CUP PRODUCT ON THE PERIODIC CYCLIC COHOMOLOGY OF FOLIATION ALGEBRAS

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Abstract. We construct a ring structure on the periodic cyclic cohomology of the smooth foliation algebra when the transverse bundle of the foliation admits a spin^c structure. The linear map of the periodic cyclic cohomology of the foliation algebra into the usual cohomology ring of the base manifold becomes a ring homomorphism with respect to this structure. As an application we obtain a cup-length type lower bound estimate for the transverse Lusternik-Schnirelmann category of the foliation.

1. Introduction

Let M be a smooth manifold. By Connes-Kostant-Rosenberg-Hochsild's theorem, the periodic cyclic cohomology group $HP^*(C^\infty(M))$ is naturally isomorphic to the de Rham homology group $H_*^{\mathrm{dR}}(M;\mathbb{C})$. Via the Poincaré duality $HP^*(C^\infty(M))$ is isomorphic to the cohomology ring $H^*(M;\mathbb{C})$ of M.

Let $\mathcal{A}$ and $\mathcal{B}$ be locally convex algebras. In the bivariant formalism [4], $HP^*(\mathcal{A})$ is identified to $HP(\mathcal{A},\mathbb{C})$ and we have the bivariant Chern-Connes character $kk(\mathcal{A},\mathcal{B}) \to HP(\mathcal{A},\mathcal{B})$. In particular, any element of $kk(\mathcal{A},\mathcal{B})$ defines a homomorphism of $HP^*(\mathcal{B})$ into $HP^*(\mathcal{A})$.

A smooth deformation [5] of $\mathcal{A}$ into $\mathcal{B}$ is a locally convex algebra C over the closed interval $[0,1]$ with $C_0 = \mathcal{A}$ and $C|_{(0,1]} = (0,1] \times \mathcal{B}$.

Any such smooth deformation of $\mathcal{A}$ into $\mathcal{B}$ determines an algebra homomorphism of $J^2A \to B \otimes \mathcal{K}$. This determines an element of $kk(\mathcal{A},\mathcal{B})$. Thus, any smooth deformation of a smooth groupoid G into another H determines a kk-morphism in $kk(C^\infty(G), C^\infty(H))$.

We apply this to the projection $p\colon M \to M/F$ onto the space of the leaves of a foliated manifold $(M;F)$, and the diagonal maps $\Delta\colon M \to M \times M$ and $\Delta\colon M/F \to M/F \times M/F$. When these maps are K-oriented, we obtain the associated wrong way kk-morphisms $p_!$ and $\Delta_!$. The pullback maps of these induced on the periodic cyclic cohomology groups describe the cup products and the index theory homomorphism $HP^*(C^\infty(M;F)) \to H^*(M)$.

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2. Ring structure of the periodic cyclic cohomology

2.1. **Diagonal map in the kk-category.** Let $(M;F)$ be a compact C^∞ foliated manifold. In the following we assume that M and F are oriented and that there is a spin^c structure on the transverse bundle $\tau = TM/F$.

For each open set U of M, let F_U be the foliation on U obtained as the restriction of F. The locally convex algebras $\mathcal{A}_U = C^\infty(U;F_U)$ of [1] give a covariant functor on the category of open sets with the inclusions as its morphisms.

In the following two sections we assume that the codimension $q = \operatorname{codim} F$ of F is even.

We are going to construct a kk-morphism $\Delta_!$ of $\mathcal{A}_M$ into $\mathcal{A}_M \otimes \mathcal{A}_M$. Let T be a closed complete transversal of F, G_T be the restricted holonomy groupoid on. Thus the algebra $C^\infty(G_T)$ is kk-equivalent to $\mathcal{A}$. The restriction τ_T of τ to T is identified to the tangent bundle of T.

The transverse spin^c structure on τ gives a kk-equivalence between $C_c^\infty(T)$ of $C_c^\infty(\tau_T)$. Let V be a tubular neighborhood of the image of the diagonal embedding of τ_T into $\tau_T \times \tau_T$. Then V is identified to the pullback $p^*\tau_\mathbb{C}$ of the complexification of τ along the bundle projection $\tau_T \to T$. Hence we have a kk-equivalence of $C^\infty(\tau_T)$ to $C_c^\infty(V)$ corresponding the Thom isomorphism for the complex vector bundle $p^*\tau_\mathbb{C}$.

The kk-morphism $C_c^\infty(\tau_T) \to C^\infty(V)$ can be extended to a kk-morphism of the groupoid algebra $C_c^\infty(G_T \ltimes \tau_T)$ into the algebra $C_c^\infty(\Delta(G_{\tau T})$ of the following subgroupoid $\Delta(G_{\tau T}) \subset (G_T \times G_T) \ltimes (\tau_T \times \tau_T)$. For each $(\gamma, \xi) \in G_T \times_T \tau_T$ ($s\gamma = p\xi$) and transverse vectors $\eta, \zeta \in \tau_{p\xi}$, we associate a local homeomorphism of $(p\xi + \eta, p\xi - \zeta, \xi + \zeta, \xi - \zeta)$ to $(\gamma(p\xi + \eta), \gamma(p\xi - \eta), d\gamma(p\xi - \zeta), d\gamma(p\xi + \zeta))$. Thus obtained kk-morphism $C_c^\infty(G_T \ltimes \tau_T) \to C_c^\infty(\Delta(G_{\tau T})$ and the inclusion $C_c^\infty(\Delta(G_{\tau T}) \subset C^\infty((G_T \times G_T) \ltimes (\tau_T \times \tau_T))$ gives the required wrong way map $\Delta_!$.

2.2. **Product structure of the periodic cyclic cohomology.** The bivariant Chern-Connes character $kk(\mathcal{A}_M, \mathcal{A}_M \otimes \mathcal{A}_M) \to HP(\mathcal{A}_M, \mathcal{A}_M \otimes \mathcal{A}_M)$ determines an associative product structure on $HP^*(\mathcal{A}_M)$.

Recall the "exterior" cup product

$$HP^*(\mathcal{A}) \otimes HP^*(\mathcal{A}) \to HP^*(\mathcal{A} \otimes \mathcal{A}), \phi \otimes \psi \to \phi \# \psi$$

defined in [3].

Definition 1. This composition

$$HP^*(\mathcal{A}_M) \otimes HP^*(\mathcal{A}_M) \to HP^*(\mathcal{A}_M)$$

of the exterior cup product map with the pullback by $\Delta_! \in HP(\mathcal{A}_M, \mathcal{A}_M \otimes \mathcal{A}_M)$ is called the *interior cup product* on $HP^*(\mathcal{A}_M)$.

There is a distinguished cyclic cocycle on $\mathcal{A}_M$, the transverse fundamental class $\psi \in HP^0(\mathcal{A}_M)$ given by the transverse integration of transverse q-forms.

Proposition 2. *The transverse fundamental class becomes the multiplicative unit of $HP^*(\mathcal{A}_M)$*

2.3. **Chern character.** We have a linear map $\lambda\colon HP^*(\mathcal{A}_M) \to H^*(M)$ related to the index theory [3]. This map becomes a ring homomorphism with respect to the cup product on $H^*(M)$.

When the foliation is trivial, i.e. $\mathcal{A}_U = C_c^\infty(U)$ for each open set U of M, this reduces to the Poincaré duality map as follows. We have the identification of (B, b)-bicomplex of $C^\infty(M)$ with an iterate of the complex $(\Omega_*(M))$ of de Rham currents as in [2]. The localization map $\Omega^{*-*}(M) \to \Omega^{*-*}(U_*)$ reduces to the localization $\Omega^{\mathrm{dR}}_{*-*}(M) \to \Omega^{\mathrm{dR}}_{*-*}(U_*)$ of currents. On the other hand, we have a subcomplex $\Omega^{*-*}_{\mathrm{dR}}$ of $\Omega^{\mathrm{dR}}_{*-*}$, which consists of smooth differential forms with (possibly) noncompact support.

Let V be a tubular neighborhood image of the diagonal embedding $\Delta\colon F \to F \times F$. Then V is identified to the pullback $p^*(\tau \oplus F_{\mathbb{C}})$ of $\tau \oplus F_{\mathbb{C}}$ along $p\colon F \to M$. By assumption we have a spin^c structure on τ, we have an spin^c structure on $p^*(\tau \oplus F_{\mathbb{C}}$. Hence Δ induces a kk-morphism

$$\Delta_! \in kk(C_c^\infty(F), C_c^\infty(F) \otimes C_c^\infty(F)).$$

Consequently, we obtain a multiplication map

$$m\colon \Delta_!^*\colon HP^*(C_c^\infty(F)) \otimes HP^*(C_c^\infty(F)) \to HP^*(C_c^\infty(F)).$$

We have the Chern character map $\mathrm{Ch}\colon HP^*(C_c^\infty(F)) \to H_*^{\mathrm{dR}}(F)$ compatible with the K-theory Chern character $\mathrm{Ch}\colon K^*F \to H_c^*F$. Since we have the Thom isomorphism $H^*M \to H_c^*(F)$, we obtain an isomorphism $H_*^{\mathrm{dR}}F \to H_*M$ as its transposition. Finally, with the Poincaré duality iromorphism $H_*M \to H^*M$, we obtain a linear map α of $HP^*(C^\infty(F))$ into H^*M.

Lemma 3. *We have the following compatibility among the cup product on H^*M, the product on $HP^*(C_c^\infty(F))$ and the linear map α:*

$$\alpha(x) \cup \alpha(y) = \mathrm{Tod}(\tau)\,\mathrm{Tod}(F_{\mathbb{C}})\alpha(\Delta_!^*(x \otimes y)).$$

Proof. The "exterior product" maps

$$HP^*(C_c^\infty(F)) \otimes HP^*(C_c^\infty(F)) \to HP^*(C_c^\infty(F) \otimes C_c^\infty(F)),$$
$$H_*^{\mathrm{dR}}(F) \otimes H_*^{\mathrm{dR}}(F) \to H_*^{\mathrm{dR}}(F),$$
$$H^*M \otimes H^*M \to H^*M$$

are compatible with the transformations $HP^*(C^\infty(F)) \to H_*^{\mathrm{dR}}(F)$, $H_*^{\mathrm{dR}}(F) \to H^*(M)$, etc. Thus $\alpha(x) \cup \alpha(y)$ is equal to the pullback $\Delta^*(\alpha(x \# y))$ along the diagonal map $\Delta : M \to M \times M$. Hence it remains to show the equality

$$\Delta^*\alpha(z) = \mathrm{Tod}(\tau)\,\mathrm{Tod}(F_{\mathbb{C}}) \cup \alpha\Delta_!^*(z)$$

for $z \in HP^*(C^\infty(F \times F))$.

$$\begin{array}{ccccccccc}
HP^*C_c^\infty(F\times F) & & \longrightarrow & & HP^*(C_c^\infty(V)) & \xrightarrow{\phi_{HP}} & & & HP(C_c^\infty(F)) \\
\downarrow & & & & \downarrow & & & & \downarrow \\
H_*^{\mathrm{dR}}(F\times F) & & \longrightarrow & & H_*^{\mathrm{dR}}(V) & \xrightarrow{\phi_H} & & & H_*^{\mathrm{dR}}(F) \\
\downarrow & \searrow & & & \downarrow & & & \swarrow & \downarrow \\
H^*(M\times M) & & H^*(F\times F) & \longrightarrow & H^*V & \longrightarrow & H^*(F) & & H^*M \\
\downarrow & \swarrow & & & & & & \searrow & \downarrow \\
H^*(M\times M) & & & & \longrightarrow & & & & H^*M
\end{array}$$

The difference of the Thom isomorphisms ϕ_{HP} and ϕ_H is given by the Todd class of $\tau \oplus F_{\mathbb{C}}$:

$$\phi_H \operatorname{Ch}(z) = \operatorname{Tod}(\tau) \operatorname{Tod}(F_{\mathbb{C}}) \operatorname{Ch} \phi_{HP}(z).$$

For each manifold X, the de Rham homology groups $H_*^{\mathrm{dR}}(X)$, are isomorphic to the non-compact support cohomology groups $H^*(X)$. The pullback $H^*(V) \to H^*(F)$ along the embedding of F into V is compatible with ϕ_H. On the other hand, the non-compact cohomology groups $H^*(F \times F)$ and $H^*(F)$ are respectively isomorphic to $H^*(M \times M)$ and $H^*(M)$. □

Theorem 4. *The homomorphism* $\lambda\colon HP^*(C^\infty(M;F)) \to H^*M$ *satisfies the following compatibility with respect to the products on* $HP^*(C^\infty(M;F))$ *and* $H^*(M)$*:*

$$\lambda(x) \cup \lambda(y) = \operatorname{Tod}(\tau) \operatorname{Tod}(F_{\mathbb{C}}) \lambda(\Delta_!^*(x \otimes y)).$$

Proof. By the longitudinal index theorem for foliated space, the index map $p_! \in kk(C^\infty(M), C^\infty(M;f))$ is identified to the following topological index morphism:

Let $\iota\colon M \to \mathbb{R}^n$ be an injective immersion of M into an Euclidean space. Put $N = M \times T\mathbb{R}^n$, $F' = F \times 0$. The smooth algebra $C^\infty(N;F')$ of the foliated manifold $(N;F')$ is identified to $C^\infty(M;F) \otimes C_c^\infty(\mathbb{R}^n)$. The total space of F is mapped into N as the subspace $\{(x, \iota(x), d\iota(\xi)) : x \in M, \xi \in F_x\}$. Its tubular neighborhood W can be identified to the pullback $p^*\tau_{\mathbb{C}}$ of the complexification of τ. On the other hand W is a transversal for F'. The Thom isomorphism $C_c^\infty(F) \to C_c^\infty(W)$ and the inclusion morphism $C_c^\infty(W) \to C^\infty(N;F')$ gives the required topological index morphism $t!\colon C_c^\infty(F) \to C^\infty(N;F')$.

In the same way, using $V \times V$ as the tubular neighborhood, we have the topological index map $(t \times t)_!\colon C_c^\infty(F) \times C_c^\infty(F) \to C^\infty(N;F') \otimes C^\infty(N;F')$. The diagonal inclusion morphisms $C_c^\infty(F) \to C_c^\infty(F) \times C_c^\infty(F)$ and $C^\infty(N;F') \to C^\infty(N;F') \otimes C^\infty(N;F')$ are compatible with respect to the topological index morphisms.

Consequently it is enough to show the equality $(t \times t)_! \Delta_! = \Delta_! t_!$. This is true, since we have the equality of the morphisms as the ones from $C_c^\infty(F)$ to $C_c^\infty(V \times V) \subset C^\infty(N;F') \otimes C^\infty(N;F')$. □

The Chern character of the dual fundamental class $[M/F]^*$ of [3] is equal to $1 \in H_0(M)$.

Notation. Let $I_{(M;F)}$ denote the subspace $\{\phi : \langle \phi, [M/F]^* \rangle = 0\}$ of $HP^*(\mathcal{A}_M)$.

2.4. **Reduction of the odd codimension case to the even case.** Put $N = M \times S^1$ and $F' = F \times 0 \subset TN$. Then the foliation F' on N has even codimension and the transverse bundle of F' admits a spin^c structure. The foliation algebra $C^\infty(N; F') \simeq C^\infty(M; F) \otimes C^\infty(S^1)$ contains $C^\infty(M; F) \otimes C^\infty(0, 1)$ as a closed ideal.

We have the splittings

$$HP^*(C^\infty(N; F')) = HP^*(C^\infty(M; F) \otimes C^\infty(0, 1)) \oplus HP^*(C^\infty(M; F)),$$
$$H^*(N) = H^*(N; M) \oplus H^*(M).$$

The subspace $HP^{*+1}(C^\infty(M; F)) \simeq HP^*(C^\infty(M; F) \otimes C^\infty(0, 1))$ of $HP^*(C^\infty(N; F'))$ is closed under multiplication. It is mapped to the subring $H^*(M)$ of $H^*(N)$. Thus obtained homomorphism $HP^{*+1}(C^\infty(M; F)) \to H^*(M)$ is exactly the linear map λ.

3. Application to transverse Lusternik-Schnirelmann category

Let $(M; F)$ be a foliated smooth manifold. We have the following analogue of Lusternik-Schnirelmann category.

Definition 5. (cit. needed) An open set U of M is said to be *transversely categorical* when there is a smooth map $H : U \times [0, 1] : M$ such that

- Each H_t maps a leaf of F_U into a leaf of F.
- The initial map H_0 is the tautological inclusion map of U.
- The image of H_1 is contained in a leaf of F.

The (smooth) *transverse Lusternik-Schnirelmann category* $\mathrm{Cat}^{\pitchfork}(M; F)$ is the smallest natural number n such that there exist a covering of M by $n+1$ transversely categorical open sets $(U_i)_{i=0}^n$.

Remark 6. When the leaves of F are contractible, $\mathrm{Cat}^{\pitchfork}(M; F)$ is bounded from above by the Lusternik-Schnirelmann category $\mathrm{Cat}(M)$ of M. In such case M has the homotopy type of the homotopy classifying space BG of the holonomy groupoid of F. Thus this quantity is closely related to the homotopy type of BG.

Lemma 7. *Let $H : N \times [0, 1] \to M$ be a smooth map such that each H_t is a proper immersion satisfying $H_t^* F = H_0^* F$ for any $t \in [0, 1]$. The pullbacks*

$$\lambda H_t^* : HP^*(C^\infty(M; F)) \to H) \quad (t \in [0, 1])$$

does not depend on t.

Proof. We have the induced smooth map $dH_t : F_N \to F$ for each t. The maps $dH_t^* H^* F \to H^* F_N$ induced on the cohomology of the total spaces do not depend on t. □

Lemma 8. *Let U be a transversely contractible open set of M. The image of $I_{(M;F)}$ under the localization map $C^\infty(M; F)$ is zero.*

Proof. Let H_t $(0 \le t \le 1)$ be a smooth transverse contraction of U in M. We obtain a smooth map $H' : U \times [0, 1] \to M$ satisfying:

- Each H'_t maps a leaf of F_U into a leaf of F.
- Each H'_t is proper.
- The initial map H'_0 is the tautological inclusion map of U.
- The image of H'_1 is contained in a foliation chart of F.

Thus the foliation algebra of $H'_1(U)$ is strongly Morita equivalent to the smooth function algebra of an ordinary (non-compact) manifold of dimension q.

The pullback : $HP^*(C^\infty(M;F)) \to HP^*(C^\infty(U;F_U))$ by H'_1 factors through the natural localization map $HP^*(C^\infty(M;F)) \to HP^*(C^\infty(V;F_V))$ that sends $I_{(M;F)}$ to zero. □

We obtain a cup-length type estimate for the transverse Lusternik-Schnirelmann category of the foliation.

Theorem 9. *Let $x_0, \ldots, x_k$ be k elements of $I_{(M;F)}$ such that the Chern character $ch(x_0 \cdot x_1 \cdots x_k)$ is not zero in $H^*(M)$. Then we have $\mathrm{Cat}^{\pitchfork}(M;F) > k$.*

Proof. The image of $\lambda\colon HP^*(\mathcal{A}_M) \to H^*(M)$ restrict to the zero class in $H^*(U)$ when U is transversely contractible.

Let $U_0, \ldots, U_k$ be smooth and transversely contractible open sets. For each i we may find a class $\tilde{x}_i \in H^{>0}(M;U)$ which is mapped to x_i. Then the product $\tilde{x}_0 \cup \cdots \cup \tilde{x}_k$ in $H^*(M; U_0 \cup \cdots U_k)$, which is a multiple of $ch(x_0 \cdot x_1 \cdots x_k)$ by an invertible element, is not zero. Hence $U_0 \cup \cdots \cup U_k$ cannot be equal to M. □

Remark 10. Suppose the leaves of the foliation F are given as the orbits of an action α by a locally free action of a solvable group G. In that case the foliation algebra $C^\infty(M;F)$ is kk-equivalent to the crossed product $C^\infty(M) \rtimes_\alpha G$. By the Connes-Thom isomorphism in the periodic cyclic cohomology [5], $HP^*(C\infty(M) \rtimes_\alpha G) \simeq HP^*(C^\infty(M))$. In that case the transverse LS category of $(M;f)$ is bounded below by the cup-length of $H^*(M)$.

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