# LOCAL SYSTEMS ON CONFIGURATION SPACES, KZ CONNECTIONS AND CONFORMAL BLOCKS

TOSHITAKE KOHNO

Abstract. We describe a relationship between homological representations of the braid groups due to R. Lawrence, D. Krammer and S. Bigelow and the monodromy representations of the KZ connection. First, we describe the comparison theorem in the case of generic parameters. Then we discuss the non-generic case from the viewpoint of conformal field theory.

## 1. Introduction

The purpose of this paper is to establish a relation between the homological representations of the braid groups and the monodromy representations of the Knizhnik-Zamolodchikov (KZ) connection. The homological representations of the braid groups are defined as the action of the braid groups on the homology of abelian coverings of certain configuration spaces. They were studied by Lawrence [16] in relation with Hecke algebra representations of the braid groups and were extensively investigated by Bigelow [2] and Krammer [15]. It was shown independently by Bigelow and Krammer that they provide faithful representations of braid groups.

On the other hand, it was shown by Schechtman-Varchenko [21] and others that the solutions of the KZ equation are expressed by hypergeometric integrals. There are two parameters $\lambda$ and $\kappa$, which are related to the highest weight and the KZ connection respectively. We consider the KZ equation with values in the space of null vectors in the tensor product of Verma modules of $sl_2(\mathbf{C})$ and show that a specialization of the homological representation is equivalent to the monodromy representation of such KZ equation for generic parameters $\lambda$ and $\kappa$. A complete statement is given in Theorem 7.1. We describe a sufficient condition for the parameters to be generic so that the statement of the theorem holds. This result was studied in [12], [13] and [14]. We first review the case of generic parameters investigated in these works.

There is other approach due to Marin [17] expressing representations of the braid groups and their generalizations such as Artin groups as the monodromy of integrable connections by an infinitesimal method. Our approach depends on integral representations of the solutions of the KZ equation and is different from Marin's method.

The case of non-generic parameters is important from the viewpoint of conformal field theory (see [7], [22] and [23]). There is a period integral map from the homology of local systems over the configuration spaces to the space of conformal blocks. We describe the kernel of this map to describe the fusion rule in conformal field theory by means of homology of local systems.

---

2000 *Mathematics Subject Classification.* 52C35, 20F36, 33C60, 17B37, 17B67.

*Key words and phrases.* braid groups, hyperplane arrangements, configuration spaces, hypergeometric integrals, quantum groups, conformal field theory.

The paper is organized in the following way. In Section 2 we review fundamental results on the homology of local systems on the complement of hyperplane arrangements. In Section 3 we recall the construction of the homological representations of braid groups. In Section 4 we describe the homology of local systems on configuration spaces including the case of non-generic parameters. We recall the definition of the KZ equation in Section 5 and describe its solutions by hypergeometric integrals in Section 6. Section 7 is devoted to the comparison theorem for homological representations and the holonomy of KZ connections in the case of generic parameters. Finally in Section 8 we describe the case of non-generic parameters in relation with conformal field theory.

## 2. Local systems on the complement of hyperplane arrangement

Let $\mathcal{A} = \{H_1, \cdots, H_\ell\}$ be an arrangement of affine hyperplanes in the complex vector space $\mathbf{C}^n$. We consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

Let us assume that the hyperplanes $H_1, \cdots, H_\ell$ are defined over $\mathbf{R}$. In this case $H \in \mathcal{A}$ is regarded as a complexification of the real hyperplane $H_{\mathbf{R}}$ in $V_{\mathbf{R}} = \mathbf{R}^n$. The complement $V_{\mathbf{R}} \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbf{R}}$ consists of finitely many connected components called chambers.

The above real hyperplane arrangement $\{H_{\mathbf{R}}\}_{H \in \mathcal{A}}$ determines a natural stratification $S$ of $\mathbf{R}^n$, whose stratum is called a facet. For facets $E$ and $F$ we shall say that $E > F$ if and only if $\overline{E} \supset F$ holds. For an increasing sequence of facets $F_{j_0} < \cdots < F_{j_p}$ we take a point $v_{j_k}$ in each facet $F_{j_k}$, $0 \leq k \leq p$, and consider the simplex spanned by the vertices $v_{j_k}$, $0 \leq k \leq p$. This simplex defined for $F_{j_0} < \cdots < F_{j_p}$ is denoted by

$$\sigma(F_{j_0} < \cdots < F_{j_p}).$$

For a facet $F$ the dual cell is defined by

$$D(F) = \bigcup \sigma(F^i < F^{i-1} < \cdots < F^0)$$

where the union is for all the increasing sequences of facets $F^i < F^{i-1} < \cdots < F^0$ with $F^i = F$ and $\operatorname{codim} F^j = j$.

Let $\pi : M(\mathcal{A}) \longrightarrow \mathbf{R}^n$ be the projection corresponding to the real part. A facet decomposition of the complexified complement $M(\mathcal{A})$ is given by

$$\bigcup_F \pi^{-1}(F).$$

The associated dual complex is called the Salvetti complex $S(\mathcal{A})$, which is an $n$ dimensional CW complex. It was shown by M. Salvetti [20] that the inclusion

$$S(\mathcal{A}) \longrightarrow M(\mathcal{A})$$

is a homotopy equivalence.

First, we recall some basic definition for local systems. Let $M$ be a smooth manifold and $V$ a complex vector space. Given a linear representation of the fundamental group

$$r : \pi_1(M, x_0) \longrightarrow GL(V)$$

there is an associated flat vector bundle $E$ over $M$. The local system $\mathcal{L}$ associated to the representation $r$ is the sheaf of horizontal sections of the flat bundle $E$. Let $\pi : \widetilde{M} \to M$ be the universal covering. We denote by $\mathbf{Z}\pi_1$ the group ring of the fundamental group $\pi_1(M, x_0)$. We consider the chain complex

$$C_*(\widetilde{M}) \otimes_{\mathbf{Z}\pi_1} V$$

with the boundary map defined by $\partial(c \otimes v) = \partial c \otimes v$. Here $\mathbf{Z}\pi_1$ acts on $C_*(\widetilde{M})$ via the deck transformations and on $V$ via the representation $r$. The homology of this chain complex is called the homology of $M$ with coefficients in the local system $\mathcal{L}$ and is denoted by $H_*(M, \mathcal{L})$.

Let $\mathcal{L}$ be a complex rank one local system over $M(\mathcal{A})$ associated with a representation of the fundamental group

$$r : \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*.$$

For an arrangement $\mathcal{A} = \{H_1, \cdots, H_\ell\}$ We denote by $f_j$ be a linear form defining the hyperplane $H_j$, $1 \le j \le \ell$. We associate a complex number $a_j = a(H_j)$ called an exponent to each hyperplane and consider a multivalued function

$$\Phi = f_1^{a_1} \cdots f_\ell^{a_\ell}.$$

The homology $H_1(M(\mathcal{A}); \mathbf{Z})$ is isomorphic to $\mathbf{Z}^{\oplus \ell}$, where each generator corresponds to a hyperplane. By associating to the generator of $H_1(M(\mathcal{A}); \mathbf{Z})$ corresponding to the hyperplane $H_j$ the complex number $e^{2\pi\sqrt{-1}a_j}$ we obtain a homomorphism $H_1(M(\mathcal{A}); \mathbf{Z}) \to \mathbf{C}^*$. Combining with the abelianization map $\pi_1(M(\mathcal{A}), x_0) \to H_1(M(\mathcal{A}); \mathbf{Z})$ we obtain a homomorphism

$$\rho_\Phi : \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*.$$

The associated local system is denoted by $\mathcal{L}_\Phi$.

We shall investigate $H_*(M(\mathcal{A}), \mathcal{L})$ the homology of $M(\mathcal{A})$ with coefficients in the local system $\mathcal{L}$. For our purpose the homology of locally finite chains $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$ also plays an important role. It was shown by Z. Chen [4] that the complex associated with the facet decomposition $\bigcup_F \pi^{-1}(F)$ of $M(\mathcal{A})$ can be used to compute the homology of locally finite chains $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$.

We briefly summarize basic properties of the above homology groups. Let $\mathcal{A}$ be an essential hyperplane arrangement. Namely, we suppose that maximal codimension of a non-empty intersection of some subfamily of $\mathcal{A}$ is equal to $n$. We choose a smooth compactification $i : M(\mathcal{A}) \longrightarrow X$. Namely, $M(\mathcal{A})$ is written as $X \setminus D$, where $X$ is a smooth projective variety and $D$ is a divisor with normal crossings. For the local system $\mathcal{L}$ we consider the Leray spectral sequence.

$$E_2^{p,q} = H^p(X, R^q i_* \mathcal{L}) \Longrightarrow H^{p+q}(M(\mathcal{A}), \mathcal{L}).$$

We shall say that the local system $\mathcal{L}$ is generic if and only if there is an isomorphism

$$i_*\mathcal{L} \cong i_!\mathcal{L}$$

where $i_*$ is the direct image and $i_!$ is the extension by 0. This means that the monodromy of $\mathcal{L}$ along any divisor at infinity is not equal to 1. The following theorem was shown in [9].

**Theorem 2.1.** *If the local system $\mathcal{L}$ is generic in the above sense, then there is an isomorphism*

$$H_*(M(\mathcal{A}),\mathcal{L}) \cong H_*^{lf}(M(\mathcal{A}),\mathcal{L})$$

*We have $H_k(M(\mathcal{A}),\mathcal{L}) = 0$ for any $k \neq n$.*

*Proof.* By the hypothesis that the local system $\mathcal{L}$ is generic we have

$$R^q i_*\mathcal{L} = 0$$

for $q > 0$. The Leray spectral sequence degenerates at $E_2$-term and we have

$$E_2^{p,0} \cong E_\infty^{p,0} = H^p(M(\mathcal{A}),\mathcal{L}),$$

where $E_2^{p,0} = H^p(X, i_*\mathcal{L})$. Thus we obtain an isomorpshim

$$H^*(X, i_*\mathcal{L}) \cong H^*(M(\mathcal{A}),\mathcal{L}).$$

On the other hand, there is an isomorpshim

$$H^*(X, i_!\mathcal{L}) \cong H_c^*(M(\mathcal{A}),\mathcal{L})$$

where $H_c^*$ denotes cohomology with compact supports.

There are Poincaré duality isomorphisms:

$$H_k^{lf}(M(\mathcal{A}),\mathcal{L}) \cong H^{2n-k}(M(\mathcal{A}),\mathcal{L})$$
$$H_k(M(\mathcal{A}),\mathcal{L}) \cong H_c^{2n-k}(M(\mathcal{A}),\mathcal{L}).$$

By the hypothesis $i_*\mathcal{L} \cong i_!\mathcal{L}$ we obtain an isomorphism

$$H_k^{lf}(M(\mathcal{A}),\mathcal{L}) \cong H_k(M(\mathcal{A}),\mathcal{L}).$$

It follows from the above Poincaré duality isomorpshims and the fact that $M(\mathcal{A})$ has a homotopy type of a CW complex of dimension $n$ we have

$$H_k^{lf}(M(\mathcal{A}),\mathcal{L}) \cong 0, \quad k < n$$
$$H_k(M(\mathcal{A}),\mathcal{L}) \cong 0, \quad k > n.$$

Therefore we obtain $H_k(M(\mathcal{A}),\mathcal{L}) = 0$ for any $k \neq n$. □

Let us suppose that each hyperplane in $\mathcal{A}$ is defined over $\mathbf{R}$. We set $M(\mathcal{A})_\mathbf{R} = M(\mathcal{A}) \cap \mathbf{R}^n$ and denote by $\Delta_\nu$, $1 \leq \nu \leq s$, the bounded chambers in $M(\mathcal{A})_\mathbf{R}$. We denote by $\overline{\Delta}_\nu$ the closure of $\Delta_\nu$ in $X \setminus D$. Let

$$j : M(\mathcal{A}) \setminus \cup_\nu \Delta_\nu \longrightarrow X$$

be the inclusion map. We denote by $\mathcal{L}_0$ the restriction of the local system $\mathcal{L}$ on $M(\mathcal{A}) \setminus \cup_j \Delta_j$. In this situation we have the following theorem.

**Theorem 2.2.** *In addition to the condition that the local system $\mathcal{L}$ is generic we suppose that there is an isomorphism*

$$j_*\mathcal{L}_0 \cong j_!\mathcal{L}_0.$$

*Then the homology with locally finite chains $H_n^{lf}(M(\mathcal{A}),\mathcal{L})$ is spanned by the homology class of bounded chambers $\Delta_\nu$, $1 \leq \nu \leq s$.*

*Proof.* We put $\overline{\Delta} = \cup_\nu \overline{\Delta}_\nu$. Let us consider the homology exact sequence of the triple $(X, D \cup \overline{\Delta}, D)$ with the local system coefficient $\mathcal{L}$:

$$\longrightarrow H_p(D \cup \overline{\Delta}, D) \longrightarrow H_p(X, D) \longrightarrow H_p(X, D \cup \overline{\Delta}) \longrightarrow H_{p-1}(D \cup \overline{\Delta}, D) \longrightarrow \cdots$$

where $\mathcal{L}$ is extended by 0 on $D \cup \overline{\Delta}$. Since we have

$$H_p(X, D) \cong H_p^{lf}(M(\mathcal{A}))$$

there is a long exact sequence

$$\longrightarrow H_p^{lf}(\cup_\nu \Delta_\nu) \longrightarrow H_p^{lf}(M(\mathcal{A})) \longrightarrow H_p^{lf}(M(\mathcal{A}) \setminus \cup_j \Delta_j) \longrightarrow H_{p-1}^{lf}(\cup_\nu \Delta_\nu) \longrightarrow \cdots$$

with the local system coefficient $\mathcal{L}$. By the same argument as in the proof of Theorem 2.1 we have an isomorphism

$$H_k(M(\mathcal{A}) \setminus \cup_\nu \Delta_\nu, \mathcal{L}_0) \cong H_k^{lf}(M(\mathcal{A}) \setminus \cup_\nu \Delta_\nu, \mathcal{L}_0)$$

for any $k$ and the vanishing

$$H_k(M(\mathcal{A}) \setminus \cup_\nu \Delta_\nu, \mathcal{L}_0) \cong 0$$

for any $k$ with $k \neq n$. Here we use the theorem of Zaslawsky saying that the number of bounded chambers is equal to the absolute value of the Euler-Poincaré characteristic $|\chi(M(\mathcal{A})|$ to conclude that the above vanishing holds for any $k$. Combining with the above exact sequence, we obtain an isomorphism

$$H_n^{lf}(\cup_\nu \Delta_\nu, \mathcal{L}) \cong H_n^{lf}(M(\mathcal{A}), \mathcal{L}).$$

This leads to the statement of the theorem. □

## 3. Homological representations of braid groups

We denote by $B_n$ the braid group with $n$ strands. We fix a positive integer $n$ and a set of distinct $n$ points in $\mathbf{R}^2$ as

$$Q = \{(0,0), \cdots, (n-1, 0)\},$$

where we set $p_\ell = (\ell - 1, 0)$, $\ell = 1, \cdots, n$. We take a 2-dimensional disk in $\mathbf{R}^2$ containing $Q$ in the interior. We fix a positive integer $m$ and consider the configuration space of ordered distinct $m$ points in $\Sigma = D \setminus Q$ defined by

$$\mathcal{F}_m(\Sigma) = \{(t_1, \cdots, t_m) \in \Sigma \ ; \ t_i \neq t_j \text{ if } i \neq j\},$$

which is also denoted by $\mathcal{F}_{n,m}(D)$. The symmetric group $\mathfrak{S}_m$ acts freely on $\mathcal{F}_m(\Sigma)$ by the permutations of distinct $m$ points. The quotient space of $\mathcal{F}_m(\Sigma)$ by this action is by definition the configuration space of unordered distinct $m$ points in $\Sigma$ and is denoted by $\mathcal{C}_m(\Sigma)$. We also denote this configuration space by $\mathcal{C}_{n,m}(D)$.

In the original papers by Bigelow [2], [3] and by Krammer [15] the case $m = 2$ was extensively studied, but for our purpose it is convenient to consider the case when $m$ is an arbitrary positive integer such that $m \geq 2$.

We identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$. The quotient space $\mathbf{C}^m/\mathfrak{S}_m$ defined by the action of $\mathfrak{S}_m$ by the permutations of coordinates is analytically isomorphic to $\mathbf{C}^m$ by means of the elementary symmetric polynomials. Now the image of the hyperplanes defined by $t_i = p_\ell$, $\ell = 1, \cdots, n$, and the diagonal hyperplanes $t_i = t_j$, $1 \leq i \leq j \leq m$, are complex codimension one irreducible subvarieties of the quotient space $D^m/\mathfrak{S}_m$. This allows us to give a description of the first homology group of $\mathcal{C}_{n,m}(D)$ as

$$H_1(\mathcal{C}_{n,m}(D);\mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z} \tag{3.1}$$

where the first $n$ components correspond to meridians of the images of hyperplanes $t_i = p_\ell$, $\ell = 1, \cdots, n$, and the last component corresponds to the meridian of the image of the diagonal hyperplanes $t_i = t_j$, $1 \le i \le j \le m$, namely, the discriminant set. We consider the homomorphism

$$\alpha : H_1(\mathcal{C}_{n,m}(D);\mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \tag{3.2}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$. Composing with the abelianization map $\pi_1(\mathcal{C}_{n,m}(D), x_0) \to H_1(\mathcal{C}_{n,m}(D);\mathbf{Z})$, we obtain the homomorphism

$$\beta : \pi_1(\mathcal{C}_{n,m}(D), x_0) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}. \tag{3.3}$$

Let $\pi : \widetilde{\mathcal{C}}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$ be the covering corresponding to $\operatorname{Ker}\beta$. Now the group $\mathbf{Z} \oplus \mathbf{Z}$ acts as the deck transformations of the covering $\pi$ and the homology group $H_*(\widetilde{\mathcal{C}}_{n,m}(D);\mathbf{Z})$ is considered to be a $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$-module. Here $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ stands for the group ring of $\mathbf{Z} \oplus \mathbf{Z}$. We express $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ as the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. We consider the homology group

$$H_{n,m} = H_m(\widetilde{\mathcal{C}}_{n,m}(D);\mathbf{Z})$$

as an $R$-module by the action of the deck transformations.

As is explained in the case of $m = 2$ in [2] it can be shown that $H_{n,m}$ is a free $R$-module of rank

$$d_{n,m} = \begin{pmatrix} m+n-2 \\ m \end{pmatrix}. \tag{3.4}$$

A basis of $H_{n,m}$ as a free $R$-module is discussed in relation with the homology of local systems in the next sections. Let $\mathcal{M}(D, Q)$ denote the mapping class group of the pair $(D, Q)$, which consists of the isotopy classes of homeomorphisms of $D$ which fix $Q$ setwise and fix the boundary $\partial D$ pointwise. The braid group $B_n$ is naturally isomorphic to the mapping class group $\mathcal{M}(D, Q)$. Now a homeomorphism $f$ representing a class in $\mathcal{M}(D, Q)$ induces a homeomorphism $\tilde{f} : \mathcal{C}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$, which is uniquely lifted to a homeomorphism of $\widetilde{\mathcal{C}}_{n,m}(D)$. This homeomorpshim commutes with the deck transformations.

Therefore, for $m \ge 2$ we obtain a representation of the braid group

$$\rho_{n,m} : B_n \longrightarrow \operatorname{Aut}_R H_{n,m} \tag{3.5}$$

which is called the homological representation of the braid group or the Lawrence-Krammer-Bigelow (LKB) representation. Let us remark that in the case $m = 1$ the above construction gives the reduced Burau representation over $\mathbf{Z}[q^{\pm 1}]$.

## 4. Homology of local systems on configuration spaces

Let us consider the configuration space of ordered distinct $n$ points in the complex plane defined by

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}.$$

The configuration space $X_n$ is also denoted by $\mathcal{F}_n(\mathbf{C})$ as in the previous section. The fundamental group of $X_n$ is the pure braid group with $n$ strands denoted by $P_n$. For a positive integer $m$ we consider the projection map

$$\pi_{n,m} : X_{n+m} \longrightarrow X_n \tag{4.1}$$

given by $\pi_{n,m}(z_1, \cdots, z_n, t_1, \cdots, t_m) = (z_1, \cdots, z_n)$, which defines a fiber bundle over $X_n$. For $p \in X_n$ the fiber $\pi_{n,m}^{-1}(p)$ is denoted by $X_{n,m}$, which is also written as $\mathcal{F}_{n,m}(\mathbf{C})$ Let $(z_1, \cdots, z_n)$ be the coordinates for $p$. Then, $X_{n,m}$ is the complement of hyperplanes defined by

$$t_i = z_\ell, \quad 1 \leq i \leq m, \ 1 \leq \ell \leq n, \quad t_i = t_j, \quad 1 \leq i < j \leq m. \tag{4.2}$$

We call these hyperplanes $H_{i\ell}$, $1 \leq i \leq m, \ 1 \leq \ell \leq n$, and $D_{ij}$, $1 \leq i < j \leq m$. Such arrangement of hyperplanes is called a discriminantal arrangement. The symmetric group $\mathfrak{S}_m$ acts on $X_{n,m}$ by the permutations of the coordinates functions $t_1, \cdots, t_m$. We put $Y_{n,m} = X_{n,m}/\mathfrak{S}_m$, which is also denoted by $\mathcal{C}_{n,m}(\mathbf{C})$.

Identifying $\mathbf{R}^2$ with the complex plane $\mathbf{C}$, we have the inclusion map

$$\iota : \mathcal{F}_{n,m}(D) \longrightarrow X_{n,m}, \tag{4.3}$$

which is a homotopy equivalence. By taking the quotient by the action of the symmetric group $\mathfrak{S}_m$, we have the inclusion map

$$\bar{\iota} : \mathcal{C}_{n,m}(D) \longrightarrow Y_{n,m}, \tag{4.4}$$

which is also a homotopy equivalence.

We fix $p = (z_1, z_2 \cdots, z_n)$ as a base point. We consider a rank one local system $\mathcal{L}$ associated with a representation

$$r : \pi_1(X_{n,m}, x_0) \longrightarrow \mathbf{C}^*.$$

Let us consider the compactification

$$i_0 : X_{n,m} \longrightarrow (\mathbf{C}P^1)^m = \underbrace{\mathbf{C}P^1 \times \cdots \times \mathbf{C}P^1}_{m}.$$

We denote by $H_{i\infty}$ the hyperplane defined by $t_i = \infty$ for $1 \leq \ell \leq n$. We have

$$X_{n,m} = (\mathbf{C}P^1)^m \setminus (\cup_{1 \leq i \leq m, \ 1 \leq \ell \leq n} H_{i\ell}) \cup (\cup_{1 \leq i \leq m} H_{i\infty}) \cup (\cup_{1 \leq i \leq m} D_{ij})$$

Then we take blowing-ups at multiple points $\pi : \widehat{(\mathbf{C}P^1)^m} \longrightarrow (\mathbf{C}P^1)^m$ and obtain a smooth compactification $i : X_{n,m} \to \widehat{(\mathbf{C}P^1)^m}$ with normal crossing divisors. We are able to write down the condition $i_*\mathcal{L} \cong i_!\mathcal{L}$ explicitly by computing the monodromy of the local system $\mathcal{L}$ along divisors at infinity.

We consider the local system associated with the multivalued function of the form

$$\Phi = \prod_{1 \leq i \leq m, 1 \leq \ell \leq n} (t_i - z_\ell)^{\alpha_\ell} \prod_{1 \leq i < j \leq m} (t_i - t_j)^{2\gamma}. \tag{4.5}$$

The local system $\mathcal{L}$ on $X_{n,m}$ is invariant under the action of the symmetric group $\mathfrak{S}_m$ and induces the local system $\overline{\mathcal{L}}$ on $Y_{n,m}$.

We have the following proposition.

**Proposition 4.1.** *There is an open dense subset $V$ in $\mathbf{C}^{\ell+1}$ such that for $(\alpha_1, \cdots \alpha_\ell, \gamma) \in V$ the associated local system $\overline{\mathcal{L}}$ on $Y_{n,m}$ saitisfies*

$$H_*(Y_{n,m}, \overline{\mathcal{L}}) \cong H_*^{lf}(Y_{n,m}, \overline{\mathcal{L}})$$

*and $H_k(Y_{n,m},\overline{\mathcal{L}})=0$ for any $k\neq m$. Moreover, we have*

$$\dim H_m(Y_{n,m},\overline{\mathcal{L}}^*)=d_{n,m}, \tag{4.6}$$

*where we use the same notation as in equation (3.4) for $d_{n,m}$.*

*Proof.* We see that $Y_{n,m}$ is the complement of hypersurfaces in $\mathbf{C}^m$. We consider the embedding

$$i_0 : Y_{n,m} \longrightarrow \mathcal{S}^m\mathbf{C}P^1 \tag{4.7}$$

where $\mathcal{S}^m\mathbf{C}P^1$ is the symmetric product defined as $(\mathbf{C}P^1)^m/\mathfrak{S}_m$. We observe that $\mathcal{S}^m\mathbf{C}P^1$ is a smooth complex manifold. Now by taking blowing-ups we have a smooth compactification

$$i : Y_{n,m} \longrightarrow \widehat{\mathcal{S}^m\mathbf{C}P^1} \tag{4.8}$$

with normal crossing divisors. Let us remark that the argument of the proof of Theorem 2.1 can be applied to this situation and we have an isomorphism $H_*(Y_{n,m},\overline{\mathcal{L}})\cong H_*^{lf}(Y_{n,m},\overline{\mathcal{L}})$ and the vanishing $H_k(Y_{n,m},\mathcal{L})=0$ for $k\neq m$ if the condition $i_*\overline{\mathcal{L}}\cong i_!\overline{\mathcal{L}}$ is satisfied. Actually, by the Lefschetz hyperplane section theorem it is enough to verify the condition for a generic 2 dimensional section. In this case by expressing the monodromy along divisors with normal crossings at infinity by the parameter $(\alpha_1,\cdots\alpha_\ell,\gamma)$ we can verify that the condition $i_*\overline{\mathcal{L}}\cong i_!\overline{\mathcal{L}}$ is satisfied for $(\alpha_1,\cdots\alpha_\ell,\gamma)\in\mathbf{C}^{\ell+1}$ in an open dense subset of $\mathbf{C}^{\ell+1}$. The dimension formula for $H_m(Y_{n,m},\overline{\mathcal{L}}^*)$ follows from the calculation of the Euler-Poincaré characteristic of $Y_{n,m}$. □

For the purpose of describing the homology group $H_m^{lf}(X_{n,m},\mathcal{L})$ and $H_m^{lf}(Y_{n,m},\overline{\mathcal{L}})$ we introduce the following notation. We take the base point $p=(1,\cdots,n)$. For non-negative integers $m_1,\cdots,m_{n-1}$ satisfying

$$m_1+\cdots+m_{n-1}=m$$

we define a bounded chamber $\Delta_{m_1,\cdots,m_{n-1}}$ in $\mathbf{R}^m$ by

$$\begin{aligned}
&1<t_1<\cdots<t_{m_1}<2\\
&2<t_{m_1+1}<\cdots<t_{m_1+m_2}<3\\
&\cdots\\
&n-1<t_{m_1+\cdots+m_{n-2}+1}+\cdots+t_m<n.
\end{aligned}$$

We put $M=(m_1,\cdots,m_{n-1})$ and we write $\Delta_M$ for $\Delta_{m_1,\cdots,m_{n-1}}$. We denote by $\overline{\Delta}_M$ the image of $\Delta_M$ by the projection map $\pi_{n,m}$. The bounded chamber $\Delta_M$ defines a homology class $[\Delta_M]\in H_m^{lf}(X_{n,m},\mathcal{L})$ and its image $\overline{\Delta}_M$ defines a homology class $[\overline{\Delta}_M]\in H_m^{lf}(Y_{n,m},\overline{\mathcal{L}})$. We shall show in Section 7 that under certain generic conditions $[\overline{\Delta}_M]$ for $M=(m_1,\cdots,m_{n-1})$ with $m_1+\cdots+m_{n-1}=m$ form a basis of $H_m^{lf}(Y_{n,m},\overline{\mathcal{L}})$.

As we have shown in Theorem 2.1 there is an isomorphism $H_m(X_{n,m},\mathcal{L})\cong H_m^{lf}(X_{n,m},\mathcal{L})$ if the condition $i_*\mathcal{L}\cong i_!\mathcal{L}$ is satisfied. In this situation we denote by $[\widetilde{\Delta}_M]$ the homology class in $H_m(X_{n,m},\mathcal{L})$ corresponding to $[\Delta_M]$ in the above isomorphism and call $[\widetilde{\Delta}_M]$ the regularized cycle for $[\Delta_M]$. In general regularized cycles can be constructed by means of the boundary of the tubular neighborhood of divisors at infinity. We refer the reader to [1] for more details about this subject.

For the purpose of dealing with non-generic case we need some refined criteria for the vanishing of cohomology of local systems. For the multivalued function $\Phi$ given in equation (4.5) we set

$$\xi_\ell = e^{2\pi\sqrt{-1}\alpha_\ell}, 1 \le \ell \le n, \quad \delta = e^{4\pi\sqrt{-1}\gamma}.$$

We consider the local system $\mathcal{L}$ over $X_{n,m}$ associated with $\Phi$ The monodromy of $\mathcal{L}$ around the hyperplanes $H_{i\ell}$ and $D_{ij}$ is $\xi_\ell$ and $\delta$ respectively. We denote by $\xi_\infty$ the monodromy around $H_{i\infty}$.

The following proposition is due to R. Silvotti [22].

**Proposition 4.2** (Silvotti [22]). *If there exists at least one $\ell \in \{1, \cdots, n, \infty\}$ such that*

$$\xi_\ell^k \delta^{\frac{1}{2}k(k-1)} \ne 1$$

*holds for $1 \le k \le m$, then*

$$H^i(X_{n,m}, \mathcal{L}) \cong 0$$

*for $i \ne m$.*

We consider the case $n = 1$, namely, the configuration space $X_{1,m} = \mathcal{F}_m(\mathbf{C} \setminus \{0\})$. We write $\xi$ for $\xi_1$. The following Proposition4.3 and Proposition4.4 are also due to R. Silvotti [22]. We provide the proofs of the reader's convenience.

**Proposition 4.3.** *The vanishing of cohomology*

$$H^i(X_{1,m}, \mathcal{L}) \cong 0$$

*holds for any $i$ if and only if the condition*

$$\xi^m \delta^{\frac{1}{2}m(m-1)} \ne 1$$

*is satisfied.*

*Proof.* The case $m = 1$ is clear. We write $\mathbf{C}^*$ for $\mathbf{C} \setminus \{0\}$ and consider the map

$$\sigma : \mathbf{C}^* \times \mathcal{F}_{m-1}(\mathbf{C} \setminus \{0, 1\}) \longrightarrow \mathcal{F}_m(\mathbf{C} \setminus \{0\})$$

defined by

$$\sigma(w; w_1, \cdots w_{m-1}) = (w, ww_1, \cdots, ww_{m-1}).$$

We see that the map $\sigma$ is a diffeomorphism and there is an isomorphism

$$H^*(\mathcal{F}_m(\mathbf{C} \setminus \{0\}), \mathcal{L}) \cong H^*(\mathcal{F}_{m-1}(\mathbf{C} \setminus \{0, 1\}), \sigma^*\mathcal{L})$$

The monodromy of the local system $\sigma^*\mathcal{L}$ around $w = 0$ is $\xi^m \delta^{\frac{1}{2}m(m-1)}$. Let us suppose that $\xi^m \delta^{\frac{1}{2}m(m-1)} \ne 1$. We denote the inclusion maps by

$$i_1 : \mathbf{C}^* \longrightarrow \mathbf{C}^* \times \mathcal{F}_{m-1}(\mathbf{C} \setminus \{0, 1\})$$
$$i_2 : \mathcal{F}_{m-1}(\mathbf{C} \setminus \{0, 1\}) \longrightarrow \mathbf{C}^* \times \mathcal{F}_{m-1}(\mathbf{C} \setminus \{0, 1\})$$

By the Künneth formula there is an isomorphism

$$\begin{aligned} &H^j(\mathbf{C}^* \times \mathcal{F}_{m-1}(\mathbf{C} \setminus \{0, 1\}), \sigma^*\mathcal{L}) \\ &\cong \otimes_{p+q=j} H^p(\mathbf{C}^*, i_1^*\sigma^*\mathcal{L}) \otimes H^q(\mathcal{F}_{m-1}(\mathbf{C} \setminus \{0, 1\}), i_2^*\sigma^*\mathcal{L}). \end{aligned}$$

By the hypothesis $\xi^m \delta^{\frac{1}{2}m(m-1)} \ne 1$ the local system $i_1^*\sigma^*\mathcal{L}$ on $\mathbf{C}^*$ is non-trivial and we have

$$H^p(\mathbf{C}^*, i_1^*\sigma^*\mathcal{L}) \cong 0$$

for any $p$. This shows that
$$H^i(X_{1,m},\mathcal{L}) \cong 0$$
for any $i$. Conversely, if $\xi^m \delta^{\frac{1}{2}m(m-1)} = 1$, then the local system $i_1^*\sigma^*\mathcal{L}$ on $\mathbf{C}^*$ is trivial and we have
$$H^p(\mathbf{C}^*, i_1^*\sigma^*\mathcal{L}) \cong \begin{cases} \mathbf{C}, & p = 0,1 \\ 0, & p \neq 0,1. \end{cases}$$
Since $\chi(\mathcal{F}_{m-1}(\mathbf{C}\backslash\{0,1\}) \neq 0$ the cohomology $H^*(\mathcal{F}_m(\mathbf{C}\backslash\{0\}),\mathcal{L})$ cannot vanish. $\square$

**Proposition 4.4.** *If* $\xi^m \delta^{\frac{1}{2}m(m-1)} = 1$ *and*
$$\xi^k \delta^{\frac{1}{2}k(k-1)} \neq 1$$
*for any $k$ with $1 \leq k \leq m-1$, then*
$$H^i(X_{1,m},\mathcal{L}) \cong 0, \quad i \neq m-1, m$$
*holds. Moreover, we have*
$$\dim H^{m-1}(X_{1,m},\mathcal{L}) = \dim H^m(X_{1,m},\mathcal{L}) = (m-1)!.$$

*Proof.* We use the notation in the proof of the previous proposition. By the hypothesis the local system $i_1^*\sigma^*\mathcal{L}$ on $\mathbf{C}^*$ is trivial. By using the fact that $|\chi(X_{2,m-1}| = (m-1)!$ and Proposition 4.2, we obtain
$$H^q(\mathcal{F}_{m-1}(\mathbf{C}\setminus\{0,1\}), i_1^*\sigma^*\mathcal{L}) \cong \begin{cases} \mathbf{C}^{(m-1)!}, & q = m-1 \\ 0, & q \neq m-1 \end{cases}$$
Combining with the Künneth formula, we obtain the desired statement. $\square$

Under the same hypothesis as in Proposition 4.4 we can show for $Y_{1,m} = X_{1,m}/\mathfrak{S}_m$ that
$$H^i(Y_{1,m},\overline{\mathcal{L}}) \cong 0, \quad i \neq m-1, m$$
and that
$$\dim H^{m-1}(Y_{1,m},\overline{\mathcal{L}}) = \dim H^m(Y_{1,m},\overline{\mathcal{L}}) = 1$$
by a similar argument.

## 5. KZ connection

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. We set $\Omega = \sum_\mu I_\mu \otimes I_\mu$. Let $r_i : \mathfrak{g} \to \mathrm{End}(V_i)$, $1 \leq i \leq n$, be representations of the Lie algebra $\mathfrak{g}$. We denote by $\Omega_{ij}$ the action of $\Omega$ on the $i$-th and $j$-th components of the tensor product $V_1 \otimes \cdots \otimes V_n$. It is known that the Casimir element $c = \sum_\mu I_\mu \cdot I_\mu$ lies in the center of the universal enveloping algebra $U\mathfrak{g}$. Let us denote by $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ the coproduct, which is defined to be the algebra homomorphism determined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$. Since $\Omega$ is expressed as $\Omega = \frac{1}{2}(\Delta(c) - c \otimes 1 - 1 \otimes c)$ we have the relation

$$[\Omega, x \otimes 1 + 1 \otimes x] = 0 \tag{5.1}$$

for any $x \in \mathfrak{g}$ in the tensor product $U\mathfrak{g} \otimes U\mathfrak{g}$. By means of the above relation it can be shown that the infinitesimal pure braid relations:

$$[\Omega_{ik}, \Omega_{ij} + \Omega_{jk}] = 0, \quad (i,j,k \ \text{ distinct}), \tag{5.2}$$

$$[\Omega_{ij}, \Omega_{k\ell}], \quad (i,j,k,\ell \ \text{ distinct}) \tag{5.3}$$

hold. Let us briefly explain the reason why we have the above infinitesimal pure braid relations. For the first relation it is enough to show the case $i = 1, j = 3, k = 2$. Since we have

$$[\Omega \otimes 1, (I_\mu \otimes 1 + 1 \otimes I_\mu) \otimes I_\mu] = 0$$

by the equation (5.1) we obtained the desired relation. The equation (4.3) in the infinitesimal pure braid relations is clear from the definition of $\Omega$ on the tensor product.

We define the Knizhnik-Zamolodchikov (KZ) connection as the 1-form

$$\omega = \frac{1}{\kappa} \sum_{1 \le i < j \le n} \Omega_{ij} d \log(z_i - z_j) \tag{5.4}$$

with values in $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ for a non-zero complex parameter $\kappa$.

We set $\omega_{ij} = d \log(z_i - z_j)$, $1 \le i, j \le n$. It follows from the above infinitesimal pure braid relations among $\Omega_{ij}$ together with Arnold's relation

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{k\ell} + \omega_{k\ell} \wedge \omega_{ij} = 0 \tag{5.5}$$

that $\omega \wedge \omega = 0$ holds. This implies that $\omega$ defines a flat connection for a trivial vector bundle over the configuration space $X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}$ with fiber $V_1 \otimes \cdots \otimes V_n$. A horizontal section of the above flat bundle is a solution of the total differential equation

$$d\varphi = \omega\varphi \tag{5.6}$$

for a function $\varphi(z_1, \cdots, z_n)$ with values in $V_1 \otimes \cdots \otimes V_n$. This total differential equation can be expressed as a system of partial differential equations

$$\frac{\partial \varphi}{\partial z_i} = \frac{1}{\kappa} \sum_{j, j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \varphi, \quad 1 \le i \le n, \tag{5.7}$$

which is called the KZ equation. The KZ equation was first introduced in [8] as the differential equation satisfied by $n$-point functions in Wess-Zumino-Witten conformal field theory.

Let $\phi(z_1, \cdots, z_n)$ be the matrix whose columns are linearly independent solutions of the KZ equation. By considering the analytic continuation of the solutions with respect to a loop $\gamma$ in $X_n$ with base point $x_0$ we obtain the matrix $\theta(\gamma)$ defined by

$$\phi(z_1, \cdots, z_n) \mapsto \phi(z_1, \cdots, z_n)\theta(\gamma). \tag{5.8}$$

Since the KZ connection $\omega$ is flat the matrix $\theta(\gamma)$ depends only on the homotopy class of $\gamma$. The fundamental group $\pi_1(X_n, x_0)$ is the pure braid group $P_n$. As the above holonomy of the connection $\omega$ we have a one-parameter family of linear representations of the pure braid group

$$\theta : P_n \to \mathrm{GL}(V_1 \otimes \cdots \otimes V_n). \tag{5.9}$$

The symmetric group $\mathfrak{S}_n$ acts on $X_n$ by the permutations of coordinates. We denote the quotient space $X_n/\mathfrak{S}_n$ by $Y_n$. The fundamental group of $Y_n$ is the braid group $B_n$. In the case $V_1 = \cdots = V_n = V$, the symmetric group $\mathfrak{S}_n$ acts diagonally on the trivial vector bundle over $X_n$ with fiber $V^{\otimes n}$ and the connection $\omega$ is invariant by this action. Thus we have one-parameter family of linear representations of the braid group

$$\theta : B_n \to \mathrm{GL}(V^{\otimes n}). \tag{5.10}$$

It is known by [5] and [10] that this representation is described by means of quantum groups. We call $\theta$ the quantum representation of the braid group.

## 6. Solutions of KZ equation by hypergeometric integrals

In this section we describe solutions of the KZ equation for the case $\mathfrak{g} = sl_2(\mathbf{C})$ by means of hypergeometric integrals following Schechtman and Varchenko [21]. A description of the solutions of the KZ equation was also given by Date, Jimbo, Matsuo and Miwa [6]. We refer the reader to [1] and [19] for general treatments of hypergeometric integrals.

Let us recall basic facts about the Lie algebra $sl_2(\mathbf{C})$ and its Verma modules. As a complex vector space the Lie algebra $sl_2(\mathbf{C})$ has a basis $H, E$ and $F$ satisfying the relations:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{6.1}$$

For a complex number $\lambda$ we denote by $M_\lambda$ the Verma module of $sl_2(\mathbf{C})$ with highest weight $\lambda$. Namely, there is a non-zero vector $v_\lambda \in M_\lambda$ called the highest weight vector satisfying

$$Hv_\lambda = \lambda v_\lambda, \ Ev_\lambda = 0 \tag{6.2}$$

and $M_\lambda$ is spanned by $F^j v_\lambda$, $j \geq 0$. The elements $H, E$ and $F$ act on this basis as

$$\begin{cases} H \cdot F^j v_\lambda = (\lambda - 2j) F^j v_\lambda \\ E \cdot F^j v_\lambda = j(\lambda - j + 1) F^{j-1} v_\lambda \\ F \cdot F^j v_\lambda = F^{j+1} v_\lambda. \end{cases} \tag{6.3}$$

It is known that if $\lambda \in \mathbf{C}$ is not a non-negative integer, then the Verma module $M_\lambda$ is irreducible.

For $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$ we put $|\Lambda| = \lambda_1 + \cdots + \lambda_n$ and consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$. For a non-negative integer $m$ we define the space of weight vectors with weight $|\Lambda| - 2m$ by

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \ ; \ Hx = (|\Lambda| - 2m)x\} \tag{6.4}$$

and consider the space of null vectors defined by

$$N[|\Lambda| - 2m] = \{x \in W[|\Lambda| - 2m] \ ; \ Ex = 0\}. \tag{6.5}$$

The KZ connection $\omega$ commutes with the diagonal action of $\mathfrak{g}$ on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, hence it acts on the space of null vectors $N[|\Lambda| - 2m]$.

For parameters $\kappa$ and $\lambda$ we consider the multi-valued function

$$\Phi_{n,m} = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{2\kappa}} \prod_{1 \leq i \leq m, 1 \leq \ell \leq n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \prod_{1 \leq i < j \leq m} (t_i - t_j)^{\frac{2}{\kappa}} \tag{6.6}$$

defined over $X_{n+m}$. The function $\Phi_{n,m}$ is called the master function. Let $\mathcal{L}$ denote the local system associated to the multi-valued function $\Phi_{n,m}$.

The symmetric group $\mathfrak{S}_m$ acts on $X_{n,m}$ by the permutations of the coordinate functions $t_1, \cdots, t_m$. The function $\Phi_{n,m}$ is invariant by the action of $\mathfrak{S}_m$. The local system $\mathcal{L}$ over $X_{n,m}$ defines a local system on $Y_{n,m}$, which we denote by $\overline{\mathcal{L}}$. The local system dual to $\mathcal{L}$ is denoted by $\mathcal{L}^*$.

We put $v = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$ and for $J = (j_1, \cdots, j_n)$ set $F^J v = F^{j_1} v_{\lambda_1} \otimes \cdots \otimes F^{j_n} v_{\lambda_n}$, where $j_1, \cdots, j_n$ are non-negative integers. The weight space $W[|\Lambda| - 2m]$ has a basis $F^J v$ for each $J$ with $|J| = j_1 + \cdots + j_n = m$. For the sequence of integers $(i_1, \cdots, i_m) = (\underbrace{1, \cdots, 1}_{j_1}, \cdots, \underbrace{n, \cdots, n}_{j_n})$ we set

$$S_J(z,t) = \frac{1}{(t_1 - z_{i_1}) \cdots (t_m - z_{i_m})} \tag{6.7}$$

and define the rational function $R_J(z,t)$ by

$$R_J(z,t) = \frac{1}{j_1! \cdots j_n!} \sum_{\sigma \in \mathfrak{S}_m} S_J(z_1, \cdots, z_n, t_{\sigma(1)}, \cdots, t_{\sigma(m)}). \tag{6.8}$$

For example, we have

$$R_{(1,0,\cdots,0)}(z,t) = \frac{1}{t_1 - z_1}, \quad R_{(2,0,\cdots,0)}(z,t) = \frac{1}{(t_1 - z_1)(t_2 - z_1)}$$
$$R_{(1,1,0,\cdots,0)}(z,t) = \frac{1}{(t_1 - z_1)(t_2 - z_2)} + \frac{1}{(t_2 - z_1)(t_1 - z_2)}$$

and so on.

Since $\pi_{n,m} : X_{m+n} \to X_n$ is a fiber bundle with fiber $X_{n,m}$ the fundamental group of the base space $X_n$ acts naturally on the homology group $H_m(X_{n,m}, \mathcal{L}^*)$. Thus we obtain a representation of the pure braid group

$$r_{n,m} : P_n \longrightarrow \operatorname{Aut} H_m(X_{n,m}, \mathcal{L}^*) \tag{6.9}$$

which defines a local system on $X_n$ denoted by $\mathcal{H}_{n,m}$. In the case $\lambda_1 = \cdots = \lambda_n$ there is a representation of the braid group

$$r_{n,m} : B_n \longrightarrow \operatorname{Aut} H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \tag{6.10}$$

which defines a local system $\overline{\mathcal{H}}_{n,m}$ on $Y_{n,m}$. For any horizontal section $c(z)$ of the local system $\mathcal{H}_{n,m}$ we consider the hypergeometric type integral

$$\int_{c(z)} \Phi_{n,m} R_J(z,t) \; dt_1 \wedge \cdots \wedge dt_m \tag{6.11}$$

for the above rational function $R_J(z,t)$.

The twisted de Rham complex $(\Omega^*(X_{n,m}), \nabla)$ is a complex with differential $\nabla : \Omega^j(X_{n,m}) \to \Omega^{j+1}(X_{n,m})$ defined by

$$\nabla \omega = d\omega + d \log \Phi_{n,m} \wedge \omega.$$

for $\omega \in \Omega^j(X_{n,m})$. There is a pairing between the homology of the local system $\mathcal{L}^*$ and the cohomology of the twisted de Rham complex

$$H_m(X_{n,m}, \mathcal{L}^*) \times H^m(\Omega^*(X_{n,m}), \nabla) \to \mathbf{C}$$

defined by

$$(c, \omega) \mapsto \int_c \Phi_{n,m} \omega.$$

Such integrals are called hypergeometric integrals. We refer the reader to [19] for a detailed treatment of hypergeometric integrals in the more general situation of hyperplane arrangements.

We define a map

$$\rho : W[\lambda - 2m] \to \Omega^m(X_{n,m})$$

given by

$$\rho(w) = R_J(t,z) dt_1 \wedge \cdots \wedge dt_m$$

for $w = F^J v$ using the rational function $R_J(t,z)$. It turns out that $\rho$ induces a map to the cohomology of the twisted de Rham complex

$$N[\lambda - 2m] \longrightarrow H^m(\Omega^*(X_{n,m}), \nabla).$$

By this construction we obtain a map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \longrightarrow N[\lambda - 2m]^*$$

defined by

$$\langle \phi(c), w \rangle = \int_c \Phi \rho(w).$$

A lot of works have been done on the expression of the solutions of the KZ equation by means of hypergeometric type integrals (see [6] and [21]). According to the formulation due to V. Schechtman and A. Varchenko [21] the integral

$$\int_c \Phi \rho(w)$$

is a horizontal section of the KZ connection with values in $N[\lambda - 2m]$.

**Theorem 6.1** (Schechtman and Varchenko [21])**.** *The integral*

$$\sum_{|J|=m} \left( \int_{c(z)} \Phi_{n,m} R_J(z,t)\ dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

*lies in the space of null vectors $N[|\Lambda| - 2m]$ and is a solution of the KZ equation.*

## 7. Relation between homological representation and KZ connection

We fix a complex number $\lambda$ and consider the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}$$

by putting $\lambda_1 = \cdots = \lambda_n = \lambda$ in the definition of Section 6. As the monodromy of the KZ connection

$$\omega = \frac{1}{\kappa} \sum_{1 \le i < j \le n} \Omega_{ij} d \log(z_i - z_j)$$

with values in $N[n\lambda - 2m]$ we obtain the linear representation of the braid group

$$\theta_{\lambda,\kappa} : B_n \longrightarrow \operatorname{Aut} N[n\lambda - 2m].$$

We put

$$F(z_1, \cdots, z_n) = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{\lambda^2}{2\kappa}}.$$

The multivalued function $F$ gives an abelian representation of the braid group.

$$a_n : B_n \longrightarrow \mathbf{C}^*$$

and the representation $\theta_{\lambda,\kappa}$ is expressed in the form $a_n \otimes \tilde{\theta}_{\lambda,\kappa}$. The next theorem describes a relationship between a specialization of the homological representation $\rho_{n,m}$ and the representation $\tilde{\theta}_{\lambda,\kappa}$.

**Theorem 7.1.** *There exists an open dense subset $U$ in $(\mathbf{C}^*)^2$ such that for $(\lambda,\kappa)\in U$ the homological representation $\rho_{n,m}$ with the specialization*

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$$

*is equivalent to the monodromy representation of the KZ connection $\tilde{\theta}_{\lambda,\kappa}$ with values in the space of null vectors*

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

We assume the conditions $i_*\mathcal{L} \cong i_!\mathcal{L}$ and $i_*\overline{\mathcal{L}} \cong i_!\overline{\mathcal{L}}$ in the following. By means of the argument in Section 4 these conditions are satisfied for $(\lambda,\kappa)$ in an open dense subset in $(\mathbf{C}^*)^2$. By the assumption we have an isomorphism $H_m(X_{n,m},\mathcal{L}) \cong H_m^{lf}(X_{n,m},\mathcal{L})$ and we can take the regularized cycles $[\widetilde{\Delta}_M] \in H_m(X_{n,m},\mathcal{L})$ for the bounded chamber $\Delta_M$.

We will consider the integral

$$\sum_{|J|=m} \left( \int_{\Delta_M} \Phi_{n,m} R_J(z,t)\ dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

in the space of null vectors $N[|\Lambda| - 2m]$. In general the above integral is divergent. We replace the integration cycle by the regularized cycle $[\widetilde{\Delta}_M]$ to obtain the convergent integral. This is called the regularized integral. We refer the reader to [1] for details on this aspect.

The rest of this section is devoted to the proof of the above theorem. We first show the following proposition.

**Proposition 7.1.** *There exists an open dense subset $U$ in $(\mathbf{C}^*)^2$ such that for $(\lambda,\kappa)\in U$ the following properties (1) and (2) are satisfied.*

1. *The integrals in Theorem 6.1 over $[\widetilde{\Delta}_M]$ for $M = (m_1,\cdots,m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ are linearly independent.*
2. *The homology classes $[\overline{\Delta}_M]$ for $M = (m_1,\cdots,m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ form a basis of $H_m^{lf}(Y_{n,m},\overline{\mathcal{L}}^*) \cong H_m(Y_{n,m},\overline{\mathcal{L}}^*)$.*

*Here $m_1,\cdots,m_{n-1}$ are non-negative integers.*

*Proof.* We prepare notation for a basis of $N[|\Lambda| - 2m]$. We suppose that $\lambda_1$ is not a non-negative integer. Let us observe that for $\Lambda = (\lambda_1,\cdots,\lambda_n)$ the space of null vectors $N[|\Lambda| - 2m]$ has dimension $d_{n,m}$. This can be shown as follows. First, let us consider the weight space

$$\begin{aligned} &M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n}[\lambda_2 + \cdots + \lambda_n - 2m] \\ &= \{x \in M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n}\ ;\ Hx = (\lambda_2 + \cdots + \lambda_n - 2m)x\}. \end{aligned}$$

There is an isomorphism

$$\xi : M_{\lambda_2} \otimes \cdots \otimes M_{\lambda_n}[\lambda_2 + \cdots + \lambda_n - 2m] \longrightarrow N[|\Lambda| - 2m]$$

defined by

$$u \mapsto v_{\lambda_1} \otimes u - \frac{1}{\lambda_1} F v_{\lambda_1} \otimes Eu + \frac{1}{\lambda_1(\lambda_1 - 1)} F^2 v_{\lambda_1} \otimes E^2 u - \cdots$$

This shows that $N[|\Lambda| - 2m]$ has a basis indexed by $J' = (j_1, j_2, \cdots, j_n)$ with $j_1 = 0$ and $j_2 + \cdots + j_n = m$, where $j_2,\cdots,j_n$ are non-negative integers. Let us denote by

$S_{n,m}$ the set of such indices $J'$. The above weight space has a basis $u_{J'}$ indexed by $J' \in S_{n,m}$. We have the corresponding basis $\xi(u_{J'})$ of $N[|\Lambda| - 2m]$.

We put

$$\widetilde{\Phi}_{n,m} = \prod_{1\le i\le m, 1\le \ell\le n} (t_i - z_\ell)^{\alpha_\ell} \prod_{1\le i<j\le m} (t_i - t_j)^{2\gamma} \tag{7.1}$$

and for $J' \in S_{n,m}$ put

$$\alpha'_J = \prod_{k=2}^{n} (j_k)!\alpha_k(\alpha_k + \gamma)\cdots(\alpha_k + (j_k - 1)\gamma). \tag{7.2}$$

We assume that $\alpha_1, \cdots, \alpha_n$ and $\gamma$ are positive. We express the integral in Theorem 6.1 over the cycle $\Delta_M$ in the linear combination for the basis $\xi(u_{J'})$ of $N[|\Lambda| - 2m]$ and we donote by $\widetilde{R}_{J'}(z,t)$ the corresponding rational function. In [24] Varchenko gave a formula for the determinant

$$\det_{M,J'} \left( \alpha_{J'} \int_{\Delta_M} \widetilde{\Phi}_{n,m} \widetilde{R}_{J'}(z,t) dt_1 \wedge \cdots \wedge dt_m \right), \tag{7.3}$$

where $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ and $J' \in S_{n,m}$. According to Varchenko's formula the above determinant is expressed as a non-zero constant times the gamma factor given by

$$\prod_{i=0}^{m-1} \left( \frac{\Gamma((i+1)\gamma + 1)^{n-1}}{\Gamma(\gamma+1)^{n-1}} \, \frac{\Gamma(\alpha_1 + i\gamma + 1)\cdots(\alpha_n + i\gamma + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_n + (2m-2-i)\gamma + 1)} \right)^{\nu_i} \tag{7.4}$$

where $\nu_i$ is defined by

$$\nu_i = \begin{pmatrix} m+n-i-3 \\ m-i-1 \end{pmatrix}. \tag{7.5}$$

Since the gamma function does not has zeros and has only poles of order one at non-positive integers, it is clear that the determinant is zero only when the denominator of the gamma factor has a pole. Considering the regularized integrals over the cycles $[\widetilde{\Delta}_M]$ we can analytically continue the determinant formula to complex numbers $\alpha_1, \cdots, \alpha_n$ and $\gamma$.

Let us recall that we deal with the case

$$\alpha_\ell = -\frac{\lambda}{\kappa}, \quad 1 \le \ell \le n, \quad \gamma = \frac{1}{\kappa}.$$

From the determinant formula we observe that the linearly independence for the solutions of the KZ equation in (1) in the statement of the proposition is satisfied for $(\lambda, \kappa)$ in an open dense subset in $(\mathbf{C}^*)^2$. Under the same condition we have the linear independence for the homology classes $[\overline{\Delta}_M]$ for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$. Since we have $\dim H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}}^*) = d_{m,n}$ we obtain the property (2). This completes the proof of our proposition. $\square$

Let us consider the specialization map

$$s : R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}] \longrightarrow \mathbf{C} \tag{7.6}$$

defined by the substitutions $q \mapsto e^{-2\pi\sqrt{-1}\lambda/\kappa}$ and $t \mapsto e^{2\pi\sqrt{-1}/\kappa}$. This induces in a natural way a homomorphism

$$H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z}) \longrightarrow H_m(Y_{n,m}, \overline{\mathcal{L}}^*). \tag{7.7}$$

We take a basis $[c_M]$ of $H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ as the $R$-module for $M = (m_1, \cdots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$ in such a way that $[c_M]$ maps to the regularized cycle for $[\overline{\Delta}_M]$ by the above specialization map. We observe that the homological representation specialized at $q \mapsto e^{-2\pi\sqrt{-1}\lambda/\kappa}$ and $t \mapsto e^{2\pi\sqrt{-1}/\kappa}$ is identified with the linear representation of the braid group $r_{n,m} : B_n \to \mathrm{Aut}\, H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$.

Since the basis of $N[n\lambda - 2m]$ is indexed by the set $S_{n,m}$ we have an isomorphism

$$H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \cong N[n\lambda - 2m].$$

Now the fundamental solutions of the KZ equation with values in $N[n\lambda - 2m]$ is give by the matrix of the form

$$\left( \int_{\widetilde{\Delta}_M} \omega_{M'} \right)_{M,M'}$$

with $M = (m_1, \cdots, m_{n-1})$ and $M' = (m'_1, \cdots, m'_{n-1})$ such that $m_1 + \cdots + m_{n-1} = m$ and $m'_1 + \cdots + m'_{n-1} = m$. Here $\omega_{M'}$ is a multivalued $m$-form on $X_{n,m}$. The column vectors of the above matrix form a basis of the solutions of the KZ equation with values in $N[n\lambda - 2m]$. Thus the representation $r_{n,m} : B_n \to \mathrm{Aut}\, H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$ is equivalent to the action of $B_n$ on the solutions of the KZ equation with values in $N[n\lambda - 2m]$. This completes the proof of Theorem 7.1.

## 8. Space of conformal blocks

We take distinct $n+1$ points $p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with $p_{n+1} = \infty$ and we associate to these points the highest weights $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}$. Then the space of coinvariants

$$(M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \otimes M^*_{\lambda_{n+1}})/\mathfrak{g}$$

is identified with

$$N[\lambda_{n+1}] = N[\lambda - 2m]$$

for

$$m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda_{n+1}),$$

where $M^*_{\lambda_{n+1}}$ denotes the dual representation of $M_{\lambda_{n+1}}$.

Let us briefly discuss a relation between the space of conformal blocks in conformal field theory on the Riemann sphere and the space of coinvariants $N[\lambda - 2m]$. We deal with the Lie algebra $\mathfrak{g} = sl_2(\mathbf{C})$.

First, we recall the definition of the space of conformal blocks. We refer the reader to [11] for an introductory treatment of this subject. We fix a positive integer $K$ called the level. and put $\kappa = K + 2$. We suppose that the highest weights $\lambda_1, \cdots, \lambda_{n+1}$ associated with the points $p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$ with $p_{n+1} = \infty$ are non-negative integers and satisfy $0 \leq \lambda_1, \cdots, \lambda_{n+1} \leq K$.

For a non-negarive integer $\lambda$ we denote by $V_\lambda$ the irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$. Namely, $V_\lambda$ is an irreducible representation containing a non-zero vector $v$ such that $Ev = 0$. A basis of $V_\lambda$ is given by $\{v, Fv, \cdots, F^\lambda v\}$. The representation space $V_\lambda$ is obtained as a quotient of the Verma module $M_\lambda$. We consider the representations $V_{\lambda_1}, \cdots, V_{\lambda_{n+1}}$ associated with the above $n + 1$ points $p_1, \cdots, p_{n+1} \in \mathbf{C}P^1$.

Let us recall the notion of affine Lie algebras. We start from the loop algebra $\mathfrak{g}\otimes \mathbf{C}((\xi))$, where $\mathbf{C}((\xi))$ denotes the ring of Laurent series. We consider the central extension $\widehat{\mathfrak{g}} = \mathfrak{g}\otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$ defined by

$$[X\otimes f, Y\otimes g] = [X,Y]\otimes fg + \langle X,Y\rangle \ \mathrm{Res}_{\xi=0}(df\, g)$$

where $\langle X,Y\rangle = \mathrm{Tr}(XY)$ is the Cartan-Killing form. We call $\widehat{\mathfrak{g}}$ the affine Lie algebra. We denote by $A_+$ the subalgebra of $\mathbf{C}((\xi))$ consisting of the series with only positive powers. Similarly, $A_-$ denotes the subalgebra consisting of the series with only negative powers. We define Lie subalgebras $N_+, N_0, N_-$ by

$$N_+ = [\mathfrak{g}\otimes A_+]\oplus \mathbf{C}E, \quad N_0 = \mathbf{C}H \oplus \mathbf{C}c, \quad N_+ = [\mathfrak{g}\otimes A_-]\oplus \mathbf{C}F.$$

We have a direct sum decomposition

$$\widehat{\mathfrak{g}} = N_+ \oplus N_0 + N_-$$

as Lie algebras.

Let $\lambda$ be an integer with $0\leq \lambda \leq K$. We construct an irreducible representation $\mathcal{H}_\lambda$ starting from the finite dimensional irreducible representation $V_\lambda$ of $\mathfrak{g}$. We consider the Verma module $\mathcal{M}_\lambda$ defined as $\mathcal{M}_\lambda = U(N_-)V_\lambda$ satisfying $N_+V_\lambda = 0$, where the action of $U(N_-)$ is free and the central elements $c$ acts as the multiplication by $K$. It turns out that the Verma module $\mathcal{M}_\lambda$ contains a null vector, which means that there exists a non-zero vector $\chi \in \mathcal{M}_\lambda$ such that $N_+\chi = 0$. We consider the quotient module

$$\mathcal{H}_\lambda = \mathcal{M}_\lambda / U(N_-)\chi$$

and it can be shown that $\mathcal{H}_\lambda$ is an irreducible $\widehat{\mathfrak{g}}$-module. We call $\mathcal{H}_\lambda$ the integral highest weight module of $\widehat{\mathfrak{g}}$ with highest weight $\lambda$ of level $K$.

We regard the Riemann sphere $\mathbf{C}P^1$ as the one point compactification $\mathbf{C}\cup\{\infty\}$ and fix an affine coordinate function $z$ for $\mathbf{C}$. We take local coordinates around $p_j$, $1\leq j\leq n$ as $\xi_j = z - z(p_j)$ and take $\xi_{n+1} = 1/t$ as a local coordinate around $p_{n+1} = \infty$. We denote by $\mathcal{M}_p$ the set of meromorphic functions on $\mathbf{C}P^1$ with poles of any order at most at $p_1, \cdots, p_{n+1}$. Then $\mathfrak{g}\otimes\mathcal{M}_p$ has a structure of a Lie algebra and acts diagonally on the tensor product $\mathcal{H}_{\lambda_1}\otimes\cdots\otimes\mathcal{H}_{\lambda_n}\otimes\mathcal{H}^*_{\lambda_{n+1}}$ by means of the Laurent expansions of a meromorphic function at the points $p_1,\cdots,p_{n+1}\in\mathbf{C}P^1$ with respect to the above local coordinates. Here we notice that this action is well-defined since the affine Lie algebra is defined by means of a central extension given by a 2-cocycle expressed by the residue of a 1-form and the sum of the residues is zero on $\mathbf{C}P^1$.

The space of conformal blocks is defined as the space of coinvariants

$$\mathcal{H}(p,\lambda) = (\mathcal{H}_{\lambda_1}\otimes\cdots\ \otimes\mathcal{H}_{\lambda_n}\otimes\mathcal{H}^*_{\lambda_{n+1}})/(\mathfrak{g}\otimes\mathcal{M}_p).$$

There is also a dual formulation as follows. We define $\mathcal{H}(p,\lambda)^*$ as the space of invariant multilinear forms by

$$\mathcal{H}(p,\lambda)^* = \mathrm{Hom}_{\mathfrak{g}\otimes\mathcal{M}_p}(\mathcal{H}_{\lambda_1}\otimes\cdots\ \otimes\mathcal{H}_{\lambda_n}\otimes\mathcal{H}^*_{\lambda_{n+1}}, \mathbf{C})$$

where the action of $\mathfrak{g}\otimes\mathcal{M}_p$ on $\mathbf{C}$ is supposed to be trivial. This means that the dual space of conformal blocks $\mathcal{H}(p,\lambda)^*$ is defined as the space of invariant multilinear forms by the action of $\mathfrak{g}\otimes\mathcal{M}_p$.

It is a basic result in conformal field theory that the spaces of conformal blocks form a vector bundle over the configuration space $X_n$ equipped with a flat connection. This connection is explicitly give by the KZ connection. Therefore, the pure braid group $P_n$ acts on the space of conformal blocks $\mathcal{H}(p,\lambda)^*$ by means of the holonomy of this connection.

It turns out that there is a surjective map

$$(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \otimes V^*_{\lambda_{n+1}})/\mathfrak{g} \longrightarrow \mathcal{H}(p,\lambda)$$

and the kernel is described by some algebraic equations coming from the definition of the space of conformal blocks. The reason that the above map is not an isomorphism is that the integrable highest module $\mathcal{H}_\lambda$ is not a Verma module and there exists a null vector in $\mathcal{M}_\lambda$. The existence of such null vectors yields the above algebraic equations.

In general the dimension of the space of conformal blocks is given by the Verlinde formula. The most fundamental one is the case $n=2$, namely the Riemann sphere with 3 marked points. In this case it can be shown that the space of conformal blocks $\mathcal{H}(p_1,p_2,p_3;\lambda_1,\lambda_2,\lambda_3)$ is isomorphic to $\mathbf{C}$ if the condition

$$\begin{aligned} &|\lambda_1-\lambda_2| \leq \lambda_3 \leq \lambda_1+\lambda_2 \\ &\lambda_1+\lambda_2+\lambda_3 \in 2\mathbf{Z} \\ &\lambda_1+\lambda_2+\lambda_3 \leq 2K \end{aligned}$$

is satisfied and is 0 otherwise. The above condition is called the quantum Clebsch-Gordan condition. The first two lines correspond to the usual Clebsch-Gordan condition. It is a necessary and sufficient condition so that the tensor product $V_{\lambda_1}\otimes V_{\lambda_2}$ contains $V_{\lambda_3}$ as a $\mathfrak{g}$-submodule. We notice that in our case there is an extra condition coming from the level $K$. The above dimension of the space of conformal blocks describes so called the fusion rule in Wess-Zumino-Witten conformal field theory.

We apply the construction of solutions of the horizontal section of the KZ connection by hypergeometric integrals described in Section 7. It was shown by B. Feigin, V. Schechtman and A. Varchenko [7] that the map to the twisted de Rham complex $\rho : W[\lambda-2m] \to \Omega^m(Y_{n,m})$ and induces a map

$$\mathcal{H}(p,\lambda) \to H^m(\Omega^*(X_{n,m}),\nabla).$$

This means that the algebraic equations appearing in the construction of the space of conformal blocks correspond to exact forms. Therefore we obtain a map

$$\phi : H_m(Y_{n,m},\overline{\mathcal{L}}^*) \to \mathcal{H}(p,\lambda)^*$$

defined by

$$\langle \phi(c), w\rangle = \int_c \Phi\rho(w).$$

The map $\phi$ is equivariant with respect to the action of the pure braid group $P_n$ and is considered to be a period integral for the space of conformal blocks. It was shown by A. Varchenko [23] that the above map $\phi$ is surjective.

We will see that the map $\phi : H_m(Y_{n,m},\overline{\mathcal{L}}^*) \to \mathcal{H}(p,\lambda)^*$ is not in general injective. Our local system $\mathcal{L}$ might not be generic in the sense of previous sections since $\kappa$ and the highest weights are integers and we have rational numbers as exponents in the master function $\Phi_{n,m}$ in (6.6). We investigate such non-generic situation especially in the case $n=2$.

In the following we consider the compactification $X_{n,m} \to (\mathbf{C}P^1)^m$ and

$$i : Y_{n,m} \longrightarrow \mathcal{S}^m\mathbf{C}P^1$$

by means of the symmetric product. As in Section 4 there are hyperplanes

$$H_{i\ell},\ 1 \le i \le m,\ 1 \le \ell \le n, \quad H_{i\infty},\ 1 \le i \le m, \quad D_{ij},\ 1 \le i < j \le m$$

and we associate the exponents

$$\begin{aligned} a(H_{i\ell}) &= -\frac{\lambda_\ell}{\kappa} \\ a(H_{i\infty}) &= \frac{1}{\kappa}\left(\lambda_1 + \cdots + \lambda_n - 2(m-1)\right) \\ a(D_{ij}) &= \frac{2}{\kappa} \end{aligned}$$

so that the monodromy of the local system $\mathcal{L}$ associated with $\Phi_{n,m}$ along the above hyperplanes are given by

$$\xi_\ell = e^{2\pi\sqrt{-1}a(H_{i\ell})},\ \xi_\infty = e^{2\pi\sqrt{-1}a(H_{i\infty})},\ \delta = e^{2\pi\sqrt{-1}a(D_{ij})}$$

respectively.

A non-empy intersection of subfamily of the above hyperplane is called an edge. For a subset $\{i_1, \cdots, i_p\} \subset \{1, \cdots, m\}$ we consider the edge

$$L_{i_1\cdots i_p} = H_{i_1\infty} \cap \cdots \cap H_{i_p\infty}.$$

By blowing up $(\mathbf{C}P^1)^m$ we consider the situation where the complement of $X_{n,m}$ is a divisor with normal crossings. We say that the edge $L$ is resonant if the monodromy of the local system $\mathcal{L}$ along the divisor corresponding to $L$ is trivial.

In the following we consider the case $n = 2$.

**Proposition 8.1.** *If the inequality*

$$\lambda_1 + \lambda_2 + \lambda_3 > 2K$$

*holds, there exists a resonant edge of the form* $L_{i_1\cdots i_p}$.

*Proof.* We set

$$a(L_{i_1\cdots i_p}) = a(H_{i_1\infty}) + \cdots + a(H_{i_p\infty}) + \sum_{\{i,j\}\subset\{i_1,\cdots,i_p\}} a(D_{ij}).$$

The the monodromy of the local system $\mathcal{L}$ along the divisor corresponding to $L_{i_1\cdots i_p}$ is given by $e^{2\pi\sqrt{-1}a(L_{i_1\cdots i_p})}$. We have

$$a(L_{i_1\cdots i_p}) = \frac{p}{K+2}(\lambda_1 + \lambda_2 - 2m + p + 1).$$

We put $s = K - \lambda_3$. The the inequality $\lambda_1 + \lambda_2 + \lambda_3 > 2K$ is equivalent to $m > s$. If we set $p = s + 1$, then we have $p \le m$ and

$$a(L_{i_1\cdots i_p}) = s + 1,$$

which is an integer. In this case the monodromy of the local system $\mathcal{L}$ along the divisor corresponding to the edge $L_{i_1\cdots i_p}$ is trivial. This completes the proof. $\square$

In the above situation we say that the local system $\mathcal{L}$ has a resonance at infinity. In [22] R. Silvotti investigated the structure of $E_\infty^{m,0}$ for the Leray spectral sequence associated with the compactification $i : Y_{n,m} \longrightarrow \mathcal{S}^m\mathbf{C}P^1$. We consider the case $n = 2$. First we describe the generic case.

**Proposition 8.2.** *If the condition*

$$\xi_\ell^k \delta^{\frac{1}{2}k(k-1)} \neq 1, \quad \ell = 1, 2, \infty, \ 1 \leq k \leq m$$
$$\delta^{\frac{1}{2}k(k-1)} \neq 1, \quad 2 \leq k \leq m$$

*is satisfied, then the Leray spectral sequence degenerates at $E^2$ term and there is an isomorphism.*

$$E_\infty^{m,0} \cong H^m(Y_{2,m}, \overline{\mathcal{L}}).$$

*Furthermore, we have* $\dim E_\infty^{m,0} = 1$.

*Proof.* Under the condition we have $R^q i_* \overline{\mathcal{L}} \cong 0$ for $q > 0$ and $i_* \overline{\mathcal{L}} \cong i_! \overline{\mathcal{L}}$. Thus the Leray spectral sequence degenerates at $E_2$ term and we have

$$E_2^{m,0} \cong H^m(\mathcal{S}^m \mathbf{C}P^1, i_! \overline{\mathcal{L}}) \cong H_c^m(Y_{2,m}, \overline{\mathcal{L}}).$$

We have an isomorphism $H_c^m(Y_{2,m}, \overline{\mathcal{L}}) \cong H^m(Y_{2,m}, \overline{\mathcal{L}})$. We have a vanishing

$$H^i(Y_{2,m}, \overline{\mathcal{L}}) \cong 0, \quad i \neq m$$

and by means of the calculation of the Euler-Poincaré characteristic of $Y_{2,m}$ we conclude that $\dim H^m(Y_{2,m}, \overline{\mathcal{L}}) = 1$. This completes the proof. □

**Proposition 8.3.** *Let us suppose that the condition in Proposition 8.2 is satisfied with the only one exception*

$$\xi_\infty^k \delta^{\frac{1}{2}k(k-1)} = 1$$

*for some $1 \leq k \leq m$. Then we have $E_\infty^{m,0} \cong 0$.*

*Proof.* By means of an explicit computation together with Proposition 4.3 and Proposition 4.4 we have the following description of the direct images. In the following we suppose $k > 1$. The argument for the case $k = 1$ is similar.

(1) $R^q i_* \overline{\mathcal{L}} \cong 0, \quad q \neq 0, k-1, k$,
(2) We have $i_* \overline{\mathcal{L}} \cong i_! \overline{\mathcal{L}}$,
(3) $R^{k-1} i_* \overline{\mathcal{L}} \cong R^k i_* \overline{\mathcal{L}}$ is a rank one local system $\mathcal{L}'$ supported on a subset $Z$ homeomorphic to $Y_{2,m-k}$.

We have

$$E_2^{p,0} \cong H_c^p(Y_{2,m}, \overline{\mathcal{L}})$$

and $E_2^{p,0} \cong 0$ if $p \neq m$. There are isomorphisms

$$E_2^{p,k-1} \cong E_2^{p,k} \cong H^p(Z, \mathcal{L}') \cong 0$$

if $p \neq m - k$. Therefore only possible non-vanishing $E_2$ terms are

$$E_2^{m,0}, \ E_2^{m-k,k-1}, \ E_2^{m-k,k}$$

and the only non-trivial portion in the spectral sequence occurs at $E_k$ term:

$$d_k : E_k^{m-k,k-1} \longrightarrow E_k^{m,0}$$

so that $E_\infty^{m-k,k-1} \cong \operatorname{Ker} d_k$ and $E_\infty^{m,0} \cong E_k^{m,0} / \operatorname{Im} d_k$. Since $H^{m-1}(Y_{2,m}, \overline{\mathcal{L}}) \cong 0$ we conclude that $d_k$ is injective. Since $|\chi(Y_{2,m})| = 1$ we have $\dim E_2^{m,0} = \dim E_k^{m,0} = 1$. Similarly by using $|\chi(Y_{2,m-k})| = 1$ we have $\dim E_k^{m-k,k-1} = 1$. Therefore $d_k$ is an isomorphism. This shows that $E_\infty^{m,0} = 0$. □

Let us consider the natural map

$$\alpha : H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \to H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}}^*).$$

It is shown by Z. Chen [4] that $\operatorname{Im} \alpha$ is generated by cycles represented by bounded chambers. Let us describe the space of conformal blocks in terms of homology of local systems in the case $n = 2$. In the following we assume that $\kappa$ is a prime number.

**Theorem 8.1.** *The period integral*

$$\phi : H_m(Y_{2,m}, \overline{\mathcal{L}}^*) \to \mathcal{H}(p, \lambda)^*$$

*induces an isomorpshim*

$$H_m(Y_{2,m}, \overline{\mathcal{L}}^*)/\operatorname{Ker} \alpha \cong \mathcal{H}(p, \lambda)^*.$$

*Proof.* We have seen that the period integral

$$H_m(Y_{2,m}, \overline{\mathcal{L}}^*) \to \mathcal{H}(p, \lambda)^*$$

is well-defined and is surjective. First, we suppose that the highest weights $\lambda_1, \lambda_2, \lambda_3$ satisfy the quantum Clebsch-Gordan condition. We observe that the exponents

$$a(H_{i\ell}) = -\frac{\lambda_\ell}{\kappa},\ a(H_{i\infty}) = \frac{1}{\kappa}\left(\lambda_1 + \lambda_2 - 2(m-1)\right),\ a(D_{ij}) = \frac{2}{\kappa}$$

satisfy the condition of Proposition 8.2. In this case

$$\dim H_m(Y_{2,m}, \overline{\mathcal{L}}^*) = \dim \mathcal{H}(p, \lambda)^* = 1$$

and $\alpha$ is an isomorphism. We conclude that the period integral $\phi$ is an isomorphism. In the case the quantum Clebsch-Gordan condition is not satisfied it is enough to deal with the case of resonance at infinity as in Proposition 8.1. In this case the situation of Proposition 8.3 happens and we have $E_\infty^{m,0} \cong 0$. We have

$$E_2^{m,0} \cong H_c^m(Y_{2,m}, \overline{\mathcal{L}}^*)$$

and $E_\infty^{m,0}$ is identified with the image of the natural map

$$H_c^m(Y_{2,m}, \overline{\mathcal{L}}^*) \to H^m(Y_{2,m}, \overline{\mathcal{L}}^*).$$

Considering the dual homomorphim it follows from $E_\infty^{m,0} \cong 0$ that

$$H_m(Y_{2,m}, \overline{\mathcal{L}}^*)/\operatorname{Ker} \alpha \cong 0.$$

This shows that

$$H_m(Y_{2,m}, \overline{\mathcal{L}}^*)/\operatorname{Ker} \alpha \cong \mathcal{H}(p, \lambda)^*.$$

is an isomorphism in both cases. This completes the proof. □

We put $\operatorname{Im} \alpha = H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}}^*)_{reg}$. We call $H_m^{lf}(Y_{n,m}, \overline{\mathcal{L}}^*)_{reg}$ the space of regularizable cycles. We conclude that the period integral $\phi$ induces an isomorphism

$$H_m^{lf}(Y_{2,m}, \overline{\mathcal{L}}^*)_{reg} \cong \mathcal{H}(p_1, p_2, p_3; \lambda_1, \lambda_2, \lambda_3)^*.$$

**Acknowledgment.** The author is partially supported by Granr-in-Aid for Scientific Research, Japan Society of Promotion of Science and by World Premier Research Center Initiative, MEXT, Japan.

## References


[1] K. Aomoto and M. Kita, Theory of hypergeometric functions, Springer Monographs in Mathematics, 2011.

[2] S. Bigelow, Braid groups are linear, J. Amer. Math. Soc. **14** (2001), 471 – 486.

[3] S. Bigelow, The Lawrence-Krammer representation, Topology and geometry of manifolds (Athens, GA, 2001), 51–68, Proc. Sympos. Pure Math. **71** Amer. Math. Soc., 2003.

[4] Z. Chen, Stratifications of complementary spaces of hyperplane arrangements and homology groups with local coefficients, Thesis, the University of Tokyo, 2005.

[5] V. G. Drinfel'd, Quasi-Hopf algebras, Leningrad Math. J. **1** (1990), 1419–1457.

[6] E. Date, M Jimbo, A. Matsuo and T. Miwa, Hypergeometric type integrals and the $sl(2, \mathbf{C})$ Knizhnik-Zamolodchikov equations, Intern. J. Modern Phys. **B4** (1990), 1049–1057.

[7] B. Feigin, V. Schechtman and A. Varchenko, On algebraic equations satisfied by hyergeometric correlators in WZW models. I, Comm. Math. Phys. **163** (1994), 173–184.

[8] V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and Wess-Zumino models in two dimensions, Nuclear Phys. **B247** (1984), 83–103.

[9] T. Kohno, Homology of a local system on the complement of hyperplanes, Proc. Japan Acad. Ser. A **62** (1986), 144–147.

[10] T. Kohno, Monodromy representations of braid groups and Yang-Baxter equations, Ann. Inst. Fourier, **37-4** (1987), 139–160.

[11] T. Kohno, Conformal field theory and topology, Translations of Mathematical Monographs **210**, American Mathematical Society, 2002.

[12] T. Kohno, Hyperplane arrangements, local system homology and iterated integrals, Advanced Studies in Pure Mathematics, 62, (2012), 157–174.

[13] T. Kohno, Quantum and homological representations of braid groups, "Configuration Spaces - Geometry, Combinatorics and Topology, Edizioni della Normale, 14, (2012), 355–372.

[14] T. Kohno, Homological representations of braid groups and KZ connections, Journal of Singularities, 5, (2012), 94–108.

[15] D. Krammer, Braid groups are linear, Ann. of Math. **155** (2002), 131–156.

[16] R. J. Lawrence, Homological representations of the Hecke algebra, Comm. Math. Phys. **135** (1990), 141 – 191.

[17] I. Marin, Sur les représentations de Krammer génériques, Ann. Inst. Fourier **57** (2007), 1883 – 1925.

[18] P. Orlik and H. Terao, Arrangements of Hyperplanes, Springer Verlag, 1992.

[19] P. Orlik and H. Terao, Arrangements and hypergeometric integrals, MSJ Memoirs **9**, Mathematical Society of Japan, 2001.

[20] M. Salvetti, Topology of the complement of real hyperplanes in $C^N$. Invent. Math. **88** (1987), 603–618.

[21] V. Schechtman and A. Varchenko, Hypergeometric solutions of the Knizhnik-Zamolodchikov equation, Lett. in Math. Phys. **20** (1990), 93 – 102.

[22] R. Silvotti, Local systems on the complement of hyperplanes and fusion rules in conformal field theory, Internat. Math. Res. Notices 1994, no. 3, 111 ff., approx. 17 pp. (electronic).

[23] A. Varchenko, Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups, Advances in Math. Phys. **21**, World Scientific, 1995.

[24] A. Varchenko, Determinant formula for Selberg type integrals, Funct. Analysis and Its Appl. **4** (1991), 65 – 66.



Kavli IPMU (WPI), Graduate School of Mathematical Sciences, the University of Tokyo, Tokyo 153-8914 Japan

*E-mail address*: `kohno@ms.u-tokyo.ac.jp`