# Geometric proof of the local Jacquet-Langlands correspondence for GL($n$) for prime $n$

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Abstract. In this paper, we give a purely geometric proof of the local Jacquet-Langlands correspondence for $\mathrm{GL}(n)$ over a $p$-adic field, under the assumption that $n$ is prime and the invariant of the division algebra is $1/n$. We use the $\ell$-adic étale cohomology of the Lubin-Tate tower to construct the correspondence. We need neither a global automorphic technique nor detailed classification of supercuspidal representations of $\mathrm{GL}(n)$.

## 1 Introduction

Let $F$ be a $p$-adic field, i.e., a finite extension of $\mathbb{Q}_p$. Let $n \geq 1$ be an integer and $D$ a central division algebra over $F$ such that $\dim_F D = n^2$. The famous local Jacquet-Langlands correspondence gives a natural bijective correspondence between irreducible discrete series representations of $\mathrm{GL}_n(F)$ and irreducible smooth representations of $D^\times$. Let us recall its precise statement. Write $\mathbf{Irr}(D^\times)$ for the set of isomorphism classes of irreducible smooth representations of $D^\times$. We denote by $\mathbf{Disc}(\mathrm{GL}_n(F))$ the set of isomorphism classes of irreducible discrete series representations of $\mathrm{GL}_n(F)$. For $\rho \in \mathbf{Irr}(D^\times)$ (resp. $\pi \in \mathbf{Disc}(\mathrm{GL}_n(F))$), we denote the character of $\rho$ (resp. $\pi$) by $\theta_\rho$ (resp. $\theta_\pi$). Here $\theta_\rho$ is a locally constant function on $D^\times$, and $\theta_\pi$ is a locally integrable function on $\mathrm{GL}_n(F)$ which is locally constant on $\mathrm{GL}_n(F)^{\mathrm{reg}}$, the set of regular elements of $\mathrm{GL}_n(F)$. The precise statement of the local Jacquet-Langlands correspondence is the following:

**Theorem 1.1 (the local Jacquet-Langlands correspondence)** *There exists a unique bijection*

$$JL\colon \mathbf{Irr}(D^\times) \xrightarrow{\cong} \mathbf{Disc}\bigl(\mathrm{GL}_n(F)\bigr)$$

*satisfying the following character relation: for every regular element $h$ of $D^\times$, $\theta_\rho(h) = (-1)^{n-1}\theta_{JL(\rho)}(g_h)$, where $g_h$ is an arbitrary element of $\mathrm{GL}_n(F)$ whose minimal polynomial is the same as that of $h$.*

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The original proof of this theorem, due to Deligne-Kazhdan-Vigneras [DKV84] and Rogawski [Rog83], was accomplished by using a global automorphic method. In some cases, we can find more explicit studies in [Hen93], [BH00], [BH05], which are based on the theory of types. However, apart from the case of GL(2), a purely local proof of Theorem 1.1 seems not to be known yet (*cf.* [Hen06, p. 1173]).

In this article, under the assumption that $n$ is prime and the invariant of $D$ is $1/n$, we will give a simple geometric proof of the local Jacquet-Langlands correspondence. In particular, the local Jacquet-Langlands correspondence for $\mathrm{GL}_2(F)$ and $\mathrm{GL}_3(F)$ are fully recovered. The geometric object we use is the Lubin-Tate tower for $\mathrm{GL}_n(F)$. It is a tower of universal deformation spaces of formal $\mathcal{O}$-modules of height $n$ with Drinfeld level structures, where $\mathcal{O}$ denotes the ring of integers of $F$ (for a precise definition, see Section 3). Thanks to Carayol [Car86] and Harris-Taylor [HT01], it is now well-known that the local Jacquet-Langlands correspondence is realized in the $\ell$-adic cohomology of the Lubin-Tate tower. Their proofs are again global and automorphic. However, recent works by Strauch [Str08] and the author [Mie11] enable us to study the cohomology in a purely local manner. They used the Lefschetz trace formula to observe that the character relation in Theorem 1.1 appears naturally. In this work, we use both of the two results to obtain a bijection between $\mathbf{Irr}(D^\times)$ and $\mathbf{Disc}(\mathrm{GL}_n(F))$.

We sketch the outline of this paper. In Section 2, we give some preliminaries on representation theory. In Section 3, we recall the definition of the Lubin-Tate tower and main results of [Str08] and [Mie11] which play important roles in the proof. In Section 4, we construct the Jacquet-Langlands correspondence for $\mathrm{GL}_n(F)$ by using the $\ell$-adic cohomology of the Lubin-Tate tower, and prove its expected properties.

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**Notation** As in the introduction above, let $F$ be a $p$-adic field and $\mathcal{O}$ its ring of integers. We denote the normalized valuation of $F$ by $v_F$ and the cardinality of the residue field of $\mathcal{O}$ by $q$. Fix a uniformizer $\varpi$ of $\mathcal{O}$. Denote the completion of the maximal unramified extension of $\mathcal{O}$ by $\breve{\mathcal{O}}$ and the fraction field of $\breve{\mathcal{O}}$ by $\breve{F}$.

Throughout this paper, we fix an integer $n \geq 1$. In Section 4, it will be assumed to be a prime. Let $D$ be the central division algebra over $F$ with invariant $1/n$.

For simplicity, put $G = \mathrm{GL}_n(F)$. We denote by $G^{\mathrm{reg}}$ (resp. $G^{\mathrm{ell}}$) the set of regular (resp. regular elliptic) elements of $G$. Write $Z_G$ for the center of $G$. We apply these notations to other groups. For example, we write $(D^\times)^{\mathrm{reg}}$ for the set of regular elements of $D^\times$. As in Theorem 1.1, for $h \in (D^\times)^{\mathrm{reg}}$, let $g_h$ be an element of $G^{\mathrm{ell}}$ whose minimal polynomial is the same as that of $h$. Such an element always exists, and is unique up to conjugacy. Moreover, it induces a bijection between conjugacy classes in $(D^\times)^{\mathrm{reg}}$ and those in $G^{\mathrm{ell}}$.

For a field $k$, we denote its algebraic closure by $\overline{k}$. Let $\ell$ be a prime which is invertible in $\mathcal{O}$. We fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$ and identify them. Every representation is considered over $\mathbb{C}$.

# 2 Preliminaries on representation theory

Here we collect preliminary lemmas on representation theory which seem to be well-known.

**Lemma 2.1** *The map* $JL\colon \mathbf{Irr}(D^\times) \longrightarrow \mathbf{Disc}(G)$ *satisfying the character relation* $\theta_\rho(h) = (-1)^{n-1}\theta_{JL(\rho)}(g_h)$ *in Theorem 1.1 is unique, injective and preserves central characters.*

*Proof.* The injectivity is clear, since $\rho \in \mathbf{Irr}(D^\times)$ can be determined from $\theta_\rho|_{(D^\times)^{\mathrm{reg}}}$. Let us prove that $JL$ preserves central characters. For $\rho \in \mathbf{Irr}(D^\times)$, let $\omega_\rho$ (resp. $\omega_{JL(\rho)}$) be the central character of $\rho$ (resp. $JL(\rho)$). Then, for $h \in (D^\times)^{\mathrm{reg}}$ and $z \in F^\times = Z_G = Z_{D^\times}$, we have

$$\begin{aligned}\omega_\rho(z)\theta_\rho(h) = \theta_\rho(zh) = (-1)^{n-1}\theta_{JL(\rho)}(zg_h) &= (-1)^{n-1}\omega_{JL(\rho)}(z)\theta_{JL(\rho)}(g_h)\\ &= \omega_{JL(\rho)}(z)\theta_\rho(h).\end{aligned}$$

Since $(D^\times)^{\mathrm{reg}}$ is dense in $D^\times$, there exists $h \in (D^\times)^{\mathrm{reg}}$ such that $\theta_\rho(h) \neq 0$. Thus we conclude that $\omega_\rho = \omega_{JL(\rho)}$.

Now the uniqueness of $JL$ is a direct consequence of the orthogonality relation of characters ([Rog83, Lemma 5.3], see also [SS97, Theorem III.4.21]). ∎

Let $\mathbf{Irr}_\varpi(D^\times)$ be the subset of $\mathbf{Irr}(D^\times)$ consisting of isomorphism classes of representations on which $\varpi \in F^\times = Z_{D^\times}$ acts trivially. Define $\mathbf{Disc}_\varpi(G)$ similarly.

**Lemma 2.2** *To construct JL, it suffices to construct a surjection*

$$JL\colon \mathbf{Irr}_\varpi(D^\times) \longrightarrow \mathbf{Disc}_\varpi(G)$$

*satisfying the character relation* $\theta_\rho(h) = (-1)^{n-1}\theta_{JL(\rho)}(g_h)$ *for* $h \in (D^\times)^{\mathrm{reg}}$.

*Proof.* For $\rho \in \mathbf{Irr}(D^\times)$, let $\omega_\rho$ be its central character. Take $c \in \mathbb{C}^\times$ such that $c^n = \omega_\rho(\varpi)$, and consider the character $\chi_c\colon z \longmapsto c^{v_F(z)}$ of $F^\times$. Then $\rho\otimes(\chi_c^{-1}\circ\mathrm{Nrd}) \in \mathbf{Irr}_\varpi(D^\times)$. Extend $JL$ to $\mathbf{Irr}(D^\times)$ by

$$JL(\rho) = JL\bigl(\rho\otimes(\chi_c^{-1}\circ\mathrm{Nrd})\bigr)\otimes(\chi_c\circ\det).$$

It is easy to see that this satisfies the character relation. In particular it is independent of the choice of $c$ and injective (*cf.* Lemma 2.1). Similar argument shows that the extended $JL$ is surjective. ∎

Let $B \subset G$ be the Borel subgroup consisting of upper triangular matrices. Recall that the Steinberg representation $\mathbf{St}$ is the unique irreducible quotient of the unnormalized induction $\mathrm{Ind}_B^G \mathbf{1}$ from the trivial character $\mathbf{1}$ on $B$. For a character $\chi$ of $F^\times$, put $\mathbf{St}_\chi = \mathbf{St}\otimes(\chi\circ\det)$. A representation of the form $\mathbf{St}_\chi$ is called a twisted Steinberg representation. It is a discrete series representation of $G$. The following lemma is very well-known:

**Lemma 2.3** *The map $\chi \circ \mathrm{Nrd} \longmapsto \mathbf{St}_\chi$ gives a bijection between characters in $\mathbf{Irr}_\varpi(D^\times)$ and twisted Steinberg representations in $\mathbf{Disc}_\varpi(G)$. It satisfies the character relation $\theta_{\chi\circ\mathrm{Nrd}}(h) = (-1)^{n-1}\theta_{\mathbf{St}_\chi}(g_h)$ for $h \in (D^\times)^{\mathrm{reg}}$.*

*Proof.* Only the character relation is non-trivial. In the Grothendieck group of finite length smooth representations of $G$, $[\mathbf{St}_\chi] - (-1)^{n-1}[\chi \circ \det]$ is the alternating sum of parabolically induced representations (*cf.* [Dat07, Remarque 2.1.14]). As the character of a parabolically induced representation vanishes on $G^{\mathrm{ell}}$, we have

$$\begin{aligned}\theta_{\mathbf{St}_\chi}(g_h) &= (-1)^{n-1}\theta_{\chi\circ\det}(g_h) = (-1)^{n-1}\chi(\det g_h) = (-1)^{n-1}\chi(\mathrm{Nrd}\, h)\\ &= (-1)^{n-1}\theta_{\chi\circ\mathrm{Nrd}}(h),\end{aligned}$$

as desired. ∎

Let us denote by $\mathbf{Irr}^0_\varpi(D^\times)$ the subset of $\mathbf{Irr}_\varpi(D^\times)$ consisting of representations which are not one-dimensional, and by $\mathbf{Cusp}_\varpi(G)$ the subset of $\mathbf{Disc}_\varpi(G)$ consisting of supercuspidal representations. If $n$ is a prime, a discrete series representation of $G$ is supercuspidal if and only if it is not a twisted Steinberg representation (*cf.* classification of discrete series representations [Zel80, Theorem 9.3]). Thus, the local Jacquet-Langlands correspondence for a prime $n$ can be reduced to the following theorem:

**Theorem 2.4** *Assume that $n$ is a prime number. Then we can construct purely geometrically a surjection*

$$JL\colon\ \mathbf{Irr}^0_\varpi(D^\times) \longrightarrow \mathbf{Cusp}_\varpi(G)$$

*satisfying the character relation $\theta_\rho(h) = (-1)^{n-1}\theta_{JL(\rho)}(g_h)$ for $h \in (D^\times)^{\mathrm{reg}}$.*

Before going on, we shall recall the orthogonality relation of characters for representations in $\mathbf{Disc}_\varpi(G)$. For locally constant class functions $\varphi_1$, $\varphi_2$ on $G^{\mathrm{ell}}$ satisfying $\varphi_i(\varpi g) = \varphi_i(g)$, put

$$\langle \varphi_1, \varphi_2\rangle_{\mathrm{ell}} = \sum_T \frac{1}{\# W_T}\int_{\varpi^{\mathbb{Z}}\backslash T} D(t)\varphi_1(t)\overline{\varphi_2(t)}\, dt,$$

where $T$ runs through conjugacy classes of elliptic maximal tori of $G$, $W_T$ denotes the rational Weyl group of $T$ and $D(t)$ denotes the Weyl denominator (*cf.* [Rog83, p. 185]). The measure $dt$ on $\varpi^{\mathbb{Z}}\backslash T$ is normalized so that the volume of $\varpi^{\mathbb{Z}}\backslash T$ is one.

**Lemma 2.5** *For $\pi_1, \pi_2 \in \mathbf{Disc}_\varpi(G)$, we have*

$$\langle \theta_{\pi_1}, \theta_{\pi_2}\rangle_{\mathrm{ell}} = \begin{cases} 1 & \pi_1 \cong \pi_2,\\ 0 & \textit{otherwise.}\end{cases}$$

*Proof.* Let $\omega_1$, $\omega_2$ be the central characters of $\pi_1$, $\pi_2$, respectively. Then they are unitary, since $F^\times/\varpi^{\mathbb{Z}}$ is compact. If $\omega_1 = \omega_2$, then the lemma follows immediately from [Rog83, Lemma 5.3]. Otherwise,

$$\begin{aligned}\int_{\varpi^{\mathbb{Z}}\backslash T} D(t)\theta_{\pi_1}(t)\overline{\theta_{\pi_2}(t)}\,dt &= \int_{Z_G\backslash T}\Bigl(\int_{\varpi^{\mathbb{Z}}\backslash Z_G} D(zt)\theta_{\pi_1}(zt)\overline{\theta_{\pi_2}(zt)}\,dz\Bigr)dt\\ &= \int_{Z_G\backslash T}\Bigl(\int_{\varpi^{\mathbb{Z}}\backslash Z_G} \omega_1(z)\overline{\omega_2(z)}\,dz\Bigr)D(t)\theta_{\pi_1}(t)\overline{\theta_{\pi_2}(t)}\,dt = 0,\end{aligned}$$

as desired. ∎

For locally constant functions $\phi_1$, $\phi_2$ on $D^\times$ satisfying $\phi_i(\varpi h) = \phi_i(h)$, put

$$\langle \phi_1, \phi_2\rangle = \int_{\varpi^{\mathbb{Z}}\backslash D^\times} \phi_1(h)\overline{\phi_2(h)}\,dh,$$

where the measure $dh$ is normalized so that the volume of the compact group $\varpi^{\mathbb{Z}}\backslash D^\times$ is one.

**Lemma 2.6** *Let $\varphi_1$, $\varphi_2$ be locally constant class functions on $G^{\mathrm{ell}}$ satisfying $\varphi_i(\varpi g) = \varphi_i(g)$, and $\phi_1$, $\phi_2$ locally constant class functions on $D^\times$ satisfying $\phi_i(\varpi h) = \phi_i(h)$. Assume that $\varphi_i(g_h) = \phi_i(h)$ for every $h \in (D^\times)^{\mathrm{reg}}$. Then, we have*

$$\langle \varphi_1, \varphi_2\rangle_{\mathrm{ell}} = \langle \phi_1, \phi_2\rangle.$$

*Proof.* Clear from Weyl's integral formula for $D^\times$. ∎

# 3 Lubin-Tate tower

Let us recall briefly the definition of the Lubin-Tate tower. See [Str08, §2.1] for more detail. Let $\mathbb{X}$ be a formal $\mathcal{O}$-module over $\overline{\mathbb{F}}_q$ with $\mathcal{O}$-height $n$ (such $\mathbb{X}$ is unique up to isomorphism). For integers $m \geq 0$ and $j$, let $\mathcal{M}_m^{(j)}$ denotes the following functor from the category of complete noetherian local $\breve{\mathcal{O}}$-algebras with residue field $\overline{\mathbb{F}}_q$ to the category of sets; $\mathcal{M}_m^{(j)}(A)$ consists of isomorphism classes of triples $(X, \rho, \eta)$, where $X$ is a formal $\mathcal{O}$-module over $A$, $\rho$ is an $\mathcal{O}$-quasi-isogeny of $\mathcal{O}$-height $j$ from $\mathbb{X}$ to $X \otimes_A A/\mathfrak{m}_A$ and $\eta$ is a Drinfeld $m$-level structure of $X$. This functor is represented by a complete noetherian local ring $R_m^{(j)}$. We denote $\operatorname{Spf} R_m^{(j)}$ by $\mathcal{M}_m^{(j)}$ again, and put $\mathcal{M}_m = \coprod_{j\in\mathbb{Z}} \mathcal{M}_m^{(j)}$. Now we get the projective system of formal schemes $\{\mathcal{M}_m\}_{m\geq 0}$, which is called the Lubin-Tate tower.

We can define a natural action of $D^\times$ on each $\mathcal{M}_m$, because $D^\times$ is isomorphic to the group of self $\mathcal{O}$-quasi-isogenies of $\mathbb{X}$. On the other hand, $G = \mathrm{GL}_n(F)$ naturally acts on the tower $\{\mathcal{M}_m\}_{m\geq 0}$ as a pro-object (the Hecke action).

Denote the rigid generic fiber of $\mathcal{M}_m$ by $M_m$. It is the generic fiber of the adic space $t(\mathcal{M}_m)$ associated to $\mathcal{M}_m$. Since we are interested in representations of $G$

and $D^\times$ on which $\varpi^{\mathbb{Z}}$ acts trivially, we take the quotient of $M_m$ by $\varpi^{\mathbb{Z}} \subset D^\times$ and consider the $\ell$-adic cohomology

$$H^i_{\mathrm{LT}} = \varinjlim_m H^i_c\bigl((M_m/\varpi^{\mathbb{Z}}) \otimes_{\breve{F}} \overline{\breve{F}}, \overline{\mathbb{Q}}_\ell\bigr).$$

It is a smooth representation of $(G/\varpi^{\mathbb{Z}}) \times (D^\times/\varpi^{\mathbb{Z}})$. In fact, we may also define the action of the Weil group of $F$ on $H^i_{\mathrm{LT}}$, but in this article we do not consider it.

For $\rho \in \mathbf{Irr}_\varpi(D^\times)$, put $H^i_{\mathrm{LT}}[\rho] = \mathrm{Hom}_{D^\times}(H^i_{\mathrm{LT}}, \rho)^{\mathrm{sm}}$, where $(-)^{\mathrm{sm}}$ denotes the subset of $G$-smooth vectors. It is easy to see that $H^i_{\mathrm{LT}}[\rho]$ is an admissible representation of $G$ which is trivial on $\varpi^{\mathbb{Z}} \subset G$ (*cf.* [Mie11, Lemma 5.2]).

We summarize fundamental properties of the representation $H^i_{\mathrm{LT}}$ in the following proposition:

**Proposition 3.1** i) *Unless $n-1 \le i \le 2n$, $H^i_{\mathrm{LT}} = 0$.*

ii) *Unless $i = n-1$, no supercuspidal representation of $G$ appears in $H^i_{\mathrm{LT}}$ as a subquotient.*

iii) *For every $\rho \in \mathbf{Irr}_\varpi(D^\times)$, the $G$-representation $H^i_{\mathrm{LT}}[\rho]$ has finite length.*

*Proof.* i) is well-known. ii) is proved in [Mie10]. For iii), see [Mie11, Theorem 5.1]. ∎

By iii) of the proposition above, we can consider the character $\theta_{H^i_{\mathrm{LT}}[\rho]}$ of $H^i_{\mathrm{LT}}[\rho]$. Put $\theta_{H_{\mathrm{LT}}[\rho]} = \sum_i (-1)^i \theta_{H^i_{\mathrm{LT}}[\rho]}$. The following is a consequence of [Mie11], whose proof does not require the local Jacquet-Langlands correspondence:

**Theorem 3.2** *For $\rho \in \mathbf{Irr}_\varpi(D^\times)$, we have the following character relation:*

$$\theta_{H_{\mathrm{LT}}[\rho]}(g_h) = n\theta_\rho(h) \quad (h \in (D^\times)^{\mathrm{reg}}).$$

*Moreover, if $\rho \in \mathbf{Irr}^0_\varpi(D^\times)$ then we have $\langle \theta_{H_{\mathrm{LT}}[\rho]}, \theta_{\mathbf{St}_\chi} \rangle_{\mathrm{ell}} = 0$.*

*Proof.* The character relation is clear from [Mie11, Theorem 4.3]. For the latter, by Lemma 2.3 and Lemma 2.6, we have $\langle \theta_{H_{\mathrm{LT}}[\rho]}, \theta_{\mathbf{St}_\chi} \rangle_{\mathrm{ell}} = (-1)^{n-1} n \langle \theta_\rho, \theta_{\chi\circ\mathrm{Nrd}} \rangle = 0$. ∎

Finally we recall the main result of [Str08].

**Theorem 3.3** *For $\pi \in \mathbf{Cusp}_\varpi(G)$, $\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}}, \pi)$ is a finite-dimensional smooth representation of $D^\times$. For this representation, we have*

$$\theta_{\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}}, \pi)}(h) = (-1)^{n-1} n\theta_\pi(g_h) \quad (h \in (D^\times)^{\mathrm{reg}}).$$

*Proof.* It follows from the proof of [Str08, Theorem 4.1.3] and Proposition 3.1 ii). We will give some remarks on it. First, although the statement of [Str08, Theorem 4.1.3] involves the local Jacquet-Langlands correspondence, the only one place we need it is the last equality in the proof. Therefore Strauch's result above is free from the local Jacquet-Langlands correspondence.

Next, to construct the function $f_\pi$ in the proof, Strauch uses the fact that $\pi$ can be written as the compact induction from a compact-mod-center subgroup, which is a part of detailed classification of supercuspidal representations of $G$. However, we can simply use a matrix coefficient $\phi_\pi$ of $\pi$ satisfying $\phi_\pi(1) = d(\pi)$ as a substitute of $f_\pi^*$, where $d(\pi)$ is the formal degree of $\pi$ (under the fixed measure on $G$). In [DKV84, §A.3.e, §A.3.g], the following are proved:

(a) $\mathrm{Tr}(\phi_\pi;\pi) = 1$ and $\mathrm{Tr}(\phi_\pi;\pi') = 0$ for any irreducible admissible representation $\pi'$ of $G/\varpi^{\mathbb{Z}}$ with $\pi' \not\cong \pi$.

(b) The orbital integrals of $\phi_\pi$ over regular non-elliptic conjugacy classes vanish.

(c) For every $g \in G^{\mathrm{ell}}$, $\int_{G/\varpi^{\mathbb{Z}}} \phi_\pi(x^{-1}gx)dx = \overline{\theta_\pi(g)} = \theta_\pi(g^{-1})$ (note that $\pi$ is unitary, as its central character is unitary).

Kazhdan's theorem [Kaz86, Theorem A], which requires a global argument, can be replaced by (b). Moreover, we can substitute Harish-Chandra's character formula stated in [Str08, 4.1.1] by (c).

Finally, thanks to Proposition 3.1 iii), for every compact open subgroup $K'$ of $D^\times$, $(H^i_{\mathrm{LT}})^{K'}$ is a $G$-representation of finite length. Therefore, to compute the trace $\mathrm{Tr}(f_\pi \cdot f; \mathrm{Hom}(H^{n-1}_{\mathrm{LT}}, \pi))$ for an element $f$ of the Hecke algebra of $D^\times$, we do not need to take $W^i$ in [Str08, p. 927]. This simplifies Strauch's proof considerably. ∎

# 4 Proof of the main theorem

Now we assume that $n$ is a prime, and prove Theorem 2.4. First construct a map $JL\colon \mathbf{Irr}^0_\varpi(D^\times) \longrightarrow \mathbf{Cusp}_\varpi(G)$.

**Proposition 4.1** *For $\rho \in \mathbf{Irr}^0_\varpi(D^\times)$, there exists a unique representation $\pi \in \mathbf{Cusp}_\varpi(G)$ which appears in $H^{n-1}_{\mathrm{LT}}[\rho]$. It satisfies the character relation $\theta_\rho(h) = (-1)^{n-1}\theta_\pi(g_h)$ for every $h \in (D^\times)^{\mathrm{reg}}$.*

*Proof.* First note that $\pi \in \mathbf{Cusp}_\varpi(G)$ is projective and injective in the category of smooth representations of $G/\varpi^{\mathbb{Z}}$, and thus the following are equivalent:

- $\pi$ is a subrepresentation of a smooth representation $V$ of $G/\varpi^{\mathbb{Z}}$,
- $\pi$ is a quotient representation of $V$, and
- $\pi$ is a subquotient of $V$.

Consider the Grothendieck group $R(G/\varpi^{\mathbb{Z}})$ of finite length smooth representations of $G/\varpi^{\mathbb{Z}}$ and take the alternating sum $H_{\mathrm{LT}}[\rho] = \sum_i (-1)^i [H^i_{\mathrm{LT}}[\rho]]$ in $R(G/\varpi^{\mathbb{Z}})$. Write

$$H_{\mathrm{LT}}[\rho] = \sum_{\pi \in \mathbf{Cusp}_\varpi(G)} a_\pi[\pi] + \sum_{\pi'} b_{\pi'}[\pi'] + \sum_{\pi''} c_{\pi''}[\pi''],$$

where $\pi'$ runs through non-supercuspidal elliptic irreducible representations of $G$ which are trivial on $\varpi^{\mathbb{Z}}$ (*cf.* [Dat07, Lemme 2.1.6]), and $\pi''$ runs through non-elliptic irreducible representations of $G$ which are trivial on $\varpi^{\mathbb{Z}}$. By Proposition 3.1 ii), we know that $(-1)^{n-1}a_\pi \in \mathbb{Z}_{\geq 0}$ for every $\pi \in \mathbf{Cusp}_\varpi(G)$. Taking characters of both sides and restricting them to $G^{\mathrm{ell}}$, we have

$$\theta_{H_{\mathrm{LT}}[\rho]} = \sum_{\pi\in\mathbf{Cusp}_\varpi(G)} a_\pi\theta_\pi + \sum_{\chi} b'_\chi \theta_{\mathbf{St}_\chi}$$

for some $b'_\chi \in \mathbb{Z}$, where $\chi$ runs through characters of $F^\times$ which are trivial on $\varpi^{\mathbb{Z}}$; indeed, by [Dat07, Lemme 2.1.6], every non-supercuspidal elliptic representation $\pi'$ has the same cuspidal support as that of $\mathbf{St}_\chi$ for some $\chi$, and $\theta_{\pi'} = \pm\theta_{\mathbf{St}_\chi}$ on $G^{\mathrm{ell}}$.

By Lemma 2.5 and Theorem 3.2, we have $0 = \langle \theta_{H_{\mathrm{LT}}[\rho]}, \theta_{\mathbf{St}_\chi}\rangle_{\mathrm{ell}} = b'_\chi$. Hence we have an equality of functions over $G^{\mathrm{ell}}$:

$$\theta_{H_{\mathrm{LT}}[\rho]} = \sum_{\pi\in\mathbf{Cusp}_\varpi(G)} a_\pi\theta_\pi.$$

Since $\theta_{H_{\mathrm{LT}}[\rho]}(g_h) = n\theta_\rho(h)$ by Theorem 3.2, the left hand side is not zero. Therefore $a_\pi \neq 0$ for at least one $\pi \in \mathbf{Cusp}_\varpi(G)$. By definition, such $\pi$ appears in $H^{n-1}_{\mathrm{LT}}[\rho]$.

Next we prove the uniqueness of such $\pi$. In the proof of [Mie11, Lemma 5.2], the author constructed an injection of $G$-representations $H^{n-1}_{\mathrm{LT}}[\rho] \longrightarrow \mathrm{Hom}_{D^\times}(\rho, H^{n-1}_{\mathrm{LT}})^\vee$, where $(-)^\vee$ denotes the contragredient (in fact, we can prove that it is an isomorphism). Therefore, if $\pi$ appears in $H^{n-1}_{\mathrm{LT}}[\rho]$, then $\pi^\vee$ appears in $\mathrm{Hom}_{D^\times}(\rho, H^{n-1}_{\mathrm{LT}})$, and thus $\pi^\vee \otimes \rho$ appears in $H^{n-1}_{\mathrm{LT}}$. Now assume that $\pi, \pi' \in \mathbf{Cusp}_\varpi(G)$ appear in $H^{n-1}_{\mathrm{LT}}[\rho]$. Then two representations $\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}}, \pi^\vee)$, $\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}}, \pi'^\vee)$ of $D^\times$ share the same quotient $\rho^\vee$. Hence we have $\langle \theta_{\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}},\pi^\vee)}, \theta_{\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}},\pi'^\vee)}\rangle > 0$. By Lemma 2.6 and Theorem 3.3, we obtain

$$\langle \theta_{\pi^\vee}, \theta_{\pi'^\vee}\rangle_{\mathrm{ell}} = \frac{1}{n^2}\langle \theta_{\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}},\pi^\vee)}, \theta_{\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}},\pi'^\vee)}\rangle > 0.$$

Therefore, from Lemma 2.5 we conclude that $\pi^\vee \cong \pi'^\vee$ and $\pi \cong \pi'$.

Now we have found a unique representation $\pi \in \mathbf{Cusp}_\varpi(G)$ such that $a_\pi \neq 0$. It satisfies $\theta_{H_{\mathrm{LT}}[\rho]}(g_h) = a_\pi\theta_\pi(g_h)$ for every $h \in (D^\times)^{\mathrm{reg}}$. Again by Theorem 3.2, we have $a_\pi\theta_\pi(g_h) = n\theta_\rho(h)$. Lemma 2.6 tells us that $a_\pi^2\langle\theta_\pi,\theta_\pi\rangle_{\mathrm{ell}} = n^2\langle\theta_\rho,\theta_\rho\rangle = n^2$. Since $(-1)^{n-1}a_\pi \geq 0$, we conclude that $a_\pi = (-1)^{n-1}n$ and $\theta_\rho(h) = (-1)^{n-1}\theta_\pi(g_h)$, as desired. ∎

**Definition 4.2** For $\rho \in \mathbf{Irr}^0_\varpi(D^\times)$, define $JL(\rho)$ as the representation $\pi \in \mathbf{Cusp}_\varpi(G)$ in Proposition 4.1. We have the character relation $\theta_\rho(h) = (-1)^{n-1}\theta_{JL(\rho)}(g_h)$ for $h \in (D^\times)^{\mathrm{reg}}$.

To show Theorem 2.4, it suffices to prove the surjectivity of $JL$.

**Proposition 4.3** *The map $JL\colon \mathbf{Irr}^0_\varpi(D^\times) \longrightarrow \mathbf{Cusp}_\varpi(G)$ constructed above is surjective.*

*Proof.* Take $\pi \in \mathbf{Cusp}_{\varpi}(G)$. Theorem 3.3 tells us that $\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}}, \pi^\vee) \neq 0$, since $\theta_{\pi^\vee}|_{G^{\mathrm{ell}}} \neq 0$ by Lemma 2.5. Let $\rho$ be an irreducible representation of $D^\times$ appearing in $\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}}, \pi^\vee)$. Then $\rho \in \mathbf{Irr}^0_{\varpi}(D^\times)$. Indeed, if $\rho = \chi \circ \mathrm{Nrd}$ for a character $\chi$ of $F^\times/\varpi^{\mathbb{Z}}$, by Lemma 2.3, Lemma 2.5, Lemma 2.6 and Theorem 3.3, we have

$$\langle \theta_{\mathrm{Hom}_G(H^{n-1}_{\mathrm{LT}}, \pi^\vee)}, \theta_{\chi\circ\mathrm{Nrd}} \rangle = n \langle \theta_{\pi^\vee}, \theta_{\mathbf{St}_\chi} \rangle_{\mathrm{ell}} = 0.$$

By the same argument as in the proof of Proposition 4.1, we can conclude that $\pi$ appears in $H^{n-1}_{\mathrm{LT}}[\rho^\vee]$. Namely, $\pi = JL(\rho^\vee)$. ∎

This completes our proof of the local Jacquet-Langlands correspondence for $\mathrm{GL}_n(F)$ and $D^\times$.

# References


[BH00] C. J. Bushnell and G. Henniart, *Correspondance de Jacquet-Langlands explicite. II. Le cas de degré égal à la caractéristique résiduelle*, Manuscripta Math. **102** (2000), no. 2, 211–225.

[BH05] ———, *Local tame lifting for* GL($n$). *III. Explicit base change and Jacquet-Langlands correspondence*, J. Reine Angew. Math. **580** (2005), 39–100.

[Car86] H. Carayol, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 409–468.

[Dat07] J.-F. Dat, *Théorie de Lubin-Tate non-abélienne et représentations elliptiques*, Invent. Math. **169** (2007), no. 1, 75–152.

[DKV84] P. Deligne, D. Kazhdan, and M.-F. Vignéras, *Représentations des algèbres centrales simples p-adiques*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, pp. 33–117.

[Hen93] G. Henniart, *Correspondance de Jacquet-Langlands explicite. I. Le cas modéré de degré premier*, Séminaire de Théorie des Nombres, Paris, 1990–91, Progr. Math., vol. 108, Birkhäuser Boston, Boston, MA, 1993, pp. 85–114.

[Hen06] ———, *On the local Langlands and Jacquet-Langlands correspondences*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1171–1182.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[Kaz86] D. Kazhdan, *Cuspidal geometry of p-adic groups*, J. Analyse Math. **47** (1986), 1–36.

[Mie10] Y. Mieda, *Non-cuspidality outside the middle degree of $\ell$-adic cohomology of the Lubin-Tate tower*, Adv. Math. **225** (2010), no. 4, 2287–2297.

[Mie11] ———, *Lefschetz trace formula and $\ell$-adic cohomology of Lubin-Tate tower*, to appear in Mathematical Research Letters, 2011.

[Rog83] J. D. Rogawski, *Representations of* $\mathrm{GL}(n)$ *and division algebras over a* $p$*-adic field*, Duke Math. J. **50** (1983), no. 1, 161–196.

[SS97] P. Schneider and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 85, 97–191.

[Str08] M. Strauch, *Deformation spaces of one-dimensional formal modules and their cohomology*, Adv. Math. **217** (2008), no. 3, 889–951.

[Zel80] A. V. Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*-adic groups. II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210.