HIGHER HOLONOMY OF FORMAL HOMOLOGY CONNECTIONS AND BRAID COBORDISMS

TOSHIKAKE KOHNO

Abstract. We construct a representation of the homotopy 2-groupoid of a manifold by means of K.-T. Chen’s formal homology connections. By using the idea of this 2-holonomy map, we describe a method to obtain a representation of the category of braid cobordisms.

1. Introduction

The notion of formal homology connections was developed by K.-T. Chen in the framework of the theory of iterated integrals of differential forms. The original motivation of K.-T. Chen was to describe the homology group of the loop space of a manifold $M$ by the chain complex formed by the tensor algebra of the homology group of $M$ equipped with a derivation appearing in the formal homology connection (see [6], [7] and [8]). By means of the formal homology connection we obtain a chain map from the singular chain complex of the loop space to the above complex obtained from the tensor algebra of the homology group of $M$ of positive degrees.

The formal homology connection can be used to construct a holonomy map from the homotopy path groupoid. In particular, we obtain representations of fundamental groups. This was used to describe the holonomy of KZ connection in [16] and [17]. The purpose of this paper is to show that the notion of the holonomy can be extended to a 2-holonomy map from the homotopy 2-groupoid by means of formal homology connections. In order to formulate the 2-holonomy we employ the notion of 2-categories. We refer the reader to [4] for an introduction to 2-categories from the viewpoint of higher gauge theory.

Then we apply such method to construct a holonomy representation of the category of braid cobordisms. There is a work by Cirio and Martins [12] on the categorification of the KZ connection by means of 2-Yang-Baxter operator for $\mathfrak{sl}_2(\mathbb{C})$ (see also [10], [11] and [21]). In this paper we propose a universal construction based on the formal homology connections.

The paper is organized in the following way. In Section 1 we briefly review K.-T. Chen’s iterated integrals and their basic properties. In Section 2 we describe the notion of formal homology connections. In Section 3 we construct representations of homotopy 2-groupoids by means of the formal homology connection. In Section 4 is we describe a method to construct a representation of the category of braid cobordisms.

2. Preliminaries on K.-T. Chen’s iterated integrals

First, we briefly recall the notion of iterated integrals of differential forms due to K.-T. Chen. We refer the reader to [6], [7] and [8] for details. Let $M$ be a smooth manifold and $\omega_1, \cdots, \omega_k$ be differential forms on $M$. We fix two points $x_0$ and $x_1$ in
and consider the space of piecewise smooth paths \( \gamma : [0, 1] \to M \) with \( \gamma(0) = x_0 \) and \( \gamma(1) = x_1 \). We denote by \( \mathcal{P}(M; x_0, x_1) \) the above space of paths. In particular, in the case \( x_0 = x_1 \) the path space \( \mathcal{P}(M; x_0, x_1) \) is called the based loop space of \( M \). We consider the simplex

\[
\Delta_k = \{ (t_1, \ldots, t_k) \in \mathbb{R}^k : 0 \leq t_1 \leq \cdots \leq t_k \leq 1 \}
\]

and the evaluation map

\[
\varphi : \Delta_k \times \mathcal{P}(M; x_0, x_1) \to \underbrace{M \times \cdots \times M}_k
\]

defined by \( \varphi(t_1, \ldots, t_k; \gamma) = (\gamma(t_1), \ldots, \gamma(t_k)) \). The iterated integral of \( \omega_1, \ldots, \omega_k \) is defined as

\[
\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)
\]

where the expression

\[
\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)
\]

is the integration along the fiber with respect to the projection

\[
p : \Delta_k \times \mathcal{P}(M; x_0, x_1) \to \mathcal{P}(M; x_0, x_1).
\]

The above iterated integral is considered as a differential form on the path space \( \mathcal{P}(M; x_0, x_1) \) with degree \( p_1 + \cdots + p_k - k \), where we set \( p_j = \deg \omega_j \). To justify differential forms on the path space \( \mathcal{P}(M; x_0, x_1) \) we use the notion of plots. A plot \( \alpha : U \to \mathcal{P}(M; x_0, x_1) \) is a family of piecewise linear paths smoothly parametrized by a convex open set \( U \) in a finite dimensional Euclidean space. Given a plot \( \alpha \) we denote the corresponding iterated integral

\[
\left( \int \omega_1 \cdots \omega_k \right)_\alpha
\]

as a differential form on \( U \) obtained by pulling back by the iterated integral \( \int \omega_1 \cdots \omega_k \) by the plot \( \alpha \). In particular, in the case \( \omega_1, \ldots, \omega_k \) are 1-forms, the iterated integral \( \int \omega_1 \cdots \omega_k \) is a function on the path space \( \mathcal{P}(M; x_0, x_1) \) and its value on a path \( \gamma : [0, 1] \to M \) is the iterated line integral

\[
\int_{\gamma} \omega_1 \cdots \omega_k = \int_{\Delta_k} f_1(t_1) \cdots f_k(t_k) \, dt_1 \cdots dt_k
\]

where \( \gamma^*\omega_j = f_j(t) \, dt, 1 \leq j \leq k \).

Let us go back to the iterated integral of differential forms of arbitrary degrees. We take an extra point \( x_2 \) in \( M \) and consider the plots

\[
\alpha : U \to \mathcal{P}(M; x_0, x_1), \quad \beta : U \to \mathcal{P}(M; x_1, x_2).
\]

The composition of the plots \( \alpha \) and \( \beta \)

\[
\alpha \beta : U \to \mathcal{P}(M; x_0, x_2)
\]

is defined by

\[
\alpha \beta(x)(t) = \begin{cases} 
\alpha(x)(2t), & 0 \leq t \leq \frac{1}{2} \\
\beta(x)(2t - 1), & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

for \( x \in U \). As is shown by K.-T. Chen, we have the following rule for the composition of plots.
Proposition 2.1. The relation

\[
\left( \int \omega_1 \cdots \omega_k \right)_{\alpha \beta} = \sum_{0 \leq i \leq k} \left( \int \omega_1 \cdots \omega_i \right)_{\alpha} \wedge \left( \int \omega_{i+1} \cdots \omega_k \right)_{\beta}
\]

holds.

For a path \(\alpha\) we define its inverse path \(\alpha^{-1}\) by

\[\alpha^{-1}(t) = \alpha(1-t)\]

For the composition \(\alpha \alpha^{-1}\) we have

\[
\left( \int \omega_1 \cdots \omega_i \right)_{\alpha \alpha^{-1}} = 0
\]

As a differential form on the path space \(\mathcal{P}(M; x_0, x_1)\) we have the following.

Proposition 2.2. For the iterated integral \(\int \omega_1 \cdots \omega_k\) we have

\[
d \int \omega_1 \cdots \omega_k = \sum_{j=1}^{k} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \omega_{j+1} \cdots \omega_k + \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k
\]

where we put \(\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j\).

3. Formal homology connections

For a smooth manifold \(M\) we put \(H_+(M) = \bigoplus_{q>0} H_q(M; \mathbb{R})\) and consider the tensor algebra

\[
TH_+(M) = \bigoplus_{k \geq 0} \left( \bigotimes^k H_+(M) \right).
\]

We suppose that \(\dim H_+(M)\) is finite. We denote by \(\Omega^*(M)\) the algebra of differential forms on \(M\) and consider the tensor product \(\Omega^*(M) \otimes TH_+(M)\). We suppose that the differential \(d\) acts trivially on \(TH_+(M; \mathbb{R})\) and we extend naturally the wedge product and iterated integrals on \(\Omega^*(M) \otimes TH_+(M)\). the powers of the augmentation ideal. When \(H_+(M)\) has a basis \(X_1, \cdots, X_m\), the tensor algebra \(TH_+(M)\) is the ring of non-commutative polynomials \(\mathbb{R}\langle X_1, \cdots, X_m \rangle\). We denote by \(J\) the ideal of the above ring generated by \(X_1, \cdots, X_m\), which is called the augmentation ideal. We consider the completion \(\widehat{TH}_+(M)\) with respect to the powers of the augmentation ideal \(J\). We see that \(\widehat{TH}_+(M)\) is regarded as the ring of non-commutative formal power series \(\mathbb{R}\langle \langle X_1, \cdots, X_m \rangle \rangle\). We denote by \(\widehat{J}\) the completed augmentation ideal. Namely, \(\widehat{J}\) consists of the formal power series of the form

\[
\sum_{i=1}^{m} a_i X_i + \cdots + \sum_{i_1 \cdots i_k} a_{i_1 \cdots i_k} X_{i_1} \cdots X_{i_k} + \cdots
\]

with zero constant term.
Then $\Omega^*(M) \otimes \overline{TH_+(M)}$ is identified with the ring of non-commutative formal power series
\[ \Omega^*(M) \langle X_1, \cdots, X_m \rangle \]
over $\Omega^*(M)$. For a differential operator $\omega$ we define the parity operator $\varepsilon$ as $\varepsilon(\omega) = \omega$ when $\omega$ is of even degree and $\varepsilon(\omega) = -\omega$ when $\omega$ is of odd degree. This operator is naturally extended to $\Omega^*(M) \otimes \overline{TH_+(M)}$. Namely, for a differential form $\tau$ and a monomial $Z$ in $X_1, \cdots, X_m$ we set $\varepsilon(\tau Z) = \varepsilon(\tau)Z$. We define a generalized curvature $\kappa$ by
\[ \kappa = d\omega - \varepsilon(\omega) \wedge \omega. \]
According to K.-T. Chen a formal homology connection $\omega \in \Omega^*(M) \otimes \overline{TH_+(M)}$ is an expression
\[ \omega = \sum_{i=1}^{m} \omega_i X_i + \cdots + \sum_{i_1 \cdots i_k} \omega_{i_1 \cdots i_k} X_{i_1} \cdots X_{i_k} + \cdots \]
with differential forms of positive degrees $\omega_{i_1 \cdots i_k}$ together with a derivation $\delta$ satisfying the following properties. We put $\deg x_i = p_i - 1$ for $x_i \in H_{p_i}(M)$.
\begin{itemize}
  \item $[\omega_i]$, $1 \leq i \leq m$ is the dual basis of $X_i$, $1 \leq i \leq m$.
  \item $\deg \omega_{i_1 \cdots i_k} = \deg X_{i_1} \cdots X_{i_k} + 1$.
  \item $\delta + \kappa = 0$.
  \item $\delta$ is a derivation of degree $-1$.
  \item $\delta X_j \in \hat{J}^2$ where $\hat{J}$ is the completed augmentation ideal.
\end{itemize}
Here we suppose that the derivation $\delta$ satisfies the Leibniz rule
\[ \delta(uv) = (\delta u)v + (-1)^{\deg u}u(\delta v). \]
From the above condition we can show that $\delta \circ \delta = 0$ and $(\overline{TH_+(M)}, \delta)$ forms a complex. We denote by $TH_+(M)_k$ the degree $k$ part of $\overline{TH_+(M)}$ with respect to the above degrees. We denote by $TH_+(M)_{\leq k}$ the completed subalgebra of $\overline{TH_+(M)}$ generated by the homogeneous elements of degree less than or equal to $k$. For the formal homology connection $\omega$ we define its transport by
\[ T = 1 + \sum_{k=1}^{\infty} \int \omega \cdots \omega_k. \]
The following proposition plays a key role of for the construction of holonomy maps.

**Proposition 3.1.** Given a formal homology connection $(\omega, \delta)$ for a manifold $M$ the transport $T$ satisfies $dT = \delta T$.

**Proof.** By Proposition 2.2 we have
\[ dT = -\int \kappa + \left( -\int \kappa \omega + \int \varepsilon(\omega) \kappa \right) + \cdots \]
\[ = \sum_{k=0}^{\infty} \sum_{i=0}^{k} (-1)^{i+1} \int \varepsilon(\omega) \cdots \varepsilon(\omega) \kappa \omega \cdots \omega_i. \]
Substituting $\kappa = -\delta \omega$ in the above equation and applying the Leibniz rule for $\delta$, we obtain the equation $dT = \delta T$. \qed

Although the formal homology connection $\omega$ with the derivation $\delta$ is not uniquely determined, we can construct it inductively starting from the initial term $\sum_{i=1}^{m} \omega_i X_i$. Here are some examples.

**Examples**: (1) Let $T = S^1 \times S^1$ be the 2-dimensional torus. Let $p_i : S^1 \times S^1$, $i = 1, 2$, the projection to the $i$-th factor. We denote by $v$ a volume form of $S^1$. The de Rham cohomology $H^*(T)$ has a basis represented by $p_1^* v, p_2^* v, p_1^* v \wedge p_2^* v$ and we put $X_1, X_2, Y$ its dual basis of the homology. The formal homology connection is given as

$$\omega = p_1^* v X_1 + p_2^* v X_2 + (p_1^* v \wedge p_2^* v) Y$$

with the derivation defined by

$$\delta(X_1) = 0, \quad \delta(X_2) = 0, \quad \delta(Y) = -[X_1, X_2].$$

(2) Let $G$ be the unipotent Lie group consisting of the matrices

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R}$$

and $G_{\mathbb{Z}}$ its subgroup consisting of the above matrices with $x, y, z \in \mathbb{Z}$. We denote by $M$ the quotient space of $G$ by the left action of $G_{\mathbb{Z}}$. We see that $M$ has a structure of a compact smooth 3-dimensional manifold. The 1-forms

$$\omega_1 = dx, \quad \omega_2 = dy, \quad \omega_{12} = -xdy + dz$$

on $G$ are invariant under the left action of $G_{\mathbb{Z}}$ and define 1-forms on $M$. There is a relation

$$\omega_1 \wedge \omega_2 = -d\omega_{12}. $$

We observe that $H^1(M)$ has a basis represented by $\omega_1, \omega_2$ and $H^2(M)$ has a basis represented by $\omega_1 \wedge \omega_1, \omega_2 \wedge \omega_1$. These are typical examples of non-trivial Massey product. We denote by $X_1, X_2 \in H_1(M)$ the dual basis of $[\omega_1], [\omega_2]$ and by $Y_1, Y_2 \in H_2(M)$ the dual basis of $[\omega_1 \wedge \omega_1], [\omega_2 \wedge \omega_1]$. By means of the condition $\delta \omega + d\omega = \varepsilon(\omega) \wedge \omega$ we obtain that the derivation $\delta$ is given by

$$\delta(X_1) = 0, \quad \delta(X_2) = 0, \quad \delta(Y_1) = [[X_1, X_2], X_1], \quad \delta(Y_2) = [[X_1, X_2], X_2].$$

4. **Path groupoids, 2-path groupoids and their representations**

We introduce the path groupoid $P_1(M)$ and its 2-category extension $P_2(M)$. The path groupoid $P_1(M)$ is a category whose objects are points in $M$ and whose morphisms are piecewise smooth paths between points up to reparametrization and a thin homotopy. Here a thin homotopy is a homotopy sweeping on the path. We see that $P_1(M)$ has a structure of a groupoid since there is an associativity and each morphism has its inverse by means of the invariance of iterated integrals under the thin homotopy.

Now we discuss its extension to 2-categories. In general, a 2-category consists of objects, morphims and 2-morphims, which are morphims between morphims. There are two kinds of compositions for 2-morphims, horizontal compositions and vertical compositions and there are several consistency conditions among them. We do not
give here a full definition of a 2-category. We refer the reader to [4] for an introduction to the notion of 2-categories. The path 2-groupoid $\mathcal{P}_2(M)$ is a 2-category whose morphisms are piecewise smooth paths between points up to reparametrization and a thin homotopy and whose 2-morphisms are piecewise smooth discs $[0,1]^2 \to M$ spanning 2 paths up to reparametrization and a thin homotopy. As in the case of the path groupoid, a thin homotopy is a homotopy sweeping on the disc.

The homotopy equivalence classes of the path groupoid $\mathcal{P}_1(M)$ is the homotopy path groupoid denote by $\Pi_1(M)$. In a similar way, we define the homotopy 2-groupoid $\Pi_2(M)$ whose 2-morphisms are relative piecewise smooth homotopy classes of piecewise smooth homotopies between paths. We refer the reader to [13] for a general construction of a homotopy 2-groupoid of a topological space.

Let $\omega$ be a formal homology connection for $M$ with the derivation $\delta$. We decompose $\omega$ as

$$\omega = \omega^1 + \omega^2 + \cdots + \omega^p + \cdots$$

where $\omega^p$ is the sum consisting of $p$-forms and is called the $p$-form part of $\omega$. First, we consider the 1-form part $\omega^1$. For a piecewise smooth path $\gamma$ in $M$ the holonomy of the connection $\omega^1$ is given the transport as

$$\text{Hol}(\gamma) = 1 + \sum_{k=1}^{\infty} \int_{\gamma} \omega^1 \cdots \omega^1_k$$

which is an element of $\widehat{TH}^+_+(M)_0$. For the composition of paths we have

$$\text{Hol}(\alpha \beta) = \text{Hol}(\alpha) \text{Hol}(\beta)$$

by Proposition 2.1. Moreover, the relation

$$\text{Hol}(\alpha^{-1}) = \text{Hol}(\alpha)^{-1}$$

holds. Therefore, we obtain a representation of the path groupoid

$$\text{Hol} : \mathcal{P}_1(M) \to \widehat{TH}^+_+(M)_0.$$ 

We denote by $\widehat{TH}^+_+(M)_0^\times$ the group of invertible elements in $\widehat{TH}^+_+(M)_0$. The above $\text{Hol}$ is considered to be a map of groupoids from $\mathcal{P}_1(M)$ to $\widehat{TH}^+_+(M)_0^\times$. Here the map $\text{Hol}$ is regarded as a functor.

Let us consider the homotopy path groupoid $\Pi_1(M)$. In this case we have a holonomy map

$$\text{Hol} : \Pi_1(M) \to \widehat{TH}^+_+(M)_0/\mathcal{I}_0$$

where $\mathcal{I}_0$ is the ideal generated by the image of the derivation

$$\delta : \widehat{TH}^+_+(M)_1 \to \widehat{TH}^+_+(M)_0.$$ 

This can be verified by means of Proposition 3.1 and the Stokes theorem. Here the curvature of $\omega^1$ is

$$\kappa = d\omega^1 + \omega^1 \wedge \omega^1,$$

which is zero modulo the ideal $\mathcal{I}_0$. The above holonomy functor is a categorical formulation of the holonomy of Chen's formal homology connection. By fixing a base point $x_0 \in M$ we have a holonomy map

$$\text{Hol} : \pi_1(M, x_0) \to \widehat{TH}^+_+(M)_0.$$ 

6
and one of the main results due to K.-T. Chen is that the holonomy map induces an isomorphism
\[ \hat{R} \pi_1(M, x_0) \cong \hat{TH}_+(M)_0 \]
where \( \hat{R} \pi_1(M, x_0) \) is the completion of the group ring \( \mathbb{R} \pi_1(M, x_0) \) with respect to the powers of the augmentation ideal. The algebra \( \hat{R} \pi_1(M, x_0) \) is called the Malcev completion of the fundamental group \( \pi_1(M, x_0) \).

Now we construct representations of the homotopy 2-groupoid \( P_2(M) \). For two paths \( \gamma_0 \) and \( \gamma_1 \) in \( \mathcal{P}(M; x_0, x_1) \) we consider a piecewise smooth disc \( F : [0, 1]^2 \to M \) with
\[
F(t, 0) = \gamma_0(t), \quad F(t, 1) = \gamma_1(t) \\
F(0, s) = x_0, \quad F(1, s) = x_1,
\]
which is considered to be a 2-morphism between \( \gamma_0 \) and \( \gamma_1 \). Putting \( c(s)(t) = F(t, s) \), we obtain a family of paths
\[
c : [0, 1] \to \mathcal{P}(M; x_0, x_1),
\]
which is considered to be a 1-chain in \( \mathcal{P}(M; x_0, x_1) \). For the formal homology connection we consider the transport
\[
T = 1 + \sum_{k=1}^{\infty} \int_{\omega_k} \omega
\]
and denote by \( \langle T, c \rangle \) its integration on the 1-chain \( c \). We define the 2-holonomy
\[
Hol_2 : \mathcal{P}_2(M) \to \hat{TH}_+(M)_{\leq 1}
\]
by \( Hol_2(c) = \langle T, c \rangle \). The above holonomy map is additive with respect to the sum as 1-chains and for the composition of paths we have
\[
Hol_2(\alpha \beta) = Hol_2(\alpha)Hol_2(\beta)
\]
by means of Proposition 2.1. The above two types of compositions correspond to horizontal and vertical compositions of 2-morphisms in the 2-category. We obtain that the 2-holonomy map \( Hol_2 \) gives a representation of the path 2-groupoid \( \mathcal{P}_2(M) \).

**Theorem 4.1.** The above 2-holonomy map gives a representation of the homotopy 2-groupoid
\[
Hol_2 : \Pi_2(M) \to \overline{TH}_+(M)_{\leq 1}/\mathcal{I}_1
\]
where \( \mathcal{I}_1 \) is the ideal generated by the image of the derivation
\[
\delta : \overline{TH}_+(M)_2 \to \overline{TH}_+(M)_1
\]

**Proof.** As is shown in the above argument we have a representation of the path 2-groupoid given by
\[
Hol_2 : \mathcal{P}_2(M) \to \overline{TH}_+(M)_{\leq 1}.
\]
Suppose that for paths \( \gamma_0 \) and \( \gamma_1 \) in \( \mathcal{P}(M; x_0, x_1) \) piecewise smooth discs \( F_j : [0, 1]^2 \to M, j = 1, 2 \) with
\[
F_j(t, 0) = \gamma_0(t), \quad F_j(t, 1) = \gamma_1(t) \\
F_j(0, s) = x_0, \quad F_j(1, s) = x_1,
\]
are connected by a piecewise smooth homotopy preserving the above boundary conditions. This gives homologous 1-chains $c_1$ and $c_2$ in $\mathcal{P}(M; x_0, x_1)$ and there is a 2-chain $y$ such that $c_1 - c_2 = \partial y$. We have

$$\text{Hol}_2(c_1) - \text{Hol}_2(c_2) = \text{Hol}_2(\partial y)$$

which is by definition $\langle T, \partial y \rangle$. By Stokes theorem we have

$$\langle T, \partial y \rangle = \langle dT, y \rangle.$$

On the other hand we have $dT = \delta T$ by Proposition 3.1. This shows that $\text{Hol}_2(c_1) = \text{Hol}_2(c_2)$ in $\hat{TH}_+(M)_{\leq 1}/\mathcal{I}_1$ and the 2-holonomy map from the homotopy 2-groupoid $\Pi_2(M)$ is well-defined. The fact that this give a representation of the 2-groupoid $\Pi_2(M)$ follows from the corresponding properties such as

$$\text{Hol}_2(\alpha\beta) = \text{Hol}_2(\alpha)\text{Hol}_2(\beta)$$

for the path 2-groupoid $\mathcal{P}_2(M)$. This completes the proof. □

We refer the reader to [1] and [2] for a different approach to higher holonomies based on iterated integrals.

5. Holonomy of braids and its extension to braid cobordisms

We apply a method explained in the previous sections to holonomy of braids and representation of the category of braid cobordisms. We start by recalling basic facts on hyperplane arrangements. Let $\mathcal{A} = \{H_1, \cdots, H_\ell\}$ be a collection of finite number of complex hyperplanes in $\mathbb{C}^n$. We call $\mathcal{A}$ a hyperplane arrangement. Let $f_j$, $1 \leq j \leq \ell$, be linear forms dining the hyperplanes $H_j$. We consider the complement $M(\mathcal{A}) = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$ and denote by $\Omega^*(M(\mathcal{A}))$ the algebra of differential forms on $M(\mathcal{A})$ with values in $\mathbb{C}$. The Orlik-Solomon algebra $OS(\mathcal{A})$ is the subalgebra of $\Omega^*(M(\mathcal{A}))$ generated by the logarithmic forms $\omega_j = d \log f_j$, $1 \leq j \leq \ell$. We refer the reader to [22] and [23] for basic properties of the Orlik-Solomon algebra. The fundamental fact is that the inclusion map

$$i : OS(\mathcal{A}) \rightarrow \Omega^*(M(\mathcal{A}))$$

induces an isomorphism of cohomology, where the differential on $OS(\mathcal{A})$ is trivial. In particular, we have an isomorphism of algebras

$$OS(\mathcal{A}) \cong H^*(M(\mathcal{A}); \mathbb{C}).$$

A formal homology connection for $M(\mathcal{A})$ is given as follows. Let $\{Z_j\}$ be a basis of $H_+(M(\mathcal{A}); \mathbb{C})$ and $\{\varphi_j\}$ be its dual basis in the Orlik-Solomon algebra $OS(\mathcal{A})$. Then we can take a formal homology connection given as

$$\omega = \sum_{j=1}^m \varphi_j Z_j$$

where the derivation $\delta : \hat{TH}_+(M(\mathcal{A}))_p \rightarrow \hat{TH}_+(M(\mathcal{A}))_{p-1}$ is the dual of the wedge product. More explicitly, as is described in [18], when the wedge product is given by

$$\varepsilon(\varphi_i) \wedge \varphi_j = \sum_k e_{ij}^k \varphi_k$$

we have

$$\text{Hol}_2(\alpha\beta) = \text{Hol}_2(\alpha)\text{Hol}_2(\beta).$$
the derivation $\delta$ is defined as
$$\delta Z_k = \sum_{i,j} c_{ij}^k [Z_i, Z_j].$$

This is a consequence of the formality of $M(\mathcal{A})$ in the sense of rational homotopy theory. There are no non-trivial Massey products and the derivation $\delta$ is completely determined by the product structure of the Orlik-Solomon algebra.

We consider the configuration space of ordered distinct $n$ points in the complex plane $\mathbb{C}$. Namely, we put
$$X_n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ if } i \neq j\}.$$  

The configuration space $X_n$ is the complement of the union of big diagonal hyperplanes $H_{ij}$ defined by $z_i = z_j$ in $\mathbb{C}^n$ for $1 \leq i < j \leq n$. By considering the action of the symmetric group $\mathfrak{S}_n$ by the permutation of coordinates, we set
$$Y_n = X_n / \mathfrak{S}_n.$$  

We have a covering map
$$\pi : X_n \longrightarrow Y_n$$
and the fundamental group $\pi_1(Y_n)$ is the braid group of $n$ strings denoted by $B_n$ and $\pi_1(X_n)$ is the pure braid group of $n$ strings denoted by $P_n$.

We denote by $OS(X_n)$ the Orlik-Solomon algebra for the above arrangement of hyperplanes $\{H_{ij}\}_{1 \leq i < j \leq n}$. We set
$$\omega_{ij} = d \log(z_i - z_j), \quad 1 \leq i < j \leq n.$$  

Then the Orlik-Solomon algebra $OS(X_n)$ is generated by $\omega_{ij}$, $1 \leq i < j \leq n$. We have
$$\omega_{ij} \land \omega_{jk} + \omega_{jk} \land \omega_{ik} + \omega_{ik} \land \omega_{ij} = 0, \quad 1 \leq i < j < k \leq n$$
and it was shown by Arnol’d [3] that these are actually fundamental relations. Namely, the Orlik-Solomon algebra $OS(X_n)$ is isomorphic to the exterior algebra generated by $e_{ij}$, $1 \leq i < j \leq n$, modulo the ideal generated by $e_{ij}e_{jk} + e_{jk}e_{ik} + e_{ik}e_{ij}$.

It turns out that the degree $q$ part of $OS(X_n)$ has a basis represented by
$$\omega_{i_1 j_1} \land \cdots \land \omega_{i_q j_q}, \quad j_1 < \cdots < j_q.$$  

This is called the normal form of a basis of $OS(X_n)$. We denote by $X_{i_1 j_1, \cdots, i_q j_q}$ its dual basis of the homology $H_q(X_n)$. The formal homology connection is given by
$$\omega = \sum_{j_1 < \cdots < j_q, 1 \leq q \leq n} \omega_{i_1 j_1} \land \cdots \land \omega_{i_q j_q} X_{i_1 j_1, \cdots, i_q j_q}.$$  

Since $d\omega = 0$ the generalized curvature $\kappa = d\omega - \varepsilon(\omega) \land \omega$ is decomposed as
$$\kappa = \omega^1 \land \omega^1 + (\omega^1 \land \omega^2 - \omega^2 \land \omega^1) + \cdots$$
according the degrees of differential forms.

The 1-form part of the formal homology connection is
$$\omega^1 = \sum_{i < j} \omega_{ij} X_{ij}$$
where $X_{ij}$, $1 \leq i < j \leq n$, is a basis of $H_1(X_n; \mathbb{C})$ corresponding to the hyperplanes $H_{ij}$ and the representation of the path groupoid described in the previous section is give as
$$Hol : P_1(X_n) \longrightarrow \mathbb{C}\langle\langle X_{ij}\rangle\rangle$$
where \( C(\langle X_{ij} \rangle) \) is the ring of non-commutative formal power series with indeterminates \( X_{ij}, 1 \leq i < j \leq n \). This induces the representation of the homotopy path groupoid

\[
Hol : \Pi_1(X_n) \rightarrow C(\langle X_{ij} \rangle)/I_0.
\]

The generators of the ideal \( I_0 \) are determined in the following way. We express the 2-form part of \( \kappa \) by the normal form of the basis of \( OS(X_n) \) as

\[
\omega^1 \wedge \omega^1 = \sum_{j_1 < j_2} \omega_{i_1,j_1} \wedge \omega_{i_2,j_2} Z_{i_1,j_1,i_2,j_2}.
\]

Then by the condition \( \delta \omega + \kappa = 0 \) we have \( \delta(X_{i_1,j_1,i_2,j_2}) = -Z_{i_1,j_1,i_2,j_2} \). It turns out that the generators of \( I_0 \) are infinitesimal pure braid relations:

\[
[X_{i_1,k}, X_{j_1} + X_{j,k}], [X_{i_1} + X_{i,j}, X_{j,k}] \ (i,j,k \text{ distinct}),
\]

\[
[X_{i,j}, X_{k,l}] \ (i,j,k,l \text{ distinct}).
\]

In particular, we obtain a holonomy homomorphism

\[
Hol : P_n \rightarrow C(\langle X_{ij} \rangle)/I_0
\]

which is a prototype of the Kontsevich integral [19] for knots and gives a universal finite type invariants for pure braids (see [16], [17] and [9]).

Now we consider the 2-holonomy map

\[
Hol_2 : \Pi_2(X_n) \rightarrow T\overline{H}(X_n)_{\leq 1}/I_1.
\]

We deal with the 1-form and the 2-form

\[
\omega^1 = \sum_{i<j} \omega_{i,j} X_{i,j}, \quad \omega^2 = \sum_{j_1 < j_2} \omega_{i_1,j_1} \wedge \omega_{i_2,j_2} X_{i_1,j_1,i_2,j_2}.
\]

In the expression of \( \omega^2 \) we consider the sum for the normal basis of the degree 2 part of \( OS(X_n) \) and our formulation is slightly different from the one by Cirio and Martins ([10], [11] and [12]). Although we do not give an explicit form here, we explain a method to determine the generators of the ideal \( I_1 \). We express the 3-form part of the generalized curvature \( \kappa \) by the normal form of a basis of \( OS(X_n) \) as

\[
\omega^1 \wedge \omega^2 - \omega^2 \wedge \omega^1 = \sum_{j_1 < j_2 < j_3} \omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \wedge \omega_{i_3,j_3} Z_{i_1,j_1,i_2,j_2,i_3,j_3}
\]

Then we have \( \delta(X_{i_1,j_1,i_2,j_2,i_3,j_3}) = -Z_{i_1,j_1,i_2,j_2,i_3,j_3} \) and the ideal \( I_1 \) is generated by \( Z_{i_1,j_1,i_2,j_2,i_3,j_3} \), which are expressed by Lie brackets of \( X_{i,j} \) and \( X_{i,j,k} \).

Based on the idea of the construction of the 2-holonomy map we discuss a method to construct a representation of the category of braid cobordisms. First, we describe the notion of the category of braid cobordisms. Let us recall that a braid is an embedding of a 1-manifold which is a disjoint union of closed intervals into \( \mathbb{C} \times [0,1] \) so that the projection onto \([0,1]\) has no critical points, and the boundary of the 1-manifold is mapped to \(2n\) points

\[
(1,0), (2,0), \ldots, (n,0), (1,1), (2,1), \ldots, (n,1) \in \mathbb{C} \times [0,1].
\]

The isotopy classes of braids fixing the boundary form the braid group \( B_n \). A braid cobordism between braids \( g \) and \( h \) is a compact surface \( S \) with boundary and corners, smoothly and properly embedded in \( \mathbb{C} \times [0,1]^2 \), such that the following conditions are satisfied.
(1) The boundary of $S$ is the union of 1-manifolds

\[
S \cap (C \times [0, 1] \times \{0\}) = g,  \\
S \cap (C \times [0, 1] \times \{1\}) = h,  \\
S \cap (C \times \{0\} \times [0, 1]) = \{1, 2, \ldots, n\} \times \{0\} \times [0, 1],  \\
S \cap (C \times \{1\} \times [0, 1]) = \{1, 2, \ldots, n\} \times \{1\} \times [0, 1].
\]

(2) The projection of $S$ onto $[0, 1]^2$ is a branched covering with simple branch points only.

Considering the set of braids as a category, we can equip the set of braid cobordisms with a structure of a 2-category, which is denoted by $BC_n$. Here the 2-morphisms are equivalence classes of braid cobordisms with the isotopies fixing the boundary. A braid cobordism is also called a braided surface (see [5] and [14]).

To extend the 2-holonomy map to $BC_n$ we consider the integration of the transport $T$ on one-parameter deformation family of singular braids with double points associated with a braid cobordism. To get a finite value we need to regularize the integral at branched points. This regularization was described in a slightly different setting in [12]. By a regularization we obtain a representation of the category of braid cobordism

\[ Hol_2 : BC_n \longrightarrow \widehat{T\bar{H}}^{+}(X_n)_{\leq 1}/I_1. \]

An approach for a regularization is as follows. In the expression of the transport $T$ an infinite sum of iterated integrals of 1-forms and 2-forms appear, but they are convergent for a one-parameter deformation family of non-singular braids. A possible divergence for a braid with double points for such iterated integrals can be regularized by a method similar to the one used by Le and Murakami [20]. Details of this construction will be discussed in a separate publication. Finally, we refer the reader to Khovanov and Thomas [15] for interesting problems concerning the extension of actions of braids to representations of the category of braid cobordisms.

Acknowledgment. The author would like to thank the referee for valuable comments. The author is partially supported by Grant-in-Aid for Scientific Research, Japan Society of Promotion of Science and by World Premier Research Center Initiative, MEXT, Japan.

References


Kavli IPMU, Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914 Japan

E-mail address: kohno@ms.u-tokyo.ac.jp