

ORBIT EQUIVALENCE AND OPERATOR ALGEBRAS

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ABSTRACT. This treatise came out of the lecture given by Narutaka Ozawa at the University of Tokyo during the winter semester 2006-2007. It includes an elementary theory of orbit equivalence via type II_1 von Neumann algebras, Lück's dimension theory [4] and its application to L^2 Betti numbers [3], convergence of the semigroup associated to a derivation and a theorem of Popa on embeddability of subalgebras.

CONTENTS

1. Introduction	1
1.1. Orbit equivalence	1
1.2. Preliminaries on von Neumann algebra	3
1.3. Crossed products	4
1.4. von Neumann algebra from orbit equivalence	4
2. Elementary theory of orbit equivalence	5
2.1. Essentially free action	5
2.2. Inclusion of von Neumann algebras	5
2.3. Theorem of Connes-Ferdman-Weiss	8
3. L^2 -Betti numbers	11
3.1. Introduction	11
3.2. Operators affiliated to a type II_1 algebra	12
3.3. Projective modules over a finite von Neumann algebra	13
3.4. Application to orbit equivalence	16
4. Derivation on von Neumann algebras	19
4.1. Densely defined derivation	19
4.2. Semigroup associated to a derivation	21
Appendix A. Embeddability of subalgebras	27
References	29

1. INTRODUCTION

1.1. Orbit equivalence.

Definition 1. Let Y be a topological space. Let B_Y denote the σ -algebra of the Borel sets of Y . When Y is a separable complete metric space, (Y, B_Y) is said to be a standard σ -algebra.

Remark 1. When X is a standard Borel space, X is either (at most) countable or isomorphic to the closed interval $[0, 1]$.

2000 *Mathematics Subject Classification.* 46L10;37A20.

Definition 2. A standard Borel space with a Borel probability measure is said to be a (standard) probability space. A point $x \in X$ of probability space (X, μ) is said to be an atom of (X, μ) when $\mu(x) > 0$. A probability space (X, μ) is said to be diffuse when there it has no atom.

Example 1. (Examples of standard probability space)

- (1) The infinite product $(\prod_{n \in \mathbb{N}} \{0, 1\}, \otimes_n \mu_n)$ where μ_n is a probability measure on $\{0, 1\}$ for each $n \in \mathbb{N}$.
- (2) When G is a separable compact group, the Haar measure on G makes G into a standard probability space.

When a probability space (X, μ) is given, we obtain a (w^*) separable von Neumann algebra $L^\infty(X)$ and a normal state μ on it. When $\phi : (X, \mu) \rightarrow (Y, \nu)$ is an isomorphism of probability spaces, we obtain an isomorphism $\phi_* : L^\infty(Y) \rightarrow L^\infty(X)$ such that $\mu \circ \phi_* = \nu$.

Theorem 1. (*von Neumann*)

- (1) When (X, μ) and (Y, ν) are diffuse probability spaces, $(L^\infty(X, \mu), \mu) \simeq (L^\infty(Y, \nu), \nu)$.
- (2) For each isomorphism $\sigma : L^\infty(Y) \rightarrow L^\infty(X)$ with $\mu\sigma = \nu$, there exists a Borel isomorphism $\phi : X \rightarrow Y$ such that $\phi_*\mu = \nu$ and $\phi_* = \sigma$.

Proof. (Outline): (1) We may assume that $Y = \prod_{\mathbb{N}} \{0, 1\}$, $\mu = \otimes(\frac{1}{2}, \frac{1}{2})$. Since X is diffuse, we have a decomposition $X = X_0 \amalg X_1$ with $\mu(X_0) = \frac{1}{2}$. We can continue this as $X_0 = X_{00} \amalg X_{01}$, $\mu(X_{00}) = \frac{1}{4}$, so on. Note that there is a separating family $(B_n)_{n \in \mathbb{N}}$ in B_X to make the partition by $X_{**\dots}$ fine enough, which will imply the desired isomorphism between $L^\infty X$ and $L^\infty Y$.

(2) We assume that $Y = [0, 1]$ here. For each $r \in \mathbb{Q} \cap [0, 1]$, put $E_r = \sigma(\chi_{[0, r]})$. Define $\phi : X \rightarrow [0, 1]$ by $\phi(x) = \inf \{r : x \in E_r\}$. The inverse image $\phi^{-1}([0, t]) = \cup_{r < t} E_r$ is Borel, which means that ϕ is a Borel map. By $\sigma(\chi_{[0, r]}) = \phi_*(\chi_{[0, r]})$ for $r \in \mathbb{Q} \cap [0, 1]$, we have $\sigma = \phi_*$ and $\phi_*\mu = \text{Lebesgue measure}$.

It remains to replace ϕ by a Borel isomorphism. Let $(B_n)_{n \in \mathbb{N}}$ be a separating family of X . For each n , there exists $F_n \in B_Y$ such that $\phi_*\chi_{F_n} = \chi_{B_n}$. Thus $N = \cup_n B_n \Delta \phi^{-1}F_n$ is a null set. On $X \setminus N$, $x \in B_n$ is equivalent to $\phi(x) \in F_n$. If x and y are distinct points of $\mathbb{C}N$, there exists an integer n such that $x \in B_n$ while $y \notin B_n$. Thus $\phi(x) \neq \phi(y)$ and ϕ is injective on $\mathbb{C}N$. We may assume that N and $Y \setminus \phi(\mathbb{C}N)$ are uncountable so that $N \simeq \mathbb{C}\phi(\mathbb{C}N)$. \square

Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action by a discrete countable group. (We may assume that it acts by Borel isomorphisms.) Then we consider an action $\alpha \curvearrowright L^\infty(X, \mu)$ preserving the state μ .

Definition 3. Two actions $\Gamma \curvearrowright (X, \mu)$ and $\Gamma \curvearrowright (Y, \nu)$ are said to be conjugate when there exists a probability space isomorphism $\phi : (X, \mu) \rightarrow (Y, \nu)$ a.e. Γ -equivariant. This is equivalent to the existence of a Γ -equivariant state preserving isomorphism $\sigma : L^\infty(Y, \nu) \rightarrow L^\infty(X, \mu)$.

Definition 4. Let $\Gamma \curvearrowright (X, \mu)$ be an action by measure preserving Borel isomorphisms. $\mathcal{R}_{\Gamma \curvearrowright (X, \mu)} = \{(sx, x) : s \in \Gamma\}$ is called the orbit equivalence relation.

Definition 5. Two actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are said to be orbit equivalent when there exists a measure preserving Borel isomorphism $\phi : Y \rightarrow X$ such that $\Gamma\phi(y) = \phi(\Lambda y)$ for a.e. $y \in Y$.

Definition 6. A partial Borel isomorphism on X is a triple (ϕ, A, B) where $A, B \in B_X$ and $\phi : A \xrightarrow{\sim} B$ is a Borel isomorphism.

Definition 7. A measure preserving standard orbit equivalence is a subset \mathcal{R} of $X \times X$ satisfying the following conditions:

- (1) \mathcal{R} is a Borel subset in the product space structure.
- (2) \mathcal{R} is an equivalence relation.
- (3) For each $x \in X$, the class of x is at most countable.
- (4) For any partial Borel isomorphism ϕ whose graph is contained in \mathcal{R} , ϕ is measure preserving.

Theorem 2. (*Lusin*) Let X, Y be standard spaces.

- (1) If $\phi : X \rightarrow Y$ is Borel and countable-to-one, $\phi(X)$ is Borel. Moreover there exists a Borel partition $X = \coprod X_n$ such that $\phi|_{X_n}$ is a Borel isomorphism onto $\phi(X_n)$.
- (2) When \mathcal{R} is a standard orbit equivalence, $\mathcal{R} = \mathcal{G}(\phi_n)$ where ϕ_n is a partial Borel isomorphism for each n .

Lemma 3. Let A be a subset of a standard space X , ϕ a mapping of A into X . ϕ and A are Borel if and only if the graph $\mathcal{G}(\phi) = \{(\phi x, x) : x \in A\}$ of ϕ is Borel in $X \times X$.

Proof. \Leftarrow is a consequence of Lusin's theorem.

\Rightarrow : Let $(B_n)_{n \in \mathbb{N}}$ be a separating family of X . $y \neq \phi(x)$ is equivalent to $(y, x) \in \cup_n (\mathbb{C}B_n \times \phi^{-1}(B_n))$. Thus $\mathcal{G}(\phi) = \mathbb{C}(\cup_n (\mathbb{C}B_n \times \phi^{-1}(B_n)))$. \square

1.2. Preliminaries on von Neumann algebra. Let A be a $*$ -subalgebra of $B(H)$. (A will generate a von Neumann algebra M of our interest.) In the following A is often assumed to admit a cyclic tracial vector $\xi_\tau \in H$, i.e. $\|\xi_\tau\| = 1$, $A\xi_\tau$ is dense in H , and that the vector state $\tau(a) = \langle a\xi_\tau, \xi_\tau \rangle$ is tracial.

Remark 2. A state τ is tracial means that two sesquilinear forms $\tau(ab^*)$ and $\tau(b^*a)$ agree. To check this, it is enough to show $\tau(aa^*) = \tau(a^*a)$ by polarization. Under the assumption above ξ_τ becomes a separating vector for A'' . Indeed, $a\xi_\tau = 0$ implies $\tau(bc^*a) = 0$ for $b, c \in A$, which means $\tau(c^*ab) = 0$ and in turn $\langle aH, H \rangle = 0$.

Notation. Put $\hat{a} = a\xi_\tau$ so that $\langle \hat{a}, \hat{b} \rangle = \tau(ab^*)$.

Remark 3. We have a conjugate linear map $J : H \rightarrow H$ defined by $\hat{a} \mapsto \hat{a}^*$. Then we have $JaJ\hat{b} = \widehat{ba^*}$ which implies $JAJ \subset'$ and $JA''J \subset A'$. On the other hand, $x \in A'$ and $a \in A$ imply

$$\langle Jx\xi_\tau, a\xi_\tau \rangle = \langle Ja\xi_\tau, x\xi_\tau \rangle = \langle a^*\xi_\tau, x\xi_\tau \rangle = \langle x^*\xi_\tau, a\xi_\tau \rangle,$$

i.e. $Jx\xi_\tau = x^*\xi_\tau$, thence ξ_τ is a cyclic tracial vector for A' . The J -operator for (A', ξ_τ) is exactly equal to the original J . Doing the same argument as above, we obtain $JA'J \subset A''$.

Theorem 4. The map $A'' \rightarrow A', a \mapsto JaJ$ is a conjugate linear $*$ -algebra antiisomorphism.

1.3. Crossed products. Let $\Gamma \curvearrowright (X, \mu)$ be a probability space with a measure preserving action of a group Γ . Let s be an element of Γ . When f is a complex Borel function defined on X , put $\alpha_s(f) : x \mapsto f(s^{-1}x)$. This induces a μ -preserving $*$ -automorphism on $L^\infty(X)$.

On the other hand, we get a unitary representation $\pi : \Gamma \curvearrowright L^2(X, \mu)$ given by $\pi_s f = \alpha_s f$ for $f \in L^2(X)$. Note that $\pi_s f \pi_s^* = \alpha_s(f)$ for $s \in \Gamma$ and $f \in L^\infty(X)$.

With the regular representation $\lambda : \Gamma \curvearrowright B(\ell_2\Gamma)$, the von Neumann algebra $L^\infty(X) \rtimes \Gamma$ on $L^2(X) \otimes \ell_2\Gamma$ is generated by the operators $\pi \otimes \lambda(s)$ for $s \in \Gamma$ and $f \otimes 1$ for $f \in L^\infty(X)$ is called the crossed product of $L^\infty(X)$ by α .

Let A denote $\{\sum_{\text{finite}} f_s \otimes 1 \cdot \pi \otimes \lambda(s)\} \subset L^\infty(X) \rtimes \Gamma$. By abuse of notation, in the following f stands for $f \otimes 1$ and $\lambda(s)$ for $\pi \otimes \lambda(s)$. Now $\xi_\tau = \mathbf{1} \otimes \delta_e$ is a cyclic tracial vector for A . Indeed, $\tau(f\lambda(s)) = \delta_{e,s}\mu(f)$ implies

$$\tau(f\lambda(s)g\lambda(t)) = \delta_{st,e}f\alpha_s(g) = \delta_{ts,e}\alpha_t(f)g = \tau(g\lambda(t)f\lambda(s)).$$

Note that the above expressions are nonzero only if $s = t^{-1}$.

Let V denote the isometry $L^2(X) \rightarrow L^2(X) \otimes \ell_2\Gamma$, $f \mapsto f \otimes \delta_e$. Then the contraction $E : L^\infty(X) \rtimes \Gamma \rightarrow B(L^2(X))$, $a \mapsto V^*aV$ has image $L^\infty(X)$, i.e. E is a conditional expectation onto $L^\infty(X)$. Note that $\tau = \mu \circ E$.

1.4. von Neumann algebra from orbit equivalence. Let \mathcal{R} be a standard orbit equivalence on X . Hence it is a countable disjoint union $\coprod_n \mathcal{G}(\phi_n)$ of the graphs of partial isometries. We may assume that $\phi_0 = \text{Id}_X$. We will define a ‘‘Borel probability measure’’ on \mathcal{R} .

Observe that when $f : \mathcal{R} \rightarrow \mathbb{C}$ is Borel, $X \rightarrow \mathbb{C}$, $x \mapsto \sum_y f(y, x) = \sum_n f(\phi_n x, x)$ is also Borel. Define a measure ν on \mathcal{R} by $\int \xi d\nu = \int \sum_y \xi(y, x) d\mu(x)$ for each Borel function $\xi : \mathcal{R} \rightarrow \mathbb{C}$. Thus when B is a Borel subset of \mathcal{R} , $\nu(B) = \int |\pi_r^{-1}(x) \cap B| d\mu(x)$ for the second projection $\pi_r : \mathcal{R} \rightarrow X$, $(y, x) \mapsto x$.

We get a pseudogroup $[[\mathcal{R}]]$ of \mathcal{R} , defined to be the set

$$\{\phi : \text{partial Borel isomorphism, } \mathcal{G}(\phi) \subset \mathcal{R}\}.$$

Thus, $\text{Id}_A \in [[\mathcal{R}]]$ for any Borel set $A \subset X$, $\phi \in [[\mathcal{R}]]$ implies $\phi^{-1} \in [[\mathcal{R}]]$, and the composition $\phi \circ \psi$ of ϕ and ψ is defined when $\text{dom}(\phi)$ contains the image of ψ .

For $\phi \in [[\mathcal{R}]]$, define $v_\phi \in B(L^2(\mathcal{R}, \nu))$ by $v_\phi \xi(y, x) = \xi(\phi^{-1}y, x)$. Thus $v_\phi v_\psi = v_{\phi \circ \psi}$. On the other hand, $\{\chi_{\mathcal{G}(\phi)} : \phi \in [[\mathcal{R}]]\}$ is total in $L^2(\mathcal{R}, \nu)$ and $v_\phi \chi_{\mathcal{G}\psi} = \chi_{\mathcal{G}\phi\omega\psi}$. Moreover, we have

$$\langle v_\phi \chi_{\mathcal{G}\psi}, \chi_{\mathcal{G}\theta} \rangle = \int \mathcal{G}(\phi\psi) \cap \mathcal{G}(\theta) d\nu = \mu\{x : \phi\psi x = \theta x\} = \langle \chi_{\mathcal{G}\psi}, v_{\phi^{-1}} \chi_{\mathcal{G}\theta} \rangle,$$

which implies $v_\phi^* = v_{\phi^{-1}}$.

Definition 8. The von Neumann algebra $\text{vN}(\mathcal{R})$ generated by $\{v_\phi : \phi \in [[\mathcal{R}]]\}$ is called the von Neumann algebra of \mathcal{R} .

$\xi_\tau = \chi_{\mathcal{G}(\text{Id}_X)}$ is a cyclic tracial vector: in fact,

$$\begin{aligned} \tau(v_{\phi\psi}) &= \mu(\{x : \phi \circ \psi(x) = x\}) \\ &= \mu(\{y : \psi\phi y = y\}) \quad (y = \phi^{-1}x) \\ &= \tau(v_{\psi\phi}). \end{aligned}$$

Note that $L^\infty X \subset \mathfrak{vN}\mathcal{R}$ “in the diagonal,” subject to the relation $v_\phi f = (f \circ \phi^{-1})v_\phi$. We have a conditional expectation $E : \mathfrak{vN}\mathcal{R} \rightarrow L^\infty X$, $a \mapsto V^* a V$ implemented by the “diagonal inclusion” isometry $V : L^2 X \rightarrow L^2 \mathcal{R}$. We have $E(v_\phi) = \chi_{\{x : \phi x = x\}}$.

2. ELEMENTARY THEORY OF ORBIT EQUIVALENCE

2.1. Essentially free action. Suppose we are given an action $\Gamma \curvearrowright (X, \mu)$. Then we get two inclusions of von Neumann algebras:

- (1) $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ in $B(L^2 X \otimes \ell_2 \Gamma)$.
- (2) $L^\infty(X) \subset \mathfrak{vN}(\mathcal{R}_{\Gamma \curvearrowright (X, \mu)})$ in $B(L^2 \mathcal{R})$.

In general these are different, e.g. when the action is trivial.

Definition 9. An action $\Gamma \curvearrowright (X, \mu)$ is said to be essentially free when $\forall s \in \Gamma \setminus \{e\} \mu(\{x : sx = x\}) = 0$.

Theorem 5. *When the action is essentially free, the above two inclusions of von Neumann algebras are equal.*

Remark 4. $J\hat{v}_\phi = v_{\hat{\phi}^{-1}}$ implies $J\xi(x, y) = \overline{\xi(y, x)}$.

Proof of the theorem. Identification of the representation Hilbert spaces is given by $U : L^2 X \otimes \ell_2 \Gamma \rightarrow L^2 \mathcal{R}$, $g \otimes \delta_t \mapsto g \cdot \chi_{\mathcal{G}(t)}$. When $s = t$, $f\chi_{\mathcal{G}(s)} = g\chi_{\mathcal{G}(t)}$ by the essential freeness assumption. Now,

$$U^* v_s U (g \otimes \delta_t) = U^* \alpha_s(g) v_s \chi_{\mathcal{G}(t)} = U^* \alpha_s(g) \chi_{\mathcal{G}(st)} = \alpha_s(g) \otimes \delta_{st}.$$

This shows $U^* v_s U = \pi \otimes \lambda(s)$. On the other hand, $U^* f U = f \otimes 1$ is trivial. \square

Definition 10. Let $A \subset M$ be a von Neumann subalgebra (in the following A is often assumed to be commutative). $\mathcal{N}(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ is called the normalizer of A . $\mathcal{N}^p(A) = \{v \in M : \text{partial isometry, } v^*v, vv^* \in A, vAv^* = Avv^*\}$ is called the partial normalizer (?) of A .

Lemma 6. *For any $v \in \mathcal{N}^p(A)$, there exist $u \in \mathcal{N}(A)$ and $e \in \text{Proj}(A)$ such that $v = ue$. For any $\phi \in \llbracket \mathcal{R} \rrbracket$, there exists a Borel isomorphism $\tilde{\phi}$ whose graph is contained in \mathcal{R} and $\tilde{\phi}|_{\text{dom } \phi} = \phi$.*

Proof. We prove the second assertion. Put $E = \text{dom } \phi$ and $F = \text{ran } \phi$. When $\mu(E \Delta F) = 0$, there is nothing to do. When $\mu(E \Delta F) \neq 0$, $\exists k > 0$ such that $\phi^k(E \setminus F) \cap (F \setminus E)$ is non-null. If not, $\phi^k(E \setminus F) \subset F \cap \mathfrak{C}(F \setminus E) = F \cap E \subset E$ up to a null set and ϕ^{k+1} can be defined a.e. on $E \setminus F$. Thus we get a sequence $(\phi^k(E \setminus F))_{k \in \mathbb{N}}$ of subsets with nonzero measure. For any pair $m < n$, $\phi^m(E \setminus F) \cap \phi^n(E \setminus F)$ is equal to $\phi^m(\phi^{n-m}(E \setminus F) \cap (E \setminus F))$ which is null. This contradicts to $\mu(X) = 1$.

Now, put $\phi_1 = \phi \prod (\phi^{-k}|_{\phi^k(E \setminus F) \cap (F \setminus E)})$. Use the maximality argument (Zorn’s lemma) to obtain a globally defined Borel isomorphism. \square

2.2. Inclusion of von Neumann algebras.

Definition 11. Let $M \subset N$ be an inclusion of von Neumann algebras. $E : N \rightarrow M$ is said to be a conditional expectation when E is a unital completely positive map satisfying $E(axb) = aE(x)b$ for $a, b \in M$ and $x \in N$.

Fact. When N is finite with a faithful tracial state τ , there exists a unique conditional expectation E that preserves τ . Then we obtain an orthogonal projection $e_M : L^2 N \rightarrow M\xi_\tau \simeq L^2 M$ extending E .

Remark 5. (Martingale) If we are given $N_1 \subset N_2 \subset \cdots \subset M$ with $N = \vee_i N_i$ or $M \supset N_1 \supset N_2 \supset \cdots$ with $N = \cap_i N_i$, together with conditional expectations $E_n : M \rightarrow N_n$ and $E : M \rightarrow N$, $e_n \rightarrow e$ in the strong operator topology implies $\|E(x) - E_n(x)\|_2 \rightarrow 0$.

For example, let $A \subset M$ be a finite dimensional commutative subalgebra, e_i ($1 \leq i \leq n$) the minimal projections of A . Then $E_{A' \cap M}(x) = \sum_{i=1}^n e_i x e_i$. If we have a sequence $A_1 \subset A_2 \subset \cdots \subset M$ of finite dimensional commutative subalgebras and $A = \vee A_i$, we have $E_{A'_n \cap M} \rightarrow E_{A' \cap M}$. The latter is equal to E_A if and only if A is a maximal abelian subalgebra.

Definition 12. $A \subset M$ is said to be a Cartan subalgebra of M when A is a maximal abelian subalgebra and $\mathcal{N}(A)'' = M$. (Then we also have $M = \mathcal{N}^p(A)''$)

Theorem 7. $L^\infty X \subset \mathfrak{vN}\mathcal{R}$ is a Cartan subalgebra.

Proof. Since the generators v_ϕ are in $\mathcal{N}(A)$, it is enough to show that $L^\infty X$ is maximal abelian in $\mathfrak{vN}\mathcal{R}$. Recall that $\mathcal{R} = \coprod \mathcal{G}(\phi_n)$ with $\phi_0 = \text{Id}_X$. Then let a be an element of the relative commutant of $L^\infty X$. \hat{a} can be written as $\sum_n f_n \chi_{\mathcal{G}(\phi_n)}$. By assumption $fa = af$ for any $f \in L^\infty X$. Thus,

$$\widehat{fa} = \sum f f_n \chi_{\mathcal{G}(\phi_n)}, \quad \widehat{af} = J \bar{f} J \hat{a} = \sum f \circ \phi_n^{-1} \cdot f_n \chi_{\mathcal{G}(\phi_n)}.$$

Hence $f f_n = f \circ \phi_n f_n$ for any n and any f , which implies $f_n = 0$ except for $n = 0$. \square

Definition 13. \mathcal{R} is said to be ergodic when any \mathcal{R} -invariant Borel subset of X is either of measure 0 or 1. An action $\Gamma \curvearrowright (X, \mu)$ is said to be ergodic when $\mathcal{R}_{\Gamma \curvearrowright X}$ is ergodic.

Corollary 8. $\mathfrak{vN}\mathcal{R}$ is a factor if and only if \mathcal{R} is ergodic.

Proof. The Cartan subalgebra $L^\infty X$ contains the center of $\mathfrak{vN}\mathcal{R}$. \square

Put $A = L^\infty X$, $M = \mathfrak{vN}\mathcal{R}$. Let $v \in \mathcal{N}^p(A)$ be given. The projections v^*v and vv^* of A are respectively denoted by χ_E and χ_F for Borel sets $E, F \subset X$. The map $L^\infty E \rightarrow L^\infty F$, $f \mapsto v f v^*$ is a *-isomorphism. Thus there exists a Borel isomorphism $\phi_v : E \rightarrow F$ such that $v f v^* = f \circ \phi_v^{-1}$. ($v = \sigma v_{\phi_v}$ for some $\sigma \in \mathcal{U}(L^\infty F)$.)

Theorem 9. $A \vee JAJ = L^\infty \mathcal{R}$ and $v \xi v^* = \xi(\phi_v^{-1}(y), x)$ ν -a.e. for any $v \in \mathcal{N}^p(A)$ and any $x \in L^\infty(\mathcal{R})$. In particular, $\phi_v \in \llbracket \mathcal{R} \rrbracket$ up to a null set.

Proof. $fJgJ \in L^\infty$ for $f, g \in A$: indeed, $fJgJ$ is the multiplication by the function $f(y)g(x)$ on \mathcal{R} .

$$v f J g J v^* = v f v^* J g J = f \circ \phi_v^{-1} J g J \quad (J M J = M').$$

Hence $v \xi v^*(y, x) = v(\phi_v^{-1} y, x)$ for $\xi \in A \vee JAJ$. It remains to show $\chi_{\mathcal{G}(\text{Id}_X)} \in A \vee JAJ$. Because, if this is satisfied, we would have $\chi_{\mathcal{G}(\phi_v)} = v \chi_{\mathcal{G}(\text{Id}_X)} v^* \in L^\infty \mathcal{R}$.

Take an increasing sequence $A_1 \subset A_2 \subset \cdots$ of finite dimensional algebras with $A = \vee A_k$. The conditional expectation $E_n : M \rightarrow A_n$ is equal to $\sum_k e_k^{(n)} J e_k^{(n)} J$ (as an operator on L^2) for the minimal projections $(e_k^{(n)})_k$ of A_n . Now, E_n converges to the conditional expectation E_A onto A in the strong operator topology, which is equal to the multiplication by $\chi_{\mathcal{G}(\text{Id}_X)}$. Hence $\chi_{\mathcal{G}(\text{Id}_X)} \in A \vee JAJ$. \square

Remark 6. (2-cocycle) Suppose we are given a map $\sigma_{\phi,\psi} : \text{ran}(\phi\psi) \rightarrow \mathbb{T}$ for each pair $\phi, \psi \in \llbracket \mathcal{R} \rrbracket$, satisfying $\sigma_{\phi,\psi}\sigma_{\phi\psi,\theta} = (\sigma_{\psi,\theta} \circ \phi^{-1})\sigma_{\phi,\psi\theta}$. Then $v_\phi^\sigma v_\psi^\sigma = \sigma_{\phi,\psi} v_{\phi\psi}^\sigma$ determines an associative product on $\mathbb{C}\llbracket \mathcal{R} \rrbracket$ with a trace τ . The GNS representation gives an inclusion $L^\infty X \subset \text{vN}(\mathcal{R}, \sigma) \subset B(L^2 \mathcal{R})$ of von Neumann algebras.

Fact. Any Cartan subalgebra of $\text{vN}(\mathcal{R}, \sigma)$ is isomorphic to $L^\infty X$.

Theorem 10. *Let \mathcal{R} (resp. \mathcal{S}) be an orbit equivalence on X (resp. Y), $F : X \rightarrow Y$ a measure preserving Borel isomorphism. The induced isomorphism $F_* : L^\infty X \rightarrow L^\infty Y$ can be extended to a normal $*$ -homomorphism $\text{vN}\mathcal{R} \rightarrow \text{vN}\mathcal{S}$ if and only if $F\mathcal{R} \subset \mathcal{S}$ up to a ν -null set.*

Proof. For simplicity we identify Y with X by means of F . If $\llbracket \mathcal{R} \rrbracket \subset \llbracket \mathcal{S} \rrbracket$, the required homomorphism is induced by the isometry $L^2 \mathcal{R} \rightarrow L^2 \mathcal{S}$. Conversely, if $\pi : \text{vN}\mathcal{R} \rightarrow \text{vN}\mathcal{S}$ is an extension of F_* , for any $\phi \in \llbracket \mathcal{R} \rrbracket$ we have

$$\pi(v_\phi)\pi(f)\pi(v_\phi)^* = \pi(f \circ \phi^{-1}) = f \circ \phi^{-1},$$

which implies $\pi(v_\phi) = \sigma_\phi v_\phi$ for some $\sigma_\phi \in L^\infty X$. \square

Let M be a finite von Neumann algebra with trace τ , identified to a subalgebra of $B(L^2 M)$. Suppose A is a von Neumann subalgebra of M . Let e_A be the projection onto the span of $A\xi_\tau$ and put $\langle M, A \rangle = (M \cup \{e_A\})''$. Note that $e_A x \xi_\tau = E_A(x)\xi_\tau$. For any $x \in M$ and $\hat{a} \in L^2 A$,

$$e_A x \hat{a} = e_A \widehat{xa} = \widehat{E_A(x)a} = \widehat{E_A(x)}a$$

which implies $e_A x e_A = E_A(x)e_A$. Moreover, $e_A \in A'$ implies

$$\langle M, A \rangle = \overline{\left\{ \sum x_j e_A y_j + z : x_j, y_j, z \in M \right\}}^{\text{wop}}.$$

Now,

$$e_A J x J e_A \hat{a} = e_A \widehat{ax^*} = \widehat{E_A(ax^*)} = a \widehat{E_A(x^*)} = J E_A(x) J \hat{a}$$

implies $\langle M, A \rangle' = M' \cap \{e_A\}' = JAJ$, consequently $\langle M, A \rangle = (JAJ)'$. Note that when A is commutative, $e_A JAJ = a^* e_A e$.

We have the ‘‘canonical trace’’: $\text{Tr} : \langle M, A \rangle \ni \sum_i x_i e_A y_i \mapsto \tau(\sum_i x_i y_i)$ which is a priori unbounded. Still, Tr is normal semifinite, and its trace property is verified as follows:

$$\begin{aligned} \left\| \sum x_i e_A y_i \right\|_{2, \text{Tr}}^2 &= \text{Tr} \left(\sum y_i^* e_A x_i^* x_j e_A y_j \right) = \sum \tau(y_i^* E_A(x_i^* x_j) y_j) \\ &= \sum \tau(E_A(y_j y_i^*) E_A(x_i^* x_j)) = \|y_i e_A x_i^*\|_{2, \text{Tr}}^2. \end{aligned}$$

Suppose $A \subset M$ is Cartan. Put $\tilde{A} = \{A, JAJ\}'' \subset \langle M, A \rangle$ and consider $\text{Tr}|_{\tilde{A}}$.

Example 2. $A = L^\infty X$, $M = \text{vN}\mathcal{R}$. Then $\tilde{A} = L^\infty \mathcal{R}$, $e_A = \chi_\Delta$ and $\text{Tr} = \int d\nu$ on $L^\infty \mathcal{R}$. Indeed,

$$\text{Tr}(f e_A) = \tau(f) = \int_\Delta f d\mu = \int f d\nu \quad (f \in L^\infty X)$$

implies

$$\text{Tr}(u f e_A u^*) = \text{Tr}(f e_A) = \int_\Delta f d\mu = \int u f e_A u^* d\nu \quad (f \in L^\infty X, u \in \mathcal{N}(A)).$$

Remark 7. When $A \subset M$ is Cartan and $p \in \text{Proj}(A)$, $A_p \subset pMp$ is also Cartan since $\mathcal{N}_{pMp}^p(A_p)p\mathcal{N}_M^p(A)p$.

Example 3. When $Y \subset X$, the restricted equivalence $\mathcal{R}|_Y = Y \times Y \cap \mathcal{R}$ gives $\mathfrak{vN}\mathcal{R}|_Y = p_Y \mathfrak{vN}\mathcal{R}_Y$.

Exercise 1. Show that when A is a Cartan subalgebra of a factor M , $\tau p_1 = \tau p_2$ for $p_1, p_2 \in \text{Proj}(A)$ implies the existence of $v \in \mathcal{N}^p(A)$ such that $p_1 \sim p_2$ via v . This implies that given an ergodic relation \mathcal{R} on X , subsets Y_1 and Y_2 of X with the same measure, one would obtain $(A_{p_{Y_1}} \subset M_{p_{Y_1}}) \simeq (A_{p_{Y_2}} \subset M_{p_{Y_2}})$ via v .

2.3. Theorem of Connes-Ferdman-Weiss.

Definition 14. A discrete group Γ is said to be amenable when $\ell_\infty \Gamma$ has an invariant state.

Example 4. Commutative groups, or more generally solvable groups are amenable. Countable union of amenable groups are again amenable.

Definition 15. A cartan subalgebra $A \subset M$ is said to be amenable when there exists a state $m : \tilde{A} \rightarrow \mathbb{C}$ invariant under the adjoint action of $\mathcal{N}(A)$. An orbit equivalence \mathcal{R} on X is said to be amenable when $L^\infty X \subset \mathfrak{vN}\mathcal{R}$ is amenable.

Remark 8. Let $\Gamma \curvearrowright X$ be an measure preserving essentially free action. Since Γ is assumed to be discrete, \mathcal{R} can be identified to $\Gamma \times X$ and invariant measure on \mathcal{R} is nothing but a product measure on $\Gamma \times X$ of an invariant measure on Γ times an arbitrary measure on X . Thus, \mathcal{R} is amenable if and only if Γ is amenable.

Definition 16. A von Neumann algebra M is said to be injective when there exists a conditional expectation $\Phi : BH \rightarrow M$.

Fact. The above condition is independent of the choice of the faithful representation $M \hookrightarrow B(H)$. Moreover, M is injective if and only if it is AFD.

Theorem 11. (Connes-Ferdman-Weiss [2]) *Let M be a factor with separable predual, A a Cartan subalgebra of M . The following conditions are equivalent:*

- (1) *The pair $A \subset M$ is amenable.*
- (2) *This pair is AFD in the sense that for any finite subset \mathcal{F} of $N(A)$ and a positive real number $\epsilon > 0$, there exists a finite dimensional subalgebra B of M such that*
 - *B has a matrix unit consisting of elements of $N^p A$.*
 - *$\|v - E_B(v)\| < \epsilon$ for any $v \in \mathcal{F}$.*
- (3) *(A, M) is isomorphic to $(D, \bar{\otimes} M_2 \mathbb{C})$ where $D = \bar{\otimes} D_2$ for the diagonal subalgebra $D_2 \subset M_2$. (Note that $N^p(D)$ essentially consists of the “matrix units” of $M_{2^\infty} = \bar{\otimes} M_2$.)*
- (4) *M is injective.*

Lemma 12. *In the assertion of (2), B may be assumed to be isomorphic to M_{2^N} for some N .*

Proof of the lemma. Perturbing a bit, we may assume $\tau(e_{ij}^{(d)}) \in 2^{-N} \mathbb{N}$ for large enough N where $(e_{ij}^{(d)})_{d, 1 \leq i, j \leq n_d}$ is a matrix unit of $B = \bigoplus_d M_{n_d}$. By taking a partition if necessary, we may assume that $\tau(e_{ii}^{(d)}) = 2^{-N}$ for any d and i . Then, since M is a factor, we have $e_{ii}^{(d)} \sim e_{jj}^{(f)}$ in M for any d, f, i and J . This means B is contained in a subalgebra of M which is isomorphic to M_{2^N} . \square

Proof of (2) \Rightarrow (3): Note that there is a total (with respect to the 2-norm) sequence $(v_k)_{k \in \mathbb{N}} \subset \mathcal{N}^p(A)$. We are going to construct an increasing sequence of subalgebras $(B_k)_k$ in M with compatible matrix units $(e_{i,j}^{(k)})_{i,j}$ satisfying $B_k \simeq M_{2N_k}$ and $\|E_{B_k}(v_l) - v_l\|^2 < \frac{1}{k}$ for $l \leq k$. Suppose we have constructed B_1, \dots, B_k . Now apply the assertion of (2) to the finite set $\mathcal{F}' = \{e_{i,r}^{(k)} v_l e_{r,1}^{(k)}\}$, we obtain a matrix units $(f_{ij})_{i,j}$ such that $\sum f_{ii} = e_{11}^{(k)}$ and

$$\|E_{\text{span}f_{ij}}(x) - x\| < \frac{1}{n(k)^2(k+1)}$$

where $n(k)$ denotes the size of B_k . By the assumption that A is a maximal abelian subalgebra of M , the projections of $\mathcal{N}^p(A)$ are actually contained in A . Thus we obtain the inclusion $D \subset A$ (hence, the equality between them) under the identification $M \simeq \otimes M_2$.

Proof of (3) \Rightarrow (4): By assumption $M = (\cup B_n)''$ where B_n are finite dimensional subalgebras of M , $M' = (\cup JB_n J)''$. Let Φ_n denote the conditional expectation of $B(H)$ onto $(JB_n J)'$: $\Phi_n(x) = \int_{\mathcal{U}(JB_n J)} x u u^* du$ where du denotes the normalized Haar measure on the compact group $\mathcal{U}(JB_n J)$. For each x , the sequence $(\|\Phi_n(x)\|)_n$ is bounded above by $\|x\|$. Thus we can take a Banach limit $\Phi(x)$ of $(\Phi_n(x))_n$, which defines a conditional expectation of $B(H)$ onto $\cap_n (JB_n J)' = (\cup JB_n J)' = M$.

Proof of (4) \Rightarrow (1): Put $H = L^2 M$ and let Φ be a conditional expectation of $B(H)$ onto M . Then $\tau\Phi$ is an $\text{Ad}\mathcal{U}(M)$ -invariant state on $B(H)$. $\mathcal{N}(A)$ is obviously contained in $\mathcal{U}(M)$ and so is \tilde{A} in $B(H)$.

Remark 9. When $A \subset M$ is an amenable Cartan subalgebra and e is a projection in A , $A_e \subset M_e$ is also amenable.

We are going to complete the proof of Theorem 11 by showing (1) \Rightarrow (2).

Lemma 13. *Let ϕ be a measure preserving partial Borel isomorphism on X and let E_0 denote the fixed point set $X^\phi = \{x \in \text{dom } \phi : \phi x = x\}$. There exist Borel sets B_1, B_2, B_3 of X satisfying $X = \coprod_{0 \leq i \leq 3} E_i$ and $\phi E_i \cap E_i$ is null for $i > 0$.*

Proof. Take E_1 to be a Borel set with a maximal measure satisfying $\phi E_1 \cap E_1 = \emptyset$. Put $E_2 = \phi E_1$. Then $\phi E_2 \cap E_2 = \emptyset$ by the injectivity of ϕ . Finally, put $E_3 = \mathcal{C}(\cup_{0 \leq i \leq 2} E_i)$. Then $\phi E_3 \cap E_3$ is null by the maximality of E_1 . \square

Corollary 14. *For any finite set \mathcal{F} of $\mathcal{N}^p(A)$, there exists projections q_1, \dots, q_m of A ($m = 4^{|\mathcal{F}|}$) satisfying $\sum q_k = 1$ and that $q_k v q_k$ is either 0 or in $\mathcal{U}(A_{q_k})$ for any $v \in \mathcal{F}$.*

Lemma 15. *(Dye) For any finite subset $\mathcal{F} \subset \mathcal{N}(A)$ and $\epsilon > 0$, there exists $a \in \tilde{A}_+$ with $\text{Tr}(a) = 1$ and $\sum_{u \in \mathcal{F}} \|u a u^* - a\|_{1, \text{Tr}} < \epsilon$. ($\|x\|_{1, \text{Tr}} = \text{Tr}(|x|)$)*

Proof. Let $m : \tilde{A} \rightarrow \mathbb{C}$ be an $\text{Ad}\mathcal{N}(A)$ -invariant state. Since L^1 is w^* -dense in $(L^\infty)^*$, there exists a net $a_i \in \tilde{A}_+$ satisfying $\text{Tr}(a_i) = 1$ and $\text{Tr}(a_i x) \rightarrow m(x)$ for any $x \in \tilde{A}$. Then

$$\text{Tr}((u a_i u^* - a_i)x) = \text{Tr}(a_i u^* x u) - \text{Tr}(a_i x) \rightarrow m(u^* x u) - m(x) = 0$$

for any $u \in \mathcal{N}(A)$ and $x \in \tilde{A}$. The element $u a_i u^* - a_i$ is weakly convergent to 0. By Hahn-Banach's theorem, taking the convex combination, we may assume $\|u a_i u^* - a_i\|_{1, \text{Tr}} \rightarrow 0$ for each $u \in \mathcal{N}(A)$. \square

Lemma 16. (Namioka) *Let \mathcal{F} , ϵ be as above. There exists a projection p of \tilde{A} satisfying $\text{Tr}(p) < \infty$ and $\sum_{u \in \mathcal{F}} \|upu^* - p\|_{2, \text{Tr}}^2 < \epsilon \|p\|_{2, \text{Tr}}^2$.*

Proof. Let $a \in \tilde{A}_+$ be an element given by Lemma 15. For each $r > 0$ put $P_r = \chi_{(r, \infty)}(a)$. We have

$$\|uau^* - u\|_{1, \text{Tr}} = \int_0^\infty \|uP_r u^* - P_r\|_{1, \text{Tr}} dr \quad 1 = \|a\|_{1, \text{Tr}} = \int_0^\infty \|P_r\|_{1, \text{Tr}} dr.$$

Hence

$$\int_0^\infty \sum_{u \in \mathcal{F}} \|uP_r u^* - P_r\|_{1, \text{Tr}} dr < \epsilon \int_0^\infty \|P_r\|_{1, \text{Tr}} dr.$$

Thus there exists r such that $p = P_r$ satisfies $\sum \|upu^* - p\|_{1, \text{Tr}} < \epsilon \|p\|_{1, \text{Tr}}$. Since the summands are differences of projections, thus $\|-\|_{1, \text{Tr}}$ is approximately equal to $\|-\|_{2, \text{Tr}}$. \square

Lemma 17. (Local AFD approximation by Popa) *Let \mathcal{F} , ϵ be as above. There exists a finite dimensional subalgebra $B \subset M$ with matrix units in $\mathcal{N}^p(A)$, satisfying $\|E_B(eue) - (u - e^\perp u e^\perp)\|_2^2 < \epsilon \|e\|_2^2$ for every $u \in \mathcal{F}$, where e denotes the multiplicative unit of B and E_B the conditional expectation $eMe \rightarrow B$.*

Proof. We may assume $1 \in \mathcal{F}$. Take $p \in \tilde{A}_+$ as in Lemma 16. Since $\text{Tr} p < \infty$, we may assume that p can be written as $\sum_{i=1}^n v_i e_A v_i^*$ for $v_i \in \mathcal{N}^p(A)$. By Corollary 14, there exist projections $(q_k)_k$ in A with $\sum q_k = 1$ and each $q_k v_i^* u v_j q_k$ is either 0 or is in $\mathcal{U}(A q_k)$ for $1 \leq i, j \leq n, u \in \mathcal{F}$. Taking finer partition if necessary, we deduce that $\text{dist}(q_k v_i^* u v_j q_k, \mathbb{C} q_k) < \sqrt{\epsilon/n}$. Thus

$$\sum_k \sum_{u \in \mathcal{F}} \|(upu^* - p) J q_k J\|_{2, \text{Tr}}^2 = \sum_{u \in \mathcal{F}} \|upu^* - p\|_{2, \text{Tr}}^2 < \epsilon \|p\|_{2, \text{Tr}}^2 = \sum_k \|p J q_k J\|_{2, \text{Tr}}^2.$$

Hence for some k , $q = q_k$ satisfies $\sum \|(upu^* - p) J q J\|^2 \leq \epsilon \|p J q J\|^2$. By $p J q J = \sum v_i e_A J q J v_i^* = \sum v_i q e_A v_i^*$, replacing v_i by $v_i q$, we may assume $v_i^* v_j = \delta_{i,j} q$ and $p J q J = p$. (Note that $p = \sum v_i e_A v_i^*$ is a projection, which means that the v_i are mutually orthogonal.) Thus we obtain $\sum \|upu^* - p\|^2 \leq \epsilon \|p\|^2$, each $v_i u v_j^* \in A_q$ is close to a constant z_{ij} by $\sqrt{\epsilon/n}$, and $(v_i)_i$ is a matrix unit in A_q . Put $e = \sum v_i v_i^*$. Thus,

$$\|p\|_{2, \text{Tr}}^2 = \text{Tr}(\sum v_i e_A v_i^*) = \tau(\sum v_i v_i^*) = \|e\|_\tau^2.$$

Consequently,

$$\begin{aligned} \|upu^* - p\|_{2, \text{Tr}}^2 &= 2 \text{Tr} p - 2 \text{Tr}(upu^* p) = 2\tau(e) - 2 \text{Tr}(\sum uv_i e_A v_i^* u^* v_j e_A v_j) \\ &= 2\tau(e) - 2\tau(\sum uv_i v_i^* u^* v_j v_j^*) = 2\tau(e) - 2\tau(ueu^* e) \\ &= \|ueu^* - e\|_{2, \tau}^2. \end{aligned}$$

Hence $\sum_{u \in \mathcal{F}} \|ue - eu\|_2^2 < \epsilon \|e\|_2^2$. Now $eue = \sum v_i v_i^* u v_j v_j^* \approx \sum z_{ij} v_i v_j^* \approx \epsilon \|e\|_2^2$ in $\|-\|_{2, \tau}$. Hence

$$\|eue - E_B(eue)\|_{2, \tau}^2 < \epsilon \|e\|_{2, \tau}^2 \quad \|E_B(eue) - (u - e^\perp u e^\perp)\|_{2, \tau}^2 < 2\epsilon \|e\|_{2, \tau}^2.$$

When we have a family (B_i) of mutually orthogonal finite dimensional algebras satisfying the assertion of the lemma, $e = \sum 1_{B_i}$ satisfies $\|E_{\oplus B_i}(eue) - (u - e^\perp u e^\perp)\|_{2,\tau}^2 < 2\epsilon \|e\|_{2,\tau}^2$. \square

Lemma 18. *In the notation of Lemma 17, $e = 1$.*

Proof. Otherwise we can apply Lemma 17 to $A_{e^\perp} \subset M_{e^\perp}$ and $\mathcal{F}' = e^\perp \mathcal{F} e^\perp$, to obtain a finite dimensional algebra $B_0 \subset M_{e^\perp}$ satisfying the assertion of Lemma 17. Use the Pythagorean equality. \square

Proof of (1) \Rightarrow (2): Take B_1, \dots, B_m satisfying $\|\sum_m 1_{B_i}\|_2^2 > 1 - \epsilon$. Put $B = \oplus_i B_i \oplus \mathbb{C}(\sum 1_{B_i})^\perp$. Then we have $\|E_B(u) - u\|_2^2 < 3\epsilon$ for $u \in \mathcal{F}$. \square

3. L^2 -BETTI NUMBERS

3.1. Introduction. Let Ω, X be sets. Put $\mathfrak{F}(\Omega, X)$ denote the set of the mappings of Ω into X . Let Γ be a discrete group. Let λ denote the left regular representation of Γ on $\ell_2\Gamma$. We have the ‘‘standard complex’’ of right Γ modules

$$0 \longrightarrow \ell_2\Gamma \xrightarrow{\partial} \mathfrak{F}(\Gamma, \ell_2\Gamma) \xrightarrow{\partial} \mathfrak{F}(\Gamma^2, \ell_2\Gamma) \longrightarrow \dots$$

Let $H_i(\Gamma, \ell_2\Gamma)$ denote the i th (co)homology group of this complex. Note that this complex is a complex of $R\Gamma$ modules. The space of 1-cocycles

$$Z_1 = \{b \in \mathfrak{F}(\Gamma, \ell_2\Gamma) : b(st) = b(s) + \lambda(s)b(t)\}$$

is identified with the space of derivations from Γ to $\ell_2\Gamma$ with respect to the trivial right action. When $b \in Z_1$ the map

$$s \mapsto \begin{pmatrix} \lambda(s) & b(s) \\ 0 & 1 \end{pmatrix}$$

of Γ into $B(\ell_2\Gamma \oplus \mathbb{C})$ becomes multiplicative. On the other hand the space of 1-boundaries

$$B_1 = \{b \in \mathfrak{F}(\Gamma, \ell_2\Gamma) : \exists f \in \ell_2\Gamma, b(s) = \lambda(s)f - f\}$$

is identified with the space of the inner derivations. Note that for any $b \in Z_1$, there is a function $f \in \mathfrak{F}(\Gamma, \mathbb{C})$ such that $b(s) = \lambda(s)f - f$ if we do not require the square summability of f . Indeed, a vector system $(b(s))_{s \in \Gamma}$ is a derivation if and only if we have $\langle b(s), \delta_t \rangle = \langle b(st) - b(t), \delta_e \rangle$ for any $s, t \in \Gamma$, and in such a case we may put $f(s) = \langle b(s), \delta_s \rangle$ to obtain $b(s) = \lambda(s)f - f$.

Remark 10. The 0-th homology group $H_0 = Z_0$ is the space of the Γ -invariant vectors in $\ell_2\Gamma$. Thus this becomes 0-module when the group is infinite.

In the following we assume that the group Γ admits a finite generating set \mathcal{S} . Let $D\Gamma$ denote the space Z_1 of the derivations, $\text{Inn } D\Gamma$ the space B_1 of the inner derivations. Let $\mathcal{O}_{\mathcal{S}}$ denote the mapping $b \mapsto (b(s))_{s \in \mathcal{S}}$ of $D\Gamma$ into $\oplus_{\mathcal{S}} \ell_2\Gamma$. This is an injective $R\Gamma$ -module map. Note that the range of $\mathcal{O}_{\mathcal{S}}$ is closed. Indeed, $(f(s))_{s \in \mathcal{S}}$ is in $\text{ran } \mathcal{O}_{\mathcal{S}}$ if and only if

$$f(s_1) + \lambda(s_1)f(s_2) + \dots + \lambda(s_1 \cdots s_{n-1})f(s_n) = 0$$

holds for each relation $s_1 \cdots s_n = e$ among elements of \mathcal{S} .

A sequence $(f_n)_{n \in \mathbb{N}}$ of unit vectors is said to be an approximate kernel of the restriction $\mathcal{O}_{\mathcal{S}}|_{\text{Inn } D\Gamma}$ when $\lambda(s)f_n - f_n$ tends to zero (in norm) for any $s \in$

\mathcal{S} . $\mathcal{O}_{\mathcal{S}}|_{\text{Inn } D\Gamma}$ has an approximate kernel if and only if Γ is amenable. Thus $\mathcal{O}_{\mathcal{S}}(\text{Inn } D\Gamma)$ is closed if and only if Γ is finite or non-amenable.

Let P, Q denote the orthogonal projections onto $\mathcal{O}_{\mathcal{S}}(D\Gamma)$ and $\mathcal{O}_{\mathcal{S}}(\text{Inn } D\Gamma)$. These commute with the diagonal action of $\mathbb{R}\Gamma$ on $\oplus_{\mathcal{S}}\ell_2\Gamma$, i.e. $P, Q \in M_{\mathcal{S}}\mathbb{L}\Gamma$. We can measure them by the trace $\tilde{\tau} = \text{Tr} \otimes \tau$. The first Betti number $\beta_1^{(2)} = \dim_{\mathbb{L}\Gamma} H_1(\Gamma, \ell_2)$ is equal to the difference $\tilde{\tau}(P) - \tilde{\tau}(Q)$.

Example 5. When Γ is a finite group, $\beta_0^{(2)} = \frac{1}{|\Gamma|}$ while $\beta_i^{(2)} = 0$ for $0 < i$ because any $\mathbb{C}\Gamma$ module is projective. On the other hand when Γ is equal to the free group \mathbb{F}_n generated by a set \mathcal{S} consisting of n elements, $\text{ran } \mathcal{O}_{\mathcal{S}} = \oplus_{\mathcal{S}}\ell_2\Gamma$ and $\beta_1^{(2)} = n - 1$.

We omit the injection $\mathcal{O}_{\mathcal{S}}$ and identify $D\Gamma$ with a subspace of $\oplus_{\mathcal{S}}\ell_2\Gamma$. Thus $\partial^0 : \ell_2\Gamma \rightarrow \mathfrak{F}(\Gamma, \ell_2\Gamma)$ factors through $\oplus_{\mathcal{S}}\ell_2\Gamma$ and $\partial^0 : \ell_2\Gamma \rightarrow \oplus_{\mathcal{S}}\ell_2\Gamma$ is written as $f \mapsto (\lambda(s)f - f)_{s \in \mathcal{S}}$.

Let $\epsilon_1^{(2)} : \oplus_{\mathcal{S}}\ell_2\Gamma \rightarrow \ell_2\Gamma$ denote the adjoint of ∂ . Thus $\epsilon_1^{(2)}$ is expressed as $(\xi_s)_{s \in \mathcal{S}} \mapsto \sum_{s \in \mathcal{S}} (\lambda(s^{-1}) - 1)\xi_s$ and the orthogonal complement of $\ker \epsilon_1^{(2)}$ is equal to the closure of $\text{ran } \partial = \text{Inn } D\Gamma$.

Proposition 19. *When we identify $\mathbb{C}\Gamma$ with the space of vectors with finite support in $\ell_2\Gamma$, we have $D\Gamma = (\ker \epsilon_1^{(2)} \cap \oplus_{\mathcal{S}}\mathbb{C}\Gamma)^\perp$.*

Proof. The space $\mathbb{C}\Gamma$ has $\mathfrak{F}(\Gamma, \mathbb{C})$ as its algebraic dual. A vector system $b \in \oplus_{\mathcal{S}}\ell_2$ is in $D\Gamma$ if and only if there is an $f \in \mathfrak{F}(\Gamma, \mathbb{C})$ such that $b(s) = \lambda(s)f - f$. The latter implies

$$\forall \xi \in \ker \epsilon_1^{(2)} \cap \oplus_{\mathcal{S}}\mathbb{C}\Gamma, \langle \xi, b \rangle = \sum_s \langle \xi(s), b(s) \rangle = \sum_s \langle (\lambda(s^{-1}) - 1)\xi(s), f \rangle = 0.$$

Conversely, when $(b(s))_{s \in \mathcal{S}}$ is orthogonal to $\ker \epsilon_1^{(2)} \cap \oplus_{\mathcal{S}}\mathbb{C}\Gamma$, the functional $\langle b, - \rangle$ on $\oplus_{\mathcal{S}}\mathbb{C}\Gamma$ is induced by a functional f on the kernel of the map $\mathbb{C}\Gamma \rightarrow \mathbb{C}$. This f can be extended to a linear map on the whole $\mathbb{C}\Gamma$, and we have $b(s) = \lambda(s)f - f$, i.e. $b \in D\Gamma$. \square

Remark 11. The i -th cohomology group $H^i(\Gamma, \ell_2\Gamma)$ is dimension isomorphic to $\text{Tor}_i^{\mathbb{C}\Gamma}(\mathbb{C}, \ell_2\Gamma)$. This is seen by considering the exact functors $E \rightarrow E^*$ on the category of $\mathbb{L}\Gamma$ -modules and that of $\mathbb{L}\Gamma$ -bimodules, where E^* denotes the dual module of the weak closure of E . We have functors $(A, B) \rightarrow A \otimes_{\mathbb{C}\Gamma} B$ and $(A, B) \rightarrow \text{Hom}_{\mathbb{C}\Gamma}(A, B)$ of $\mathbb{C}\Gamma\text{-mod} \times \mathbb{L}\Gamma\text{-bimod}$ into $\mathbb{L}\Gamma\text{-mod}$. Then the functor equivalence $(A \otimes_{\mathbb{C}\Gamma} B)^* \simeq \text{Hom}_{\mathbb{C}\Gamma}(A, B^*)$ up to dimension implies the dimension equivalence between the derived functors $\text{Tor}_p(A, B)^* \simeq \text{Ext}^p(A, B^*)$. The case $A = \mathbb{C}$ and $B = \ell_2\Gamma$ describes the desired isomorphism.

For example, we have a flat resolution P of the trivial Γ -module \mathbb{C} with $P_0 = \mathbb{C}\Gamma$ and $P_1 = \mathbb{C}\Gamma \otimes_{\mathbb{C}} \mathbb{C}\mathcal{S}$, with $d_1(a \otimes b) = ab - a$. The first torsion group $\text{Tor}_1^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C})$ is by definition the quotient $\ker(\text{id}_{\ell_2\Gamma} \otimes d_1) / \ell_2\Gamma \otimes \ker d_1$. Now $\text{id}_{\ell_2\Gamma} \otimes d_1 = \epsilon_1^{(2)}$ implies $\ker(\text{id}_{\ell_2\Gamma} \otimes d_1) = \text{Inn } D\Gamma^\perp$ while $\ell_2\Gamma \otimes \ker d_1 = \ker \epsilon_1^{(2)} \cap \oplus_{\mathcal{S}}\mathbb{C}\Gamma$ implies $\ell_2\Gamma \otimes \ker d_1 = D\Gamma^\perp$.

3.2. Operators affiliated to a type II_1 algebra. Let (M, τ) be a finite von Neumann algebra with a faithful normal tracial state (τ is unique if M is a factor), L^2M the induced Hilbert M - M module. For each $n \in \mathbb{N}$ put $\tilde{\tau} = \tau \otimes \text{Tr}$ on $M \otimes M_n\mathbb{C} \simeq M_nM$.

Definition 17. Let H be a left Hilbert module over M . A densely defined closed operator T on H is said to be affiliated to M , written as $T \sim M$, when we have $uT = Tu$ for any $u \in \mathcal{U}(M')$. Here the equality entails the agreement of the domains, i.e. $u \operatorname{dom} T = \operatorname{dom} T$.

Remark 12. An operator T is affiliated to M if and only if for the polar decomposition $T = v|T|$ the partial isometry v and the spectral projections of $|T|$ are in M . Note that in such cases τ takes the same value on the left support $l(T) = vv^*$ of T and the right support $r(T) = v^*v$.

We consider the case $H = L^2M$. Suppose $T \sim M$. It is said to be square integrable when $\hat{1} \in \operatorname{dom} T$. This condition is equivalent to

$$\tau(|T|^2) = \|T\hat{1}\|^2 = \int t^2 d\tau(E) < \infty$$

for the spectral measure $T = \int t dE$ of T . For each $\xi \in L^2M$ let L_ξ° denote the unbounded operator defined by $\operatorname{dom} L_\xi^\circ = \hat{M} \subset L^2M$ and $L_\xi^\circ x = \xi x$.

Proposition 20. *The operator $L_\xi^\circ x$ is closable and its closure L_ξ is affiliated to M . Moreover we have $L_\xi^* = L_{J\xi}$. If T is affiliated to M and square integrable, $T = L_{T\hat{1}}$.*

Proof. We show the inclusion $L_{J\xi}^\circ \subset (L_\xi^\circ)^*$. For any elements $x, y \in M$,

$$\langle L_\xi^\circ \hat{x}, \hat{y} \rangle = \langle \xi x, y \rangle = \langle J\hat{y}, J(\xi x) \rangle = \langle \hat{1}y^*, x^* J\xi \rangle = \langle \hat{x}, (J\xi)y \rangle.$$

On the other hand, when $u \in \mathcal{U}\mathcal{R}_M$, $uL_\xi^\circ = L_\xi^\circ u$ implies $uL_\xi = L_\xi u$.

Next we show the inclusion $(L_\xi)^* \subset L_{J\xi}$. Let $\eta \in \operatorname{dom}(L_\xi)^*$. Consider the polar decomposition $L_{J\xi} = v|L_{J\xi}|$ and the spectral decomposition $|L_{J\xi}| = \int_0^\infty \lambda de_\lambda$. Then $e_\lambda v^* L_{J\xi} = e_\lambda |L_{J\xi}|$ is bounded (i.e. is in M_+) for any λ . By definition, $L_{J\xi}(y\hat{1}) = (J\xi)y$ for $y \in M$. Hence $e_\lambda v^* L_{J\xi}(y\hat{1}) = e_\lambda v^*((J\xi)y) = (e_\lambda v^* J\xi)y$. Putting $y = 1$, we obtain $e_\lambda v^* L_{J\xi}\hat{1} = e_\lambda v^* J\xi \in M.\hat{1}$ for any $\lambda > 0$.

Thus, by definition of $(L_\xi)^*$, we have

$$\begin{aligned} \langle (L_\xi)^* \eta, (e_\lambda v^*)^* y\hat{1} \rangle &= \langle \eta, L_\xi (e_\lambda v^*)^* y\hat{1} \rangle = \langle \eta, \xi (e_\lambda v^*)^* y \rangle = \langle \eta, Jy^* (e_\lambda v^*) J\xi \rangle \\ &= \langle \eta, Jy^* e_\lambda v^* L_{J\xi} \hat{1} \rangle \quad (\text{by using above}) \\ &= \langle \eta, (e_\lambda v^* L_{J\xi})^* y\hat{1} \rangle. \end{aligned}$$

Hence $e_\lambda v^* (L_\xi)^* \eta = e_\lambda v^* L_{J\xi} \eta = |L_{J\xi}| e_\lambda \eta$ for any $\lambda > 0$. By letting $\lambda \rightarrow \infty$, $e_\lambda \eta \rightarrow \eta$ and $|L_{J\xi}| e_\lambda \eta \rightarrow v^* (L_\xi)^* \eta$. Since $|L_{J\xi}|$ is a closed operator, $\eta \in \operatorname{dom}(|L_{J\xi}|) = \operatorname{dom}(L_{J\xi})$. Hence $(L_\xi)^* \subset L_{J\xi}$ and $|L_{J\xi}| = v^* (L_\xi)^*$.

Finally, let us prove the last part. Let $T \sim M$ with the polar decomposition $v|T| = T$. Note that $\hat{v}^* = \hat{1}v^* \in \operatorname{dom} T$, $\hat{1} \in \operatorname{dom} T^*$, $T^*\hat{1} = |T|\hat{v}^*$. Put $\xi = T\hat{1}$, $\eta = T^*\hat{1}$. Since $T \sim M$, $L_\xi^\circ \subset T$, $L_\eta^\circ \subset T^*$ and we obtain $L_\xi \subset T \subset L_{J\eta}$. \square

3.3. Projective modules over a finite von Neumann algebra. Let $m, n \in \mathbb{N}$. We have an isomorphism $\operatorname{Mor}(M^{\oplus m}, M^{\oplus n}) = M_{m,n}(M)$ by multiplication of matrices on column vectors.

Definition 18. An left M -module V is said to be finitely generated projective module when it is a projective object in the category of the M -modules and has a finite set generating itself.

Remark 13. Any finitely projective M module is isomorphic to some $M^{\oplus m}.P$ for a natural number m and an idempotent matrix P in $M_m M$.

Lemma 21. *In the above we may replace P with an orthogonal projection $P^* = P$ without changing the value of $\tilde{\tau}(P)$.*

Proof. Let P_0 be the right support of P . $P(P - P_0) = 0$ implies $P_0(P - P_0) = 0$. Thus $S = \text{Id} + (P - P_0)$ is invertible. With respect to the orthogonal decomposition $\text{Id} = P_0 \oplus P_0^\perp$, these operators are expressed as

$$P_0 = \begin{pmatrix} \text{Id} & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} \text{Id} & 0 \\ ? & 0 \end{pmatrix}, \quad S = \begin{pmatrix} \text{Id} & 0 \\ ? & \text{Id} \end{pmatrix}.$$

The operator $SP_0 = SP_0S^{-1}$ is Hermitian. \square

Remark 14. When $M^{\oplus m}P$ and $M^{\oplus n}Q$ are isomorphic, $\tilde{\tau}(P) = \tilde{\tau}(Q)$.

Definition 19. For each finitely projective M -module V isomorphic to $M^{\oplus m}P$ where P is a orthogonal projection in $M_m M$, $\dim_M V - \tilde{\tau}(P)$ is called the τ -dimension

Lemma 22. *Let V be a submodule of $M^{\oplus n}$. When V is closed $M^{\oplus n}$ with respect to the L^2 -norm (V is weakly closed), V is finitely generated and projective.*

Proof. The L^2 completion $\bar{V}^{\|\cdot\|_2} \subset L^2 M^{\oplus n}$ is written as $L^2 M^{\oplus n} P$ for an orthogonal projection P . Then V is equal to $M^{\oplus n} P$. \square

Lemma 23. *For each $T \in \text{Mor}(M^{\oplus m}, M^{\oplus n})$, its kernel and range are finitely generated projective modules.*

Proof. Obviously the kernel of T is weakly closed in $M^{\oplus m}$. On the other hand for the projection P such that $\ker T = M^{\oplus m} P$, T induces an isomorphism $M P^\perp \rightarrow \text{ran } T$. \square

Remark 15. When a submodule $V \subset M^{\oplus m}$ is finitely generated, V is projective. In fact, $V = M^{\oplus m} A$ for some $A \in M_{m,n}(M)$. Thus we have

$$V \simeq M^{\oplus n} l(A) \simeq M^{\oplus m} r(A) \simeq \bar{V}.$$

Hence $\dim_M V = \dim_M \bar{V}$.

Remark 16. If $W \subset V$ are finitely generated projective modules, $\dim_M W \leq \dim_M V$.

Definition 20. Let V be an M -module. Put

$$\dim_M V = \sup \{ \dim_M W : W \subset V, W \text{ is projective} \} \in [0, \infty].$$

Remark 17. Note that the above definition of \dim_M is compatible with the previous one for finitely generated projective modules. In general, $W \subset V$ implies $\dim_M W \leq \dim_M V$ and $(V_i)_{i \in I} \uparrow V$ ($V = \cup_{i \in I} V_i$) implies $\dim_M V = \lim_i \dim_M V_i$.

Theorem 24. (Lück [4]) *When*

$$0 \longrightarrow V_0 \xrightarrow{\iota} V_1 \xrightarrow{\pi} V_2 \longrightarrow 0$$

is exact, we have $\dim_M V_1 = \dim_M V_0 + \dim_M V_2$.

The cokernel in the lower row has dimension 0, thus so does the one in the upper row. \square

Definition 22. $\beta_n^{(2)}(\Gamma) = \dim_{\mathbb{L}\Gamma} \text{Tor}_n^{\mathbb{C}\Gamma}(\mathbb{L}\Gamma, \mathbb{C}_{\text{triv}})$ is called the n -th L^2 -Betti number of Γ .

Remark 18. $\beta_n^{(2)}(\Gamma)$ is equal to $\dim_{\mathbb{L}\Gamma} \text{Tor}_n^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C}_{\text{triv}})$.

Example 6. $\beta_n^{(2)}(\mathbb{F}_r) = r - 1$ when $n = 2$, 0 otherwise. This is seen as follows: let g_1, \dots, g_r be the standard generators of \mathbb{F}_r . A free resolution of the trivial $\mathbb{C}[\mathbb{F}_r]$ -module \mathbb{C} is given by

$$0 \longrightarrow (\mathbb{C}[\mathbb{F}_r])^r \xrightarrow{d_1} \mathbb{C}[\mathbb{F}_r] \xrightarrow{\alpha} \mathbb{C}$$

where $d_1 : (\xi_k)_{k=1}^r \mapsto \sum (\lambda_{g_k}^* - 1)\xi_k$ and α is the augmentation map. Now, d_1 is injective: let $\chi_j \in \ell_\infty \mathbb{F}_r$ be the characteristic function of $\mathbb{F}_r g_j$. Then $(\lambda_{g_k} - 1)\chi_j = \delta_{j,k}\delta_e$ and $(\xi_k)_k \in \ker d_1$ implies

$$0 = \left\langle \sum_k (\lambda_{g_k}^* - 1)\xi_k, \chi_j \right\rangle = \sum_k \langle \xi_k, (\lambda_{g_k} - 1)\chi_j \rangle = \xi_j(e).$$

Replacing χ_j by $\chi_j^t = \chi_j(-t^{-1})$ for $t \in \Gamma$, we have $\xi_j(t) = 0$ for any j and t . Thus, the torsion group is the cohomology of the complex

$$0 \longrightarrow (L^2\mathbb{F}_r)^r \xrightarrow{d_1} L^2\mathbb{F}_r \longrightarrow 0.$$

Let R be a ring. Recall that a right R -module N is flat if and only if the tensor product functor $N \otimes_R -$ preserves injections $V \hookrightarrow F$ where F is a finitely generated free module. The latter holds if and only if $N \otimes_R -$ preserves the injectivity of inclusion $I \hookrightarrow R$ of the left ideals.

Theorem 30. *Let $M \hookrightarrow N$ be a trace preserving inclusion of finite von Neumann algebras. Then N is flat over M and $\dim_N N \otimes_M V = \dim_M V$ for any M -module V .*

Proof. Recall that any finitely generated submodule of a free M -module is projective. (That is, M is semihereditary.) To see this, let V be a finitely generated submodule of a finitely generated free module $M^{\oplus m}$. $V \simeq M^{\oplus n}A$ for some (m, n) -matrix A . Then V is projective, being isomorphic to $M^{\oplus}l(A)$. Now,

$$N \otimes V \simeq N^{\oplus n}.l(A) \simeq N^{\oplus m}A \hookrightarrow N^{\oplus} \simeq N \otimes M^{\oplus m}.$$

Thence N is flat over M .

Let V be a finitely generated M -module. Suppose we had an inclusion $\Phi : M^{\oplus m}.P \hookrightarrow V$ of a projective module. Then $N^{\oplus m}.P \hookrightarrow N \otimes V$ by the flatness of N . This shows that $\dim_N N \otimes_M V \leq \dim_M V$. On the other hand, for any surjection $\pi M^{\oplus n} \twoheadrightarrow V$, the induced homomorphism $\pi_* : N^{\oplus n} \rightarrow N \otimes V$ is surjective and $\dim N \otimes V = n - \dim \pi_*$, thus $\dim N \otimes V \leq \dim V$. \square

3.4. Application to orbit equivalence.

Notation. Let $\alpha : \Gamma \curvearrowright (X, \mu)$ be a probability measure preserving essentially free action. Put $A = L^\infty(X, \mu)$, $M = \mathbb{L}\Gamma$, $N = L^\infty(X, \mu) \rtimes \Gamma = \text{vN}(\mathcal{R}_{\Gamma \curvearrowright X})$. Let R_0 denote the linear span $\text{alg}(L^\infty(X, \mu), \Gamma)$ of $f\lambda(s)$ for the $f \in A, s \in \Gamma$. Let R denote the linear span $\text{alg}(N(A))$ of fv_ϕ for the $f \in A, \phi \in [[\mathcal{R}]]$.

Remark 19. R_0 is free over $\mathbb{C}\Gamma$ and $\mathbb{R}_0 \otimes_{\mathbb{C}\Gamma} \mathbb{C} \simeq L^\infty(X)$. The induced left R_0 -structure on $L^\infty(X)$ is given by $\sum f_s \lambda_s \cdot g = \sum f_s \alpha_s(g)$ thus $R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{C} \simeq A$ and we have $\text{Tor}_*^{R_0}(N, A) \simeq \text{Tor}_*^{\mathbb{C}\Gamma}(N, \mathbb{C})$. The latter is isomorphic to $N \otimes_M \text{Tor}_*^{\mathbb{C}\Gamma}(M, \mathbb{C})$ by the flatness of N . Note that $\dim_N N \otimes_M \text{Tor}_n^{\mathbb{C}\Gamma}(M, \mathbb{C}) = \dim_M \text{Tor}_n^{\mathbb{C}\Gamma}(M, \mathbb{C}) = \beta_n^{(2)}(\Gamma)$.

Our goal is to show the equality $\dim_N \text{Tor}_n^{R_0}(N, A) = \dim_N \text{Tor}_n^R(N, A)$. Note that the latter only depends on the orbit equivalence $\mathcal{R}_{\Gamma \curvearrowright}$.

Lemma 31. *For any $x \in R$ and $\epsilon > 0$, there is a projection p in A such that $\tau p > \epsilon$ and $xp^\perp \in R_0$.*

Proof. When x is of the form $v_\phi f$, the assertion is trivial by the expression $v_\phi = \sum \lambda(g_k) e_k$. The general case reduces to the above by $\tau(p \vee q) \leq \tau p + \tau q$. \square

For the time being let A denote an arbitrary finite von Neumann algebra.

Definition 23. Let V be a left A -module. For $\xi \in V$,

$$[\xi] = \inf \{ \tau p : p \in \text{Proj } A, p\xi = \xi \}$$

is called the rank norm of ξ .

Remark 20. $[\xi]$ is subadditive and scalar invariant. $V_t = \{ \xi : [\xi] = 0 \}$ is the largest submodule with $\dim_A V_t = 0$. Any A -module homomorphism $\phi : V \rightarrow W$ contracts $[\xi]$. Moreover for any $\eta \in \ker \phi$ and $\epsilon > 0$, there is an element $\xi \eta \in \phi^{-1} \eta$ such that $[\xi] \leq [\eta] + \epsilon$.

Definition 24. Let V be an A -module. Consider a metric on V defined by $d(\xi, \eta) = [\xi - \eta]$. Let $C(V)$ denote the completion of V with respect to d .

Remark 21. $C(V)$ admits a left action of A : the continuity with respect to d follows from $[a\xi] \leq \min[a, [\xi]]$: $p\xi = \xi$ implies

$$ap\xi = l(ap)ap\xi - l(ap)a\xi \Rightarrow [a\xi] \leq \tau(l(ap)) = \tau(r(ap))$$

$C(V)$ contains V/V_t as a dense subspace.

Remark 22. $V \subset W$ is dense if and only if for any $\xi \in W$ and $\epsilon > 0$, there exists $p \in A$ such that $\tau p < \epsilon$ such that $p^\perp \xi \in V$, which, in turn, happens if and only if $\dim W/V = 0$.

Lemma 32. *The functor $V \mapsto CV$ is exact.*

Proof. Right exactness: consider an exact sequence $V_1 \rightarrow V_0 \rightarrow 0$. Let $\xi \in CV_0$, $(\xi_n)_{n \in \mathbb{N}} \subset V_0$ a sequence convergent to ξ . We may assume that $d(\xi, \xi_n) \leq 2^{-(n+1)}$. We can inductively lift (ξ_n) to (η_n) in V_1 such that $d(\eta_n, \eta_{n+1}) \leq 2^{-n}$.

General exactness: let

$$V_2 \xrightarrow{g} V_1 \xrightarrow{f} V_0$$

be an exact sequence, ξ an element of $\ker C(f)$. Choose a sequence $(\xi_n)_n$ convergent to ξ . Then $f(\xi_n) \rightarrow C(f)(\xi) = 0$. This implies the existence of a sequence $(\eta_n)_n$, convergent to 0 and $f\eta_n = f\xi_n$. $\xi = \lim \xi_n - \eta_n$ is in the closure of the image of g , which, by the right exactness of C , is equal to the image of $C(g)$. \square

Now we turn to the orbit equivalence situation: $A \subset R_0 \subset R \subset N$. We consider A -rank metric on R_0 -modules.

Lemma 33. *When V is an R_0 (resp. R) module, CV admits an R_0 (resp. R) module structure.*

Proof. If $x = \sum_{n=1}^N v_{\phi_n} f_n$, for any $\xi \in V$ we have the estimate $[x\xi] \leq n[\xi]$. \square

Lemma 34. *When V is an R_0 module, CV admits an R -module structure.*

Proof. Let $x \in R$, $(x_n)_n$ be a sequence in R_0 convergent to x . For any $\xi \in V$, $x_n \xi$ is A -rank convergent to $x\xi$. \square

Lemma 35. *When V is a left R_0 -module. $N \otimes_{R_0} V \rightarrow N \otimes_{R_0} CV$ is a dimension isomorphism.*

Proof. Suppose $x = \sum a_i \otimes \xi_i$ ($a_i \in N, \xi_i \in V$) represents 0 in $N \otimes_{R_0} CV$. In the tensor product over \mathbb{C} ,

$$\sum a_i \otimes \xi_i = \sum (b_j v_j \otimes \eta_j - b_j \otimes v_j \eta_j)$$

for $b_j \in N, v_j \in R_0, \eta_j \in CV$. For each j , there is a projection p_j such that $\tau(p_j) \sim 0$ and $p_j^\perp \eta_j \in V$. Thus we get a representative of x given by

$$\sum (b_j v_j \otimes p_j \eta_j - b_j \otimes v_j p_j \eta_j) + \sum (b_j v_j \otimes p_j^\perp \eta_j - b_j \otimes v_j p_j^\perp \eta_j)$$

The second summand becomes 0 in $N \otimes_{R_0} V$. Now, choose the smallest projection p in N such that $p v_j p_j = v_j p_j, p_j \leq p$. Then $x = (1 \otimes p)x$ and $[x]_N \sim 0$. Hence $N \otimes V \rightarrow N \otimes CV$ is an isometry. When $\xi_n \in V$ converges to $\xi \in CV$, $a \otimes \xi_n$ converges to $a \otimes \xi$ in $[-]_N$. \square

Remark 23. For any R -module W , $N \otimes_{R_0} W \rightarrow N \otimes_R W$ is an \dim_N -isomorphism.

Theorem 36. $\dim_N \operatorname{Tor}_n^{R_0}(N, A) = \dim_N \operatorname{Tor}_n^R(N, A)$.

Proof. Consider projective resolutions of A : $P_* \rightarrow A$ as an R_0 -module, $Q_* \rightarrow A$ as an R -module. We have morphisms $\phi_* : P_* \rightarrow Q_*$ and $\psi_* : Q_* \rightarrow CP_*$. Thus we get a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_n & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & A \\ & & \downarrow \phi_n & & & & \downarrow \phi_0 & & \parallel \\ \cdots & \longrightarrow & Q_n & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & A \\ & & \downarrow \psi_n & & & & \downarrow \psi_0 & & \downarrow \\ \cdots & \longrightarrow & CP_n & \longrightarrow & \cdots & \longrightarrow & CP_0 & \longrightarrow & CA \\ & & \downarrow C\phi_n & & & & \downarrow C\phi_0 & & \parallel \\ \cdots & \longrightarrow & CQ_n & \longrightarrow & \cdots & \longrightarrow & CQ_0 & \longrightarrow & CA. \end{array}$$

By the uniqueness of projective resolution up to homotopy, compositions of two homomorphisms $\psi_n \phi_n$ and $C\phi_n \psi_n$ are homotopic to the standard inclusion isomorphisms. Thus the theorem follows once we know that $P_* \rightarrow CP_*$ induces a \dim_N -isomorphism between the cohomology groups.

The torsion N -modules $\{V : \dim_N V = 0\}$ form a Serre subcategory of N -mod. Thus, in general, when a morphism $V_* \rightarrow W_*$ of complexes is a dimension isomorphism, the induced homomorphism between the cohomology groups is also a dimension isomorphism because it factors through an isomorphism in the localization category of the N -module complex category over the torsion module category. \square

Corollary 37. *Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ be essentially free probability measure preserving actions. If $\mathcal{R}_{\Gamma \curvearrowright X} \simeq \mathcal{R}_{\Lambda \curvearrowright Y}$, $\beta_n^{(2)}(\Gamma) = \beta_n^{(2)}(\Lambda)$.*

Remark 24. Put $\beta_*^2(A, N) = \dim_N \text{Tor}_*^R(N, A)$. For any nonzero projection p in A , $\beta_*^2(A, N) = \tau(p)\beta_*^2(pA, pNp)$.

4. DERIVATION ON VON NEUMANN ALGEBRAS

In the following we only consider normal Hilbert (bi)modules. Examples include the identity bimodule L^2N and the coarse (M, N) -module $L^2M \otimes L^2N$.

Let Γ be a countable discrete group, (π, H_0) a unitary representation of Γ . A map $b : \Gamma \rightarrow H_0$ is said to be a derivation when it satisfies $b(st) = b(s) + \pi(s)b(t)$ i. e. a derivation for the (π, triv) -bimodule structure. A derivation b is said to be inner when there exists $\xi \in H_0$ such that $b(s) = \pi(s)\xi - \xi$. Put

$$H^1(\Gamma, \pi) = \{\text{derivations}\} / \{\text{inner derivations}\}.$$

When b is a derivation, $\phi_r(s) = e^{-r\|b(s)\|^2}$ for $r \geq 0$ determines a positive semidefinite semigroup. Our goal is to show that it extends to a semigroup $\tilde{\phi}_r : \text{LF} \rightarrow \text{LF}$ of τ preserving completely positive maps.

4.1. Densely defined derivation. Let M denote $L^2\Gamma$. Consider $H = M \otimes H_0$. A left action $M \rightarrow B(H)$ is defined by $\lambda(f) \mapsto \lambda \otimes \pi(f)$ (this is possible by the Fell absorption.) On the other hand we have a right action $M^o \rightarrow B(H)$ is defined by $\rho(g) \mapsto \rho(g) \otimes id$. Put $\delta(s) = \delta_s \otimes b(s) \in \ell_2\Gamma \otimes H_0$. By

$$\delta(st) = \delta_{st} \otimes (b(s) + \pi(s)b(t)) = \rho \otimes 1(t^{-1})\delta(s) + \lambda \otimes \pi(s)\delta(t),$$

δ extends to a (possibly unbounded) derivation $\mathbb{C}\Gamma \rightarrow H$ satisfying $\delta(xy) = x\delta(y) + \delta(x)y$.

Notation. Let (M, τ) be a finite von Neumann algebra with a faithful normal tracial state, \mathcal{D} a weak*-dense *-subalgebra of M . Let H be a Hilbert bimodule over M , $\delta : M \rightarrow H$ a derivation defined on \mathcal{D} which is closable as a densely defined operator $L^2M \rightarrow H$. Let $\bar{\delta}$ denote its closure.

We are going to show that the domain of $\bar{\delta}$ is a *-subalgebra of $\mathcal{L}(H)$ and that $\bar{\delta}$ is a derivation.

Notation. Let $\|\cdot\|_{\text{Lip}}$ denote the 1-Lipschitz norm:

$$\|f\|_{\text{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Let Lip_0 denote the space of 1-Lipschitz continuous functions which map 0 to 0.

For any $x \in L^2M_{sa}$, regarded as a self adjoint unbounded operator on L^2M , we can consider its functional calculus $f(x)$.

Proposition 38. *When $x, y \in L^2M_{sa}$ and $f \in \text{Lip}_0$, the functional calculus $f(x), f(y)$ is in L^2M and*

$$\|f(x) - f(y)\|_2 \leq \|f\|_{\text{Lip}} \|x - y\|_2.$$

Proof. For the spectral measure $E(t)$ of x , $x = \int t dE(t)$ and $\|x\|_2^2 = \int |x|^2 d\tau E(T)$. Thus $\|f(x)\|_2^2 = \int |f(t)|^2 d\tau E(t) \leq \|f\|_{\text{Lip}}^2 \int |t|^2 d\tau E(t)$ and $f(x)$ is in L^2M . For the second assertion, consider the bilinear map

$$C_0(\mathbb{R})^2 \ni (f, g) \mapsto \tau(f(x)g(x)) = \langle f(x)\hat{1}f(y), \hat{1} \rangle.$$

This defines a linear form $C_0(\mathbb{R} \times \mathbb{R}) \rightarrow \mathbb{C}$, i.e. $\tau(f(x)g(y)) = \int fgd\mu$ for some measure μ on $\mathbb{R} \times \mathbb{R}$. Thus, $\tau(|f(x) - f(y)|^2)$ is equal to

$$\int |f(s) - f(t)|^2 d\mu(s, t) \leq \|f\|_{\text{Lip}}^2 \int |s - t|^2 d\mu(s, t) = \|f\|_{\text{Lip}}^2 \|x - y\|_2^2. \quad \square$$

Definition 25. Let I be a bounded closed interval in \mathbb{R} , $f \in C^1(I)$. The function

$$\tilde{f}(x, y) = \begin{cases} \frac{f(x) - f(y)}{x - y} & (x \neq y) \\ f'(x) & (x = y) \end{cases}$$

is called the difference quotient of f .

Note that $\|\tilde{f}\|_{\infty} = \|f'\|_{\infty}$. When $a \in M_{s_a}$ and $[-\|a\|, \|a\|] \subset I$, we have $\pi_a : C(I \times I) \rightarrow B(H)$ by $\pi_a(f \otimes g)\xi = f(a)\xi g(a)$.

Lemma 39. For any $a \in \mathcal{D}$ and $f \in C^1(I)$, the operator $f(a)$ is in $\text{dom } \bar{\delta}$ and $\bar{\delta}(f(a)) = \pi_a(\tilde{f})\delta(a)$.

Proof. The assertion is obvious for polynomial functions. The equality for the general C^1 -functions follows from it because it is compatible with the C^1 -norm. \square

Remark 25. When T is a closed operator on H , $x_n \rightarrow x$ ($n \rightarrow \infty$) in H and $\sup_n \|Tx_n\| < \infty$ imply that $x \in \text{dom } T$ and that $Tx \in \bigcap_{m=0}^{\infty} \overline{\text{conv}}\{Tx_n : n \geq m\}$, where $\overline{\text{conv}}$ denotes the closed convex span. This is because, taking a suitable subsequence if necessary, we may assume that the bounded sequence Tx_n is weakly convergent to some y . Taking the convex closure, we can find a sequence $(z_n)_{n \in \mathbb{N}}$ such that $Tz_n \rightarrow y$ in norm and that z_n is in the algebraic convex closure of $\{x_k : k \geq n\}$. By construction, $(z_n)_{n \in \mathbb{N}}$ converges to x .

Lemma 40. Let x be an unbounded self adjoint operator on L^2M which is in $\text{dom } \bar{\delta}$, $f \in \text{Lip}_0$. Then $f(x) \in \text{dom } \bar{\delta}$ and $\|\bar{\delta}(f(x))\| \leq \|f\|_{\text{Lip}} \|\bar{\delta}(x)\|$.

Proof. Choose a mollifier $(\phi_n)_n$ and set $f_n = f * \phi_n$. Thus f_n is of C^1 class and $f_n \rightarrow f$ uniformly on I . By

$$|f_n(y) - f_n(z)| = \int |f(y-r) - f(z-r)| \phi_n(r) dr \leq \|f\|_{\text{Lip}} |y - z|,$$

we have $\|f_n\|_{\text{Lip}} \leq \|f\|_{\text{Lip}}$. Now take a sequence $(a_n)_{n \in \mathbb{N}}$ in \mathcal{D}_{s_a} which is convergent to x in $\|\cdot\|_2$ -norm. Then

$$\|\bar{\delta}(f_n(a))\| = \left\| \pi_a(\tilde{f}_n)\delta(a) \right\| \leq \|f_n\|_{\text{Lip}} \|\delta a\|.$$

This shows $f(x) \in \text{dom } \bar{\delta}$. \square

Definition 26. A derivation $\delta : M \rightarrow H$ is said to be real when we have

$$\langle \delta(x), \delta(y)z \rangle = \langle z^* \delta(y^*, x^*) \rangle$$

for any $x, y, z \in M$.

Remark 26. We summarize a few properties of real derivations.

- When M is the group von Neumann algebra $L\Gamma$ of a group Γ , the above condition is equivalent to $\langle \delta(s), \delta(t) \rangle \in \mathbb{R}$.
- In general, when we have a J -operator, δ is real if and only if $Jx\delta(y)z = z^*\delta(y^*)x^*$, since, by definition, $\langle \delta(x), \delta(y)z \rangle$ is equal to $\langle z^*J\delta(y), J\delta(x) \rangle$.
- When δ is real, $\text{dom } \bar{\delta}$ is self adjoint.

Let $\bar{\mathcal{D}}$ denote $\text{dom } \bar{\delta}$.

Lemma 41. *Let δ be a real derivation. When $x \in \bar{\mathcal{D}}$, $|x|$ is also in $\bar{\mathcal{D}}$ and $M \cap \bar{\mathcal{D}}$ is a $*$ -subalgebra of M .*

Proof. Consider the linear map $\delta^{(2)} : M_2\mathcal{D} \rightarrow M_2H \simeq H^{\oplus 4}$. Then $\delta^{(2)} = \bar{\delta}^{(2)}$ and for any $z \in \bar{\mathcal{D}}$,

$$w = \begin{bmatrix} 0 & z^* \\ z & 0 \end{bmatrix} \in \text{dom } \delta^{(2)} \Rightarrow w^2 = \begin{bmatrix} |z|^2 & 0 \\ 0 & |z^*|^2 \end{bmatrix} \in \text{dom } \bar{\delta}^{(2)}.$$

Thus $|z|^2$ is in $\bar{\mathcal{D}}$.

Let $x, y \in \bar{\mathcal{D}}$. The polarization

$$x^*y = \frac{1}{4} \sum i^k |x + i^k y|$$

shows $x^*y \in \bar{\mathcal{D}}$, and in particular $x^* \in \bar{\mathcal{D}}$ follows from $1 \in \bar{\mathcal{D}}$. \square

Lemma 42. *For any $x \in \bar{\mathcal{D}} \cap M_{sa}$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in \mathcal{D}_{sa} such that*

$$\|x_n - x\|_2 \rightarrow 0, \|\delta(x_n) - \bar{\delta}(x)\| \rightarrow 0 \text{ and } \|x\|_\infty \leq \|x\|_\infty.$$

In particular, $x_n \rightarrow x$ in the ultrastrong topology.

Proof. The only nontrivial part is the last inequality. This is achieved by the functional calculus with respect to the function

$$f(t) = \begin{cases} \|x\|_\infty & (\|x\|_\infty < t) \\ t & (|t| \leq \|x\|_\infty) \\ -\|x\|_\infty & (t < -\|x\|_\infty). \end{cases} \quad \square$$

Theorem 43. *The restriction of $\bar{\delta}$ to $\bar{\mathcal{D}} \cap M$ is a derivation.*

Proof. Let $x \in \bar{\mathcal{D}} \cap M$. Choose a sequence $(x_n)_{n \in \mathbb{N}}$ in \mathcal{D} weakly convergent to x and $\delta(x_n) \rightarrow \bar{\delta}(x)$. For each $y \in \bar{\mathcal{D}} \cap M$, we have $x_n y \rightarrow xy$ in the $\|\cdot\|_2$ -norm. Since y is bounded, we have $\delta(x_n)y \rightarrow \bar{\delta}(y)$. On the other hand, the representation of M on H is normal, which implies $x_n \delta(y) \rightarrow x \delta(y)$. Thus we have $\bar{\delta}(xy) = x \delta(y) + \bar{\delta}(x)y$. Similar approximation in y shows that $\bar{\delta}(xy) = x \bar{\delta}(y) + \bar{\delta}(x)y$. for any $y \in \bar{\mathcal{D}} \cap M$. \square

4.2. Semigroup associated to a derivation. In the following we assume $M \cap \bar{\mathcal{D}} = \mathcal{D}$. Put $\Delta = \delta^* \bar{\delta}$. This is a positive self adjoint operator on $L^2 M$ satisfying $\Delta \hat{1} = \hat{1}$ and commutes with the J operator so that we have “ $\Delta(x^*) = (\Delta x)^*$.” Put $\phi_t = e^{-t\Delta}$. This is a semigroup of positive contractions satisfying $\phi_t \hat{1} = \hat{1}$ and $\phi_t \nearrow \text{Id}$ as $t \searrow 0$. The normalized resolvents

$$\eta_\alpha = \frac{\alpha}{\alpha + \Delta}$$

for $\alpha > 0$ are again positive contractions on L^2M with $\eta_\alpha \nearrow \text{Id}$ as $\alpha \nearrow \infty$. These operators are connected by

$$\begin{array}{ccc} \Delta & \xrightleftharpoons[\text{der.}]{\text{exp.}} \phi_t & \xrightarrow{\text{Laplace trans.}} \eta_\alpha \\ & \underbrace{\hspace{10em}}_{\text{inverse}} & \end{array}$$

where the Laplace transform is given by

$$\eta_\alpha = \alpha \int_0^\infty e^{-\alpha t} \phi_t dt = \int_0^\infty e^{-t} \phi_{\frac{t}{\alpha}} dt.$$

Recall that any unital completely positive map $\phi : M \rightarrow M$ is expressed as $V^*\pi(x)V$ for some representation $\pi : M \rightarrow B(K)$ and an isometry $V : L^2M \rightarrow K$ (Stein-spring's theorem). When ϕ is normal, π can be taken as a normal representation (we may take the normal part of general π given by Steinespring's theorem). Thus,

- (1) For any $x \in M$, $\phi(x^*x) - \phi(x^*)\phi(x) = V^*\pi x^*(1 - VV^*)\pi xV \geq 0$. When ϕ preserves τ , $\|\phi(x)\|_2 \leq \|x\|$.
- (2) When ϕ preserves τ , $\|\phi(x^*y) - \phi(x^*)\phi(y)\|_2 = \|V^*\pi x(1 - VV^*)\pi yV\hat{1}\|$ is bounded from above by

$$\|\phi(x^*x) - \phi x^*\phi x\|_\infty^{\frac{1}{2}} (\tau(\phi(y^*y) - \phi y^*\phi y))^{\frac{1}{2}} \leq 2 \|x\|_\infty \|y - \phi(y)\|_2$$

$$\text{by } \tau(\phi(y^*y) - \phi y^*\phi y) = \|y\|_2^2 - \|\phi y\|_2^2, \text{ etc.}$$

Fact. Consider the 1-norm $\|x\|_1 = \sup\{|\tau(xy)| : \|y\|_\infty \leq 1\}$ for $x \in M$. $x \in L^2M$ is in M if and only if $\sup\{|\tau(xy)| : \|y\|_1 \leq 1, xy \in M\}$ is finite.

Theorem 44. (*Sauvageot*, [1]?) *The contractions ϕ_t and η_α map M into M , are unital completely positive and τ -symmetric, i. e. $\tau(\phi_t(x)y) = \tau(x\phi_t(y))$ etc.*

Proof. Observe that $\phi_t^{(n)} = e^{-t\Delta^{(n)}}$ where $\Delta^{(n)} = \delta^{(n)*}\delta^{(n)}$ for $\delta^{(n)} : M_n\mathcal{D} \rightarrow M_nH$. Thus, it is enough to show that the maps are positive to conclude that they are actually completely positive. Put

$$\Delta_\alpha = \frac{\alpha\Delta}{\alpha + \Delta} = \alpha(1 - \eta_\alpha).$$

Then

$$\phi_t = e^{-t\Delta}e = \lim_{\alpha \nearrow \infty} e^{-t\Delta_\alpha} = \lim_{\alpha \nearrow \infty} e^{-t\alpha} \sum_{n=0}^{\infty} \frac{t\alpha\eta_\alpha}{n!}$$

where the limit is taken in the strong operator topology (note: this might be the norm topology, as we are using c_0 functions converging from below). The last expression is compatible with the $x \mapsto \tau(xy)$ ($\|y\|_1 \leq 1$) functionals. Thus it reduces to show that η_α restricts to a positive map on M .

By scaling δ , we may assume that $\alpha = 1$. Let $x \in M_+$ and put $y = (1 + \Delta)^{-1}x \in \text{dom } \Delta$. We have

$$\|\delta y\|^2 + \|y\|_2^2 = \langle y, \Delta y \rangle + \langle y, y \rangle = \langle y, x \rangle$$

Then the function $\Phi(z) = \|\bar{\delta}(z)\|^2 + \|z - x\|_2^2$ for $z \in \overline{\mathcal{D}}_{sa}$ satisfies

$$\begin{aligned} \|\bar{\delta}(z - y)\|^2 + \|z - y\|_2^2 &= \|\bar{\delta}(z)\|^2 - 2\langle z, \Delta y \rangle + \|\bar{\delta}(y)\|^2 + \|z\|^2 - 2\langle z, y \rangle + \|y\|^2 \\ &= \|\bar{\delta}(z)\|^2 + \|z\|^2 - 2\langle z, x \rangle + \|x\|^2 \\ &\quad - (\|\bar{\delta}(y)\|^2 + \|y\|^2 - 2\langle y, x \rangle + \|x\|^2) \\ &= \Psi(z) - \Psi(y). \end{aligned}$$

Consider a function

$$f(t) = \begin{cases} \|x\|_\infty & (\|x\|_\infty < t) \\ t & (0 \leq t \leq \|x\|_\infty) \\ 0 & (t < 0). \end{cases}$$

of Lip_0 class with $\|f\|_{\text{Lip}} = 1$. Then

$$\Psi(f(z)) = \|\bar{\delta}(f(z))\|^2 + \|f(z) - f(x)\| \leq \Psi(z).$$

Take a sequence $(z_n)_{n \in \mathbb{N}}$ in \mathcal{D}_{sa} with $\|z_n - y\|_2 \rightarrow 0$ and $\|\delta z_n - \bar{\delta}y\| \rightarrow 0$. Then we have

$$\|fz_n - y\|_2^2 \leq \Psi(fz_n) - \Psi(y) \leq \Psi(z_n) - \Psi(y) \rightarrow 0.$$

Thus $y = \lim fz_n$ and $0 \leq y \leq \|x\|$ and η_1 is shown to be unital positive. \square

Let B be a von Neumann subalgebra of M . Then we are interested in “when ϕ_t converges uniformly on B_1 ?” Roughly, this means “ δ is inner on B .”

Lemma 45. *Let $\Omega \subset M_1$. Then $\phi_t \rightarrow id$ uniformly on Ω as $t \rightarrow 0$ if and only if $\eta_\alpha \rightarrow id$ uniformly on Ω as $\alpha \rightarrow \infty$.*

Proof. \Rightarrow : We have

$$\|x - \eta_\alpha x\|_2 \leq \int_0^\infty e^{-s} \|x - \phi_{\frac{s}{\alpha}}(x)\|_2 ds,$$

but $\|x - \phi_{\frac{s}{\alpha}}(x)\|_2$ does not exceed 2.

\Leftarrow : Suppose ϕ_s did not converge uniformly on Ω . Then there is a constant c such that for any t there exists an element x_t of Ω satisfying $\langle x_t - \phi_t x_t, x_t \rangle \geq c$. Then

$$\begin{aligned} \langle x_t - \eta_{\frac{1}{t}} x_t, x_t \rangle &= \int_0^\infty e^{-s} \langle x_t - \phi_{st} x_t, x_t \rangle ds \\ &\geq \int_0^1 e^{-s} \langle x_t - \phi_t(x_t), x_t \rangle ds \\ &\geq c(1 - e^{-1}) \end{aligned}$$

and η_α is not uniformly convergent on Ω . \square

Lemma 46. *For the latter convenience we record the following equalities:*

(1) In $B(L^2M)$,

$$\eta_{\frac{1}{\alpha}}^{\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t} \eta_{\frac{\alpha(1+t)}{t}} dt.$$

(2) In $B(L^2M)$,

$$(\text{Id} - \eta_\alpha)^{\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \frac{t^{-\frac{1}{2}}}{1+t} (1 - \eta_{\frac{\alpha(1+t)}{t}}) dt = \text{Id} - \theta_\alpha$$

where θ_α restricts to a unital completely positive map on M .

(3) $\psi_t = e^{-t\Delta^{\frac{1}{2}}}$ is τ -symmetric and unital completely positive on M .

Proof. (1): we have

$$s^{\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \frac{s}{s+t} t^{-\frac{1}{2}} dt \Rightarrow \eta_\alpha^{\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \frac{\eta_\alpha}{t + \eta_\alpha} t^{-\frac{1}{2}} dt.$$

On the other hand,

$$\frac{\eta_\alpha}{t + \eta_\alpha} = \frac{\alpha}{\alpha(1+t) + t\Delta} = \frac{1}{1+t} \eta_{\frac{\alpha(1+t)}{t}}.$$

(3): We have $\Delta_\alpha^{\frac{1}{2}} = \alpha^{\frac{1}{2}}(\text{Id} - \eta_\alpha)^{\frac{1}{2}} = \alpha^{\frac{1}{2}}(\text{Id} - \theta_\alpha)$. Thus ψ_t can be written as

$$\lim_{\alpha \rightarrow \infty} e^{-t\Delta_\alpha^{\frac{1}{2}}} = \lim_{\alpha \rightarrow \infty} e^{-\alpha^{\frac{1}{2}}t} e^{t\alpha^{\frac{1}{2}}\theta_\alpha}. \quad \square$$

Lemma 47. For $x, y \in \mathcal{D}$, put $\Gamma(x^*, y) = \Delta^{\frac{1}{2}}(x^*)y + x^*\Delta^{\frac{1}{2}}(y) - \delta^{\frac{1}{2}}(x^*y)$. Then we have

$$\|\Gamma(x^*, y)\|_2 \leq 4 \|\delta(x)\| \|x\|_\infty \|\delta(y)\| \|y\|_\infty.$$

Proof. First we have

$$\Gamma(x^*, y) = \left. \frac{d}{dt} (\psi_t(x^*y) - \psi_t(x^*)\psi_t(y)) \right|_{t=0}.$$

Note that $\|\psi_t x\| \leq \|x\|$. Define a sesquilinear form on $\mathcal{D} \otimes M$ by $\langle y \otimes b, x \otimes a \rangle = \tau(a^* \Gamma(x^*, y)b)$. This is positive semidefinite by

$$\langle \sum x_i \otimes a_i, \sum x_i \otimes a_i \rangle = \lim_{t \rightarrow 0} \tau \left(\sum a_i \frac{\psi_t(x_i^* x_j) - \psi_t x_i^* \psi_t(x_j)}{t} a_j \right) \leq 0.$$

For $z = v|z| \in M$, we have

$$|\tau(\Gamma(x^*, y)z)| = |\langle y \otimes v|z|^{\frac{1}{2}}, x \otimes |z|^{\frac{1}{2}} \rangle| \leq \langle y \otimes v|z|^{\frac{1}{2}}, y \otimes v|z|^{\frac{1}{2}} \rangle^{\frac{1}{2}} \langle x \otimes |z|^{\frac{1}{2}}, x \otimes |z|^{\frac{1}{2}} \rangle^{\frac{1}{2}}.$$

Here, $\langle x \otimes a, x \otimes a \rangle \leq \|aa^*\|_2 \|\Gamma(x^*, x)\|_2$ and

$$\begin{aligned} \|\Gamma(x^*, x)\| &\leq \left\| \Delta^{\frac{1}{2}} x^* \right\|_2 \|x\|_\infty + \|x^*\|_\infty \left\| \Delta^{\frac{1}{2}} x \right\|_2 + \left\| \Delta^{\frac{1}{2}}(x^*x) \right\|_2 \\ &\leq 4 \|\delta(x)\| \|x\|_\infty. \end{aligned}$$

(Here we used the fact that $\left\| \Delta^{\frac{1}{2}}(x^*x) \right\|_2 = \|\delta(x^*)x + x^*\delta(x)\|$.) Hence we arrive at

$$|\tau(\Gamma(x^*, y)z)|^2 \leq \|\Gamma(x^*, x)\|_2 \|z\|_2 \|(y^*, y)\| \|z\|_2,$$

thus $\|\Gamma(x^*, y)\|^2 \leq \|\Gamma(x^*, x)\|_2 2 \|(y^*, y)\|$. \square

Put $\zeta_\alpha = \eta_\alpha^{\frac{1}{2}}$. $\Delta^{\frac{1}{2}} \zeta_\alpha = \Delta_\alpha^{\frac{1}{2}} = (\text{Id} - \eta_\alpha)^{\frac{1}{2}}$ (hence bounded) and $\left\| \Delta_\alpha^{\frac{1}{2}} x \right\|_2^2 = \alpha \langle (\text{Id} - \eta_\alpha)x, x \rangle$. Put $\tilde{\delta}_\alpha = \alpha^{\frac{1}{2}} \delta \zeta_\alpha$. Thus $\left\| \tilde{\delta}_\alpha(x) \right\| = \langle (\text{Id} - \eta_\alpha)x, x \rangle$ and $\|\delta\|_\alpha(x) \rightarrow 0$ if and only if $\|x - \eta_\alpha x\|_2 \rightarrow 0$.

Theorem 48. (*Peterson?*) Let $\Omega \subset M_1$ and suppose $\eta_\alpha \rightarrow \text{Id}$ uniformly on Ω . Then we have $\left\| \tilde{\delta}_\alpha(ax) - \zeta_\alpha(a)\tilde{\delta}_\alpha(x) \right\| \rightarrow 0$ ($\alpha \rightarrow \infty$) uniformly for $a \in \Omega$ and $x \in M_1$.

Proof. By assumption ζ_α and θ_α converge uniformly to Id on Ω , by e.g. .

$$\theta_\alpha = \frac{1}{\pi} \int_0^\infty \frac{t^{\frac{1}{2}}}{1+t} \eta_{\frac{\alpha t}{1+t}} dt.$$

In particular, $\theta_\alpha(ax) \approx \theta_\alpha(a)\theta_\alpha(x) \approx a\theta_\alpha(x)$ where \approx means the 2-norm convergence under $\alpha \rightarrow \infty$. Now,

$$\begin{aligned} \alpha^{-\frac{1}{2}} \Delta^{\frac{1}{2}} \zeta_\alpha(ax) &= \alpha^{-\frac{1}{2}} (\text{Id} - \theta_\alpha)(ax) \approx \alpha^{-\frac{1}{2}} a (\text{Id} - \theta_\alpha)(x) \approx \alpha^{-\frac{1}{2}} \zeta_\alpha(a) (\text{Id} - \theta_\alpha)(x) \\ &= \alpha^{-\frac{1}{2}} \zeta_\alpha(a) \Delta^{\frac{1}{2}} \zeta_\alpha(x) \approx \alpha^{-\frac{1}{2}} \Delta^{\frac{1}{2}} (\zeta_\alpha(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x) \end{aligned}$$

where the last approximation is given by applying Lemma 47 to get the error estimate

$$4\sqrt{\alpha^{-\frac{1}{2}} \left\| \delta^{\frac{1}{2}}(\zeta_\alpha(a)) \right\| \left\| \alpha^{\frac{1}{2}} \delta \zeta_\alpha x \right\|}.$$

Here, $\alpha^{-\frac{1}{2}} \left\| \delta^{\frac{1}{2}}(\zeta_\alpha(a)) \right\| \sim 0$ and $\left\| \alpha^{\frac{1}{2}} \delta \zeta_\alpha x \right\|$ is bounded by 1.

Finally we arrive at

$$\tilde{\delta}_\alpha(ax) \approx \alpha^{-\frac{1}{2}} \delta(\zeta_\alpha(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x) = \zeta_\alpha(a)\tilde{\delta}_\alpha(x). \quad \square$$

Theorem 49. (*Haagerup*) Let M be a von Neumann algebra. M is finite injective if and only if for any nonzero central projection p of M , there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathcal{U}(pM)$ such that $\left\| \sum_{i=1}^n u_i \otimes u_i \right\|_\infty = n$.

Proof. (Outline) \Rightarrow : By Connes' theorem, $M \otimes_{\min} \bar{M} \rightarrow B(L^2M)$ can be defined by $(a \otimes b) \cdot \hat{x} = \widehat{ab^*}$. Now, $(\sum_{i=1}^n u_i \otimes \bar{u}_i) \cdot \hat{1} = n\hat{1}$ when $u_i \in \mathcal{U}(M)$.

\Leftarrow : The minimal tensor product $M \otimes_{\min} \bar{M}$ acts on $H \hat{\otimes} \bar{H}$ i.e. the Hilbert-Schmidt space of H . For any finite set $F \subset \mathcal{U}(M)$ containing 1 and $\left\| \sum_{u \in F} u \otimes \bar{u} \right\| = |F|$, there exists $T \in HS(H)$ of 2-norm 1, $\left\| \sum_{u \in F} uTu^* \right\| \approx |F|$. Then $uTu^* \approx T$. Now, define $\phi_F(x) = \text{Tr}(T^*xT)$. Then $\phi_F(uxu^*) \approx \phi_\alpha(x)$ for $u \in F$. We obtain an ultrafilter convergence $\phi_F \rightarrow \phi \in S(B(H))$ such that $\phi(uxu^*) = \phi(x)$ for any $u \in \mathcal{U}(M)$. This holds under any central projection, which means M is injective. \square

Recall that we are investigating closable real derivations on M . Thus, H is an M -bimodule with a J -operator: $J(a\delta(x)b) = b^*\delta(x^*)a^*$. We have the operators

$$\eta_\alpha = \frac{\alpha}{\alpha + \delta^* \delta}, \zeta_\alpha = \eta_\alpha^{\frac{1}{2}}, \tilde{\delta}_\alpha = \alpha^{-\frac{1}{2}} \delta \zeta_\alpha : M \rightarrow H.$$

As $\alpha \rightarrow \infty$, we have $\left\| \tilde{\delta}_\alpha(a) \right\|^2 = \left\| (\text{Id} - \eta_\alpha)^{\frac{1}{2}} a \right\|_2^2 = \tau((a - \eta_\alpha a)a^*) \searrow 0$.

Theorem 50. Let $(M; \tau)$ be a finite von Neumann algebra, $H = (L^2M \otimes L^2M)^{\oplus \mathbb{N}}$. Suppose $Q \subset M$ is a von Neumann subalgebra without injective summand. Then $\phi_t \rightarrow \text{Id}$ uniformly on $(Q' \cap M)_1$.

Proof. It is enough to show that for any nonzero central projection $p \in Q$, there exists a central projection $q \leq p$ in Q such that $\phi_t \rightarrow \text{Id}$ on $q(Q' \cap M)_1$. In fact, then by the maximal argument we would get a family $(p_i)_{i \in I}$ of nonzero central

projections such that $\sum_{i \in I} p_i = 1$ and $\phi_t \rightarrow \text{Id}$ on $p_i(Q' \cap M)_1$ for each i . Taking a finite subset $I_0 \subset I$ such that $\tau(\sum_{i \in I_0} p_i) < \frac{\epsilon}{3}$, we find t_0 such that $t > t_0$ implies $\|\phi_t(a) - a\|_2 < \frac{\epsilon}{3}$ for $a \in p_{I_0}(Q' \cap M)_1$. On the other hand, for any $a \in p_{I_0}(Q' \cap M)_1$ $\tau(a - p_{I_0}a) < \frac{\epsilon}{3}$.

Thus we are going to prove the negation of the above claim leads to that pQ is injective. Let $q \leq p$ be a nonzero central projection in Q , $u_1, \dots, u_n \in \mathcal{U}(qQ)$. As ϕ_t does not converge uniformly on $q(Q' \cap M)_1$, there exists $x_\alpha \in q(Q' \cap M)_1$ for any α such that $\liminf \|\tilde{\delta}_\alpha(x_\alpha)\| > 0$.

Applying Theorem 48 to the finite subset $\Omega = \{u_1, \dots, u_n\}$ on which ϕ_t is uniformly convergent, for any $x \in q(Q' \cap M)$, as $\alpha \rightarrow \infty$,

$$\sum_i \zeta_\alpha(u_i) \tilde{\delta}_\alpha(x) \zeta_\alpha(u_i^*) \approx \sum_i \tilde{\delta}_\alpha(u_i x u_i^*) = n \tilde{\delta}_\alpha(x).$$

Thus, $\left\| \sum_i \zeta_\alpha(u_i) \otimes \overline{\zeta_\alpha(u_i)} \right\|_{\min} \rightarrow n$ as $\alpha \rightarrow \infty$. On the other hand, since ζ_α is a normal unital completely positive map, $\left\| \sum_i \zeta_\alpha(u_i) \otimes \overline{\zeta_\alpha(u_i)} \right\|_{\min}$ is always bounded by $\|\sum u_i \otimes \bar{u}_i\|$, which shows that $\|\sum u_i \otimes \bar{u}_i\| = n$. Thus we have the injectivity of pQ by Theorem 49. \square

Remark 27. If a 1-cocycle $b : \mathbb{F}_r \rightarrow \ell_2 \mathbb{F}_r^{\oplus n}$ satisfies $\|b(s)\|_2^2 = |s|$, we obtain a derivation δ on $\ell_2 \mathbb{F}_r \otimes \ell_2 \mathbb{F}_r^{\oplus n}$ given by $\delta(s) = \delta_\Delta \otimes b$ where δ_Δ is the ‘‘diagonal’’ operator on $\ell_2 \mathbb{F}_r$ which multiplies the standard base δ_s by $|s|$. The semigroup ϕ_t associated to this derivation is written as $\phi_t(\lambda(s)) = e^{-t|s|} \lambda(s)$, thus it is in $\mathbb{K}(L^2 M)$.

When B is a von Neumann subalgebra of $\mathbb{L}\mathbb{F}_r$, $\phi_t \rightarrow \text{Id}$ uniformly on B_1 if and only if B is a direct sum $\oplus M_{n_i}$ of finite dimensional algebras.

Corollary 51. *Let Q be a von Neumann subalgebra of $\mathbb{L}\mathbb{F}_r$ without injective summand. Then the relative commutant $Q' \cap \mathbb{L}\mathbb{F}_r$ is completely atomic. In particular, $Q \otimes L^\infty([0, 1]) \not\subset \mathbb{L}\mathbb{F}_r$.*

Theorem 52. *Let $(M; \tau)$ be a finite von Neumann algebra, $H = (L^2 M \otimes L^2 M)^\mathbb{N}$, δ a closable real derivation. If $B \subset M$ is diffuse (i.e. without minimal projection) von Neumann subalgebra such that ϕ_t converges to Id uniformly on B_1 , one has $\phi_t \rightarrow \text{Id}$ uniformly on $N(B)_1''$.*

Proof. Since B is diffuse, there exists a sequence $(v_n)_{n \in \mathbb{N}}$ in $\mathcal{U}(B)$ ultraweakly convergent to 0 (e.g. $e^{2\pi i n t} \in L^\infty[0, 1]$ for $n \in \mathbb{N}$). For any $u \in N(B)$,

$$\begin{aligned} \|\tilde{\delta}_\alpha(u)\| &\leq \liminf \left\| \tilde{\delta}_\alpha(u) - \zeta_\alpha(v_n) \tilde{\delta}_\alpha(u) \zeta_\alpha(u^* v_n^* u) \right\| \\ &\rightarrow \left\| \tilde{\delta}_\alpha(u) - \tilde{\delta}_\alpha(v_n u u^* v_n^* u) \right\| = 0 \quad (n \rightarrow \infty) \end{aligned}$$

The convergence holds uniformly for u . It remains to apply the following lemma to $N(B) = G$. \square

Lemma 53. *When $\phi_t \rightarrow \text{Id}$ uniformly on $G \subset \mathcal{U}(M)$, we have the uniform convergence $\phi_t \rightarrow \text{Id}$ on G_1'' .*

Proof of the lemma. Let $\phi : M \rightarrow M$ be a τ -symmetric unital completely positive map (hence a contraction). Consider the Stinespring construction on $M \otimes_{\text{alg}} L^2 M$ by $\langle a \otimes x, b \otimes y \rangle = \langle \phi(b^* a)x, y \rangle$. This is positive semi definite by the unital completely

positivity. The M - M -action $a.(c \otimes x).b = ac \otimes xb$ is bounded and induces an M -bimodule structure on the completion.

Now, for $\xi_0 = 1 \otimes \hat{1} \in M \otimes L^2M$,

$$\|a\xi_0 - \xi_0a\|^2 = \tau(\phi(aa^*)) + \tau(aa^*) - 2\Re\tau(\phi(aa^*)) = 2\tau((a - \phi(a))a^*).$$

On the other hand,

$$\frac{1}{2} \|a - \phi(a)\|_2^2 \leq \|a\xi_0 - \xi_0a\| \leq 2 \|a - \phi(a)\|_2 \cdot \|a\|_2.$$

Thus, if $\|u - \phi(u)\| \leq \epsilon$, we have $\|\xi_0 - u\xi_0u^*\| \leq \sqrt{2\epsilon}$. Taking the circumcenter of $\{u\xi_0u^* : u \in G\}$, we get a G -invariant vector η_0 satisfying $\|\xi_0 - \eta_0\| \leq \sqrt{2\epsilon}$ (this is possible by the Ryll-Nardzewski's fixed point theorem). Thus we obtain $\|a\xi_0 - \xi_0a\| \leq 2\sqrt{2\epsilon}$ for $a \in (G'')_1$. \square

APPENDIX A. EMBEDDABILITY OF SUBALGEBRAS

Let $A \subset M$ be an inclusion of finite von Neumann algebras with a trace τ on M . Recall that we have $e_A \in B(L^2M)$, the orthogonal projection onto $L^2A = \overline{A1}$ and the basic extension $\langle M, A \rangle$ of M :

$$\langle M, A \rangle = \vee\mathcal{N}\{M, e_A\} = \left\{ \sum_{\text{finite}} x_i e_A y_i : x_i, y_i \in M \right\}''$$

and the semifinite trace $\text{Tr}(\sum x_i e_A y_i) = \sum \tau(x_i y_i)$ on $\langle M, A \rangle$.

Theorem 54. (Popa) *Let $A \subset M$ be an inclusion of separable finite von Neumann algebras, p a nonzero projection in M , $B \subset pMp$ a von Neumann subalgebra. The followings are equivalent:*

- (1) *There are no sequence $(w_n)_n$ in $\mathcal{U}(B)$ such that $\|E_A(y^*w_nx)\|_2 \rightarrow 0$ for any $x, y \in M$.*
- (2) *There exists a nonzero positive element $d \in \langle M, A \rangle$ of finite trace such that $0 \notin \overline{\text{conv}}^w \{wdw^* : w \in \mathcal{U}(B)\}$*
- (3) *There exists a closed nonzero $B=A$ submodule H of pL^2M such that $\dim_A H_A < \infty$.*
- (4) *There exists a projection e in A , another $0 \neq f$ in B and a normal $*$ -homomorphism $\theta : fBf \rightarrow eAe$ such that there exists a nonzero partial isometry $v \in M$ satisfying $xv = v\theta(x)$ for any $x \in fBf$, and $vv^* \in (fBf)' \cap fMf$, $v^*v \in \theta(fBf)' \cap eMe$.*

Proof. (1) \Rightarrow (2): By assumption there exists a finite set $\mathcal{F} \subset M$ and $\epsilon > 0$ such that

$$\inf_{w \in \mathcal{U}(B)} \sum_{x, y \in \mathcal{F}} \|E_A(y^*wx)\|_2^2 \geq \epsilon.$$

Now, put $d = \sum_{y \in \mathcal{F}} ye_Ay^* \in \langle M, A \rangle_+$. By definition $\text{Tr}(d) < \infty$ and we have

$$\sum_{x \in \mathcal{F}} \langle w^*dw\hat{x}, \hat{x} \rangle = \sum_{x, y \in \mathcal{F}} \langle e_A y^* \widehat{wx}, \widehat{y^*wx} \rangle = \sum_{x, y \in \mathcal{F}} \|E_A(y^*wx)\|_2^2 \geq \epsilon$$

for any $w \in \mathcal{U}(B)$.

(2) \Rightarrow (3): Let \mathcal{C} denote the closed convex hull of $\{wdw^* : w \in \mathcal{U}(B)\}$ in $L^2\langle M, A \rangle$. We can take the circumcenter d_0 of \mathcal{C} which is not equal to zero by (2). Then d_0 is in $B' \cap p\langle M, A \rangle p$ and $\text{Tr}(d_0) \leq \text{Tr}(d) < \infty$. Thus we can take a

nonzero spectral projection q of d_0 such that $\text{Tr}(q) < \infty$. Now, $H = qL^2M$ is a B - A submodule with $\dim_A H_A = \text{Tr}(q)$.

(3) \Rightarrow (4): Fact. When H is a B - A module with $\dim_A H_A < \infty$, there exists a nonzero projection f of B , an fBf - A module $K \subset fH$ such that $K_A \hookrightarrow L^2A_A$ as a right A -module.

Thus, let V denote such an injection $K_A \rightarrow L^2A_A$. When $x \in fBf$, $VxV^* \in \text{End}_A(L^2A_A) = A$. Thus, putting $\theta(x) = VxV^*$, we obtain a normal $*$ -homomorphism (since V is injective) θ of fBf into eAe for $e = VV^*$. Put $\xi = V^*\hat{1} \in K$. Since $V\xi = VV^*\hat{1} = \hat{e}$, $\xi \neq 0$. On the other hand, for any $x \in fBf$,

$$\begin{aligned} x\xi &= V^*VxV^*\hat{1} = V^*\theta(x)\hat{1} \\ &= V^*\hat{1}\theta(x) \quad (\theta(x) \in eAe) \\ &= \xi\theta(x). \end{aligned}$$

Now we are going to investigate

$$\xi \in K \subset fH \subset pL^2M \subset L^2M$$

as a square integrable operator affiliated with M . By above we have $xL^\xi = L_\xi\theta(x)$ for any $x \in \mathcal{U}(fBf)$. Let $v|L_\xi|$ be the polar decomposition of L_ξ . Then

$$|L_\xi|^2 = (xL_\xi)^*(xL_\xi) = (L_\xi\theta(x))^*L_\xi\theta(x) = \theta(x)^*|L_\xi|^2\theta(x)$$

for $x \in \mathcal{U}(fBf)$. Thus $|L_\xi|$ commutes with $\theta(fBf)$. In particular $v^*v = s(|L_\xi|) \in \theta(fBf)' \cap eMe$. Finally,

$$xv|L_\xi| = xL_\xi = L_\xi\theta(x) = v|L_\xi|\theta(x) = v\theta(x)|L_\xi|,$$

which implies $xvv^*v = v\theta(x)v^*v$, i.e. $xv = v\theta(x)$ for any $x \in fBf$.

(4) \Rightarrow (1): Take e, f, v as in (4). Let E_θ denote the conditional expectation $eMe \rightarrow \theta(fBf)$. Then $0 \neq E_\theta(v^*v) \in Z(\theta(fBf))$, $vE_\theta(v^*v)^2v^* \in (fBf)' \cap fMf$. Let $(f_i)_{i \in I}$ be a maximal family of mutually orthogonal nonzero projections satisfying $f_0 = f$ and $f_i \lesssim f$ in B . Thus, $\sum f_i$ is equal to the central support $z_B(f)$ of f in B . Put $u_0 = f$. For each i , take a partial isometry u_i satisfying $u_i u_i^* = f_i$ and $u_i^* u_i \leq f$. Put $v_i = u_i v$. Now we have, for $w \in \mathcal{U}(B)$,

$$\sum_i \|E_A(v_i^* w v_0)\|_2^2 \geq \sum_i \|v v^* E_\theta(v_i^* w v_0)\|_2^2 = \dots = \tau(E_\theta(v^* v)^3) > 0.$$

Since $\sum \|v_i^*\|_2^2 \leq 1$ and $\|E_A(v_i^* w v_0)\|_2 \leq \|v_i^*\|_2$, there exists a finite subset \mathcal{F} of $\{v_i : i \in I\}$ containing v_0 and $\sum_{v_i \notin \mathcal{F}} \|v_i^*\|_2^2 < \tau(E_\theta(v^* v)^3)/2$. \square

Definition 27. Let A and B be von Neumann subalgebras of M . B is said to embed into A inside M when the equivalent conditions of Theorem 54 hold for B and A .

Corollary 55. *If B does not embed into A inside M , there exists a commutative von Neumann subalgebra B_0 of B which does not embed into A inside M . Equivalently, if any commutative subalgebra of B embeds into A , B also embeds into A .*

Remark 28. The above theorem is useful when we have τ -symmetric unital completely positive maps $\phi_i : M \rightarrow M$ which restrict to the identity map on A , giving $\hat{\phi}_i \in \langle M, A \rangle \cap A'$. Often one has $\hat{\phi}_i \in \mathbb{K}\langle M, A \rangle = C^*(x e_A y : x, y \in M)$.

$B \subset M$ is said to be rigid when $\phi_i \rightarrow \text{Id}$ uniformly on the unit ball of B_1 . Then, taking $\phi = \phi_{i_0}$ satisfying

$$\|\phi(b) - b\|_2 < \frac{1}{3} \quad (\forall b \in B_1),$$

$d = \chi_{[\frac{1}{2}, 1]}(\hat{\phi})$ satisfies $\text{Tr}(d) < \infty$ and

$$\|wdw^*\hat{1} - \hat{1}\| \leq \frac{1}{2} + \|w\hat{\phi}w^*\hat{1} - \hat{1}\| = \frac{1}{2} + \|\phi(w^*) - w^*\|_2 \leq \frac{5}{6}.$$

Hence $\overline{\text{conv}}^2 \{wdw^*\}$ does not contain 0, and B embeds into A inside M .

REFERENCES

- [1] F. Cipriani and J.-L. Sauvageot, Derivations as square roots of Dirichlet forms, *J. Funct. Anal.* **201** (2003) 78–120.
- [2] A. Connes, J. Feldman, and B. Weiss, An amenable equivalence relation is generated by a single transformation, *Ergodic Theory Dynamical Systems* **1** (1981) 431–450 (1982).
- [3] D. Gaboriau, On orbit equivalence of measure preserving actions, in *Rigidity in dynamics and geometry (Cambridge, 2000)*, Springer, Berlin, 2002, 167–186.
- [4] W. Lück, Dimension theory of arbitrary modules over finite von Neumann algebras and L^2 -Betti numbers. II. Applications to Grothendieck groups, L^2 -Euler characteristics and Burnside groups, *J. Reine Angew. Math.* **496** (1998) 213–236.