ORBIT CONFIGURATION SPACES ASSOCIATED TO DISCRETE SUBGROUPS OF $PSL(2,\mathbb{R})$

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Abstract. The purpose of this article is to analyze several Lie algebras associated to "orbit configuration spaces" obtained from a group G acting freely, and properly discontinuously on the upper 1/2-plane $\mathbb{H}^2$. The Lie algebra obtained from the descending central series for the associated fundamental group is shown to be isomorphic, up to a regrading, to

1. the Lie algebra obtained from the higher homotopy groups of "higher dimensional arrangements" modulo torsion, as well as
2. the Lie obtained from horizontal chord diagrams for surfaces.

The resulting Lie algebras are similar to those studied in [13, 14, 15, 2, 7, 8, 6]. The structure of a related graded Poisson algebra defined below and obtained from an analogue of the infinitesimal braid relations parametrized by G is also addressed.

1. Introduction

The purpose of this article is to consider certain Lie algebras which are described next.

1. One Lie algebra is obtained from the descending central series for the fundamental group of orbit configuration spaces associated to discrete subgroups of $PSL(2,\mathbb{R})$ acting freely, and properly discontinuously on the upper 1/2-plane $\mathbb{H}^2$ by fractional linear transformations.
2. A second Lie algebra is obtained from the classical higher homotopy groups modulo torsion for loop spaces of orbit configuration spaces of points in $\mathbb{H}^2 \times \mathbb{C}^q$.
3. A third Lie algebra is obtained from horizontal chord diagrams for surfaces.

The main results here are that these Lie algebras, apart from a "trivial" degree shift, are isomorphic. In addition, there are natural Poisson algebras obtained from the homology of the iterated loop spaces of these orbit configuration spaces, a structure given below.

Orbit configuration spaces were studied by the third author in [19], and are defined next. Given a manifold M on which the discrete group G acts properly discontinuously, let $\mathrm{Conf}^G(M,n)$ denote the orbit configuration space

$$\mathrm{Conf}^G(M,n) = \{(m_1,\ldots,m_n) \in M^n \mid G\cdot m_i \cap G\cdot m_j = \varnothing \text{ if } \ i \neq j\}.$$

It will be assumed throughout this article that $M \to M/G$ is the projection map for a covering space.

If G is a discrete group, then by [19], there are fibrations

$$\mathrm{Conf}^G(M,n) \to \mathrm{Conf}^G(M,i)$$

with fibre over the point $(p_1,p_2,\cdots,p_i)$ in $\mathrm{Conf}^G(M,i)$ given by

$$\mathrm{Conf}^G(M - Q_i^G, n-i)$$

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where $G \cdot p$ denotes the G-orbit of p, and

$$Q_i^G = \amalg_{1 \leq j \leq i} G \cdot p_j.$$

The groups, and manifolds addressed in this article are given as follows.

1. The space M is either the upper 1/2-plane $\mathbb{H}^2 = SL(2,\mathbb{R})/SO(2)$ or a product $\mathbb{H}^2 \times \mathbb{C}^q$.
2. The group G is a discrete subgroup of $PSL(2,\mathbb{R})$ where it is assumed that G acts freely on $\mathbb{H}^2$ by fractional linear transformations, trivially on $\mathbb{C}^q$, and diagonally on the product $\mathbb{H}^2 \times \mathbb{C}^q$. In particular, the natural projection
$$\mathbb{H}^2 \rightarrow \mathbb{H}^2/G$$
is the projection for a covering space.

Next consider the Lie algebra obtained from the descending central series for a discrete group G. For each strictly positive integer q, there is a canonical (and trivially defined) graded Lie algebra $E_0^*(G)_q$ attached to the one obtained from the descending central series for G, and which is defined as follows.

1. Fix a strictly positive integer q.
2. Let $\Gamma^m(G)$ denote the m-th stage of the descending central series for G.
3. $E_0^{2mq}(G)_q = \Gamma^m(G)/\Gamma^{m+1}(G)$,
4. $E_0^i(G)_q = \{0\}$, if $i \not\equiv 0 \mod 2q$, and
5. the Lie bracket is induced by that for the associated graded for the $\Gamma^m(G)$.

Restrict attention to path-connected topological spaces X which either have torsion free homology or where homology is taken with field coefficients. This technical condition implies the strong form of the Künneth theorem gives that the homology of $X \times X$ is isomorphic to $H_*X \otimes H_*X$. In this case, the module of primitive elements in the homology of X, $\operatorname{Prim} H_*(X)$, is the module generated by elements α which have trivial coproduct. The universal enveloping algebra of a graded Lie algebra L is denoted $U[L]$. Fix a base-point $\mathbf{x} = (x_1, \cdots, x_n)$ in $\operatorname{Conf}(S_g, n)$. Let $\mathbf{y}$ denote a fixed choice of base-point for $\operatorname{Conf}^G(\mathbb{H}^2, n)$, and x_0 a base-point for S_g. The fundamental group of the surface S_g is denoted $\pi_1(S_g, x_0) = G$, and the fundamental group of the orbit configuration space is denoted $\pi_1(\operatorname{Conf}^G(\mathbb{H}^2, n), \mathbf{y}) = P_n(S_g)^0$.

Theorem 1.1. *Let n, and q be fixed natural numbers. There are isomorphisms of Lie algebras*
$$E_0^*(P_n(S_g)^0)_q \rightarrow \operatorname{Prim} H_*\Omega(\operatorname{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)),$$
and
$$U[E_0^*(P_n(S_g)^0)_q] \rightarrow H_*\Omega(\operatorname{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)).$$
Furthermore, these Lie algebras are prescribed as follows:

1. *There are sub-Lie algebras of $E_0^*(P_n(S_g)^0)_q$ given by $L[i]$ the free Lie algebra generated by elements $B_{i,j}^\sigma$ for fixed i with $1 \leq j < i \leq n$ of degree $2q$ and where σ runs over the elements of the group G.*
2. *There is an isomorphism of abelian groups given by the natural additive extension*
$$\bigoplus_{2 \leq i \leq n} L[i] \rightarrow E_0^*(P_n(S_g)^0)_q.$$
3. *A complete set of relations are as follows.*
   (a) *If $\{i,j\} \cap \{s,t\} = \phi$, then $[B_{i,j}^\sigma, B_{s,t}^\tau] = 0$.*
   (b) *If $1 \leq j < s < i \leq k$, then $[B_{i,j}^\tau, B_{i,s}^{\tau\sigma^{-1}} + B_{s,j}^\sigma] = 0$.*
   (c) *If $1 \leq j < s < i \leq k$, then $[B_{s,j}^\sigma, B_{i,j}^\tau + B_{i,s}^{\tau\sigma^{-1}}] = 0$.*
   (d) *The antisymmetry relation, and Jacobi identity for a graded Lie algebra.*

The symmetric group on n-letters Σ_n acts naturally on the configuration space, and thus on the cohomology ring. On the other hand, the symbol $B^{\gamma}_{j,i}$ has not been defined above in the cases for which $j < i$. This element is defined to be $\tau(i,j)(B^{\gamma}_{i,j})$ where $\tau(i,j)$ is the permutation which switches i, and j. The element $\tau(i,j)(B^{\gamma}_{i,j})$ will be shown to be equal to $B^{\gamma^{-1}}_{i,j}$ in Lemma 3.1 of section 3 below. In addition the full action of the symmetric group is also specified in Lemma 3.1. The following additional properties are satisfied for $\mathrm{Conf}^G(\mathbb{H}^2,n)$.

Theorem 1.2. *Let n be fixed natural number. The following properties are satisfied.*

1. *The orbit configuration space $\mathrm{Conf}^G(\mathbb{H}^2,n)$ is a $K(P_n(S_g)^0,1)$. The wreath product of the symmetric group Σ_n with G, $\Sigma_n \wr G$, acts properly discontinuously on $\mathrm{Conf}^G(\mathbb{H}^2,n)$. The orbit space*
$$\mathrm{Conf}^G(\mathbb{H}^2,n)/\Sigma_n \wr G$$
is homeomorphic to $\mathrm{Conf}(\mathbb{H}^2/G,n)/\Sigma_n$, a $K(\Pi,1)$ where Π is the n-stranded braid group for the surface $\mathbb{H}^2/G$.
2. *The fibration $\mathrm{Conf}^G(\mathbb{H}^2,n) \to \mathrm{Conf}^G(\mathbb{H}^2,n-1)$ has (i) trivial local coefficients in homology, and (ii) a cross-section.*
3. *The Lie algebra $E^*_0(\pi_1(\mathrm{Conf}^G(\mathbb{H}^2,n))$ is isomorphic to that given in Theorem 1.1.*
4. *The integral homology of $\mathrm{Conf}^G(\mathbb{H}^2,n)$ is given additively by*
$$H_*\mathrm{Conf}^G(\mathbb{H}^2,n) \cong H_*(C_1)\otimes H_*(C_2)\otimes\cdots\otimes H_*(C_{n-1})$$
where C_i is the infinite bouquet of circles $\bigvee_{|Q^G_i|} S^1$ where Q^G_i as defined as above.

Horizontal chord diagrams of a closed oriented surface of genus g, S_g, are described in section 2 of this article. In case $g > 0$, these horizontal diagrams give analogous constructions as those in genus 0 as given in [14, 15, 16]. Let $\mathcal{A}_n(S_g)^0$ and $\mathcal{A}_n(S_g)$ denote the algebras of horizontal chord diagrams of S_g as introduced, and studied in [11] (see section 2 for the notation).

Theorem 1.3. *The algebra*
$$\mathcal{A}_n(S_g)^0$$
is isomorphic to the universal enveloping algebra of $E^*_0(\pi_1\mathrm{Conf}^G(\mathbb{H}^2,n),\mathbf{y}) = E^*_0(P_n(S_g)^0)$ where
$$S_g = \mathbb{H}^2/G.$$

Work of T. Kohno [13, 14, 15], as well as subsequent work of M. Falk and R. Randell [9] gives the structure of the Lie algebra associated to the descending central series for the k-stranded pure braid group, the fundamental group of $\mathrm{Conf}(\mathbb{R}^2,n)$. Work in [7] gives the associated Lie algebras for loop spaces of $\mathrm{Conf}^L(\mathbb{C}\times\mathbb{C}^q,n)$ where L is the the standard lattice of integral points in $\mathbb{C}$ acting by translations on $\mathbb{C}$. Work of J. González-Meneses, and L. Paris [11] gives the structure of the Lie algebras associated to the descending central series for the fundamental group of $\mathrm{Conf}^G(\mathbb{H}^2,n)$.

These Lie algebras arise in an analysis of Vassiliev invariants for pure braids on a surface. The theorem above gives a comparison between these Lie algebras, and their analogues for certain choices of loop spaces. Analogous groups were shown to be given by morphisms of coalgebras in [6]. In addition, recent work of Papadima, and Suciu [17] give related structures when the associated Lie algebras are of finite type. The ones which occur here for a surface of genus greater than 0 are not of finite type, and it is as yet unclear whether there is an extension of the work in [17] to these choices of Lie algebras.

These Lie algebras also occur in a second convenient context. Namely, by [19] there is a principal G^n bundle

$$\mathrm{Conf}^G(\mathbb{H}^2, n) \to \mathrm{Conf}(\mathbb{H}^2/G, n),$$

and thus there is a short exact sequence of groups

$$1 \to \pi_1(\mathrm{Conf}^G(\mathbb{H}^2, n), \mathbf{y}) \to \pi_1(\mathrm{Conf}(\mathbb{H}^2/G, n), \mathbf{x}) \to G^n \to 1.$$

Hence the kernel of the map

$$\pi_1(\mathrm{Conf}(\mathbb{H}^2/G, n), \mathbf{x}) \to G^n$$

induced by the natural inclusion

$$\mathrm{Conf}(\mathbb{H}^2/G, n) \to (\mathbb{H}^2/G)^n$$

is isomorphic to the fundamental group $\pi_1(\mathrm{Conf}^G(\mathbb{H}^2, n), \mathbf{y}) = P_n(S_g)^0$. The Lie algebra obtained from the descending central series for $\pi_1(\mathrm{Conf}(\mathbb{H}^2/\mathrm{G}, \mathrm{n}), \mathbf{x})$ is not analyzed here. The structure of the analogous Lie algebra associated to $\pi_1(\mathrm{Conf}^G(\mathbb{H}^2, n), \mathbf{y})$ is handled more easily than that for $\pi_1(\mathrm{Conf}(\mathbb{H}^2/G, n), \mathbf{x})$ and is given above, as well as earlier in [11].

There is a natural structure of graded Poisson algebra structure for the homology of an iterated loop space. This Poisson algebra given by

$$H_*\Omega^k(\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n))$$

which is described in more detail in section 6 here. Loosely speaking, the resulting structure is the free Poisson algebra subject to the "Poisson analogues" of the relations in Theorem 1.1. Precise definitions in the next theorem are given in section 4.

Theorem 1.4. *Assume that k is least 1.*

1. *If $k > 1$, the homology of $\Omega^k(\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n))$, with any field coefficients, is a graded Poisson algebra with Poisson bracket given by the Browder operation $\lambda_{k-1}[-,-]$ for the homology of a k-fold loop space.*
2. *If $1 < k < 2q+1$, the homology of $\Omega^k\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$ with coefficients in a field $\mathbb{F}$ of characteristic zero, is the free Poisson algebra generated by elements*

$$B^\sigma_{i,j}$$

of degree $2q+1-k$ for $1 \le j < i \le n$, and σ in G modulo the "infinitesimal Poisson surface braid relations" given as follows:
   (a) *If $\{i,j\} \cap \{s,t\} = \phi$, then $\lambda_{k-1}[B^\sigma_{i,j}, B^\tau_{s,t}] = 0$.*
   (b) *If $1 \le j < s < i \le n$, then $\lambda_{k-1}[B^\tau_{i,j}, B^{\tau\sigma^{-1}}_{i,s} + B^\sigma_{s,j}] = 0$.*
   (c) *If $1 \le j < s < i \le n$, then $\lambda_{k-1}[B^\sigma_{s,j}, B^\tau_{i,j} + B^{\tau\sigma^{-1}}_{i,s}] = 0$.*
   (d) *The antisymmetry relation, and Jacobi identity for a graded Poisson algebra.*
3. *There is a map*

$$E^2 : \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n) \to \Omega^2\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^{q+1}, n)$$

which induces a homology isomorphism in degree $2q+1$. The associated loop map $\Omega(E^2) : \Omega\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n) \to \Omega^3\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^{q+1}, n)$ induces an isomorphism on $H_{2q}(-;\mathbb{Z})$. Furthermore, the image of the map $\Omega(E^2)$ in homology is the subalgebra generated by the classes of degree $2q$.

2. Horizontal chord diagrams for surfaces

Let S_g be a closed oriented surface of genus g. The main subject of this section is a relation between Vassiliev invariants for the braid groups of S_g and horizontal chord diagrams on S_g. We refer the reader to [1] for Vassiliev invariants of knots in S^3 and chord diagrams. Let $\mathrm{Conf}(S_g, n)$ denote the configuration space of ordered distinct n points on S_g as given in section 1. As above, fix a base-point $\mathbf{x} = (x_1, \cdots, x_n) \in \mathrm{Conf}(S_g, n)$. The fundamental group of the configuration space $\pi_1(\mathrm{Conf}(S_g, n), \mathbf{x})$ is by definition the pure braid group of S_g with n strands, and is denoted by $P_n(S_g)$. The symmetric group Σ_n acts freely on $\mathrm{Conf}(S_g, n)$ by permutation of components. The fundamental group of the quotient space $\mathrm{Conf}(S_g, n)/\Sigma_n$ is the braid group of S_g with n strands, and is denoted by $B_n(S_g)$.

Let us now assume $g > 1$. The fundamental group $\pi_1(S_g, x_0)$ with base-point x_0 is identified with a discrete subgroup G of $PSL(2,\mathbb{R})$ acting freely on $\mathbb{H}^2$. The n-fold product G^n acts on the orbit configuration space yielding a covering space projection

$$\mathrm{Conf}^G(\mathbb{H}^2, n) \to \ \mathrm{Conf}(S_g, n)$$

with the covering transformation group G^n. In addition, $\mathrm{Conf}(S_g, n)$ is naturally a subspace of the n-fold cartesian product S_g^n. Thus, the inclusion map $i : \mathrm{Conf}(S_g, n) \to S_g^n$ induces a natural homomorphism

$$i_* : P_n(S_g) \to \times_{j=1}^n \pi_1(S_g, x_j).$$

Let $P_n(S_g)^0$ denote the kernel of i_*. By the above remarks, $P_n(S_g)^0$ is isomorphic to the fundamental group of the orbit configuration space $\mathrm{Conf}^G(\mathbb{H}^2, n)$. In the case $g = 1$, we represent S_g as an elliptic curve $\mathbb{C}/L$ with a lattice L and we define $P_n(S_g)^0$ to be the fundamental group of the orbit configuration space $\mathrm{Conf}^L(\mathbb{C}, n)$. Thus, for any $g \geq 1$, we have an exact sequence

$$1 \to P_n(S_g)^0 \to P_n(S_g) \to \times_{j=1}^n \pi_1(S_g, x_j) \to 1.$$

Let us recall that any element of $P_n(S_g)$ is represented as a collection of disjoint smooth paths $\gamma = (\gamma_1, \cdots, \gamma_n)$ in $S_g \times [0, 1]$, where the i-th string γ_i runs from the point $(x_i, 0)$ to some point $(x_j, 1)$. Such notion of braids is extended to that of singular braids in the following way. We shall say that a collection of smooth paths $\gamma = (\gamma_1, \cdots, \gamma_n)$ in $S_g \times [0, 1]$, where the i-th string γ_i runs monotonically in $t \in [0, 1]$ from the point $(x_i, 0)$ to some point $(x_j, 1)$, is called a singular braid if the paths $\gamma_1, \cdots, \gamma_n$ are allowed to intersect transversely, but only at finitely many double points. The set of isotopy classes of such singular braids is denoted by $\mathcal{S}B_n(S_g)$. Let us notice that $\mathcal{S}B_n(S_g)$ is not a group, but has a structure of a monoid. We denote by $\mathbb{Z}[B_n(S_g)]$ the group ring of $B_n(S_g)$. We have a map

$$\eta : \mathcal{S}B_n(S_g) \to \mathbb{Z}[B_n(S_g)]$$

defined in the following manner. Let γ be an element of $\mathcal{S}B_n(S_g)$ and we denote by $p_1, \cdots, p_k$ its double points. For $\epsilon_1 = \pm 1, \cdots, \epsilon_k = \pm 1$, $\gamma_{\epsilon_1 \cdots \epsilon_k}$ stands for the braid obtained by replacing each double point p_i by a positive or a negative crossing according as ϵ_i is 1 or -1. For the above $\gamma \in \mathcal{S}B_n(S_g)$ we define η by

$$\eta(\gamma) = \sum_{\epsilon_1 = \pm 1, \cdots, \epsilon_k = \pm 1} \epsilon_1 \cdots \epsilon_k \ \gamma_{\epsilon_1 \cdots \epsilon_k}. \tag{2.1}$$

Let $\mathcal{S}_k B_n(S_g)$ denote the set of isotopy classes of singular braids with at least k double points and we define $\mathcal{F}_k$ to be the $\mathbb{Z}$ submodule of $\mathbb{Z}[B_n(S_g)]$ generated by $\eta(\mathcal{S}_k B_n(S_g))$. We have $\mathcal{F}_i \mathcal{F}_j \subset \mathcal{F}_{i+j}$ and it can be shown that each $\mathcal{F}_k$ is a two-sided ideal of $\mathbb{Z}[B_n(S_g)]$. Thus, we obtain a decreasing filtration

$$\mathbb{Z}[B_n(S_g)] = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_k \supset \cdots,$$

which we shall call the Vassiliev filtration. In a similar way, by putting $\mathcal{F}'_k = \mathcal{F}_k \cap \mathbb{Z}[P_n(S_g)]$, we obtain the Vassiliev filtration

$$\mathbb{Z}[P_n(S_g)] = \mathcal{F}'_0 \supset \mathcal{F}'_1 \supset \cdots \supset \mathcal{F}'_k \supset \cdots ,$$

for the pure braid group of S_g. A map $v : B_n(S_g) \to \mathbb{Z}$ is extended linearly to a homomorphim of $\mathbb{Z}$-modules $v : \mathbb{Z}[B_n(S_g)] \to \mathbb{Z}$. We shall say that v is a Vassiliev invariant of order k if v vanishes on the ideal $\mathcal{F}_{k+1}$. The set of Vassiliev invariants of order k with values in $\mathbb{Z}$ is identified with

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[B_n(S_g)]/\mathcal{F}_{k+1}, \mathbb{Z})$$

and we denote this set by $\mathcal{V}_k(B_n(S_g))$. There is an increasing filtration

$$\mathcal{V}_0(B_n(S_g)) \subset \mathcal{V}_1(B_n(S_g)) \subset \cdots \subset \mathcal{V}_k(B_n(S_g)) \subset \cdots$$

A Vassiliev invariant of order k for $P_n(S_g)$ is defined in a similar way and the set of such invariants is identified with

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[P_n(S_g)]/\mathcal{F}'_{k+1}, \mathbb{Z}),$$

which is denoted by $\mathcal{V}_k(P_n(S_g))$.

Our next object is to describe the weight systems for Vassiliev invariants of braids on surfaces. For this purpose we need the notion of horizontal chord diagrams on surfaces. Let $I_1 \sqcup \cdots \sqcup I_n$ be the disjoint union of n unit intervals. Fix a parametrization

$$p_j : [0,1] \to I_j$$

for each j, $1 \le j \le n$. A horizontal chord diagram on n strands with k chords is a 1-dimensional complex constructed in the following way. Fix $t_1, \cdots, t_k \in [0,1]$ such that $0 < t_1 < \cdots < t_k < 1$. Let

$$(i_1, j_1), (i_2, j_2), \cdots, (i_k, j_k)$$

be pairs of distinct integers such that $1 \le i_p \le n$, $1 \le j_p \le n$, $p = 1, 2, \cdots, n$. Consider k copies of parametrized unit intervals $C_1, C_2, \cdots, C_k$ and attach each C_ν to $I_1 \sqcup \cdots \sqcup I_n$ in such a way that it starts at $p_{i_\nu}(t_\nu)$ and ends at $p_{j_\nu}(t_\nu)$ for $1 \le \nu \le k$. In this way, we obtain a 1-dimensional complex with n strands $I_1 \sqcup \cdots \sqcup I_n$ and chords $C_1, \cdots, C_k$ attached to them. Such a 1-dimensional complex is called a horizontal chord diagram on n strands with k chords. Here each strand and each chord are oriented by the above parametrizations. In what follows below, chord diagrams up to orientation preserving homeomorphism will be considered.

Let C_n^k denote the set of horizontal chord diagrams on n strands with k chords. For $\Gamma \in C_n^k$ consider a continuous map $f : \Gamma \to S_g$ such that

$$f(p_i(0)) = f(p_i(1)) = x_i, \ 1 \le i \le n \tag{2.2}$$

and denote by $[f]$ its homotopy class. Here, we consider the homotopy preserving the condition (2.2). Let $D_n^k(S_g)$ denote the free $\mathbb{Z}$-module spanned by pairs $(\Gamma, [f])$ for $\Gamma \in C_n^k$ and $f : \Gamma \to S_g$, a continuous map with the condition (2.2). The subspace of $D_n^k(S_g)$ spanned by $f : \Gamma \to S_g$, such that each curve $f(p_i(t)), 0 \le t \le 1$, is homotopic to the point $\{x_i\}$ is denoted by $D_n^k(S_g)^0$.

Let Γ_{ij} denote a horizontal chord diagram on n strands with one chord C_{ij} defined by the pair (i,j), $i \ne j$, $1 \le i,j \le n$. Recall the base-point $x_0 \in S_g$ and consider the fundamental group $\pi_1(S_g, x_0)$. Next fix a path in S_g connecting x_0 to x_j, and identify the set of homotopy classes of paths from x_i to x_j with $\pi_1(S_g, x_0)$. For $\gamma \in \pi_1(S_g, x_0)$ consider $(\Gamma_{ij}, [f]) \in D_n^k(S_g)^0$ such that $f(C_{ij})$ corresponds to $\gamma \in \pi_1(S_g, x_0)$ by the above identification, and denote this choice for $(\Gamma_{ij}, [f])$ by $X_{i,j}^{\gamma}$.

Notice that the direct sum

$$D_n(S_g)^0 = \bigoplus_{k\geq 0} D_n^k(S_g)^0$$

has a structure of an algebra over $\mathbb{Z}$ where the product is defined by the concatenation of chord diagrams. As an algebra $D_n^k(S_g)^0$ is generated by $X_{i,j}^{\gamma}$, $1 \leq i \neq j \leq n$, $\gamma \in \pi_1(S_g, x_0)$. From the definition it follows immediately that the relation $X_{i,j}^{\gamma} = X_{j,i}^{\gamma^{-1}}$ holds. It turns out that $D_n(S_g)^0$ is a non-commutative associative algebra freely generated by $X_{i,j}^{\gamma}$, $1 \leq i < j \leq n$ over $\mathbb{Z}$. The direct sum

$$D_n(S_g) = \bigoplus_{k\geq 0} D_n^k(S_g)$$

has a structure of an associative algebra as well. For the subspace $D_n^0(S_g)$ spanned by the chord diagrams with empty chord, there is a natural injection

$$\iota_j : \pi_1(S_g, x_j) \to D_n^0(S_g),\ 1 \leq j \leq n,$$

which induces an isomorphism of $\mathbb{Z}$ algebras

$$\mathbb{Z}[G^n] \cong D_n^0(S_g)$$

with $G = \pi_1(S_g, x_0)$. Here we fix an isomorphism $\pi_1(S_g, x_0) \cong \pi_1(S_g, x_j)$ by means of a path from x_0 to x_j. The algebra

$$\Lambda_n = \mathbb{Z}[G^n]$$

acts on $D_n(S_g)^0$ by the conjugation

$$\Gamma \mapsto \iota_j(\mu)\ \ \Gamma\ \iota_j(\mu^{-1}),\ \Gamma \in D_n(S_g)^0,\ \mu \in \pi_1(S_g, x_0).$$

Lemma 2.1. *With respect to the above action, the following holds.*

$$\begin{aligned}
&\iota_l(\mu)\ X_{i,j}^{\gamma}\ \iota_l(\mu^{-1}) = X_{i,j}^{\gamma} \quad \textit{for } l \neq i, j,\\
&\iota_i(\mu)\ X_{i,j}^{\gamma}\ \iota_i(\mu^{-1}) = X_{i,j}^{\mu\gamma},\\
&\iota_j(\mu)\ X_{i,j}^{\gamma}\ \iota_j(\mu^{-1}) = X_{i,j}^{\gamma\mu^{-1}}.
\end{aligned}$$

Proof. Consider the concatenation of the chord diagrams $\iota_l(\mu)$ and $X_{i,j}^{\gamma}$, where $\iota_l(\mu)$ is a chord diagram with empty chord and the chord of $X_{i,j}^{\gamma}$ is considered to be the element γ in $\pi_1(S_g, x_0)$. We construct a homotopy of chord diagrams such that the initial point of the above chord slides along the loop $\mu \in \pi_1(S_g, x_0)$ in the negative direction. The resulting chord diagram $X_{i,j}^{\mu\gamma}\ \iota_l(\mu)$ is homotopic to $\iota_l(\mu) X_{i,j}^{\gamma}$. This shows the second equality. The other equalities are shown in a similar way. □

The algebra $D_n(S_g)$ is considered to be the semidirect product

$$D_n(S_g)^0 \rtimes \Lambda_n$$

by the above action of Λ_n on $D_n(S_g)^0$. Write $X_{i,j}$ for $X_{i,j}^{e}$. Let $\mathcal{I}$ be the two-sided ideal of $D_n(S_g)$ generated by

$$\begin{aligned}
&[X_{i,j}, X_{s,t}], \quad i, j, s, t \text{ distinct},\\
&[X_{i,j}, X_{j,s} + X_{i,s}], \quad i, j, s \text{ distinct}.
\end{aligned}$$

Then we set

$$\mathcal{A}_n(S_g) = D_n(S_g)/\mathcal{I},\ \ \mathcal{A}_n(S_g)^0 = D_n(S_g)^0/\mathcal{I} \cap D_n(S_g)^0.$$

There is an action of Λ_n on $\mathcal{A}_n(S_g)^0$ by conjugation and the semidirect product

$$\mathcal{A}_n(S_g)^0 \rtimes \Lambda_n$$

with respect to this action is isomorphic to $\mathcal{A}_n(S_g)$.

Lemma 2.2. *The algebra $\mathcal{A}_n(S_g)^0$ is generated by $X^\gamma_{i,j}$, $1 \le i \ne j \le n$, $\gamma \in \pi_1(S_g, x_0)$, with relations:*

$$\begin{aligned} &X^\gamma_{i,j} = X^{\gamma^{-1}}_{j,i} \\ &[X^\gamma_{i,j}, X^\delta_{s,t}] = 0, \quad i,j,s,t \text{ distinct}, \\ &[X^\gamma_{i,j}, X^\delta_{j,s} + X^{\gamma\delta}_{i,s}] = 0 \quad i,j,s \text{ distinct}. \end{aligned}$$

Proof. The first two relations clearly hold from the definition. Let us show that the the relation $[X^\gamma_{i,j}, X^\delta_{j,s} + X^{\gamma\delta}_{i,s}] = 0$ holds in $\mathcal{A}_n(S_g)^0$ for distinct i,j,s. From Lemma 2.1 we obtain

$$[X^\gamma_{i,j}, X^\delta_{j,s}] = \iota_i(\gamma)\iota_s(\delta^{-1})[X_{i,j}, X_{j,s}]\iota_i(\gamma^{-1})\iota_s(\delta).$$

Similarly, we have

$$[X^\gamma_{i,j}, X^{\gamma\delta}_{i,s}] = \iota_i(\gamma)\iota_s(\delta^{-1})[X_{i,j}, X_{i,s}]\iota_i(\gamma^{-1})\iota_s(\delta).$$

Hence the relation follows from $[X_{i,j}, X_{j,s} + X_{i,s}] = 0$. Since $D_n(S_g)$ is the semidirect product $D_n(S_g)^0 \rtimes \Lambda_n$ the ideal $\mathcal{I} \cap D_n(S_g)^0$ is generated by the conjugation by Λ_n of the generators of $\mathcal{I}$. This shows the desired statement. □

The algebras $\mathcal{A}_n(S_g)^0$ and $\mathcal{A}_n(S_g)$ were introduced in [11] by the above generators and relations. These algebras are considered to be graded algebras by defining $\deg X^\gamma_{i,j} = 1$, $1 \le i \ne j \le n$, $\gamma \in \pi_1(S_g, x_0)$, and $\deg g = 0$ for any $g \in \Lambda_n$. There is a direct sum decomposition

$$\mathcal{A}_n(S_g)^0 = \bigoplus_{k \ge 0} \mathcal{A}^k_n(S_g)^0, \quad \mathcal{A}_n(S_g) = \bigoplus_{k \ge 0} \mathcal{A}^k_n(S_g),$$

where $\mathcal{A}^k_n(S_g)^0$ and $\mathcal{A}^k_n(S_g)$ denote the degree k parts. Let v be a Vassiliev invariant of order k for $P_n(S_g)$. Then there is a $\mathbb{Z}$-module homomorphism $w(v) : D_n(S_g) \to \mathbb{Z}$ defined in the following way. Let us take $\delta \in D^k_n(S_g)$ represented by $f : \Gamma \to S_g$, where Γ is a horizontal chord diagram on n strands with k chords. Contracting each chord on S_g, we may consider δ as a projection diagram of a singular pure braid on S_g with k transverse double points. We set

$$w(v)(\delta) = \sum_{\epsilon_1 = \pm 1, \cdots, \epsilon_k = \pm 1} \epsilon_1 \cdots \epsilon_k \; v(\delta_{\epsilon_1 \cdots \epsilon_k}), \tag{2.3}$$

where $\delta_{\epsilon_1 \cdots \epsilon_k}$ stands for the pure braid on S_g obtained from δ by replacing each double point with positive or negative crossing according as ϵ_j is 1 or -1 for $1 \le j \le k$ as in (2.1). Since v is an order k invariant the above expression is well defined. Thus, we obtain a $\mathbb{Z}$-module homomorphism

$$w : \mathcal{V}_k(P_n(S_g)) \to \mathrm{Hom}_{\mathbb{Z}}(D^k_n(S_g), \mathbb{Z}).$$

Lemma 2.3. *For $v \in \mathcal{V}_k(P_n(S_g))$ the associated map $w(v)$ vanishes on the ideal $\mathcal{I}$ and induces a homomorphism*

$$w : \mathcal{V}_k(P_n(S_g)) \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{A}^k_n(S_g), \mathbb{Z})$$

Proof. Consider the 4 chord diagrams Γ_j, $1 \le j \le 4$, in $D^k_n(S_g)$ expressed as

$$\Gamma_1 = YX_{i,j}X_{j,s}Z, \; \Gamma_1 = YX_{j,s}X_{i,j}Z$$
$$\Gamma_3 = YX_{i,j}X_{i,s}Z, \; \Gamma_4 = YX_{i,s}X_{i,j}Z$$

with some fixed $Y, Z \in D_n(S_g)$. Contracting the chords on S_g, we represent Γ_j, $1 \le j \le 4$, as singular pure braids on S_g. Replacing the double points by positive or negative crossings and applying the definition 2.3, we see that the relation

$$w(v)(\Gamma_1) - w(v)(\Gamma_2) + w(v)(\Gamma_3) - w(v)(\Gamma_4) = 0$$

holds. This completes the proof. □

The above $w(v)$ is called the weight system of the Vassiliev invariant v. To define the weight system of a Vassiliev invariant for $B_n(S_g)$, we need to extend the algebra $\mathcal{A}_n(S_g)$ by the group algebra of the symmetric group Σ_n in the following way. We define $\widetilde{\Lambda}_n$ to be the group algebra of the semidirect product

$$G^n \rtimes \Sigma_n$$

where the symmetric group acts as permutations. We define $\widetilde{\mathcal{A}}_n(S_g)$ as the semidirect product

$$\mathcal{A}_n(S_g)^0 \rtimes \widetilde{\Lambda}_n.$$

where the action of Σ_n on $\mathcal{A}_n(S_g)^0$ is defined by

$$\sigma X^\gamma_{i,j}\sigma^{-1} = X^\gamma_{\sigma(i),\sigma(j)},\ \sigma \in \Sigma_n$$

and the action of G^n is the conjugation action defined before. The algebra $\widetilde{\mathcal{A}}_n(S_g)$ has a structure of a graded algebra as an extension of the graded algebra $\mathcal{A}_n(S_g)$ by defining $\deg g = 0$ for any $g \in \widetilde{\Lambda}_n$. We denote the degree k part by $\widetilde{\mathcal{A}}^k_n(S_g)$. The weight system for $B_n(S_g)$ is defined by a natural $\mathbb{Z}$-module homomorphism

$$w : \mathcal{V}_k(B_n(S_g)) \to \mathrm{Hom}_{\mathbb{Z}}(\widetilde{\mathcal{A}}^k_n(S_g), \mathbb{Z}).$$

Let us consider the graded algebras $\mathrm{gr}\,\mathbb{Z}[B_n(S_g)]$ and $\mathrm{gr}\,\mathbb{Z}[P_n(S_g)]$ associated with the Vassiliev filtration. Namely, we set

$$\mathrm{gr}\,\mathbb{Z}[B_n(S_g)] = \bigoplus_{k\geq 0} \mathcal{F}_k/\mathcal{F}_{k+1}, \quad \mathrm{gr}\,\mathbb{Z}[P_n(S_g)] = \bigoplus_{k\geq 0} \mathcal{F}'_k/\mathcal{F}'_{k+1}.$$

The following theorem was shown by J. González-Meneses and L. Paris.

Theorem 2.4 ([11]). *There exists a homomorphism of $\mathbb{Z}$-modules $u : \mathbb{Z}[B_n(S_g)] \to \widetilde{\mathcal{A}}_n(S_g)$ such that the associated graded map gives an isomorphism of graded $\mathbb{Z}$ algebras*

$$\mathrm{gr}\,\mathbb{Z}[B_n(S_g)] \cong \widetilde{\mathcal{A}}_n(S_g).$$

The restriction of the above homomorpshism u to $\mathbb{Z}[P_n(S_g)]$ gives an isomorphism of graded $\mathbb{Z}$-algebras

$$\mathrm{gr}\,\mathbb{Z}[P_n(S_g)] \cong \mathcal{A}_n(S_g).$$

In particular, there is an isomorphism of $\mathbb{Z}$-modules

$$\mathcal{F}'_k/\mathcal{F}'_{k+1} \cong \mathcal{A}^k_n(S_g).$$

Now the statement of Theorem 1.3 follows from Theorem 1.1 together with Lemma 2.2. It can also be shown directly using an argument due to J. González-Meneses and L. Paris in the proof of Theorem 2.4 in the following way. The restriction of the Vassiliev filtration for $\mathbb{Z}[P_n(S_g)]$ on $\mathbb{Z}[P_n(S_g)^0]$ coincides with the filtration given by the powers of the augmentation ideal I of $\mathbb{Z}[P_n(S_g)^0]$. Hence, the associated graded algebra

$$\mathrm{gr}_I\mathbb{Z}[P_n(S_g)^0] = \bigoplus_{k\geq 0} I_k/I_{k+1}$$

is isomorphic to the graded algebra $\mathcal{A}_n(S_g)^0$. It can be shown that the successive quotients of the descending central series $\Gamma^m(P_n(S_g)^0)/\Gamma_{m+1}(P_n(S_g)^0)$, $m = 0, 1, \cdots$, is a free $\mathbb{Z}$-module. Therefore, using a theorem due to Quillen [18], we obtain the isomorphism of graded algebras $U[E^*_0(P_n(S_g)^0)] \cong \mathrm{gr}_I\mathbb{Z}[P_n(S_g)^0]$. This shows the isomorphism of graded algebras $U[E^*_0(P_n(S_g)^0)] \cong \mathcal{A}_n(S_g)^0$.

The exceptional case of $g = 1$ was considered in articles [15, 7]. This case arises by considering horizontal chord diagrams for elliptic curves [15]. Analogous information associated to the orbit configuration space $\mathrm{Conf}^L(\mathbb{C}, n)$ where L denotes a fixed lattice in the complex plane $\mathbb{C}$ such that $\mathbb{C}/L$ is a fixed elliptic curve is given in [7].

3. The Lie algebra relations in Theorem 1.1

The purpose of this section is to define the elements $B^{\tau}_{i,j}$, and to derive the relations among these elements as stated in Theorem 1(3). To analyze this structure, it is convenient to construct explicit cycles in $\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$ as follows.

Choose points $p_1, p_2, \cdots, p_n$ in distinct orbits of G in $\mathbb{H}^2 \times \mathbb{C}^q$. Let σ, and τ denote elements of G. Given fixed i, and j fixed with $1 \leq j < i \leq n$, and σ in G, choose a closed neighborhood U homeomorphic to a closed disk with the following properties.

1. U contains the interior points p_j, together with the points $\sigma(p_j) + z/\lambda$ for λ a fixed real scalar for all z in $\mathbb{H}^2 \times \mathbb{C}^q$ of norm 1 where the hyperbolic metric is used for $\mathbb{H}^2$,
2. U intersects each orbit exactly once.
3. U does not contain the p_t for all t with $t \neq j$.

Define maps

$$A^{\sigma}_{i,j} : S^{2q+1} \to \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$$

by the formula

$$A^{\sigma}_{i,j}(z) = (w_1, w_2, \cdots, w_k)$$

where

1. $w_t = p_t$ if $t \neq i$, and
2. $w_i = \sigma(p_j) + z/\lambda$.

Fix a choice of fundamental cycle ι for S^{2q+1} for what follows below. The image in homology ${A^{\sigma}}_{i,j*}(\iota)$ will be denoted by the "name" $A^{\sigma}_{i,j}$ in $H_{2q+1}\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$. In addition, the "name" $B^{\sigma}_{i,j}$ is also used ambiguously for

1. the analogous element in $\pi_1(\mathrm{Conf}^G(\mathbb{H}^2, n), \mathbf{x})$ denote , and
2. the image of the transgression of the spherical class $A^{\sigma}_{i,j}$ in $H_{2q}\Omega\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$.

Relations among the elements $A^{\sigma}_{i,j}$ are obtained as follows. The first relation is given by the next lemma.

Lemma 3.1. *The relation*

$$\tau_{(i,j)} A^{\sigma}_{i,j} = A^{\sigma^{-1}}_{i,j}$$

holds in

$$H_{2q+1}\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n), \textit{and}$$
$$E^*_0(\pi_1(\mathrm{Conf}^G(\mathbb{H}^2, n), \mathbf{x}))$$

where $\tau_{(i,j)}$ denotes the transposition which interchanges i, and j in the symmetric group Σ_n.

Furthermore, the action of an element γ in the symmetric group on the element $A^{\sigma}_{i,j}$ is specified by the formula

$$\gamma(A^{\sigma}_{i,j}) = \begin{cases} A^{\sigma}_{\gamma(i),\gamma(j)} & \text{if } \gamma(i) > \gamma(j), \\ A^{\sigma^{-1}}_{\gamma(j),\gamma(i)} & \text{if } \gamma(i) < \gamma(j). \end{cases}$$

Proof. First recall the standard homotopy for the classical configuration space with $G = \{identity\}$. Fix points $p_m = 4me_1$, in $\mathbb{C}^q$ for $1 \leq m \leq n$ for which e_1 is a unit vector. Define

$$h : [0,1] \times S^{2q-1} \to \mathrm{Conf}(\mathbb{C}^q, n)$$

by the formula

$$h(t, z) = (x_1, x_2, \cdots, x_n)$$

for which

1. $x_s = p_s$ if $s \neq i, j$,
2. $x_j = p_j - tz$,
3. $x_i = p_i + (1-t)z$.

Notice that the point $(x_1, x_2, \cdots, x_n)$ is in the configuration space, and that the map h is continuous. In particular, each x_s is a function of t with $x_j(0) = p_j$ with

1. $x_j(0) = p_j$,
2. $x_j(1) = p_j - z$,
3. $x_i(0) = p_i + z$, and
4. $x_i(1) = p_i$.

Thus the maps

1. $\tau(A_{i,j}) : S^{2q-1} \to \mathrm{Conf}(\mathbb{C}^q, n)$, and
2. $A_{i,j} : S^{2q-1} \to \mathrm{Conf}(\mathbb{C}^q, n)$

are homotopic.

An analogous homotopy is constructed next for $A^{\sigma}_{i,j}$ as follows. The small disk U is regarded as the standard unit disk in $\mathbb{R}^{2q+2}$. Define

$$H : [0,1] \times S^{2q+1} \to \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$$

by the formula

$$H(t,z) = (y_1, y_2, \cdots, y_n)$$

for which

1. $y_s = p_s$ if $s \neq i, j$,
2. $y_j = p_j - tz/\lambda$, and
3. $y_i = \sigma(p_j) + (1-t)z/\lambda$.

Hence H is a homotopy of $A^{\sigma}_{i,j}$ (a homotopy which is non-base-point preserving). Notice that $(\tau_{(i,j)} \circ H)(t,z) = (u_1, u_2, \cdots, u_k)$ where

1. $u_s = p_s$ if $s \neq i, j$,
2. $u_j = \sigma(p_j) + (1-t)z/\lambda$, and
3. $u_i = p_j - tz/\lambda$.

Thus $\tau_{(i,j)}(A^{\sigma}_{i,j})$ is homotopic to the map which sends z to $(u_1, u_2, \cdots, u_n)$ where

1. $u_s = p_s$ if $s \neq i, j$,
2. $u_j = \sigma(p_j)$, and
3. $u_i = p_j - z/\lambda$.

which is given by $A^{\sigma^{-1}}_{i,j}$ up to (a non-base-point preserving) homotopy.

A similar computation applies to $\gamma(A^{\sigma}_{i,j})$, and is deleted. This suffices. $\square$

Analogous maps, and homotopies are considered next. Fix integers $1 \leq j < s < i \leq n$, and define maps

$$D(\sigma, \tau, i, s, j) : S^{2q+1} \times S^{2q+1} \to \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n),$$

and

$$G(\sigma, \tau, i, s, j) : S^{2q+1} \times S^{2q+1} \to \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$$

by the following formulas:

1. $$D(\sigma, \tau, i, s, j)(u, v) = (w_1, w_2, \cdots, w_n)$$
   where
   (a) $w_t = p_t$ if $t \neq i, s, j$,
   (b) $w_j = p_j$,
   (c) $w_s = \sigma(p_j) + u/8\lambda$, and
   (d) $w_i = \tau(p_j) + v/16\lambda$.

2.
$$G(\sigma,\tau,i,s,j)(u,v) = (w_1, w_2, \cdots, w_n)$$
where
(a) $w_t = p_t$ if $t \neq i, s, j$,
(b) $w_j = p_j$,
(c) $w_s = \sigma(p_j) + v/16\lambda$, and
(d) $w_i = \tau(p_j) + u/8\lambda$.

Write $a \otimes 1$, and $1 \otimes b$ for the respective generators in $H_{2q+1}(S^{2q+1} \times S^{2q+1})$ associated to the individual fundamental cycles of each axis. Consider the induced map in homology for

$$D(\sigma,\tau,i,s,j)_* : H_{2q+1}(S^{2q+1} \times S^{2q+1}) \to \ H_{2q+1}\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n).$$

Notice that $H_{2q+1}\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$ is free abelian with basis

$$\{A^\sigma_{i,j} \mid \sigma \in G, 1 \leq j < i \leq n\}$$

and where the projection to the i, and j coordinates

$$p_{i,j} : \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n) \to \ \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, 2)$$

is non-trivial in homology when restricted to $A^\sigma_{i,j}$. Thus projecting gives the following formulas:

1. $p_{i,j}D(\sigma,\tau,i,s,j)(u,v) = (p_j, \tau(p_j) + v/16\lambda)$ which is homotopic to $A^\tau_{i,j}(v)$,
2. $p_{i,s} \circ D(\sigma,\tau,i,s,j)(u,v) = (\sigma(p_j) + u/8\lambda, \tau(p_j) + v/16\lambda)$ which is homotopic to $A^{\tau\sigma^{-1}}_{i,s}(u)$,
3. $p_{s,j}D(\sigma,\tau,i,s,j)(u,v) = (p_j, \sigma(p_j) + u/8\lambda)$ which is homotopic to $A^\sigma_{s,j}(u)$, and
4. $p_{\alpha,\beta}D(\sigma,\tau,i,s,j)(u,v)$ is null otherwise.

Lemma 3.2. *The following formulas are satisfied:*

1. $D(\sigma,\tau,i,s,j)_*(a \otimes 1) = A^{\tau\sigma^{-1}}_{i,s} + A^\sigma_{s,j}$,
2. $D(\sigma,\tau,i,s,j)_*(1 \otimes b) = A^\tau_{i,j}$, *and*
3. $[B^{\tau\sigma^{-1}}_{i,s} + B^\sigma_{s,j}, B^\tau_{i,j}] = 0$ *in* $H_*(\Omega\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n))$, *and in* $E^*_0(\pi_1\mathrm{Conf}^G(\mathbb{H}^2, n), \mathbf{x})$.

Proof. Notice that formula 1 follows from the above remarks. Furthermore, in case $n > 0$, the elements $B^\tau_{i,j}$ are defined to be the adjoint of the elements $A^\tau_{i,j}$, and thus the relation 2 holds in homology.

In addition, formula 2 is satisfied on the level of the associated graded as the homotopy D gives that $[B^{\tau\sigma^{-1}}_{i,s} \cdot B^\sigma_{s,j}, B^\tau_{i,j}]$ lies in $\Gamma^2(\pi_1\mathrm{Conf}^G(\mathbb{H}^2, n), \mathbf{x})$, the second stage of the descending central series. The lemma follows. $\square$

Similar formulas apply to $G(\sigma,\tau,i,s,j)(u,v)$.

1. $p_{i,j}G(\sigma,\tau,i,s,j)(u,v) = (p_j, \tau(p_j) + u/8\lambda)$ which is homotopic to $A^\tau_{i,j}(u)$,
2. $p_{i,s}G(\sigma,\tau,i,s,j)(u,v) = (\sigma(p_j) + v/16\lambda, \tau(p_j) + u/8\lambda)$ which is homotopic to $A^{\tau\sigma^{-1}}_{i,s}(u)$,
3. $p_{s,j}G(\sigma,\tau,i,s,j)(u,v) = (p_j, \sigma(p_j) + v/16\lambda)$ which is homotopic to $A^\sigma_{s,j}(v)$, and
4. $p_{\alpha,\beta}G(\sigma,\tau,i,s,j)(u,v)$ is null otherwise.

In addition, the following formulas are satisfied:

1. $G(\sigma,\tau,i,s,j)_*(a \otimes 1) = A^\tau_{i,j} + A^{\tau\sigma^{-1}}_{i,s}$,
2. $G(\sigma,\tau,i,s,j)_*(1 \otimes b) = A^\sigma_{s,j}$, and
3. $[B^\sigma_{s,j}, B^\tau_{i,j} + B^{\tau\sigma^{-1}}_{i,s}] = 0$.

The statement and proof of the next lemma are analogous.

Lemma 3.3. *The following formulas are satisfied:*

1. $G(\sigma,\tau,i,s,j)_*(a\otimes 1) = A^{\tau}_{i,j} + A^{\tau\sigma^{-1}}_{i,s}$,
2. $G(\sigma,\tau,i,s,j)_*(1\otimes b) = A^{\sigma}_{s,j}$, *and*
3. $[B^{\sigma}_{s,j}, B^{\tau}_{i,j}+B^{\tau\sigma^{-1}}_{i,s}] = 0$ *in* $H_*(\Omega\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n))$, *and in* $E^*_0(\pi_1\mathrm{Conf}^G(\mathbb{H}^2,n),\mathbf{x})$.

Define

$$H(\sigma,\tau,i,j,s,t) : S^{2q+1}\times S^{2q+1} \to \mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)$$

by the following formulas where $1\le j<i<s<t$:

$$H(\sigma,\tau,i,s,j)(u,v) = (w_1,w_2,\cdots,w_n)$$

where

1. $w_m = p_m$ if $m\neq i,j,s,t$,
2. $w_j = p_j$,
3. $w_i = \sigma(p_j) + u/8\lambda$,
4. $w_s = p_s$, and
5. $w_t = \tau(p_s) + v/8\lambda$.

Thus the following formulas are satisfied:

1. $H(\sigma,\tau,i,j,s,t)_*(a\otimes 1) = A^{\sigma}_{i,j}$,
2. $H(\sigma,\tau,i,j,s,t)_*(1\otimes b) = A^{\tau}_{t,s}$,and
3. $[B^{\sigma}_{i,j}, B^{\tau}_{s,t}] = 0$ in $H_*(\Omega\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n))$ if $1\le j<i<s<t$.
4. The other cases for which $\{i,j\}\cap\{s,t\}=\phi$ are omitted as the details are similar.

Lemma 3.4. *The following formulas are satisfied:*

1. $H(\sigma,\tau,i,j,s,t)_*(a\otimes 1) = A^{\sigma}_{i,j}$,
2. $H(\sigma,\tau,i,j,s,t)_*(1\otimes b) = A^{\tau}_{t,s}$, *and*
3. $[B^{\sigma}_{i,j}, B^{\tau}_{s,t}] = 0$ *in* $H_*(\Omega\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n))$ *in* $H_*(\Omega\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n))$, *and in* $E^*_0(\pi_1\mathrm{Conf}^G(\mathbb{H}^2,n),\mathbf{x})$ *if* $\{i,j\}\cap\{s,t\}=\phi$.

The proof of this lemma is similar to the previous two, and is omitted. The relations stated in Theorem 1.1 follow.

4. Proof of Theorem 1.1

Recall the following analogue of results due to Fadell and Neuwirth [10] for "usual" configuration spaces, and which apply to orbit configuration spaces [19].

Lemma 4.1. *Let M be a manifold with a free, and properly discontinuous action of a group G. Then for $j\le n$, the projection $p:\mathrm{Conf}^G(M,n)\to \mathrm{Conf}^G(M,j)$ onto the first j coordinates is a locally trivial bundle, with fibre $\mathrm{Conf}^G(M-Q^G_j,n-j)$.*

It does not suffice to use free actions in the above lemma. The extra hypothesis that the action be properly discontinuous is required for the case when G is not finite. Notice that the hypotheses are trivially satisfied when G is a finite group acting freely on a Hausdorff space M.

The next lemma guarantees cross-sections for these fibrations in favorable cases.

Lemma 4.2. *Let M be a parallelizable manifold with a free, and properly discontinuous action of a group G such that the quotient map $M\to M/G$ is the projection for a covering space. Then for $j<n$, the projection $p:\mathrm{Conf}^G(M,n)\to \mathrm{Conf}^G(M,j)$ admits a cross-section. Thus if M is simply-connected of dimension at least 3, then there are homotopy equivalences*

$$\Omega(\mathrm{Conf}^G(M,j))\times\Omega(\mathrm{Conf}^G(M-Q^G_j,n-j))\to \Omega(\mathrm{Conf}^G(M,n)).$$

Thus there is a homotopy equivalence

$$\times_{0\le j\le n-1}\Omega(M-Q^G_j)\to \Omega(\mathrm{Conf}^G(M,n)).$$

Proof. The exponential map with source the tangent bundle of M, $exp : \tau(M) \to M$, gives a local homeomorphism from $\mathbb{R}^m$ to an open set in M (where M is a manifold of dimension m). Choose n elements $y_1, y_2, \cdots, y_n$ in $\mathbb{R}^m$ in M with $y_1 = 0$, such that $x_1, x_2, \cdots, x_n$ for $x_i = exp(y_i)$ lie in n distinct orbits as the action of G is properly discontinuous. A cross-section is defined by by $\sigma(x) = (x_1, x_2, \cdots, x_n)$ where $x_i = exp(y_i)$.

Notice that there are fibrations

$$\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n) \to \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n-1)$$

with both (1) cross-sections, and (2) fibres given by

$$\mathbb{H}^2 \times \mathbb{C}^q - Q^G_{n-1}.$$

Since any multiplicative fibration with section is homotopy equivalent to a product, there are homotopy equivalances

$$\prod_{0 \leq i \leq n-1} \Omega(\mathbb{H}^2 \times \mathbb{C}^q - Q^G_i) \to \Omega(\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$$

for $q \geq 1$. The lemma follows. ∎

The previous lemma gives the proof of Theorem 1.1 in case $q > 0$. That is the maps $\Omega(\mathbb{H}^2 \times \mathbb{C}^q - Q^G_i) \to \Omega(\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$ for $q \geq 1$ induce multiplicative maps on the level of homology groups which are split monomorphisms. The only relations satisfied are those given already as the homology of $\Omega(\mathbb{H}^2 \times \mathbb{C}^q - Q^G_i)$ is a tensor algebra.

It suffices to prove the theorem in case $q = 0$. The proof in this case follows at once from Theorem 1.2 which is proven in the next section.

5. Proof of Theorem 1.2

Fix a base-point y_n in $\mathrm{Conf}^G(\mathbb{H}^2, n)$. The direct sum decomposition of Lie algebras given in Theorem 1.2 follows from the split, short exact sequence of groups with trivial local coefficients given by

$$1 \to \pi_1(\mathbb{H}^2 - Q^G_{n-1}, x) \to \pi_1(\mathrm{Conf}^G(\mathbb{H}^2, n), \mathbf{y}_n) \to \pi_1(\mathrm{Conf}^G(\mathbb{H}^2, n-1), \mathbf{y}_{n-1}) \to 1.$$

A split, short exact sequence of Lie algebras as restated in the next lemma is given in [13, 9, 20].

Lemma 5.1. *Let*

$$1 \to A \to B \to C \to 1$$

be a split short exact sequence of groups for which the conjugation of C on $H_1(A)$ is trivial. Then there is a split short exact sequence of Lie algebras

$$0 \to E_0^{i*}(A) \to E_0^{i*}(B) \to E_0^{i*}(C) \to 0.$$

Hence the proof of Theorem 1.2 rests on the proof of the triviality of the local coefficient system for the fibration

$$\mathbb{H}^2 - Q^G_{n-1}, \to \mathrm{Conf}^G(\mathbb{H}^2, n) \to \mathrm{Conf}^G(\mathbb{H}^2, n-1),$$

a proof analogous to that given in [7], and in Lemma 5.5. Since the homology of the fibre is concentrated in degree one, and the fibration has a cross-section, the Serre spectral sequence for these fibrations collapses. The next proposition gives the additive structure for the integral homology of $\mathrm{Conf}^G(\mathbb{H}^2, n)$ as stated in Theorem 1.2.

Proposition 5.2. *The integral homology of* $\mathrm{Conf}^G(\mathbb{H}^2, n)$ *is additively given by*

$$H_*\mathrm{Conf}^G(\mathbb{H}^2, n) \cong H_*(C_1) \otimes H_*(C_2) \otimes \cdots \otimes H_*(C_{n-1})$$

where C_i is the infinite bouquet of circles $\bigvee_{|Q^G_i|} S^1$.

Notice that the homology of $\mathrm{Conf}^G(\mathbb{H}^2, n)$ is not of finite type. Thus the double dualization does not give an isomorphism of vector spaces from the homology to the double dual of the homology.

Proposition 5.3. *The integral cohomology of* $\mathrm{Conf}^G(\mathbb{H}^2, n)$ *is given additively by an isomorphism*

$$H^*\mathrm{Conf}^G(\mathbb{H}^2, n) \to \; H^*(C_1) \otimes \cdots \otimes H^*(C_{n-1}).$$

In addition, for every $i = 2, \ldots n$*, there are choices of classes in* $H^*\mathrm{Conf}^G(\mathbb{H}^2, n)$ *given by* $A^{\sigma_1}_{i,1}, A^{\sigma_2}_{i,2}, \ldots, A^{\sigma_{i-1}}_{i,i-1}$ *for* $\sigma_p \in G$*, which are in a 1-1 correspondence with the generators in* $H_1(C_{i-1})$*, satisfying the following relations:*

1. $A^{\mu}_{i,j} \cdot A^{\nu}_{i,j} = 0$ *for all* μ*, and* ν *in* G*.*
2. $A^{\mu}_{i,t} \cdot A^{\nu}_{i,j} = A^{\mu\nu^{-1}}_{j,t} \cdot (A^{\nu}_{i,j} - A^{\mu}_{i,t})$ *if* $1 \le t < j < i \le k$ *with* μ *and,* ν *in* G*.*

Lemma 5.4. *Let* $p : E \to B$ *be a locally trivial bundle with* B *path-connected. Let* $x, y \in B$ *and let* F_x *and* F_y *be the corresponding fibers. Then* $\pi_1(B, x)$ *acts trivially on* $H_*(F_x)$ *if and only if* $\pi_1(B, y)$ *acts trivially on* $H_*(F_y)$*.*

Lemma 5.5. *The fibration*

$$\pi : \mathrm{Conf}^G(\mathbb{H}^2, n) \to \; \mathrm{Conf}^G(\mathbb{H}^2, n-1)$$

has trivial local coefficients.

Proof. The construction at the beginning of section 3 for points $p_1, p_2, \cdots, p_k$ in distinct orbits of G in $\mathbb{H}^2$ together with a closed neighborhood U homeomorphic to a closed disk satisfies the following properties.

1. U contains the interior points p_j, together with the points $\sigma(p_j) + z/\lambda$ for λ a fixed real scalar for all z in $\mathbb{H}^2$ of norm 1 where the hyperbolic metric is used for $\mathbb{H}^2$,
2. U intersects each orbit exactly once.
3. U does not contain the p_t for all t with $t \neq j$.

Next fix one value of z, say α. Consider the Dehn twist which "rotates" p_j to $\sigma(p_j) + \alpha/\lambda$, and fixes the boundary of U pointwise. This disk, and Dehn twist is used to describe the local coefficient system for certain fibrations below. This Dehn twist is isotopic to the identity by an isotopy which fixes the complement of U pointwise.

The proof of the lemma is by downward induction on r and is analogous to that given in [7] in the special case of $\mathrm{Conf}^L(\mathbb{C}, n)$ for the standard integral lattice L in $\mathbb{C}$. Namely, the isotopy in the previous paragraph gives a homeomorphism of $\mathbb{H}^2 - Q^G_i$ for which $Q^G_i = \amalg_{1 \le j \le i} G \cdot p_j$. A direct inspection gives that the effect of this homeomorphism on the level of $H_1(\mathbb{H}^2 - Q^G_i)$ is the identity, but is not the identity on $\pi_1(\mathbb{H}^2 - Q^G_i, x)$. $\square$

6. Graded Poisson algebra structures

There is a natural structure of graded Poisson algebra for the homology of an iterated loop space. It is the purpose of this section to describe this structure for the case of $\Omega^k \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$ for $k > 1$.

It is standard that a graded associative algebra A inherits the structure of a graded Lie algebra with Lie bracket given by the commutator

$$[a, b] = a \cdot b - (-1)^{|a| \cdot |b|} b \cdot a$$

for an element a of degree $|a|$, and an element b of degree $|b|$ in the graded associative algebra A. Thus the homology of a 1-fold loop space is naturally a graded Lie algebra.

If $k > 1$, the homology of a k-fold loop space, $\Omega^k X$, admits the structure of a graded Poisson algebra [5], pages $215 - 217$. Namely, there is a bilinear map, the Browder operation, given by

$$\lambda_{k-1} : H_i(\Omega^k X) \otimes H_j(\Omega^k X) \to H_{i+j+k-1}(\Omega^k X)$$

which satisfies the following axioms for a graded Poisson algebra for which the degree of an element x is written $|x|$:

1. (Jacobi identity)
$$\alpha\lambda_{k-1}[a, \lambda_{k-1}[b, c]] + \beta\lambda_{k-1}[b, \lambda_{k-1}[c, a]] + \gamma\lambda_{k-1}[c, \lambda_{k-1}[a, b]] = 0$$
where
   - $\alpha = (-1)^{(|a|+k-1)(|c|+k-1)}$,
   - $\beta = (-1)^{(|b|+k-1)(|a|+k-1)}$, and
   - $\gamma = (-1)^{(|c|+k-1)(|b|+k-1)}$.
2. (Antisymmetry)
$$\lambda_{k-1}[a, b] = (-1)^{|a||b|+1+(k-1)(|a|+|b|+1)}\lambda_{k-1}[b, a].$$
3. (Product formula)
$$\lambda_{k-1}[a \cdot b, c] = a \cdot \lambda_{k-1}[b, c] + (-1)^{|a|\cdot|b|} b \cdot \lambda_{k-1}[a, c].$$
4. (Commutation with homology suspension σ_*)
$$\sigma_*(\lambda_{k-1}[a, b]) = \lambda_{k-2}[\sigma_*(a), \sigma_*(b)].$$
5. (Degree of the operation) The degree of $\lambda_{k-1}[a, b]$ is $k - 1 + |a| + |b|$.

In addition, it was proven that this pairing is compatible with the Whitehead product structure for the classical Hurewicz homomorphism via the following commutative diagram

$$\begin{array}{ccc}
\pi_{m+k}(X) \otimes \pi_{n+k}(X) & \xrightarrow{W_0} & \pi_{m+n+2k-1}(X) \\
\downarrow{\scriptstyle s_* \otimes s_*} & & \downarrow{\scriptstyle s_*} \\
\pi_m(\Omega^k X) \otimes \pi_n(\Omega^k X) & \xrightarrow{W_k} & \pi_{m+n+k-1}(\Omega^k X) \\
\downarrow{\scriptstyle \phi \otimes \phi} & & \downarrow{\scriptstyle \phi} \\
H_m(\Omega^k X) \otimes H_n(\Omega^k X) & \xrightarrow{\lambda_{k-1}} & H_{m+n+k-1}(\Omega^k X)
\end{array}$$

for which

1. the map s_* is the natural isomorphism,
2. the map ϕ is the classical Hurewicz homomorphism, and
3. the map W_k is the adjoint of the classical Whitehead product
$$W_0 : \pi_{m+k}(X) \otimes \pi_{n+k}X \to \ \pi_{m+n+2k-1}(X).$$

Theorem 6.1. *Assume that k is greater than 1.*

1. *If $k > 1$, the homology of $\Omega^k(\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n))$, with any field coefficients, is a graded Poisson algebra with Poisson bracket given by the Browder operation $\lambda_{k-1}[-, -]$ for the homology of a k-fold loop space.*
2. *If $1 < k < 2q + 1$, the homology of $\Omega^k \mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$ with coefficients in a field $\mathbb{F}$ of characteristic zero, is the free Poisson algebra generated by elements*
$$B^{\sigma}_{i,j}$$
of degree $2q + 1 - k$ for $1 \leq j < i \leq n$, and σ in G modulo the "infinitesimal Poisson surface braid relations" given as follows:

(a) *If* $\{i,j\}\cap\{s,t\}=\phi$, *then* $\lambda_{k-1}[B^{\sigma}_{i,j}, B^{\tau}_{s,t}]=0$.
(b) *If* $1\leq j<s<i\leq n$, *then* $\lambda_{k-1}[B^{\tau}_{i,j}, B^{\tau\sigma^{-1}}_{i,s}+B^{\sigma}_{s,j}]=0$.
(c) *If* $1\leq j<s<i\leq n$, *then* $\lambda_{k-1}[B^{\sigma}_{s,j}, B^{\tau}_{i,j}+B^{\tau\sigma^{-1}}_{i,s}]=0$.
(d) *The antisymmetry relation, Jacobi identity, and product formula for a graded Poisson algebra are satisfied.*

3. *There is a map*

$$E^2:\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)\to\ \Omega^2\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^{q+1},n)$$

which induces a homology isomorphism in degree $2q+1$. *The associated loop map*

$$\Omega(E^2):\Omega\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)\to\ \Omega^3\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^{q+1},n)$$

induces an isomorphism on $H_{2q}(-;\mathbb{Z})$. *Furthermore, the image of the map* $\Omega(E^2)$ *in homology is the subalgebra generated by the classes of degree* $2q$.

Proof. Part (1) is a special case of results in [5], pages $215-217$. Furthermore, if $q>1$, then the characteristic zero homology of $\Omega^q(X)$ for a q-connected space X is isomorphic to the graded symmetric algebra generated by the image of the rational Hurewicz homomorphism (as is well-known from work of Milnor-Moore).

Part (2) now follows from the case $q=1$ together with the proof of Theorem 1.1. Namely, the Browder operation λ_{q-1} is precisely the commutator for the underlying associative algebra given by the homology of a 1-fold loop space $\Omega\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)$. Furthermore, the homology suspension σ_* satisfies $\sigma_*(\lambda_{k-2}[x,y])=\lambda_{k-1}[\sigma_*(x),\sigma_*(y)]$ [5, pages 215–217]. Since the homology suspension when restricted to the module of primitives

$$\sigma_*:\mathrm{Prim}\,H_n(\Omega^k(X);\mathbb{Q})\to\ \mathrm{Prim}\,H_{n+1}(\Omega^{k-1}(X);\mathbb{Q})$$

is an isomorphism when restricted to k-connected spaces X for $k>2$, it follows that the Poisson bracket relations follow at once from the case $k=1$ as given in Theorem 1.1.

Part (3) requires a construction given as follows. Notice that there is a map

$$\Theta:\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)\to\ \Pi_I S^{2q+1}$$

where the index set I is given by (i,j,σ,τ) for $1\leq j<i\leq n$ with σ, and τ elements of G. This map is gotten by sending $(z_1,\ldots,z_n)$ to $(\sigma(z_i)-\tau(z_j))/\rho$ for $1\leq j<i\leq n$, with σ, and τ in G, and $\rho=||\sigma(z_i)-\tau(z_j)||$.

This map is a split monomorphism in homology. Thus after suspending once, $\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)$ is homotopy equivalent to a bouquet of spheres for any $q\geq 0$.

Hence, there is a map $\gamma:\Sigma^2\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)\to\ \vee_I S^{2k+3}$ which induces an isomorphism in homology in degree $2k+3$. There is an induced map

$$\sigma^2:\Sigma^2\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)\to\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^{q+1},n)$$

given by the composite

$$\Sigma^2\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)\ \xrightarrow{\gamma}\ \vee_I S^{2k+3}\ \xrightarrow{\phi}\ \mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^{q+1},n)$$

where $\phi:\vee_I S^{2k+3}\to\ \mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^{q+1},n)$ is the natural map obtained from the Hurewicz homomorphism which induces an isomorphism on the first non-vanishing homology group of $\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^{q+1},n)$.

The adjoint of this composite

$$E^2:\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^q,n)\to\ \Omega^2\mathrm{Conf}^G(\mathbb{H}^2\times\mathbb{C}^{q+1},n)$$

induces an isomorphism on the first non-trivial homology group. This last map may be regarded as an analogue of the classical Freudenthal double suspension map for which the spaces $\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n)$ are replaced by single odd dimensional spheres. Looping E^2 is given by $\Omega(E^2) : \Omega\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^q, n) \to \Omega^3\mathrm{Conf}^G(\mathbb{H}^2 \times \mathbb{C}^{q+1}, n)$.

The theorem follows. $\square$

Related structures for certain complements of hyperplane arrangements are given in [3].

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