(\mathcal{M}, \mathcal{Q}_\ell)$ with a $K_0/K_m$-equivariant structure. Let $n$ be an integer with $0 \leq n \leq m$ and put $G = (p_{nm*}\mathcal{F})_{K_n/K_m}$. Then we have $H^i_c(\mathcal{M}, \mathcal{F})_{K_n/K_m} = H^i_c(\mathcal{M}, \mathcal{G})$.

**Proof.** Since the cardinality of $K_n/K_m$ is prime to $\ell$, $(-)_{K_n/K_m}$ commutes with $H^i_c$. Therefore, we have

$$H^i_c(\mathcal{M}, \mathcal{F})_{K_n/K_m} = \lim_{U \in \mathcal{Q}_m} H^i_c(U, \mathcal{F}|_U)_{K_n/K_m} = \lim_{V \in \mathcal{Q}_n} H^i_c(p_{nm}^{-1}(V), \mathcal{F}|_{p_{nm}^{-1}(V)})_{K_n/K_m} = \lim_{V \in \mathcal{Q}_n} H^i_c(V, (p_{nm*}\mathcal{F}|_{p_{nm}^{-1}(V)})_{K_n/K_m}) = H^i_c(V, G|_V) = H^i_c(\mathcal{M}, \mathcal{G}).$$

**Definition 3.6** A system of coefficients over the tower $\{\mathcal{M}_m\}_{m \geq 0}$ is the data $\mathcal{F} = \{\mathcal{F}_m\}_{m \geq 0}$ where $\mathcal{F}_m$ is an object of $D_c^b(\mathcal{M}_m, \mathcal{Q}_\ell)$ with a $K_0/K_m$-equivariant structure such that $(p_{nm*}\mathcal{F})_{K_n/K_m} = \mathcal{F}_n$ for every integers $m, n$ with $0 \leq n \leq m$. Then, by Lemma 3.5, we have $H^i_c(\mathcal{M}, \mathcal{F}_m)_{K_n/K_m} = H^i_c(\mathcal{M}, \mathcal{F}_n)$. We put $H^i_c(\mathcal{M}_\infty, \mathcal{F}) = \lim_{m \to \infty} H^i_c(\mathcal{M}_m, \mathcal{F}_m)$.

If each $\mathcal{F}_m$ is endowed with a $J$-equivariant structure which commutes with the given $K_0/K_m$-equivariant structure, and for every $0 \leq n \leq m$ the $J$-equivariant structures on $\mathcal{F}_m$ and $\mathcal{F}_n$ are compatible under the identification $(p_{nm*}\mathcal{F}_m)_{K_n/K_m} = \mathcal{F}_n$, then we say that we have a $J$-equivariant structure on $\mathcal{F}$. Such a structure naturally induces the action of $J$ on $H^i_c(\mathcal{M}_\infty, \mathcal{F})$.

By replacing “$D_c^b(\mathcal{M}_m, \mathcal{Q}_\ell)$” with “$D_c^b(\mathcal{M}, \mathcal{Z}_\ell)$”, we may also define a system of integral coefficients $\mathcal{F}$ over $\{\mathcal{M}_m\}_{m \geq 0}$, the cohomology $H^i_c(\mathcal{M}_\infty, \mathcal{F})$ and a $J$-equivariant structure on $\mathcal{F}$. 

10
\[ \ell \text{-adic cohomology of the Rapoport-Zink space for } \operatorname{GSp}(4) \]

**Corollary 3.7** Let \( \mathcal{F}^{\circ} \) be a system of integral coefficients over \( \{ \mathcal{M}_m \}_{m \geq 0} \) with a \( J \)-equivariant structure and \( \mathcal{F} \) the system of coefficients associated with \( \mathcal{F}^{\circ} \). Then \( H^i_c(\mathcal{M}_\infty, \mathcal{F}) \) is a smooth \( K_0 \times J \)-representation and \( H^i_c(\mathcal{M}_\infty, \mathcal{F})^{K_m} \) is a finitely generated smooth \( J \)-representation for every integer \( m \geq 0 \).

**Proof.** The smoothness is clear from Theorem 3.4 and the definition of \( H^i_c(\mathcal{M}_\infty, \mathcal{F}) \). Since \( H^i_c(\mathcal{M}_\infty, \mathcal{F})^{K_m} = H^i_c(\mathcal{M}_m, \mathcal{F}_m) \), the second assertion also follows from Theorem 3.4.

\[ \square \]

## 4 Shimura variety and \( p \)-adic uniformization

In this section, we introduce certain Shimura varieties (Siegel threefolds) related to our Rapoport-Zink tower. Let us fix a 4-dimensional \( \mathbb{Q} \)-vector space \( V' \) and an alternating perfect pairing \( \psi': V' \times V' \rightarrow \mathbb{Q} \). For an integer \( m \geq 0 \) and a compact open subgroup \( K^p \subset \operatorname{GSp}(V'_{\mathfrak{p},p}) = \operatorname{GSp}(\mathbb{Q}_p^\times, \mathbb{Q}_p^\times) \), consider the functor \( \operatorname{Sh}_{m,K^p} \) from the category of locally noetherian \( \mathbb{Z}_p \)-schemes to the category of sets that associates \( S \) with the set of isomorphism classes of quadruples \((A, \lambda, \eta^p, \eta_p)\) where

- \( A \) is a projective abelian surface over \( S \) up to prime-to-\( p \) isogeny,
- \( \lambda: A \rightarrow A^{\vee} \) is a prime-to-\( p \) polarization,
- \( \eta^p \) is a \( K^p \)-level structure of \( A \),
- and \( \eta_p: \mathbb{L}/p^m \mathbb{L} \rightarrow A[p^m] \) is a Drinfeld \( m \)-level structure

(for the detail, see [Kot92, §5]). Two quadruples \((A, \lambda, \eta^p, \eta_p)\) and \((A', \lambda', \eta'^p, \eta'_p)\) are said to be isomorphic if there exists a prime-to-\( p \) isogeny from \( A \) to \( A' \) which carries \( \lambda \) to a \( \mathbb{Z}_p^\times \)-multiple of \( \lambda' \), \( \eta^p \) to \( \eta'^p \) and \( \eta_p \) to \( \eta'_p \). We put \( \operatorname{Sh}_{K^p} = \operatorname{Sh}_{0,K^p} \). It is known that if \( K^p \) is sufficiently small, \( \operatorname{Sh}_{m,K^p} \) is represented by a quasi-projective scheme over \( \mathbb{Z}_p \) with smooth generic fiber. In the sequel, we always assume that \( K^p \) is enough small so that \( \operatorname{Sh}_{m,K^p} \) is representable. We denote the special fiber of \( \operatorname{Sh}_{m,K^p} \) (resp. \( \operatorname{Sh}_{K^p} \)) by \( \overline{\operatorname{Sh}}_{m,K^p} \) (resp. \( \overline{\operatorname{Sh}}_{K^p} \)).

For a compact open subgroup \( K^p \) contained in \( K^p \) and an integer \( m' \geq m \), we have the natural morphism \( \operatorname{Sh}_{m',K^p} \rightarrow \operatorname{Sh}_{m,K^p} \). This is a finite morphism and is moreover étale if \( m' = m \).

Next we recall the \( p \)-adic uniformization theorem, which gives a relation between \( \mathbf{\mathcal{M}} \) and \( \operatorname{Sh}_{K^p} \). Let us fix a polarized abelian surface \((A_0, \lambda_{A_0})\) over \( \overline{\mathbb{F}}_p \) such that \( A_0[p^\infty] \) is an isoclinic \( p \)-divisible group with slope \( 1/2 \). Note that such \((A_0, \lambda_{A_0})\) exists; for example, we can take \((A_0, \lambda_{A_0}) = (E^2, \lambda_E^2)\), where \( E \) is a supersingular elliptic curve over \( \overline{\mathbb{F}}_p \) and \( \lambda_E \) is a polarization of \( E \). By definition, the rational Dieudonné module \( D(A_0[p^\infty])_\mathbb{Q} \) is isomorphic to \( D(X)_\mathbb{Q} \). Thus, by the subsequent lemma, there is an isomorphism of isocrystals \( D(A_0[p^\infty])_\mathbb{Q} \cong D(X)_\mathbb{Q} \) which preserves the natural polarizations.

**Lemma 4.1** We use the notation in [RR96, §1]. Let \( d \geq 1 \) be an integer.

i) Let \( b \) be an element of \( B(\operatorname{GSp}_{2d}) \) and \( b' \) the image of \( b \) under the natural map \( B(\operatorname{GSp}_{2d}) \rightarrow B(\operatorname{GL}_{2d}) \). Then \( b \) is basic if and only if \( b' \) is basic.
Tetsushi Ito and Yoichi Mieda

ii) The map \( B(\text{GSp}_{2d})_{\text{basic}} \to B(\text{GL}_{2d})_{\text{basic}} \) induced from i) is an injection.

Proof. Note that the center of \( \text{GSp}_{2d} \) coincides with that of \( \text{GL}_{2d} \). Thus i) is clear, since \( b \) (resp. \( b' \)) is basic if and only if the slope morphism \( \nu_b: \mathbb{D} \to \text{GSp}_{2d} \) (resp. \( \nu_{b'}: \mathbb{D} \to \text{GL}_{2d} \)) factors through the center of \( \text{GSp}_{2d} \) (resp. \( \text{GL}_{2d} \)).

We prove ii). By [RR96, Theorem 1.15], it suffices to show that the natural map \( \pi_1(\text{GSp}_{2d}) \to \pi_1(\text{GL}_{2d}) \) is injective. Take a maximal torus \( T \) (resp. \( T' \)) of \( \text{GSp}_{2d} \) (resp. \( \text{GL}_{2d} \)) such that \( T \subset T' \). Then, since \( \text{Sp}_{2d} \) (resp. \( \text{SL}_{2d} \)) is simply connected, \( \pi_1(\text{GSp}_{2d}) \) (resp. \( \pi_1(\text{GL}_{2d}) \)) can be identified with the quotient of \( X_s(T) \) (resp. \( X_s(T') \)) induced by \( c: T \to \mathbb{G}_m \) (resp. \( \det: T' \to \mathbb{G}_m \)), where \( c \) denotes the similitude character of \( \text{GSp}_{2d} \). In particular, both \( \pi_1(\text{GSp}_{2d}) \) and \( \pi_1(\text{GL}_{2d}) \) are isomorphic to \( \mathbb{Z} \).

The commutative diagram

\[
\begin{array}{ccc}
\text{GSp}_{2d} & \xrightarrow{c} & \mathbb{G}_m \\
\downarrow & & \downarrow z \mapsto z^d \\
\text{GL}_{2d} & \xrightarrow{\det} & \mathbb{G}_m
\end{array}
\]

induces the commutative diagram

\[
\begin{array}{ccc}
X_s(T) & \xrightarrow{\times d} & X_s(\mathbb{G}_m) \\
\downarrow & & \downarrow \\
X_s(T') & \xrightarrow{\times d} & X_s(\mathbb{G}_m)
\end{array}
\]

In particular, the natural map \( \pi_1(\text{GSp}_{2d}) \to \pi_1(\text{GL}_{2d}) \) is injective.

Therefore, there is a quasi-isogeny \( X \to A[p^{\infty}] \) preserving polarizations. If we replace \( (X, \lambda_0) \) by the polarized \( p \)-divisible group \( (A_0[p^{\infty}], \lambda_{A_0}) \) associated with \( (A_0, \lambda_{A_0}) \), the \( G \)-representation \( H_{KZ} \) remains unchanged. Thus, in order to prove Theorem 3.2, we may assume that \( (X, \lambda_0) = (A_0[p^{\infty}], \lambda_{A_0}) \). In the remaining part of this article, we always assume it. Moreover, we fix an isomorphism \( H_1(A_0, A_{\infty,p}) \cong V_{\lambda_{A_0}} \) preserving alternating pairings.

Denote the isogeny class of \( (A_0, \lambda_{A_0}) \) by \( \phi \) and put \( I^\phi = \text{Aut}(\phi) \). We have natural group homomorphisms \( I^\phi \to J \) and \( I^\phi \to \text{Aut}(H_1(A_0, A_{\infty,p})) = \text{GSp}(V_{\lambda_{A_0}}) \). These are injective.

Let \( Y_{K^p} \) be the reduced closed subscheme of \( \text{Sh}_{K^p} \) such that \( Y_{K^p}(\mathbb{F}_p) \) consists of triples \( (A, \lambda, \eta^p) \) where the \( p \)-divisible group associated with \( (A, \lambda) \) is isogenous to \( (X, \lambda) \). It is the basic (or supersingular) stratum in the Newton stratification of \( \text{Sh}_{K^p} \). Note that \( (A, \lambda, \eta^p) \in \text{Sh}_{K^p}(\mathbb{F}_p) \) belongs to \( Y_{K^p}(\mathbb{F}_p) \) if and only if \( (A, \lambda) \in \phi \) ([Far04, Proposition 3.1.8], [Kot92, §7]). We denote the formal completion of \( \text{Sh}_{K^p} \) along \( Y_{K^p} \) by \( (\text{Sh}_{K^p})'_{Y_{K^p}} \).

Now we can state the \( p \)-adic uniformization theorem:
\section{\textit{$\ell$-adic cohomology of the Rapoport-Zink space for $\text{GSp}(4)$}}

**Theorem 4.2 ([RZ96, Theorem 6.30])** There exists a natural isomorphism of formal schemes:

\[
\theta_{K^p} : I^\phi \big( \tilde{\mathcal{M}} \times \text{GSp}(V'_{k_{\infty,p}})/K^p \big) \xrightarrow{\cong} (\text{Sh}_{K^p})_{/Y_{K^p}}.
\]

In the left hand side, $I^\phi$ acts on $\tilde{\mathcal{M}}$ through $I^\phi \hookrightarrow J$ and acts on $\text{GSp}(V'_{k_{\infty,p}})/K^p$ through $I^\phi \hookrightarrow \text{GSp}(V'_{k_{\infty,p}})$.

The isomorphisms $\{\theta_{K^p}\}_{K^p}$ are compatible with change of $K^p$. (It is also compatible with the Hecke action of $\text{GSp}_4(V'_{k_{\infty,p}})$, but we do not use it.)

Let us briefly recall the construction of the isomorphism $\theta_{K^p}$. Take a lift $(\tilde{X}, \tilde{\lambda}_0)$ of $(X, \lambda_0)$ over $\mathbb{Z}_{p\infty}$ (such a lift is unique up to isomorphism). Then, by the Serre-Tate theorem, the lift $(\tilde{A}_0, \tilde{\lambda}_{A_0})$ of $(A_0, \lambda_{A_0})$ is canonically determined. Let $S$ be an object of $\text{Nilp}_p(A, \rho) \in \tilde{\mathcal{M}}(S)$ and $[g] \in \text{GSp}(V'_{k_{\infty,p}})/K^p$. Then $\rho$ extends uniquely to the quasi-isogeny $\tilde{\rho} : \tilde{X} \times_{\mathbb{Z}_{p\infty}} S \rightarrow X$. We can see that there exist a polarized abelian variety $(A, \lambda)$ and a $p$-quasi-isogeny $\tilde{A}_0 \times_{\mathbb{Z}_{p\infty}} S \rightarrow A$ preserving polarizations, such that the associated quasi-isogeny $\tilde{A}_0[p^\infty] \times_{\mathbb{Z}_{p\infty}} S \rightarrow A[p^\infty]$ coincides with $\tilde{\rho}$. The fixed isomorphism $H_1(A_0, \mathbb{A}^\infty) \cong V'_{k_{\infty,p}}$ naturally induces a $K^p$-level structure $\eta$ of $A$. The morphism $\theta_{K^p}$ is given by $\theta_{K^p}((X, \rho), [g]) = (A, \lambda, \eta \circ g)$.

By composing the morphism $\tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}} \times \text{GSp}(V'_{k_{\infty,p}})/K^p$; $x \mapsto (x, [id])$, we get a morphism $\tilde{\mathcal{M}} \rightarrow (\text{Sh}_{K^p})_{/Y_{K^p}}$, which is also denoted by $\theta_{K^p}$. For $U \in \mathcal{Q}_0$, we denote the image of $U$ under $\theta_{K^p}$ by $Y_{K^p}(U)$. It is an open subset of $Y_{K^p}$.

**Proposition 4.3** Let $U$ be an element of $\mathcal{Q}_0$. Then for a sufficiently small compact open subgroup $K^p$ of $\text{GSp}(V'_{k_{\infty,p}})$, $\theta_{K^p}$ induces an isomorphism $U \cong Y_{K^p}(U)$.

Moreover, if we denote the open formal subscheme of $\tilde{\mathcal{M}}$ (resp. $(\text{Sh}_{K^p})_{/Y_{K^p}}$) whose underlying topological space is $U$ (resp. $Y_{K^p}(U)$) by $\tilde{\mathcal{M}}_{/U}$ (resp. $(\text{Sh}_{K^p})_{/Y_{K^p}(U)}$), then $\theta_{K^p}$ induces an isomorphism $\theta_{K^p} : \tilde{\mathcal{M}}_{/U} \cong (\text{Sh}_{K^p})_{/Y_{K^p}(U)}$.

**Proof.** The proof is similar to [Far04, Corollaire 3.1.4]. Put $\Gamma_{K^p} = I^\phi \cap K^p$, where the intersection is taken in $\text{GSp}(V'_{k_{\infty,p}})$. It is known that $\Gamma_{K^p}$ is discrete and torsion-free [RZ96]. By Theorem 4.2, $\theta_{K^p}$ gives an isomorphism from $\Gamma_{K^p} \setminus \tilde{\mathcal{M}}$ to an open and closed formal subscheme of $(\text{Sh}_{K^p})_{/Y_{K^p}}$. By the same method as in [Far04, Lemme 3.1.2, Proposition 3.1.3], we can see that every element $\gamma \in \Gamma_{K^p}$ other than 1 satisfies $\gamma \cdot U \cup U = \emptyset$ if $K^p$ is sufficiently small. For such $K^p$, the natural morphism $\tilde{\mathcal{M}}_{/U} \rightarrow \Gamma_{K^p} \setminus \tilde{\mathcal{M}}$ is an open immersion. Thus we have an open immersion $\tilde{\mathcal{M}}_{/U} \hookrightarrow \Gamma_{K^p} \setminus \tilde{\mathcal{M}} \xrightarrow{\theta_{K^p}} (\text{Sh}_{K^p})_{/Y_{K^p}(U)}$, whose image is $(\text{Sh}_{K^p})_{/Y_{K^p}(U)}$.

Next we consider the case with Drinfeld level structures at $p$. Let $Y_{m,K^p}$ be the closed subscheme of $\text{Sh}_{m,K^p}$ obtained as the inverse image of $Y_{K^p}$ under $\text{Sh}_{m,K^p} \hookrightarrow \text{Sh}_{K^p}$. By the construction of $\theta_{K^p}$ described above, we have the following result:
Corollary 4.4 Let \( m \geq 0 \) be an integer. We can construct naturally a morphism \( \theta_{m,K^p} : \mathcal{M}_m \rightarrow (\text{Sh}_{m,K^p})^h_{/Y_{m,K^p}} \) which makes the following diagram cartesian:

\[
\begin{align*}
\mathcal{M}_m & \xrightarrow{\theta_{m,K^p}} (\text{Sh}_{m,K^p})^h_{/Y_{m,K^p}} \\
\mathcal{M}_m & \xrightarrow{\theta_{m,K^p}} (\text{Sh}_{K^p})^h_{/Y_{K^p}}.
\end{align*}
\]

In particular, the similar result as Proposition 4.3 holds for \( \theta_{m,K^p} \); that is, for \( U \in \mathcal{Q}_m \), \( \theta_{m,K^p} \) induces \( (\mathcal{M}_m)_{/U} \xrightarrow{\cong} (\text{Sh}_{m,K^p})^h_{/Y_{m,K^p}(U)} \) if \( K^p \) is sufficiently small.

5 Proof of the non-cuspidality result

5.1 The system of coefficients \( \mathcal{F}^{[h]}, \mathcal{F}^{(h)} \)

Definition 5.1 Let \( m \geq 1 \) and \( 0 \leq h \leq 2 \) be integers. We denote by \( S_{m,h} \) the set of direct summands of \( L/p^m L \) of rank \( 4 - h \), and by \( S_{m,h}^{\text{coi}} \) the subset of \( S_{m,h} \) consisting of coisotropic direct summands (recall that \( I \in S_{m,h} \) is said to be coisotropic if \( I^\perp \subset I \)).

For \( I \in S_{m,h} \), let \( \overline{\text{Sh}}_{m,K^p,[I]} \) be the \( \overline{\mathbb{F}}_p \)-scheme defined by

\[
\overline{\text{Sh}}_{m,K^p,[I]}(S) = \{ (A, \lambda, \eta^p, \eta_p) \in \overline{\text{Sh}}_{m,K^p,[I]}(S) \mid I \subset \text{Ker} \eta_p \}.
\]

Clearly it is a closed subscheme of \( \overline{\text{Sh}}_{m,K^p} \). Similarly, we can define the closed formal subscheme \( \mathcal{M}_{m,[I]} \) of \( \mathcal{M}_m \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p \). Obviously, \( \mathcal{M}_{m,[I]} \) is stable under the action of \( J \) on \( \mathcal{M}_m \).

We denote by \( Y_{m,K^p,[I]} \) the closed subscheme of \( \overline{\text{Sh}}_{m,K^p,[I]} \) obtained as the inverse image of \( Y_{m,K^p} \). As Corollary 4.4, we have the following cartesian diagram of formal schemes:

\[
\begin{align*}
\mathcal{M}_{m,[I]} & \xrightarrow{\theta_{m,K^p}} (\overline{\text{Sh}}_{m,K^p,[I]})^h_{/Y_{m,K^p,[I]}} \\
\mathcal{M}_m & \xrightarrow{\theta_{m,K^p}} (\text{Sh}_{m,K^p})^h_{/Y_{m,K^p}}.
\end{align*}
\]

Definition 5.2 For \( I \in S_m \), we put

\[
\overline{\text{Sh}}_{m,K^p,(I)} = \overline{\text{Sh}}_{m,K^p,[I]} \setminus \bigcup_{I' \in S_m, I \subseteq I'} \overline{\text{Sh}}_{m,K^p,[I']},
\]

which is an open subscheme of \( \overline{\text{Sh}}_{m,K^p,[I]} \), and thus is a subscheme of \( \overline{\text{Sh}}_{m,K^p} \). Moreover, for an integer \( h \) with \( 0 \leq h \leq 2 \), we put \( \overline{\text{Sh}}_{m,K^p}^{[h]} = \bigcup_{I \in S_m} \overline{\text{Sh}}_{m,K^p,[I]}^{[h]} \) and
\(\text{\`etale inclusion is clear.}\)

\[\text{The latter part of iii) in Lemma 5.3 is the only place where the same}\]

\[\text{Remark 5.4} \quad \text{De} \text{\`e} \text{finition 5.5}\]

\[\text{It is clear from}\]

\[\text{Proof. Let us prove i). Put}\]

\[\text{immediate.}\]

\[\text{isogeny}\]

\[\text{nor slope 1 part, then it is a line of slope 1}\]

\[\text{Lemma 5.3}\]

\[\text{We have}\]

\[\text{For every integer}\]

\[\text{Let}\]

\[\text{has no multiplicative part. Therefore}\]

\[\text{rank}\]

\[\text{We define}\]

\[\text{with}\]

\[\text{We have}\]

\[\text{Proof.}\]

\[\text{By i),}\]

\[\text{\`etale part by i). Since}\]

\[\text{X}^\flat \cong \text{X}, \text{X has no multiplicative part. Therefore}\]

\[\text{is isoclinic of slope 1}\]

\[\text{indeed, if a Newton polygon with the terminal point (4, 2) has neither slope 0 part}\]

\[\text{nor slope 1 part, then it is a line of slope 1/2. Thus, by Lemma 4.1, there is a quasi-}\]

\[\text{isogeny}\]

\[\text{X} \longrightarrow \text{X preserving polarizations; namely,}\]

\[\text{The opposite inclusion is clear.}\]

\[\text{Remark 5.4}\]

\[\text{The latter part of iii) in Lemma 5.3 is the only place where the same}\]

\[\text{algorithm does not work in the case GSp(2d) with}\]

\[\text{Definition 5.5}\]

\[\text{Let}\]

\[\text{For}\]

\[\text{by}\]

\[\text{For an integer}\]

\[\text{For a}\]

\[\text{as}\]

\[\text{We define}\]

\[\text{and}\]

\[\text{as follows:}\]

\[\text{We define}\]

\[\text{as follows:}\]
Here $\theta_m: \mathcal{M}_m \to Y_{m,K^p}$ is the morphism induced from $\theta_{m,K^p}$ in Corollary 4.4.

These are independent of the choice of $K^p$; indeed, for another compact open subgroup $K'^p$ contained in $K^p$, the natural map $\text{Sh}_{m,K^p} \to \text{Sh}_{m,K'^p}$ is étale.

**Proposition 5.6** Let $h$ be an integer with $1 \leq h \leq 2$.

i) We have the following distinguished triangle:

$$\mathcal{F}^{[h-1]}_m \to \mathcal{F}^h_m \to \mathcal{F}^{[h]}_m \to \mathcal{F}^{[h-1]}_m[1].$$

ii) We have $\mathcal{F}^{(h)}_m = \bigoplus_{I \in S_{m,h}} \mathcal{F}_{m,I}$.

**Proof.** By the definition, i) is clear. ii) is also clear from Lemma 5.3 ii). □

**Proposition 5.7** For $I \in S_{m,h} \setminus S_{m,h}^{\text{coll}}$, we have $\mathcal{F}^o_{m,I} = \mathcal{F}_{m,I} = 0$.

**Proof.** We will prove that $Rj^1_{m,I}R\psi^1\mathbb{Z}_t = 0$. Since the dual of $Rj^1_{m,I}R\psi^1\mathbb{Z}_t$ is isomorphic to $j^1_{m,I}R\psi^1\mathbb{Z}_t(3)[6]$, it suffices to show that, for every $x \in \overline{\text{Sh}}_{m,K^p,(I)}(\mathbb{F}_p)$, no point on the generic fiber of $\text{Sh}_{m,K^p}$ specializes to $x$. In other words, for every complete discrete valuation ring $R$ with residue field $\mathbb{F}_p$ which is a flat $\mathbb{Z}_p$-algebra, and every $\mathbb{Z}_p^\times$-morphism $\bar{x}: \text{Spec} R \to \text{Sh}_{m,K^p}$, the image of the closed point of $\text{Spec} R$ under $\bar{x}$ does not lie in $\overline{\text{Sh}}_{m,K^p,(I)}$. This is a consequence of the following lemma. □

**Lemma 5.8** Let $R$ be a complete discrete valuation ring with perfect residue field $k$ and with mixed characteristic $(0,p)$, and $(X, \lambda)$ a polarized $p$-divisible group over $R$. We denote the generic (resp. special) fiber of $X$ by $X_g$ (resp. $X_s$). Then, for every $m \geq 1$, the kernel of the specialization map $X_g[p^m] \to X_s[p^m]$ is a coisotropic direct summand of $X_g[p^m]$.

**Proof.** We shall prove that the kernel of the specialization map $T_pX_g \to T_pX_s$ is a coisotropic direct summand of $T_pX_g$. Consider the exact sequence $0 \to X_g \to X_s \to X_{s,\text{ét}} \to 0$ over $k$. It is canonically lifted to the exact sequence $0 \to X_0 \to X \to X_{\text{ét}} \to 0$ over $R$, where $X_{\text{ét}}$ is an étale $p$-divisible group (cf. [Mes72, p. 76]). Thus we have the following commutative diagram, whose rows are exact:

$$
\begin{array}{cccccc}
0 & \to & T_pX_0 & \to & T_pX_g & \to & T_pX_{s,\text{ét}} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \to & 0 & \to & T_pX_s & \to & T_pX_{s,\text{ét}} & \to & 0.
\end{array}
$$

Hence the kernel of $T_pX_g \to T_pX_s$ coincides with $T_pX_{0,g}$. Therefore it suffices to show that the composite $(T_pX_{0,g})^\perp \to T_pX_g \to T_pX_{s,\text{ét}}$ is $0$.

On the other hand, by the polarization $T_pX_g \cong (T_pX_{0,g})^\vee(1)$, $(T_pX_{0,g})^\perp$ corresponds to $(T_pX_{s,\text{ét},g})^\vee(1) \cong T_pX_{s,\text{ét},g}^\vee$. Thus it suffices to prove that every Galois-equivariant homomorphism $T_pX_{s,\text{ét},g}^\vee \to T_pX_{s,\text{ét},g}$ is $0$. For this, we may replace the
Tate modules $T_p^\vee X_{s,\text{ét},q}$ and $T_p X_{s,\text{ét},q}$ by the rational Tate modules $V_p^\vee X_{s,\text{ét},q}$ and $V_p X_{s,\text{ét},q}$. These are crystalline representations and the corresponding filtered $\varphi$-modules are the rational Dieudonné modules $D(X_{s,\text{ét}}^\vee)_Q$ and $D(X_{s,\text{ét}})_Q$, respectively. Since the slope of the former is 1 and that of the latter is 0, there is no $\varphi$-homomorphism other than 0 from $D(X_{s,\text{ét}}^\vee)_Q$ to $D(X_{s,\text{ét}})_Q$. This completes the proof.

The following corollary is immediate from Proposition 5.6 ii) and Proposition 5.7.

**Corollary 5.9** For $h$ with $1 \leq h \leq 2$, we have $\mathcal{F}_m^{(h)} = \bigoplus_{I \in \mathcal{S}_m} \mathcal{F}_{m, I}$.

Let us consider the action of $K_0$. Since $K_0/K_m$ naturally acts on $\text{Sh}_{m,K}$ and the action of $g \in K_0/K_m$ maps $\text{Sh}_{m,K}^{[l]}$ to $\text{Sh}_{m,K}^{[g^{-1}l]}$, the complexes $\text{Sh}_{m,K}$ and $\text{Sh}_{m,K}^{[h]}$ are preserved by the action of $K_0/K_m$. Therefore $\mathcal{F}_m^{[h]}$, $\mathcal{F}_m$, $\mathcal{F}_m^{(h)}$ and $\mathcal{F}_m^{(h)}$ have natural $K_0/K_m$-equivariant structures. Moreover, in the same way as in [Mie10a, Proposition 2.5], we can observe that $\mathcal{F}_m^{[h]} \{ \mathcal{F}_m^{[h]} \}_{m \geq 1}$ and $\mathcal{F}_m^{(h)} \{ \mathcal{F}_m^{(h)} \}_{m \geq 1}$ form systems of coefficients (resp. integral coefficients) over $\{ \mathcal{M}_m \}_{m \geq 1}$.

Thanks to [Mie10b], we can define $J$-equivariant structures on the systems of coefficients introduced above.

**Proposition 5.10** The complexes $\mathcal{F}_m^0$, $\mathcal{F}_m^{[h]}$, $\mathcal{F}_m^{(h)}$, $\mathcal{F}_m$, $\mathcal{F}_m^{[h]}$ and $\mathcal{F}_m^{(h)}$ have natural $J$-equivariant structures. These structures are compatible with the distinguished triangles and the direct sum decompositions in Proposition 5.6.

**Proof.** We will prove the proposition for $\mathcal{F}_m^{(h)}$; other cases are similar. Put

$$\text{Sh}_{m,K}^{[h]} = (\text{Sh}_{m,K})^{[h]} \times \text{Sh}_{m,K}^{[h]} \times \text{Sh}_{m,K}^{[h]} \times \text{Sh}_{m,K}^{[h]} \times (\text{Sh}_{m,K}^{[h]} \times \text{Sh}_{m,K}^{[h]} \times \text{Sh}_{m,K}^{[h]}),$$

$$\mathcal{M}_m^{[h]} = \mathcal{M}_m \times (\text{Sh}_{m,K}^{[h]})^{[h]} \times \text{Sh}_{m,K}^{[h]} \times \text{Sh}_{m,K}^{[h]} \times \mathcal{M}_m^{[h]}.$$ 

Then, by [Mie10b, Proposition 3.11], we have the canonical isomorphism

$$(R_{j_m^{(h)}} R_{j_m^{[h]}} R_{\Psi,Q_\ell})|_{Y_m,K} \cong R_{(\text{Sh}_{m,K})^{[h]} \times \text{Sh}_{m,K}^{[h]} \times \text{Sh}_{m,K}^{[h]}} Q_\ell.$$ 

Moreover, since $\theta_{m,K}$ is étale (cf. Corollary 4.4), by [Mie10b, Proposition 3.14], we have the canonical isomorphism

$$\mathcal{F}_m^{(h)} \cong R_{\mathcal{M}_m^{[h]} \times \mathcal{M}_m^{(h)}} Q_\ell.$$ 

Since the action of $J$ on $\mathcal{M}_m$ preserves the closed formal subscheme $\mathcal{M}_m^{[h]}$ for every $I \in \mathcal{S}_m$, it also preserves the closed formal subscheme $\mathcal{M}_m^{(h)}$ for every $h$. Thus, by the functoriality [Mie10b, Proposition 3.7], $R_{\mathcal{M}_m^{[h]} \times \mathcal{M}_m^{(h)}} Q_\ell$ has a natural $J$-equivariant structure. We may import the structure into $\mathcal{F}_m^{(h)}$ by the isomorphism above.

The compatibilities with the exact sequence and the direct sum decomposition are clear from the construction (cf. [Mie10b, Remark 3.8]).
It is easy to see that the actions defined in the previous proposition give $J$-equivariant structures on the systems of (integral) coefficients $\mathcal{F}^{[h]}$, $\mathcal{F}^{[0]}$, $\mathcal{F}^{[h]}$ and $\mathcal{F}^{[h]}$. Thus we get the smooth representations $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ and $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ of $K_0 \times J$ (cf. Corollary 3.7).

**Proposition 5.11** There exists an isomorphism $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{[0]}) \cong H_{RZ}^i$, which is compatible with the action of $K_0 \times J$.

**Proof.** Let $m \geq 1$ be an integer and $U \in \mathcal{O}_m$. Then, by [Mie10b, Corollary 4.40] and Proposition 4.3, we have the $J$-equivariant isomorphism

$$H_c^i(U, \mathcal{F}^{[0]}|_U) \cong H_c^i((\mathcal{M}_m)^{rig} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p^\infty, \mathbb{Q}_\ell).$$

Since this isomorphism is functorial, we have $K_0 \times J$-equivariant isomorphisms

$$H_c^i(\mathcal{M}_m, \mathcal{F}^{[0]}) \cong \lim_{U \subset \mathcal{M}_m} H_c^i((\mathcal{M}_m)^{rig} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p^\infty, \mathbb{Q}_\ell) \cong H_c^i(\mathcal{M}_m^{rig} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p^\infty, \mathbb{Q}_\ell).$$

For the isomorphy of (*), we need [Hub98, Proposition 2.1 (iv)] and [Mie10b, Lemma 4.14].

**Remark 5.12** We can deduce from Proposition 5.11 and Corollary 3.7 that the action of $K_0 \times J$ on $H_{RZ}^i$ is smooth.

### 5.2 $G$-action on $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{[h]})$, $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{[h]})$

In this subsection, we define actions of $G$ on $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ and $H_c^i(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ by using the method in [Man05, §6]. Put $G^+ = \{ g \in G \mid g^{-1}L \subset L \}$, which is a submonoid of $G$. For $g \in G^+$, let $e(g)$ be the minimal non-negative integer such that $\text{Ker}(g^{-1}: V/L \rightarrow V/L)$ is contained in $p^{-e(g)}L/L$. Since $\text{Ker} g^{-1} = (gL+L)/L$, we have $gL \subset p^{-e(g)}L$.

In the sequel, we fix a compact open subgroup $K^p$ of $\text{GSp}(V^{\infty,p})$ and denote $\text{Sh}_{m,K^p}$, $\overline{\text{Sh}}_{m,K^p}$, $\overline{\text{Sh}}_{m,K^p,[l]}$, \ldots by $\text{Sh}_m$, $\overline{\text{Sh}}_m$, $\overline{\text{Sh}}_{m,[l]}$, \ldots, respectively. Moreover, we fix $g \in G^+$ and denote $e(g)$ by $e$ for simplicity.

Assume that $m \geq e$. Let us consider the $\mathbb{Z}_p^\infty$-scheme $\text{Sh}_{m,g}$ such that for a $\mathbb{Z}_p^\infty$-scheme $S$, the set $\text{Sh}_{m,g}(S)$ consists of isomorphism classes of quintuples $(A, \lambda, \eta^p, \eta_p, \mathcal{E})$ satisfying the following.

- The quadruple $(A, \lambda, \eta^p, \eta_p)$ gives an element of $\text{Sh}_m(S)$.
- $\mathcal{E} \subset X[p^c]$ is a finite flat subgroup scheme of order $p^{e_p(\det g^{-1})}$, where we put $X = A[p^\infty]$. It is self-dual with respect to $\lambda$, and satisfies $\eta_p(\text{Ker} g^{-1}) \subset \mathcal{E}(S)$, where $\eta_p$ denotes the composite $p^{-m}L/L \times_{p^m} L/p^mL \rightarrow X[p^m]$. 

18
$\ell$-adic cohomology of the Rapoport-Zink space for $\text{GSp}(4)$

- For $E$ as above, we have the following commutative diagram:

$$
\begin{array}{ccc}
p^{-m}L/L & \xrightarrow{\eta'_p} & X[p^m]/\mathcal{E} \\
\downarrow{g^{-1}} & & \downarrow{g^{-1}} \\
p^{-m}g^{-1}L/L & \xrightarrow{\eta'_p} & X[p^m]/\mathcal{E} \\
\uparrow{\Phi} & & \uparrow{\Phi} \\
L/p^{m+e}L & \xrightarrow{\sim} & p^{-m+e}L/L \\
\end{array}
$$

We denote the composite of the lowest row by $\eta_p \circ g$ and assume that it gives a Drinfeld $(m - e)$-level structure.

We have the two natural morphisms

$$
\text{pr}: \text{Sh}_{m,g} \longrightarrow \text{Sh}_m; (A, \lambda, \eta^p, \eta_p, \mathcal{E}) \longrightarrow (A, \lambda, \eta^p, \eta_p),
$$

$$
[g]: \text{Sh}_{m,g} \longrightarrow \text{Sh}_{m-e}; (A, \lambda, \eta^p, \eta_p, \mathcal{E}) \longrightarrow (A/\mathcal{E}, \lambda, \eta^p, \eta_p \circ g).
$$

It is known that these are proper morphisms, $\text{pr}$ induces an isomorphism on the generic fibers, and $[g]$ induces the action of $g$ on the generic fibers [Man05, Proposition 16, Proposition 17].

We can easily see that $\{\text{Sh}_{m,g}\}_{m \geq e}$ form a projective system whose transition maps are finite. Obviously, $\text{pr}$ and $[g]$ are compatible with change of $m$.

Similarly we can define the formal scheme $\hat{M}_{m,g}$ and the morphisms $\text{pr}: \hat{M}_{m,g} \longrightarrow \hat{M}_m$ and $[g]: \hat{M}_{m,g} \longrightarrow \hat{M}_{m-e}$. The former morphism induces an isomorphism on the Raynaud generic fibers and the composite $[g]^{\text{rig}} \circ (\text{pr}^{\text{rig}})^{-1}$ coincides with the action of $g$. The group $J$ naturally acts on $\hat{M}_{m,g}$ and two morphisms $\text{pr}$ and $[g]$ are compatible with the action of $J$. Moreover, if we denote by $Y_{m,g}$ the inverse image of $Y_m \subset \text{Sh}_m$ under $\text{pr}: \text{Sh}_{m,g} \longrightarrow \text{Sh}_m$, then we can construct a morphism $\theta_{m,g}: \hat{M}_{m,g} \longrightarrow (\text{Sh}_{m,g})^{\wedge}_{Y_{m,g}}$ which makes the following diagrams cartesian:

$$
\begin{array}{ccc}
\hat{M}_{m,g} & \xrightarrow{\theta_{m,g}} & (\text{Sh}_{m,g})^{\wedge}_{Y_{m,g}} \\
\downarrow{\text{pr}} & & \downarrow{\text{pr}} \\
\hat{M}_m & \xrightarrow{\theta_{m}} & (\text{Sh}_m)^{\wedge}_{Y_m}. \\
\end{array}
$$

$$
\begin{array}{ccc}
\hat{M}_{m,g} & \xrightarrow{\theta_{m,g}} & (\text{Sh}_{m,g})^{\wedge}_{Y_{m-g}} \\
\downarrow{[g]} & & \downarrow{[g]} \\
\hat{M}_{m-e} & \xrightarrow{\theta_{m-e}} & (\text{Sh}_{m-e})^{\wedge}_{Y_{m-e}}. \\
\end{array}
$$

Now let $h$ be an integer with $1 \leq h \leq 2$ and $I \in S_{m,h}$. Then we can define the subschemes $\overline{\text{Sh}}_{m,g,I}$, $\overline{\text{Sh}}_{m,g,I}$, $\overline{\text{Sh}}_{m,g}^{[h]}$ and $\overline{\text{Sh}}_{m,g}^{(h)}$ of $\text{Sh}_{m,g}$ in the same way as $\overline{\text{Sh}}_{m,I}$, $\overline{\text{Sh}}_{m,I}$, $\overline{\text{Sh}}_{m}^{[h]}$ and $\overline{\text{Sh}}_{m}^{(h)}$. The following proposition is obvious:

**Proposition 5.13** We have the commutative diagrams below:

$$
\begin{array}{ccc}
\overline{\text{Sh}}_{m,g,I} & \longrightarrow & \overline{\text{Sh}}_{m,g,I} \\
\downarrow{\text{pr}} & & \downarrow{\text{pr}} \\
\overline{\text{Sh}}_{m,I} & \longrightarrow & \overline{\text{Sh}}_{m,I} \\
\end{array}
$$

$$
\begin{array}{ccc}
\overline{\text{Sh}}_{m,g}^{[h]} & \longrightarrow & \overline{\text{Sh}}_{m,g}^{[h]} \\
\downarrow{\text{pr}} & & \downarrow{\text{pr}} \\
\overline{\text{Sh}}_{m}^{[h]} & \longrightarrow & \overline{\text{Sh}}_{m}^{[h]} \\
\end{array}
$$

$$
\begin{array}{ccc}
\overline{\text{Sh}}_{m,g}^{(h)} & \longrightarrow & \overline{\text{Sh}}_{m,g}^{(h)} \\
\downarrow{\text{pr}} & & \downarrow{\text{pr}} \\
\overline{\text{Sh}}_{m}^{(h)} & \longrightarrow & \overline{\text{Sh}}_{m}^{(h)} \\
\end{array}
$$

19
The rectangles in the left diagram is cartesian. The rectangles in the right diagram is cartesian up to nilpotent elements (namely, \( \overline{\text{Sh}}_{m,g}^{[I]} \rightarrow \overline{\text{Sh}}_{m}^{[I]} \times \overline{\text{Sh}}_{m,g} \) induces a homeomorphism on the underlying topological spaces, and so on).

Let us consider how \( \overline{\text{Sh}}_{m,g,[I]} \) are mapped by \([g]: \text{Sh}_{m,g} \rightarrow \text{Sh}_{m-e} \). For this purpose, let us introduce some notation.

**Definition 5.14** We denote by \( S_{\infty,h} \) the set of direct summands of \( L \) of rank \( 4-h \) and by \( S_{\infty,h}^{\text{coi}} \) the subset of \( S_{\infty,h} \) consisting of coisotropic direct summands. We can identify \( S_{\infty,h} \) with the set of direct summands of \( V \) of rank \( 4-h \); thus \( G \) naturally acts on \( S_{\infty,h} \) and \( S_{\infty,h}^{\text{coi}} \). Let \( g^{-1}: S_{m,h} \rightarrow S_{m-e,h} \) be the unique map which makes the following diagram commutative:

\[
\begin{array}{ccc}
S_{\infty,h} & \rightarrow & S_{m,h} \\
\downarrow g^{-1} & & \downarrow g^{-1} \\
S_{\infty,h} & \rightarrow & S_{m-e,h}.
\end{array}
\]

The existence of such \( g^{-1} \) follows from \( p^m L \subseteq p^e L \subseteq g^{-1}L \subseteq L \). Indeed, for direct summands \( I, I' \) of \( V \), we have

\[
I \cap L + p^m L = I' \cap L + p^m L \implies g^{-1}I \cap g^{-1}L + p^m L = g^{-1}I' \cap g^{-1}L + p^m L \\
\implies g^{-1}I \cap g^{-1}L \cap p^e L + p^m L = g^{-1}I' \cap g^{-1}L \cap p^e L + p^m L \\
\iff g^{-1}I \cap L + p^{m-e} L = g^{-1}I' \cap L + p^{m-e} L.
\]

Obviously \( g^{-1}: S_{m,h} \rightarrow S_{m-e,h} \) induces a map from \( S_{m,h}^{\text{coi}} \) to \( S_{m-e,h}^{\text{coi}} \).

**Proposition 5.15**

i) For \( h \in \{1,2\} \) and \( I \in S_{m,h} \), \([g]\) induces morphisms

\[
\begin{align*}
\text{Sh}_{m,g,[I]} & \rightarrow \text{Sh}_{m-e,[g^{-1}I]}, \\
\text{Sh}_{m,g}^{[h]} & \rightarrow \text{Sh}_{m-e}^{[h]}, \\
\text{Sh}_{m,g}^{[h]} & \rightarrow \text{Sh}_{m-e}^{[h]}.
\end{align*}
\]

ii) The rectangles of the following commutative diagram is cartesian up to nilpotent elements:

\[
\begin{array}{ccc}
\text{Sh}_{m,g}^{[h]} & \rightarrow & \text{Sh}_{m,g}^{[h]} \\
\downarrow & & \downarrow \quad [g] \\
\text{Sh}_{m-e}^{[h]} & \rightarrow & \text{Sh}_{m-e}^{[h]}
\end{array}
\]

**Proof.** By the definition of \([g]\), it is clear that \([g]\) induces a morphism \( \text{Sh}_{m,g,[I]} \rightarrow \text{Sh}_{m-e,[g^{-1}I]} \) for \( I \in S_{m,h} \), and thus induces a morphism \( \text{Sh}_{m,g}^{[h]} \rightarrow \text{Sh}_{m-e}^{[h]} \). On the other hand, note that, for every \((A,\lambda,\eta^p,\eta_p,E) \in \text{Sh}_{m,g}(\overline{\mathbb{F}}_p)\), the \( p \)-divisible groups \( A[p^\infty] \) and \( A[p^\infty]/E \) are isogenous, and thus have the same \( \acute{e} \text{tale heights.} \)
η-adic cohomology of the Rapoport-Zink space for $\text{GSp}(4)$

Therefore, by Lemma 5.3 i), the inverse image of $\overline{\text{Sh}}_{m-e}^{[h]}$ (resp. $\overline{\text{Sh}}_{m-e}^{(h)}$) under $[g]$ coincides with $\overline{\text{Sh}}_{m-g}^{[h]}$ (resp. $\overline{\text{Sh}}_{m-g}^{(h)}$) as sets. Therefore a morphism $\overline{\text{Sh}}_{m-g}^{(h)} \rightarrow \overline{\text{Sh}}_{m-e}^{(h)}$ is naturally induced and the rectangles in the diagram above are cartesian up to nilpotent elements. Finally, since $\overline{\text{Sh}}_{m-g,[I]} = \overline{\text{Sh}}_{m,g,[I]} \cap \overline{\text{Sh}}_{m-g}^{(h)}$ and $\overline{\text{Sh}}_{m-e,(g-1)[I]} = \overline{\text{Sh}}_{m-e,[g-1]I} \cap \overline{\text{Sh}}_{m-e}^{(h)}$, $[g]$ induces a morphism $\overline{\text{Sh}}_{m,g,[I]} \rightarrow \overline{\text{Sh}}_{m-e,(g-1)[I]}$. 

By Proposition 5.13 and Proposition 5.15, we have the natural cohomological correspondence $\gamma_g$ from $\mathcal{F}_{m-e}^{[h]}$ (resp. $\mathcal{F}_{m-e}^{(h)}$) to $\mathcal{F}_m^{[h]}$ (resp. $\mathcal{F}_{m}^{(h)}$); see §6. This cohomological correspondence induces a homomorphism $\gamma_g$ from $H_c^j(\overline{\mathcal{M}}_{m-e}, \mathcal{F}_{m-e}^{[h]})$ (resp. $H_c^j(\overline{\mathcal{M}}_{m-e}, \mathcal{F}_{m-e}^{(h)})$) to $H_c^j(\mathcal{M}, \mathcal{F}_m^{[h]})$ (resp. $H_c^j(\mathcal{M}, \mathcal{F}_m^{(h)})$). Indeed, for $U \in \mathcal{Q}_{m-e}$, we can take $U' \in \mathcal{Q}_{m}$ which contains $\text{pr}(\overline{\mathcal{M}}_{m-e})$. Then $\gamma_g$ induces $H_c^j(U, \mathcal{F}_{m-e}^{[h]}) \rightarrow H_c^j(U', \mathcal{F}_m^{[h]})$, and therefore induces $H_c^j(\mathcal{M}_{m-e}, \mathcal{F}_{m-e}^{[h]}) \rightarrow H_c^j(\mathcal{M}, \mathcal{F}_m^{[h]})$. It is easy to see that this homorphism is compatible with change of $m$; hence we get the endomorphism $\gamma_g$ on $H_c^j(\mathcal{M}, \mathcal{F}_m^{[h]})$ and $H_c^j(\mathcal{M}, \mathcal{F}_m^{(h)})$.

**Lemma 5.16** The endomorphism $\gamma_g$ commutes with the action of $J$ on $H_c^j(\mathcal{M}, \mathcal{F}_m^{[h]})$ and $H_c^j(\mathcal{M}, \mathcal{F}_m^{(h)})$.

**Proof.** We will only consider $\gamma_g$ on $H_c^j(\mathcal{M}, \mathcal{F}_m^{[h]})$, since the other case is similar. Let $U \in \mathcal{Q}_{m-e}$ and $U' \in \mathcal{Q}_m$ be as above and put $W = [g]^{-1}(U)$, $W' = \text{pr}^{-1}(U')$. It suffices to show the commutativity of the following diagram for $j \in J$:

\[
\begin{array}{ccc}
H_c^j(U, \mathcal{F}_{m-e}^{[h]}) & \xrightarrow{[g]^*} & H_c^j(W, \mathcal{F}_m^{[h]}) \\
\downarrow j & & \downarrow j \\
H_c^j(U', \mathcal{F}_{m-e}^{[h]}) & \xrightarrow{[g]^*} & H_c^j(W', \mathcal{F}_m^{[h]})
\end{array}
\]

By the construction of the $J$-actions, the left and the middle rectangles are commutative. On the other hand, since $\text{pr}$ is proper and induces an isomorphism on the generic fiber, $\text{pr}^*$ is an isomorphism and its inverse is $\text{pr}^*$. As $\text{pr}^*$ commutes with the $J$-action, the right rectangle above is also commutative. This concludes the proof.

**Lemma 5.17** i) For $g, g' \in G^+$, $\gamma_{gg'} = \gamma_g \circ \gamma_{g'}$.

ii) For $g \in K_0$, $\gamma_g$ coincides with the action of $K_0$ on $H_c^j(\mathcal{M}, \mathcal{F}_m^{[h]})$ or $H_c^j(\mathcal{M}, \mathcal{F}_m^{(h)})$, which we already introduced.

iii) The endomorphism $\gamma_{[p^{-1}]^* \text{id}}$ an isomorphism (in fact, it coincides with the action of $p^{-1} \cdot \text{id} \in J$).

**Proof.** i) follows from Corollary 6.3. ii) and iii) are consequences of [Man05, Proposition 16, Proposition 17] and the analogous properties for the Rapoport-Zink spaces (cf. [Man04, Proposition 7.4 (4), (5)]).
Note that $G$ is generated by $G^+$ and $p \cdot \text{id}$ as a monoid. Therefore, by the lemma above, we can extend the actions of $K_0$ on $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ and $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ to whole $G$. Together with Lemma 5.16, we have a smooth $G \times J$-module structures on $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ and $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]})$. We can observe without difficulty that the isomorphism in Proposition 5.11 is in fact compatible with the action of $G$:

**Proposition 5.18** The isomorphism $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[0]}) \cong H^i_{R\mathbb{Z}}$ in Proposition 5.11 is an isomorphism of $G \times J$-modules.

Next we investigate the $G$-module structure of $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ for $h \in \{1, 2\}$. Let us fix an element $\tilde{I}(h)$ of $S^\text{col}_{\infty,h}$ and denote its image under the natural map $S^\text{col}_{\infty,h} \rightarrow S^\text{col}_{m,h}$ by $\tilde{I}(h)_m$. Put $P_h = \text{Stab}_G(\tilde{I}(h))$, which is a maximal parabolic subgroup of $G$. Then we can identify $S_{\infty,h}$ with $G/P_h = K_0/(P_h \cap K_0)$ and $S_{m,h}$ with $K_m \backslash G/P_h = K_m \backslash K_0/(P_h \cap K_0)$. For $g \in G^+$ and an integer $m$ with $m \geq e := e(g)$, $g^{-1}: S_{m,h} \rightarrow S_{m-e,h}$ is identified with the map $K_m \backslash G/P_h \rightarrow K_{m-e} \backslash G/P_h$.

**Definition 5.19** We put $H^i_c(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)_m}) = \lim_{\rightarrow m} H^i_c(\mathcal{M}_m, \mathcal{F}_m, \tilde{I}(h)_m)$. Here the transition maps are given as follows: for integers $1 \leq m \leq m'$,

$H^i_c(\mathcal{M}_m, \mathcal{F}_m, \tilde{I}(h)_m) \rightarrow H^i_c(\mathcal{M}_{m'}, \mathcal{F}_{m'}, \tilde{I}(h)_m) \rightarrow \bigoplus_{I' \in S^\text{col}_{m',h}} H^i_c(\mathcal{M}_{m'}, \mathcal{F}_{m', I'})$,

$\rightarrow H^i_c(\mathcal{M}_{m'}, \mathcal{F}_{m', \tilde{I}(h)_{m'}})$.

It is easy to see that $H^i_c(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})$ has a structure of a smooth $P_h \times J$-module (use Theorem 3.4 and Proposition 5.15 i)). For each $m \geq 1$ we have the homomorphism

$H^i_c(\mathcal{M}_m, \mathcal{F}^{[h]}) = \bigoplus_{I \in S^\text{col}_{m,h}} H^i_c(\mathcal{M}_m, \mathcal{F}_m, I) \rightarrow H^i_c(\mathcal{M}_m, \mathcal{F}_m, \tilde{I}(h)_m)$,

which induces the homomorphism $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]}) \rightarrow H^i_c(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})$. By Proposition 5.15 i), we can prove that this is a homomorphism of $P_h \times J$-modules.

**Proposition 5.20** We have an isomorphism $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]}) \cong \text{Ind}_{P_h}^G H^i_c(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})$ of $G \times J$-modules.

**Proof.** By the Frobenius reciprocity, we have a $G$-homomorphism $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]}) \rightarrow \text{Ind}_{P_h}^G H^i_c(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})$. We shall observe that this is bijective. For an integer $m \geq 1$, we have

$H^i_c(\mathcal{M}_m, \mathcal{F}^{[h]}) = \bigoplus_{I \in S^\text{col}_{m,h}} H^i_c(\mathcal{M}_m, \mathcal{F}_m, I) = \bigoplus_{g \in K_0/K_0}(P_h \cap K_0) H^i_c(\mathcal{M}_m, \mathcal{F}_m, g^{-1}\tilde{I}(h)_m)$

$\cong \text{Ind}_{(P_h \cap K_0)/(P_h \cap K_0)}^{K_0/K_0} H^i_c(\mathcal{M}_m, \mathcal{F}_m, \tilde{I}(h)_m)$.
ℓ-adic cohomology of the Rapoport-Zink space for GSp(4)

where the last isomorphism, due to [Boy99, Lemme 13.2], is an isomorphism as $K_0$-modules. By taking the inductive limit, we have isomorphisms

$$H^i_c(\mathcal{M}_\infty, \mathcal{F}^{(h)}) \cong \text{Ind}_{P_h \cap K_0}^{K_0} H^i_c(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h(m))})$$

(the second isomorphism follows from the Iwasawa decomposition $G = P_h K_0$). By the proof of [Boy99, Lemme 13.2], it is easy to see that the first isomorphism above is nothing but the $K_0$-homomorphism obtained by the Frobenius reciprocity for $P_h \cap K_0 \subset K_0$. Therefore the composite of the two isomorphisms above coincides with the $G$-homomorphism introduced at the beginning of this proof. Thus we conclude the proof.

5.3 Proof of the main theorem

We begin with the following result on non-cuspidality:

**Theorem 5.21** For every $i \in \mathbb{Z}$ and $h \in \{1, 2\}$, the $G$-module $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{(h)})_{\overline{\mathbb{Q}_\ell}}$ has no quasi-cuspidal subquotient.

By Proposition 5.20 and [Ber84, 2.4], it suffices to show the following proposition:

**Proposition 5.22** Let $h \in \{1, 2\}$. The unipotent radical $U_h$ of $P_h$ acts trivially on $H^i_c(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})_{\overline{\mathbb{Q}_\ell}}$.

To prove Proposition 5.22, we need some preparations. In the sequel, let $G$ and $H$ be connected reductive groups over $\mathbb{Q}_p$, $P$ a parabolic subgroup of $G$ and $U$ the unipotent radical of $P$. We put $P = P(\mathbb{Q}_p)$, $H = H(\mathbb{Q}_p)$ and $U = U(\mathbb{Q}_p)$.

**Lemma 5.23** Let $A$ be a noetherian $\mathbb{Q}$-algebra and $V$ an $A$-module with a smooth $P$-action. Assume that $V$ is $A$-admissible in the sense of [Ber84, 1.16]. Then $U$ acts on $V$ trivially.

**Proof.** First assume that $A$ is Artinian. Then we can prove the lemma in the same way as [Boy99, Lemme 13.2.3] (we use length in place of dimension).

For the general case, we use noetherian induction. Assume that the lemma holds for every proper quotient of $A$. Take a minimal prime ideal $\mathfrak{p}$ of $A$. Then $A_\mathfrak{p}$ is Artinian and $V_\mathfrak{p}$ is an $A_\mathfrak{p}$-admissible representation of $P$ (note that $(V_\mathfrak{p})^K = (V^K)_\mathfrak{p}$ for every compact open subgroup $K$ of $P$). Therefore $U$ acts on $V_\mathfrak{p}$ trivially. Let $V'$ (resp. $V''$) be the kernel (resp. image) of $V \rightarrow V_\mathfrak{p}$. Note that $V'$ and $V''$ are $A$-admissible representations of $P$, for $A$ is noetherian.

Consider the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & V' & \rightarrow & V & \rightarrow & V'' & \rightarrow & 0 \\
& & (1) & & (2) & & (3) \\
0 & \rightarrow & V'_U & \rightarrow & V_U & \rightarrow & (V_\mathfrak{p})_U. \\
\end{array}
$$

23
It is well-known that the functor taking $U$-coinvariant $V \mapsto V_U$ is an exact functor; thus the lower row in the diagram above is exact. On the other hand, the arrow labeled $(3)$ is injective, since it is the composite of $V'' \hookrightarrow V_p \xrightarrow{\cong} (V_p)_U$. Therefore, by the snake lemma, the injectivity of $(2)$ is equivalent to that of $(1)$. In other words, we have only to prove that the action of $U$ on $V'$ is trivial.

On the other hand, by the definition, $V'$ is the union of $V_s := \{x \in V \mid sx = 0\}$ for $s \in A \setminus p$. Since $V_s$ can be regarded as an admissible $A/(s)$-representation, $U$ acts on $V_s$ trivially by the induction hypothesis. Hence $U$ acts on $V'$ trivially.

**Proposition 5.24** Let $V$ be a smooth representation of $P \times H$ over $\overline{\mathbb{Q}}_l$ and assume that for every compact open subgroup $K$ of $P$, $V^K$ is a finitely generated $H$-module. Then $U$ acts on $V'$ trivially.

**Proof.** Since $\overline{\mathbb{Q}}_l$ and $\mathbb{C}$ are isomorphic as fields, we may replace $\overline{\mathbb{Q}}_l$ in the statement by $\mathbb{C}$. Let $\mathfrak{Z}$ be the Bernstein center of $H$ [Ber84]. It is decomposed as $\mathfrak{Z} = \prod_{\theta \in \Theta} \mathfrak{Z}_\theta$, where $\Theta$ denotes the set of connected components of the Bernstein variety of $H$. For $\theta \in \Theta$, we denote the $\theta$-part of $V$ by $V_\theta$. Then we have the canonical decomposition $V = \bigoplus_{\theta \in \Theta} V_\theta$, which is compatible with the action of $P \times H$. Therefore, by replacing $V$ with $V_\theta$, we may assume that the action of $\mathfrak{Z}$ on $V$ factors through $\mathfrak{Z}_\theta$ for some $\theta \in \Theta$.

By the assumption and [Ber84, Proposition 3.3], for every compact open subgroup $K$ of $P$, $V^K$ is a $\mathfrak{Z}_\theta$-admissible $H$-module. Namely, for every compact open subgroup $K$ (resp. $K'$) of $P$ (resp. $H$), $V^K \times K'$ is a finitely generated $\mathfrak{Z}_\theta$-module. In other words, for every compact open subgroup $K'$ of $H$, $V^K$ is a $\mathfrak{Z}_\theta$-admissible $P$-module. Since $\mathfrak{Z}_\theta$ is a finitely generated $\mathbb{C}$-algebra, $U$ acts trivially on $V^K$ by Lemma 5.23. Therefore $U$ acts trivially on $V$ also.

**Proof of Proposition 5.24.** By Proposition 5.24, we have only to prove that, for every $m \geq 1$, $H^i_c(\mathcal{M}_\infty, \mathcal{F}_{(h)})^P_h \cap K_m$ is a finitely generated $J$-module (recall that a finitely generated $J$-module is noetherian [Ber84, Remarque 3.12]). As a $J$-module, it is a direct summand of $\text{Ind}_{P_h}^G H^2_c(\mathcal{M}_\infty, \mathcal{F}_{(h)})^{K_m} \cong H^2_c(\mathcal{M}_\infty, \mathcal{F}^{(h)})^{K_m}$. On the other hand, by Corollary 3.7, $H^2_c(\mathcal{M}_\infty, \mathcal{F}^{(h)})^{K_m}$ is a finitely generated $J$-module. Thus $H^i_c(\mathcal{M}_\infty, \mathcal{F}_{(h)})^P_h \cap K_m$ is also finitely generated.

**Proposition 5.25** Let $i$ be an integer. If $i \geq 5$, then $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[2]}) = 0$. On the other hand, if $i \leq 1$, then $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[0]}) = 0$.

**Proof.** By the definition, it suffices to show that for every $m \geq 1$ and every $U \in \mathcal{Q}_m$ we have $H^0_c(U, \mathcal{F}^{[2]}_m|_U) = 0$ for $i \geq 5$ and $H^0_c(U, \mathcal{F}^{[0]}_m|_U) = 0$ for $i \leq 1$. Thus the claim is reduced to the following lemma.

**Lemma 5.26** Let $S$ be the spectrum of a strict henselian discrete valuation ring and $X$ a separated $S$-scheme of finite type. We denote its special (resp. generic)
fiber by $X_s$ (resp. $X_{\eta}$). Let $Z$ be a closed subscheme of $X_s$ and denote the natural closed immersion $Z \longrightarrow X$ by $i$. Assume that $X_{\eta}$ is smooth of pure dimension $d$ and $Z$ is purely $d'$-dimensional.

Then we have $H^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = H^n_c(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = 0$ for $n > d + d'$ and $H^n_c(Z, R^\ell i^! R^\psi_X \mathbb{Q}_\ell) = 0$ for $n < d - d'$.

**Proof.** First note that $H^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = H^n_c(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = 0$ if $n > 2 \dim Z$ or $n > 2 \dim (\text{supp } R^k \psi_X \mathbb{Q}_\ell)$. By [BBD82, Proposition 4.4.2], for each $k \geq 0$ we have $\dim (\text{supp } R^k \psi_X \mathbb{Q}_\ell) \leq d - k$; therefore if $n + k > d + d'$ then we have

$$n > d' + (d - k) \geq \dim Z + \dim (\text{supp } R^k \psi_X \mathbb{Q}_\ell)$$

and thus $H^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = H^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = 0$. By the spectral sequence, we have $H^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = H^n_c(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = 0$ for $n > d + d'$.

On the other hand, by the Poincaré duality, we have

$$H^n_c(Z, R^\ell i^! R^\psi_X \mathbb{Q}_\ell) = H^{-n}(Z, D_Z(R^\ell i^! R^\psi_X \mathbb{Q}_\ell))^{\vee} = H^{-n}(Z, i^* R^\psi_X D_X, \mathbb{Q}_\ell)^{\vee} = H^{-n}(Z, i^* R^\psi_X \mathbb{Q}_\ell(d)[2d])^{\vee} = H^{2d-n}(Z, i^* R^\psi_X \mathbb{Q}_\ell)^{\vee}(-d),$$

where $D_Z$ (resp. $D_{X_{\eta}}$) denotes the dualizing functor with respect to $Z$ (resp. $X_{\eta}$). Therefore it vanishes if $2d - n > d + d'$, namely, $n < d - d'$.

Now we can prove our main theorem.

**Proof of Theorem 3.2.** By Proposition 5.11 and Proposition 5.25, we have $H^i_{RZ} = 0$ for $i \leq 1$. Therefore we may assume that $i \geq 5$.

By Proposition 5.6 i), we have the exact sequence of smooth $G$-modules

$$H^{i-1}_c(\mathcal{M}_\infty, F^{[h]})_{\mathbb{Q}_\ell} \longrightarrow H^i_c(\mathcal{M}_\infty, F^{[h-1]})_{\mathbb{Q}_\ell} \longrightarrow H^i_c(\mathcal{M}_\infty, F^{[h]})_{\mathbb{Q}_\ell}$$

for every $h$ with $1 \leq h \leq 2$. Moreover, $H^1_c(\mathcal{M}_\infty, F^{[0]})_{\mathbb{Q}_\ell} = 0$ (Proposition 5.25), we can inductively prove that $H^i_c(\mathcal{M}_\infty, F^{[h]})_{\mathbb{Q}_\ell}$ has no quasi-cuspidal subquotient; indeed, the property that a representation has no quasi-cuspidal subquotient is stable under sub, quotient and extension (use the canonical decomposition in [Ber84, 2.3.1]). In particular, $H^1_c(\mathcal{M}_\infty, F^{[0]})_{\mathbb{Q}_\ell} \cong H^1_{RZ, \mathbb{Q}_\ell}$ (cf. Proposition 5.18) has no quasi-cuspidal subquotient. This completes the proof.

**6 Appendix: Complements on cohomological correspondences**

In this section, we recall the notion of cohomological correspondences (cf. [SGA5, Exposé III], [Fuj97]) and give some results on them. These are used to define the action of $G$ on $H^{i}_c(\mathcal{M}_\infty, F^{[h]})$ and $H^i_c(\mathcal{M}_\infty, F^{[h]}).$
In this section, we change our notation. Let $k$ be a field and $\ell$ a prime number which is invertible in $k$. We denote one of $\mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Q}_\ell$ by $\Lambda$. Let $X_1$ and $X_2$ be schemes which are separated of finite type over $k$, and $L_i$ an object of $D^b_c(X_i, \Lambda)$ for $i = 1, 2$ respectively. A cohomological correspondence from $L_1$ to $L_2$ is a pair $(\gamma, c)$ consisting of a separated $k$-morphism of finite type $\gamma: \Gamma \longrightarrow X_1 \times X_2$ and a morphism $c: \gamma_1^* L_1 \longrightarrow R\gamma_2^! L_2$ in the category $D^b_c(\Gamma, \Lambda)$, where we denote $\text{pr}_i \circ \gamma$ by $\gamma_i$. For simplicity, we also write $c$ for $(\gamma, c)$, if there is no risk of confusion. If we are given a cohomological correspondence $(\gamma, c)$ where $\gamma_1$ is proper, then we have the associated morphism $R\Gamma_c(c): R\Gamma_c(X_1, L_1) \longrightarrow R\Gamma_c(X_2, L_2)$ by composing

$$
R\Gamma_c(X_1, L_1) \xrightarrow{\gamma_1} R\Gamma_c(\Gamma, \gamma_1^* L_1) \xrightarrow{R\Gamma_c(c)} R\Gamma_c(\Gamma, R\gamma_2^! L_1) = R\Gamma_c(X_2, R\gamma_2^* R\gamma_2^! L_2) \xrightarrow{\text{adj}} R\Gamma_c(X_2, L_2).
$$

We can compose two cohomological correspondences. Let $X_3$ be another scheme which is separated of finite type over $k$ and $L_3 \in D^b_c(X_3, \Lambda)$. Let $(\gamma', c')$ be a cohomological correspondence from $L_2$ to $L_3$. Consider the following diagram

$$
\begin{array}{ccc}
\Gamma \times_{X_2} \Gamma' & \xrightarrow{\text{pr}_2} & \Gamma' \\
\text{pr}_1 \downarrow & & \downarrow \gamma_1' \downarrow \\
\Gamma & \xrightarrow{\gamma_2} & X_2 \\
\downarrow \gamma_1 & & \\
X_1,
\end{array}
$$

Let $\gamma''$ be the natural morphism $\Gamma \times_{X_2} \Gamma' \longrightarrow X_1 \times X_3$ and $c'': \gamma''_1^* L_1 \longrightarrow R\gamma_2''^! L_3$ the composite of

$$
\gamma''_1^* L_1 = \text{pr}_1^* \gamma_1^* L_1 \xrightarrow{\text{pr}_1(c)} \text{pr}_1^* R\gamma_2^! L_2 \xrightarrow{\text{b.c.}} R\text{pr}_2^* \gamma_1^* L_2 \xrightarrow{R\text{pr}_2^! (c')} R\text{pr}_2^* R\gamma_2^! L_3 = R\gamma_2''^! L_3,
$$

where b.c. denotes the base change morphism. We call the cohomological correspondence $(\gamma'', c'')$ the composite of $(\gamma, c)$ and $(\gamma', c')$, and denote it by $c' \circ c$. It is not difficult to see that if $\gamma_1$ and $\gamma_1'$ are proper, then $\gamma''_1$ is also proper and $R\Gamma_c(c' \circ c) = R\Gamma_c(c') \circ R\Gamma_c(c)$.

Let us recall some operations for cohomological correspondences. Let $X_1, X_2, X_1'$ and $X_2'$ be schemes which are separated of finite type over $k$, and $\gamma: \Gamma \longrightarrow X_1 \times X_2$ and $\gamma': \Gamma' \longrightarrow X'_1 \times X'_2$ separated $k$-morphisms of finite type. Assume that the following commutative diagram is given:

$$
\begin{array}{ccc}
X_1 & \xleftarrow{\gamma_1} & \Gamma' \xrightarrow{\gamma_2} X_2' \\
\downarrow a_1 & & \downarrow a \\
X_1 & \xleftarrow{\gamma_1} & \Gamma \xrightarrow{\gamma_2} X_2.
\end{array}
$$

26
First assume that every vertical morphism is proper. Let $L'_i$ be an object of $D^b_c(X_i, \Lambda)$ for each $i = 1, 2$ and $(\gamma', c')$ a cohomological correspondence from $L'_1$ to $L'_2$. Then we can define the cohomological correspondence $(\gamma, a, c')$ from $Ra_1, L'_1$ to $Ra_2, L'_2$ by

$$\gamma'^*_1Ra_1, L'_1 \xrightarrow{\text{b.c.}} Ra_1, \gamma'^*_1L'_1 \xrightarrow{Ra_1, (c')} Ra_1, R\gamma'_2L'_2 = Ra_1, R\gamma_2^a, L'_2$$

The cohomological correspondence $(\gamma, a, c')$ is called the push-forward of $(\gamma', c')$ by $a$. It is easy to see that push-forward is compatible with composition. Moreover, we have the following lemma whose proof is also immediate:

**Lemma 6.1** In the above diagram, assume that $X_1 = X'_1$, $X_2 = X'_2$, $a_1 = a_2 = \text{id}$ and $a$ is proper. Let $L_i$ be an object of $D^b_c(X_i, \Lambda)$ for each $i = 1, 2$ and $(\gamma', c')$ a cohomological correspondence from $L_1$ to $L_2$. Then we have $R\Gamma_c(a, c') = R\Gamma_c(c')$.

Next we assume that the right rectangle in the diagram above is cartesian. Let $L'_i$ and $(\gamma', c')$ be as above. Then we have the cohomological correspondence $(\gamma, a, c')$ from $Ra_1, L'_1$ to $Ra_2, L'_2$ by

$$\gamma'^*_1Ra_1, L'_1 \xrightarrow{\text{b.c.}} Ra_1, \gamma'^*_1L'_1 \xrightarrow{Ra_1, (c')} Ra_1, R\gamma_2^b, L'_2 \xrightarrow{\text{b.c.}} R\gamma_2^b, Ra_2, L'_2.$$ 

On the other hand, let $L_i$ be an object of $D^b_c(X_i, \Lambda)$ for each $i = 1, 2$ and $(\gamma, c)$ a cohomological correspondence from $L_1$ to $L_2$. Then we have the cohomological correspondence $(\gamma, a^c)$ from $a^*_1L_1$ to $a^*_2L_2$ by

$$\gamma'^*_1a^*_1L_1 = a^*\gamma'^*_1L_1 \xrightarrow{a^*(c')} a^*R\gamma_2^b, L_2 \xrightarrow{\text{b.c.}} R\gamma_2^b, a^*_2L_2.$$ 

Finally assume that the left rectangle in the diagram above is cartesian. Let $L_i$ and $(\gamma, c)$ be as above. Then we have the cohomological correspondence $(\gamma, a^c)$ from $Ra^*_1L_1$ to $Ra^*_2L_2$ by

$$\gamma'^*_1Ra^*_1L_1 \xrightarrow{\text{b.c.}} Ra^*_1, \gamma'^*_1L_1 \xrightarrow{Ra^*_1, (c')} Ra^*_1, R\gamma_2^b, L_2 = R\gamma_2^b, Ra^*_2L_2.$$ 

These constructions are also compatible with composition.

Next we recall the specialization of cohomological correspondences. Let $S$ be the spectrum of a strict henselian discrete valuation ring on which $\ell$ is invertible. For an $S$-scheme $X$, we denote its special (resp. generic) fiber by $X_s$ (resp. $X_\eta$).

Let $X_1, X_2$ be schemes which are separated of finite type over $S$ and $\gamma : \Gamma \to X_1 \times_S X_2$ a separated $S$-morphism of finite type. Let $L_i$ be an object of $D^b_c(X_i, \Lambda)$ for each $i = 1, 2$ and $(\gamma, c)$ a cohomological correspondence from $L_1$ to $L_2$. Then we have the cohomological correspondence $(\gamma_s, R\psi(c))$ from $R\psi L_1$ to $R\psi L_2$ by

$$\gamma'^*_1, R\psi L_1 \xrightarrow{R\psi} R\psi, \gamma'^*_1, L_1 \xrightarrow{R\psi, (c')} R\psi, R\gamma_2^b, L_2 \xrightarrow{R\gamma_2^b, R\psi} R\gamma_2^b, R\psi L_2.$$ 

27
It is easy to see that this construction is compatible with composition and proper push-forward (cf. [Fuj97, Proposition 1.6.1]).

Now we will give the main result in this section. Let $X_i$, $\gamma_i$, $L_i$ be as above and $Y_i$ (resp. $Z_i$) a closed (resp. locally closed) subscheme of $X_i, s$. Assume that $\gamma_i^{-1}(Y_i) = \gamma_{2,s}^{-1}(Y_2)$ and $\gamma_i^{-1}(Z_i) = \gamma_{2,s}^{-1}(Z_2)$ as subschemes of $\Gamma_s$, and denote the former by $\Gamma_Y$ and the latter by $\Gamma_Z$. Then we have the following diagrams whose rectangles are cartesian:

\[
\begin{array}{ccc}
Y_1 & \xleftarrow{\gamma_{Y,1}} & \Gamma_Y & \xrightarrow{\gamma_{Y,2}} & Y_2 \\
| & s & | & s & |
\downarrow & & \downarrow & & \downarrow \\
X_1, s & \xleftarrow{\gamma_{1,s}} & \Gamma_s & \xrightarrow{\gamma_{2,s}} & X_{2,s}, \\
\end{array}
\quad
\begin{array}{ccc}
Z_1 & \xleftarrow{\gamma_{Z,1}} & \Gamma_Z & \xrightarrow{\gamma_{Z,2}} & Z_2 \\
| & s & | & s & |
\downarrow & & \downarrow & & \downarrow \\
X_1, s & \xleftarrow{\gamma_{1,s}} & \Gamma_s & \xrightarrow{\gamma_{2,s}} & X_{2,s}. \\
\end{array}
\]

Therefore, for a cohomological correspondence $(\gamma_{\eta}, c)$ from $L_1$ to $L_2$, the cohomological correspondence $i^* j_* j^! R\psi(c)$ from $i^* Rj_{11}^! R\psi L_1$ to $i^* Rj_{21}^! R\psi L_2$ is induced. If moreover we assume that $\gamma_1$ is proper, then we have

\[
R\Gamma_c(i^* j_* j^! R\psi(c)) : R\Gamma_c(X_{1,s}, i^* Rj_{11}^! R\psi L_1) \longrightarrow R\Gamma_c(X_{2,s}, i^* Rj_{21}^! R\psi L_2).
\]

**Proposition 6.2** The morphism $R\Gamma_c(i^* j_* j^! R\psi(c))$ depends only on the cohomological correspondence $(\gamma_{\eta}, c)$. More precisely, if another $S$-morphism $\gamma' : \Gamma' \longrightarrow X_1 \times_S X_2$ has the same generic fiber as $\gamma$ and satisfies the conditions that $\gamma_{1,s}^{-1}(Y_1) = \gamma_{2,s}^{-1}(Y_2)$, $\gamma_{1,s}^{-1}(Z_1) = \gamma_{2,s}^{-1}(Z_2)$ and $\gamma_1'$ is proper, then the morphism $R\Gamma_c(i^* j_* j^! R\psi(c))$ induced from $\gamma'$ is equal to $R\Gamma_c(i^* j_* j^! R\psi(c))$ (here $i'$ and $j'$ are defined in the same way as $i$ and $j$).

**Proof.** Since $\Gamma$ and $\Gamma'$ have the same generic fiber, there is the “diagonal” in the generic fiber of $\Gamma \times X_1 \times_S X_2 \Gamma'$. Let $\Gamma''$ be the closure of it in $\Gamma \times X_1 \times_S X_2 \Gamma'$. Then $\Gamma''$ has the same generic fiber as $\Gamma$. We have natural morphisms $\Gamma'' \longrightarrow \Gamma$ and $\Gamma'' \longrightarrow \Gamma'$, which are proper since $\gamma$ and $\gamma'$ are proper. Therefore $\Gamma'' : \Gamma'' \longrightarrow X_1 \times_S X_2$ also satisfies the same conditions as $\gamma$ and $\gamma'$. By replacing $\gamma'$ by $\gamma''$, we may assume that there exists a proper morphism $a : \Gamma' \longrightarrow \Gamma$ such that $\gamma \circ a = \gamma'$.

Then, it is easy to see that the push-forward of the cohomological correspondence $(\gamma_{\eta}', i'^* j_* j'^! R\psi(c))$ by $a$ coincides with $(\gamma_{\eta}, i^* j_* j^! R\psi(c))$. Therefore the proposition follows from Lemma 6.1. □

**Corollary 6.3** Let $X_1$, $X_2$ and $X_3$ be schemes which are separated of finite type over $S$, $Y_i$ (resp. $Z_i$) a closed (resp. locally closed) subscheme of $X_i$, and $L_i$ an object of $D_c^b(X_i, \eta, \Lambda)$ for each $i = 1, 2, 3$. Let $\gamma_i : \Gamma_i \longrightarrow X_1 \times_S X_2$ (resp. $\gamma' : \Gamma' \longrightarrow X_2 \times_S X_3$, resp. $\gamma'' : \Gamma'' \longrightarrow X_1 \times_S X_3$) be an $S$-morphism such that $\gamma_1$ (resp. $\gamma'_1$, resp. $\gamma''_1$) is proper, and $(\gamma_{\eta}, c)$ (resp. $(\gamma'_{\eta}, c')$, resp. $(\gamma''_{\eta}, c'')$) a cohomological correspondence from $L_1$ to $L_2$ (resp. from $L_2$ to $L_3$, resp. from $L_1$ to $L_3$). Moreover we assume that $\gamma_{1,s}^{-1}(Y_1) = \gamma_{2,s}^{-1}(Y_2)$, $\gamma_{1,s}(Z_1) = \gamma_{2,s}(Z_2)$, $\gamma_{1,s}(Y_2) = \gamma_{2,s}(Y_3)$, $\gamma_{1,s}(Z_2) = \gamma_{2,s}(Z_3)$,
ℓ-adic cohomology of the Rapoport-Zink space for GSp(4)

Then, as above, the morphisms

\[ R\Gamma_c(i^* j_* j! R\psi(c)) : R\Gamma_c(X_{1,s}, i_1^* Rj_1^* Rj_1^! R\psi L_1) \to R\Gamma_c(X_{2,s}, i_2^* Rj_2^* Rj_2^! R\psi L_2), \]

\[ R\Gamma_c(i^* j_* j! R\psi(c')) : R\Gamma_c(X_{2,s}, i_2^* Rj_2^* Rj_2^! R\psi L_2) \to R\Gamma_c(X_{3,s}, i_3^* Rj_3^* Rj_3^! R\psi L_3), \]

\[ R\Gamma_c(i^* j_* j! R\psi(e'')) : R\Gamma_c(X_{1,s}, i_1^* Rj_1^* Rj_1^! R\psi L_1) \to R\Gamma_c(X_{3,s}, i_3^* Rj_3^* Rj_3^! R\psi L_3) \]

are induced. Assume that the composite of \((\gamma_\eta, c)\) and \((\gamma'_\eta, c')\) coincides with \((\gamma''_\eta, c'')\). Then we have

\[ R\Gamma_c(i^* j_* j! R\psi(c')) \circ R\Gamma_c(i^* j_* j! R\psi(c)) = R\Gamma_c(i^* j_* j! R\psi(c'')). \]

**Proof.** By Proposition 6.2, we may replace \(\gamma''\) by \(\Gamma \times X_2 \Gamma' \to X_1 \times_S X_3\). Then the equality is clear, since all the operations for cohomological correspondences are compatible with composition.

\[ \square \]

**References**


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ℓ-adic cohomology of the Rapoport-Zink space for GSp(4)


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