Toward generalization of the non-abelian Lubin-Tate theory

Yoichi Mieda

The Hakubi Center for Advanced Research / Department of Mathematics, Kyoto University

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Outline

- The Lubin-Tate theory $=$
  Geometric realization of the maximal abelian extension of a p-adic field ($\fallingdotseq$ local class field theory)
  Use a 1-dimensional formal group with height 1
- The non-abelian Lubin-Tate theory $=$
  Geometric realization of the local Langlands correspondence for GL_n
  Use the universal deformation space of a 1-dimensional formal group with height n (Lubin-Tate space)
- Generalization of the non-abelian Lubin-Tate theory $=$
  Geometric realization of the local Langlands correspondence for p-adic reductive groups
  Use Rapoport-Zink spaces

Notation on Galois groups

- p: prime number
- $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$: the absolute Galois group

$$1 \longrightarrow I \longrightarrow \Gamma \xrightarrow{(*)} \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow 1$$

- $\mathrm{Frob} \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$: the geometric Frobenius element $(x \longmapsto x^{1/p})$
- $W \subset \Gamma$: the inverse image of $\mathrm{Frob}^{\mathbb{Z}} \subset \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ under $(*)$
  (the Weil group for $\mathbb{Q}_p$)

The local Langlands correspondence for GL_n

Theorem (LLC for GL_n, Harris-Taylor, Henniart)

There is a natural bijection between

- irreducible smooth representations of $\mathrm{GL}_n(\mathbb{Q}_p)$, and
- Frobenius-semisimple Weil-Deligne representations $\phi\colon W \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$

- A representation (π, V) of a topological group H is said to be smooth if for every $x \in V$, $\mathrm{Stab}_H(x) = \{h \in H \mid hx = x\}$ is open in H.
- A Weil-Deligne representation ϕ is said to be Frobenius-semisimple if for every $w \in W$, $\phi(w)$ is semisimple.
- For a prime number $\ell \neq p$, Weil-Deligne representations are in bijection with continuous ℓ-adic representations $W \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. (Grothendieck's monodromy theorem)
  Denote by $\mathrm{WD}(\sigma)$ the Weil-Deligne representation corresponding to a continuous ℓ-adic representation σ.

Example of LLC for GL_n

Example ($n = 1$)

$\mathrm{Art}\colon \mathbb{Q}_p^\times \xrightarrow{\cong} W^{\mathrm{ab}}$: isomorphism of local class field theory

$\chi\colon \mathrm{GL}_1(\mathbb{Q}_p) = \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$ corresponds to $W \twoheadrightarrow W^{\mathrm{ab}} \xrightarrow{\chi\circ\mathrm{Art}^{-1}} \mathbb{C}^\times$.

Example (general n)

- trivial rep. of $\mathrm{GL}_n(\mathbb{Q}_p) \longleftrightarrow$ n-dim. trivial rep. of $W \times \mathrm{SL}_2(\mathbb{C})$
- Steinberg rep. $\mathbf{St}_n \longleftrightarrow \phi$: irred., $\phi|_W = \mathbf{1}$ (write $\mathbf{Sp}_n$ for such ϕ)
  $\mathbf{St}_n$: irred. quotient of $\mathrm{Ind}_B^{\mathrm{GL}_n(\mathbb{Q}_p)} \mathbf{1}$ (B: upper triangular matrices)
- supercuspidal rep. $\longleftrightarrow \phi$: irred., $\phi|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$

Supercuspidal representation is a representation which does not appear in a subquotient of any proper parabolic induction.

LLC for p-adic reductive groups

G: connected reductive group over $\mathbb{Q}_p$. For simplicity assume G is split.
$\widehat{G}$: the dual group of G (the algebraic group over $\mathbb{C}$ obtained by changing roots and coroots of G)
LLC for G is not bijective in general!

Conjecture (LLC for G)

(1) There is a natural surjection with finite fibers
- from isom. classes of irred. smooth rep. of $G(\mathbb{Q}_p)$ to
- $\widehat{G}(\mathbb{C})$-conjugacy classes of L-parameters $\phi\colon W \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \widehat{G}(\mathbb{C})$

The fiber Π_ϕ^G of ϕ is called L-packet.

(2) Put $\mathcal{S}_\phi = \pi_0\bigl(\mathrm{Cent}_{\widehat{G}(\mathbb{C})}(\phi)\bigr)$.
Then there is a natural bijection $\mathrm{Irr}(\mathcal{S}_\phi) \cong \Pi_\phi^G$.

- Known for some smaller groups (e.g. SL_2, $U(3)$)
- Recent progress for classical groups (cf. Arthur's book)

LLC for $G = \mathrm{GSp}_4$

$G = \mathrm{GSp}_4 \rightsquigarrow \widehat{G} = \mathrm{GSpin}_5 = \mathrm{GSp}_4$
In this case, there is a candidate for LLC by Gan-Takeda.

Classification of L-packets

$r\colon \mathrm{GSp}_4 \hookrightarrow \mathrm{GL}_4$: natural embedding
$\phi\colon W \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C})$: L-parameter $\rightsquigarrow r \circ \phi$: Weil-Deligne rep.
Assume that Π_ϕ^G contains a supercuspidal rep. of $G(\mathbb{Q}_p)$

(I) $(r \circ \phi)|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$, $r \circ \phi$: irred. $\rightsquigarrow \Pi_\phi^G = \{\pi\}$

(II) $(r \circ \phi)|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$, $r \circ \phi = \varphi_1 \oplus \varphi_2$ (φ_i: irred., $\varphi_1 \not\cong \varphi_2$, $\dim \varphi_i = 2$)
$\rightsquigarrow \Pi_\phi^G = \{\pi_1, \pi_2\}$, π_1, π_2 are supercuspidal

(III) $r \circ \phi = \varphi \oplus (\chi \otimes \mathbf{Sp}_2)$, $\varphi|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$, φ: irred.
$\rightsquigarrow \Pi_\phi^G = \{\pi_1, \pi_2\}$, π_1 is supercuspidal, π_2 is not supercuspidal

(IV) $r \circ \phi = (\chi_1 \otimes \mathbf{Sp}_2) \oplus (\chi_2 \otimes \mathbf{Sp}_2)$ $(\chi_1 \neq \chi_2)$
$\rightsquigarrow \Pi_\phi^G = \{\pi_1, \pi_2\}$, π_1 is supercuspidal, π_2 is not supercuspidal

LLC for $GU(2, D)$

D: quaternion div. alg. over $\mathbb{Q}_p$, $J = GU(2, D)$: inner form of $\mathrm{GSp}_4(\mathbb{Q}_p)$
In this case, there is a candidate for LLC by Gan-Tantono.
$\phi\colon W \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C}) \quad \rightsquigarrow \Pi^J_\phi$: L-packet
Π^J_ϕ can be empty. $\Pi^J_\phi \neq \varnothing$ if ϕ is discrete.

Classification of L-packets

$\phi\colon W \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C})$: L-parameter.
Assume that Π^G_ϕ contains a supercuspidal rep.
Then, for previous cases (I)–(IV), we have

- (I) $\Pi^J_\phi = \{\rho\}$, ρ is supercuspidal
- (II) $\Pi^J_\phi = \{\rho_1, \rho_2\}$, ρ_1, ρ_2 are supercuspidal
- (III) $\Pi^J_\phi = \{\rho_1, \rho_2\}$, ρ_1 is supercuspidal, ρ_2 is not supercuspidal
- (IV) $\Pi^J_\phi = \{\rho_1, \rho_2\}$, ρ_1, ρ_2 are not supercuspidal

What we want to know

- Geometric reason why L-packets naturally appear.
- Characterization of L-packets of type (I)–(IV) from geometric viewpoint.

These are new problems which does not appear in the case of GL_n.

- $\mathbb{X}$: an n-dimensional p-divisible group over $\overline{\mathbb{F}}_p$ with slope $1/2$ (e.g. E: a supersingular elliptic curve over $\overline{\mathbb{F}}_p$, $\mathbb{X} = E[p^\infty]^{\oplus n}$)
  $\lambda_0 \colon \mathbb{X} \xrightarrow{\cong} \mathbb{X}^\vee$: a polarization ($\lambda_0^\vee = -\lambda_0$)
- **Nilp**: the category of $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$-algebras A in which p is unipotent

Definition (Rapoport-Zink space)

Define a functor $\mathcal{M} \colon \mathbf{Nilp} \longrightarrow \mathbf{Set}$ as follows:

$$\mathcal{M}(A) = \{(X, \lambda, \rho)\}/\cong$$

- X: p-divisible group over A, λ: polarization of X
- $\rho \colon \mathbb{X} \otimes_{\overline{\mathbb{F}}_p} A/pA \longrightarrow X \otimes_A A/pA$: quasi-isogeny ($= p^{-m} \circ$ *isogeny*)
- $\rho^{-1} \circ (\lambda \bmod p) \circ \rho \in \mathbb{Q}_p^\times \cdot \lambda_0$

$\mathcal{M}$ is represented by a formal scheme over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ called the Rapoport-Zink space.

Rapoport-Zink space for GSp_{2n}

$\mathcal{M}$ is a very large formal scheme.

- Countably many connected components (corresponding to $\deg \rho$)
- Each connected component is not quasi-compact. If $n = 2$, $\mathcal{M}^{\mathrm{red}}$ is a chain of infinitely many $\mathbb{P}^1$'s.

Group action on $\mathcal{M}$

$J = \mathrm{QIsog}(\mathbb{X}, \lambda_0)$: the group of self-quasi-isogenies $\rightsquigarrow J = GU(n, D)$
$\mathcal{M} \curvearrowleft J$ (right action): $(X, \lambda, \rho) \cdot h = (X, \lambda, \rho \circ h)$

Relation to Shimura varieties (p-adic uniformization)

Sh: Shimura variety for GSp_{2n} (the moduli space of principally polarized n-dimensional abelian varieties)
$\mathrm{Sh}^{\mathrm{ss}} \subset \mathrm{Sh}_{\overline{\mathbb{F}}_p}$: supersingular locus, $\mathrm{Sh}\,|^{\wedge}_{\mathrm{Sh}^{\mathrm{ss}}}$: formal completion along $\mathrm{Sh}^{\mathrm{ss}}$
$\rightsquigarrow \mathrm{Sh}\,|^{\wedge}_{\mathrm{Sh}^{\mathrm{ss}}}$ is uniformized by $\mathcal{M}$: $\mathrm{Sh}\,|^{\wedge}_{\mathrm{Sh}^{\mathrm{ss}}} = \coprod_{i=1}^{k} \mathcal{M}/\Gamma_i$ ($\Gamma_i \subset J$)

Rapoport-Zink tower for GSp_{2n}

M = rigid generic fiber of $\mathcal{M}$ (rigid space over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$)

Rapoport-Zink tower

$\{M_K\}_K$: proj. system of étale coverings of M (the Rapoport-Zink tower)

- K runs through compact open subgroups of $\mathrm{GSp}_{2n}(\mathbb{Z}_p)$
- Defined by using K-level str. of the universal polarized p-div. group
- $M_{\mathrm{GSp}_{2n}(\mathbb{Z}_p)} = M$
- If K is a congruence subgp $K_m = \mathrm{Ker}(\mathrm{GSp}_{2n}(\mathbb{Z}_p) \to \mathrm{GSp}_{2n}(\mathbb{Z}/p^m\mathbb{Z}))$, K_m-level str. = trivialization of p^m-torsion points (preserving pol.)

Group actions on Rapoport-Zink tower

- J acts on M_K (preserve levels)
- $G = \mathrm{GSp}_{2n}(\mathbb{Q}_p)$ acts on the tower $\{M_K\}_K$ (doesn't preserve levels) $g \in G \rightsquigarrow M_K \longrightarrow M_{g^{-1}Kg}$ (Hecke action)

Rapport-Zink tower for GSp_{2n}

Relation to Shimura varieties

$\mathrm{Sh}^{[\mathrm{ss}]} \subset \mathrm{Sh}_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$: locus consisting of abelian varieties with supersingular reduction (rigid locally closed subset)

$\mathrm{Sh}_{K,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$: Sh. var. with K-level str., $\mathrm{Sh}_K^{[\mathrm{ss}]} \subset \mathrm{Sh}_{K,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$: inv. image of $\mathrm{Sh}^{[\mathrm{ss}]}$

$\rightsquigarrow \mathrm{Sh}_K^{[\mathrm{ss}]}$ is uniformized by M_K

ℓ-adic étale cohomology of the Rapoport-Zink tower

ℓ: prime number different from p

$H^i_{\mathrm{RZ}} := \varinjlim_K H^i_c(M_K \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \widehat{\mathbb{Q}}_p^{\mathrm{ac}}, \overline{\mathbb{Q}}_\ell)$: rep. of $W \times G \times J$

<u>Goal</u>: describe H^i_{RZ} via LLC for G and J

The case of GL_n

Change the definition of the Rapoport-Zink space as follows:

- $\mathbb{X}$: 1-dim. p-div. group ($\fallingdotseq$ formal group) over $\overline{\mathbb{F}}_p$ with slope $1/n$
- forget all "polarizations"

$\rightsquigarrow$ the Lubin-Tate space, the Lubin-Tate tower, cohomology H^i_{LT}

- $G = \mathrm{GL}_n(\mathbb{Q}_p)$, $J = D_n^\times$
  (D_n is the central division algebra over $\mathbb{Q}_p$ with $\operatorname{inv} D_n = 1/n$)

Theorem (non-abelian Lubin-Tate theory, Harris-Taylor)

π: supercuspidal rep. of $\mathrm{GL}_n(\mathbb{Q}_p)$, ϕ: Weil-Deligne rep. s.t. $\Pi_\phi^{\mathrm{GL}_n} = \{\pi\}$
$\Pi_\phi^{D_n^\times} = \{\rho\}$ ($\pi \longleftrightarrow \rho$: Jacquet-Langlands correspondence)
$\sigma\colon W \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$: ℓ-adic rep. s.t. $\mathrm{WD}(\sigma) = \phi$

$$\mathrm{Hom}_{\mathrm{GL}_n(\mathbb{Q}_p)}(H^i_{\mathrm{LT}}, \pi) = \begin{cases} \sigma\left(\frac{1-n}{2}\right) \otimes \rho & (i = n-1) \\ 0 & (i \neq n-1) \end{cases}$$

The case of GSp_4

(joint work with Tetsushi Ito)

- $G = \mathrm{GSp}_4(\mathbb{Q}_p)$, $J = GU(2, D)$
- $\phi\colon W \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C})$: L-parameter
  Assume that Π^G_ϕ contains a supercuspidal rep.
- For $\rho \in \Pi^J_\phi$, put $H^i_{\mathrm{RZ}}[\rho] := \mathrm{Hom}_J(H^i_{\mathrm{RZ}}, \rho)^{G\text{-sm}}$ (rep. of $W \times G$)
- $H^i_{\mathrm{RZ}}[\rho]_{\mathrm{cusp}}$: the supercuspidal part of $H^i_{\mathrm{RZ}}[\rho]$

Classification of L-parameters (again)

$r\colon \mathrm{GSp}_4 \hookrightarrow \mathrm{GL}_4$: natural embedding

(I) $(r \circ \phi)|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$, $r \circ \phi$: irred.

(II) $(r \circ \phi)|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$, $r \circ \phi = \varphi_1 \oplus \varphi_2$ (φ_i: irred., $\varphi_1 \not\cong \varphi_2$, $\dim \varphi_i = 2$)

(III) $r \circ \phi = \varphi \oplus (\chi \otimes \mathbf{Sp}_2)$, $\varphi|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$, φ: irred.

(IV) $r \circ \phi = (\chi_1 \otimes \mathbf{Sp}_2) \oplus (\chi_2 \otimes \mathbf{Sp}_2)$ $\quad (\chi_1 \neq \chi_2)$

The case of GSp_4 (when $\phi|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$)

Consider the case where $\phi|_{\mathrm{SL}_2(\mathbb{C})} = \mathbf{1}$ (i.e. type (I) or (II))

Main theorem A

ϕ: type (I) or (II), $\rho \in \Pi^J_\phi$: supercuspidal rep.

(1) If $i \neq 3$, then $H^i_{\mathrm{RZ}}[\rho]_{\mathrm{cusp}} = 0$.

(2) $H^3_{\mathrm{RZ}}[\rho]_{\mathrm{cusp}} = \bigoplus_{\pi \in \Pi^G_\phi} \sigma_\pi \otimes \pi$

σ_π: irred. ℓ-adic rep. of W s.t. $\bigoplus_{\pi \in \Pi^G_\phi} \mathrm{WD}(\sigma_\pi) = r \circ \phi$

- If ϕ is type (II), we can determine which of φ_1 or φ_2 is equal to $\mathrm{WD}(\sigma_\pi)$.
- The above theorem is a precise version of Kottwitz's conjecture for $\sum_i (-1)^i H^i_{\mathrm{RZ}}$.

The case of GSp_4 (when $\phi|_{\mathrm{SL}_2(\mathbb{C})} \neq \mathbf{1}$)

Consider the case where $\phi|_{\mathrm{SL}_2(\mathbb{C})} \neq \mathbf{1}$ (i.e. type (III) or (IV))

Main theorem A′

ϕ: type (III), i.e. $r \circ \phi = \varphi \oplus (\chi \otimes \mathbf{Sp}_2)$
$\rho \in \Pi^J_\phi$: supercuspidal rep. (unique)

(1) If $i \neq 3$, then $H^i_{\mathrm{RZ}}[\rho]_{\mathrm{cusp}} = 0$.

(2) $H^3_{\mathrm{RZ}}[\rho]_{\mathrm{cusp}} = \sigma_\pi \otimes \pi$
$\pi \in \Pi^G_\pi$: supercuspidal rep. (unique), $\mathrm{WD}(\sigma_\pi) = \varphi$

Main theorem B

ϕ: type (III) or (IV), $\pi \in \Pi^G_\phi$: supercuspidal rep. (unique)
Then π appears as a subquotient of H^4_{RZ}.

The case of GSp_4 (when $\phi|_{\mathrm{SL}_2(\mathbb{C})} \neq \mathbf{1}$)

Our more precise expectation is the following (in progress)

Expectation

ϕ: type (III) or (IV), $\chi \otimes \mathbf{Sp}_2 \subset r \circ \phi$
$\pi \in \Pi_\phi^G$: supercuspidal rep. (unique), $\rho \in \Pi_\phi^J$: non-supercuspidal rep.

- $\chi \otimes \pi^\vee \otimes \rho$ occurs in H^3_{RZ}.
- $\chi \otimes \pi^\vee \otimes \mathrm{Zel}(\rho)^\vee$ occurs in H^4_{RZ}.
  (Zel: Zelevinsky involution, $\mathrm{Zel}(\rho)^\vee$: non-tempered rep.)

- We hope $H^2_{\mathrm{RZ}} = 0$.
- If $\Pi_\phi^J = \{\rho, \rho'\}$, then $\{\rho', \mathrm{Zel}(\rho)^\vee\}$ is a non-tempered A-packet of J.

Outline of proof

We explain the outline of our proof of main theorem B.

Main theorem B (again)

ϕ: type (III) or (IV), $\pi \in \Pi^G_\phi$: supercuspidal rep.
Then π appears as a subquotient of H^4_{RZ}.

Relate H^i_{RZ} to the cohomology of the Shimura variety.
$\mathrm{Sh}^{[\mathrm{ss}]}_K$ is uniformized by $M_K \rightsquigarrow$ the Hochschild-Serre spectral sequence

Hochschild-Serre spectral sequence (Harris, Fargues)

$$E_2^{i,j} = \mathrm{Ext}^j_{J\text{-sm}}(H^{6-j}_{\mathrm{RZ}}, \mathcal{A})(-3) \Longrightarrow \varinjlim_K H^{i+j}(\mathrm{Sh}^{[\mathrm{ss}]}_K, \overline{\mathbb{Q}}_\ell)$$

$\mathcal{A}$: space of automorphic forms on $\mathrm{GSp}_4(\mathbb{A}^{\infty,p}) \times J$

$H^i(\mathrm{Sh}^{[\mathrm{ss}]}_\infty) := \varinjlim_K H^{i+j}(\mathrm{Sh}^{[\mathrm{ss}]}_K, \overline{\mathbb{Q}}_\ell)$

Outline of proof

$\mathrm{Sh}_K^{[\mathrm{ss}]}$: supersingular reduction locus
$\mathrm{Sh}_K^{[\mathrm{good}]}$: good reduction locus

$\mathrm{Sh}_K^{[\mathrm{ss}]} \subset \mathrm{Sh}_K^{[\mathrm{good}]} \subset \mathrm{Sh}_K$

$$\rightsquigarrow IH^i(\mathrm{Sh}_\infty)_{\mathrm{cusp}} \xrightarrow[(1)]{\cong} H^i(\mathrm{Sh}_\infty)_{\mathrm{cusp}} \xrightarrow[(2)]{\cong} H^i(\mathrm{Sh}_\infty^{[\mathrm{good}]})_{\mathrm{cusp}} \xrightarrow[(3)]{\cong} H^i(\mathrm{Sh}_\infty^{[\mathrm{ss}]})_{\mathrm{cusp}}$$

(1) Use the minimal compactification over $\mathbb{C}$. True for every Shimura variety.
(2) Joint work with Naoki Imai. True for fairly general Shimura varieties.
(3) Boyer's trick. Only valid for GSp_4. (proof is not applicable to GSp_{2n} with $n \geq 3$)

Outline of proof

Another key is:

Theorem (non-cuspidality)

If $i \neq 2, 3, 4$, no supercuspidal rep. appears as a subquotient of H^i_{RZ}.

- Since $\dim M = 3$, $H^i_{\mathrm{RZ}} = 0$ unless $0 \leq i \leq 6$.
- $2 = 3 - 1 = \dim M - \dim \mathcal{M}^{\mathrm{red}}$, $\quad 4 = 3 + 1 = \dim M + \dim \mathcal{M}^{\mathrm{red}}$
- This theorem is proved by a purely local method.
- The method is similar to our purely local proof of non-cuspidality for the Lubin-Tate tower. However, we encounter many new difficulties since $\mathcal{M}$ is very large.
- For a proof, we introduce variants of formal nearby cycle functor. (in order to capture "invisible boundaries")

Outline of proof

Main theorem B (again)

ϕ: type (III) or (IV), $\pi \in \Pi_\phi^G$: supercuspidal rep. (unique)
Then π appears as a subquotient of H^4_{RZ}.

<u>Proof of main theorem B</u>

Take a cuspidal automorphic rep. Π of $\mathrm{GSp}_4(\mathbb{A})$ s.t.

- Π occurs in $IH^2(\mathrm{Sh}_\infty)$ (a condition on Π_∞)
- $\Pi_p \cong \pi^\vee$

(Need the assumption that ϕ is type (III) or (IV))

By the Hochschild-Serre spectral sequence, there are i, j with $i+j=2$ s.t. Π_p contributes to $\mathrm{Ext}^i_{J\text{-sm}}(H^{6-j}_{\mathrm{RZ}}, \mathcal{A})$.

By the non-cuspidality, we have $6-j \neq 5, 6$.
So Π_p appears in $\mathrm{Hom}_J(H^4_{\mathrm{RZ}}, \mathcal{A})$, and thus π appears in H^4_{RZ}.

Outline of proof

Our proof of main theorems A, A$'$ is similar.
Need to globalize π and ρ carefully.

Take a cuspidal automorphic rep. Π of $\mathrm{GSp}_4(\mathbb{A})$ s.t.

- Π occurs in $IH^3(\mathrm{Sh}_\infty)$ (a condition on Π_∞)
- $\Pi_p \cong \pi$
- Π satisfies the strong multiplicity one theorem at p. Namely, an autom. rep. Π' of $\mathrm{GSp}_4(\mathbb{A})$ with $\Pi^p = (\Pi')^p$ coincides with Π.

(cf. Arthur's multiplicity conjecture)

Globalize ρ to Π^J so that similar conditions and $\Pi^{\infty,p} = (\Pi^J)^{\infty,p}$ are satisfied.

Take "$\Pi^{\infty,p}$-parts" of the Hochschild-Serre spectral sequence.

Local method

Global method cannot answer our question why L-packets naturally appear. Another purely local method is usage of Lefschetz trace formula for open rigid (or adic) spaces.

Theorem (Lefschetz trace formula for open rigid spaces)

X: quasi-compact smooth adic space over $\widehat{\mathbb{Q}}_p^{\mathrm{ac}}$
$X \subset \overline{X}$: compactification
$f\colon X \longrightarrow X$: proper morphism, assumed to be extended to $\overline{f}\colon \overline{X} \longrightarrow \overline{X}$
Assume that for every $x \in \overline{X} \setminus X$, x and $\overline{f}(x)$ can be separated by closed constructible subsets of $\overline{X}$. Then,

$$\sum_i (-1)^i \operatorname{Tr}(f^*; H_c^i(X, \overline{\mathbb{Q}}_\ell)) = \# \operatorname{Fix}(f)$$

Apply this formula to quasi-compact open subsets of M_K.

Local method

Theorem

ϕ: $W \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{GSp}_4(\mathbb{C})$: L-parameter of type (I) or (II)

- Assume the character relation between Π^G_ϕ and Π^J_ϕ.
  (It is expected to be proved by the stable trace formula. It has been proved in some cases, e.g. ϕ is a TRSELP.)
- Assume that $H^i_{\mathrm{RZ}}[\rho]$ is a finitely generated G-module for $\rho \in \Pi^J_\phi$.

Then, for every elliptic regular semisimple element $g \in G$,

$$\sum_{\rho \in \Pi^J_\phi} \theta_{H_{\mathrm{RZ}}[\rho]}(g) = 4 \sum_{\pi \in \Pi^G_\phi} \theta_\pi(g)$$

- $H_{\mathrm{RZ}}[\rho] = \sum_i (-1)^i H^i_{\mathrm{RZ}}[\rho]$
- $\theta_{H_{\mathrm{RZ}}[\rho]}$, θ_π: distribution characters. They are locally constant functions over the set of regular semisimple elements of G.

Local method

- For GL_n, there are preceding works by Faltings and Strauch.
- By this method, we get no information on the action of W.
- To count fixed points in M_K, we use the p-adic period map (the map attaching the Hodge filtration of the Dieudonné module to a p-divisible group)
  $\rightsquigarrow$ by p-adic Hodge theory, stable conjugacy classes and their transfers naturally appear.
  (these are closely related to L-packets)
- With more precise study, we expect to show

$$\theta_{H_{\mathrm{RZ}}[\rho]}(g) = \frac{4}{\#\Pi_\phi^G} \sum_{\pi \in \Pi_\phi^G} \theta_\pi(g)$$

Comments on other groups

- Similar methods are applicable to $GU(2,1)$, and GL_4 with slope $1/2$.
- Boyer's trick is valid for $GU(n-1,1)$ (Mantovan, Shen)
  $\rightsquigarrow$ similar results for the alternating sum of H^i_{RZ} might be possible.
  Proof of the non-cuspidality doesn't work in this case.
- It seems important to consider the case of $GU(2,D)$ (the "dual tower" for GSp_4) simultaneously (cf. Faltings isomorphism)