NOVIKOV HOMOLOGY, JUMP LOCI AND MASSEY PRODUCTS

TOSHITAKE KOHNO AND ANDREI PAJITNOV

ABSTRACT. Let X be a finite CW complex, and $\rho : \pi_1(X) \to \mathrm{GL}(l, \mathbb{C})$ a representation. Any cohomology class $\alpha \in H^1(X, \mathbb{C})$ gives rise to a deformation γ_t of ρ defined by $\gamma_t(g) = \rho(g)\exp(t\langle\alpha, g\rangle)$. We show that the cohomology of X with local coefficients γ_{gen} corresponding to the generic point of the curve γ is computable from a spectral sequence starting from $H^*(X, \rho)$. We compute the differentials of the spectral sequence in terms of the Massey products. We show that the spectral sequence degenerates in case when X is a Kähler manifold and ρ is semi-simple.

If $\alpha \in H^1(X, \mathbb{R})$ one associates to the triple (X, ρ, α) the twisted Novikov homology (a module over the Novikov ring). We show that the twisted Novikov Betti numbers equal the Betti numbers of X with coefficients in the local system γ_{gen}. We investigate the dependence of these numbers on α and prove that they are constant in the complement to a finite number of proper vector subspaces in $H^1(X, \mathbb{R})$.

1. INTRODUCTION

Let X be a finite connected CW-complex; denote its fundamental group by G. Let $\rho : G \to \mathrm{GL}(l, \mathbb{C})$ be a representation. Any cohomology class $\alpha \in H^1(X, \mathbb{C})$ gives rise to the following deformation of ρ:

$$\gamma_t : G \to \mathrm{GL}(l, \mathbb{C}), \quad \gamma_t(g) = e^{t\langle\alpha, g\rangle}\rho(g).$$

The cohomology groups of X with local coefficients γ_t are isomorphic for all t except a subset containing only isolated points. The cohomology group $H^*(X, \gamma_{gen})$ corresponding to the generic point of the curve γ_t is the first main object of study in the present paper. We prove that there is a spectral sequence $\mathscr{E}_r^*$ starting from the homology of X with coefficients in ρ, converging to $H^*(X, \gamma_{gen})$, and the differentials in this spectral sequence are computable in terms of some special higher Massey products with α. The first differential in this spectral sequence is the homomorphism L_α of multiplication by α in the ρ-twisted cohomology of X.

This type of spectral sequences appeared in the paper of S.P. Novikov [15] in the de Rham setting. It was generalized to the case of cohomology with coefficients in a field of arbitrary characteristic in the paper [17] of the second author. See also the papers [6], [7] of M. Farber. Our present construction is close to the original ideas of S.P. Novikov. The main technical novelty of our present approach is the systematic use of *formal exponential deformations* (see §§ 3, 4). This allows to avoid the convergency issues for power series, which occur in Novikov's idea of the proof of the first main theorem of his paper (see [15], page 553).

If ρ is the trivial representation, the differentials in the spectral sequence above are the usual Massey products in the ordinary cohomology with slightly reduced indeterminacy: $d_r(x) = \langle \alpha, \dots, \alpha, x\rangle$, see § 3. Thus the spectral sequence degenerates when the space X is formal, by the classical argument of P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan [4] which applies here as well (§ 3). Thus we have

$$\text{Ker } L_\alpha/\text{Im } L_\alpha \approx H^*(X, \gamma_{gen}). \tag{1}$$

In case when ρ is not trivial the situation is more complicated. The spectral sequence degenerates, in particular, when X is a Kähler manifold and ρ is a semi-simple representation (§6, Prop. 6.8). The proof uses C. Simpson's theory of Higgs bundles [23]. Thus the isomorphism (1) holds also in this case. We introduce a class of *strongly formal spaces* for which all the spectral sequences $\mathscr{E}_r^*$ corresponding to 1-dimensional representations degenerate in their second term. An example from the work of H. Kasuya [11] shows that there exist formal spaces which are not strongly formal.

In the literature there are several other constructions of the spectral sequences related to the cohomology with twisted coefficients, such as the equivariant spectral sequence, introduced and studied by S. Papadima and A. Suciu [21]. The isomorphism (1) was obtained also in the recent work [5] of A. Dimca and S. Papadima for the case when ρ is the trivial representation. One of the advantages of our method is that it allows an explicit computation of the higher differentials of the spectral sequence and leads to the proof of the isomorphism (1) for deformations of non-trivial representations.

If α is a real cohomology class, there is another geometric construction related to ρ and α, namely, the *twisted Novikov homology* introduced in the works of H. Goda and the second author see [9], [19]. This construction associates to X, ρ and α a module over a corresponding Novikov ring $\widehat{L}_{m,\alpha}$. The rank and torsion numbers of this module are called *the twisted Novikov Betti numbers* and *twisted Novikov torsion numbers*; they provide lower bounds for numbers of zeros of any Morse form belonging to the de Rham cohomology class α (see §8 for details). These invariants detect the fibered knots in S^3 as it follows from the recent work of S. Friedl [8]. For a given space X and the representation ρ the Novikov numbers depend on the cohomology class $\alpha \in H^1(X, \mathbb{R})$. The case of the torsion numbers was studied in [18] and [19]. It is proved there that the torsion numbers are constant in the open polyhedral cones formed by finite intersections of certain half-spaces in $\mathbb{R}^n$, where $n = \mathrm{rk} H_1(X, \mathbb{Z})$. Similar analysis applies to the Novikov Betti numbers, which are of main interest to us in the present work. We prove in §8 that these numbers do not depend on α in the complement to a finite number of proper vector subspaces. In general the set of all α for which the Novikov Betti number $\widehat{b}_k^\rho(X, \alpha)$ is greater by q than the generic value (the *jump loci* for the Novikov numbers) is a union of a finite number of proper vector subspaces, see §8, Prop. 8.6.

It is known that the Novikov homology and the homology with local coefficients are related to each other. This was first observed in the paper [16] of the second author, see also Novikov [15]. Similar result holds also for the twisted Novikov homology, namely, we prove (see §8, Prop. 9.2) that for $\alpha \in H^1(X, \mathbb{R})$ we have

$$\widehat{b}_k^\rho(X, \alpha) = \beta_k(X, \gamma_{gen}).$$

This implies several corollaries about both families of numerical invariants. One corollary is that for given ρ and α the jump loci for $\beta_k(X, \rho, \alpha)$ are unions of proper vector subspaces. On the other hand the twisted Novikov Betti numbers are computable from the Massey spectral sequence.integral hyperplanes In the case of degeneracy of this spectral sequence, they equal the dimension of its second term.

2. Exact couples

In this section we recall the definition of the spectral sequence of an exact couple (following [12], [10]) and give an equivalent description of the successive terms of the spectral sequence, which will be useful in the sequel.

Let $\mathscr{C} = (D, E, i, j, k)$ be an exact couple, so that we have an exact triangle

$$\begin{array}{ccc} D & \xrightarrow{\ i\ } & D \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & E & \end{array}$$

We will usually abbreviate the notation to $\mathscr{C} = (D, E)$ and call D and E the first, respectively the second component of the exact couple. Following W. Massey we define the derived exact couple setting

$$E' = \text{Ker } (jk)/\text{Im } (jk), \quad D' = i(D)$$

and defining j', k' suitably. Iterating the process we obtain a sequence of exact couples $\mathscr{C}_r = (D_r, E_r)$, the initial couple being numbered as $\mathscr{C}_1$; this sequence is called the spectral sequence associated to the exact couple $\mathscr{C}$.

We will need an alternative description of the groups E_r and the maps j_r, k_r.

Definition 2.1. 1) For $r \geqslant 2$ let Z_r be the subgroup of all elements $x \in E$ such that $k(x) = i^{r-1}(y)$ for some $y \in D$. We put $Z_1 = E$.

2) For $r \geqslant 1$ let B_r be the subgroup of all elements $z \in E$, such that $z = j(y)$ for some $y \in D$ with $i^{r-1}(y) = 0$.

The following properties are easy to check:

$$Z_1 = E \supset Z_2 = \text{Ker } (jk) \supset Z_3 \ldots \supset Z_r \supset Z_{r+1} \ldots$$

$$B_1 = \{0\} \subset B_2 = \text{Im } (jk) \subset B_3 \subset \ldots \subset B_r \subset B_{r+1} \subset \ldots$$

$$B_i \subset Z_j \qquad \text{for every } i, j.$$

Put

$$\widetilde{E}_r = Z_r/B_r, \quad D_r = \text{Im } i^r.$$

Define a homomorphism $\widetilde{k}_r : \widetilde{E}_r \to D_r$ setting $k_r(x) = k(x)$ for every $x \in Z_r$. Define a homomorphism $\widetilde{j}_r : D_r \to \widetilde{E}_r$ as follows: if $x \in D_r$ and $x = i^r(y)$, then put $\widetilde{j}_r(x) = [j(y)]$. It is easy to check that these homomorphisms are well-defined and give rise to an exact couple $\widetilde{\mathscr{C}_r} = (D_r, \widetilde{E_r})$:

$$\begin{array}{ccc} D_r & \xrightarrow{\ i\ } & D_r \\ {\scriptstyle \widetilde{k_r}}\nwarrow & & \swarrow{\scriptstyle \widetilde{j_r}} \\ & \widetilde{E_r} & \end{array}$$

The proof of the next is in a usual diagram chasing:

Proposition 2.2. *The exact couples $\mathscr{C}_r$ and $\widetilde{\mathscr{C}_r}$ are isomorphic for any r.* □

3. Formal deformations of differential algebras and their spectral sequences

Let

$$\mathcal{A}^* = \{\mathcal{A}^k\}_{k\in\mathbb{N}} = \{\mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \dots\}$$

be a graded-commutative differential algebra (DGA) over a field $\mathbb{K}$ of characteristic zero. Let $\mathcal{N}^*$ be a graded differential module (DGM) over $\mathcal{A}^*$ (that is, $\mathcal{N}^*$ is a graded module over $\mathcal{A}^*$ endowed with a differential which satisfies the Leibniz formula with respect to the pairing $\mathcal{A}^* \times \mathcal{N}^* \to \mathcal{N}^*$). We will use the same symbol d to denote the differentials in both $\mathcal{A}^*$ and $\mathcal{N}^*$, since no confusion is possible. We denote by $\mathcal{A}^*[[t]]$ the algebra of formal power series over $\mathcal{A}^*$ endowed with the differential extended from the differential of $\mathcal{A}^*$. Let $\xi \in \mathcal{A}^1$ be a cocycle. Consider the $\mathcal{A}^*[[t]]$-module $\mathcal{N}^*[[t]]$ and endow it with the differential

$$D_t x = dx + t\xi x.$$

Then $\mathcal{N}^*[[t]]$ is a DGM over $\mathcal{A}^*[[t]]$, and we have an exact sequence of DGMs:

$$0 \longrightarrow \mathcal{N}^*[[t]] \xrightarrow{t} \mathcal{N}^*[[t]] \xrightarrow{\pi} \mathcal{N}^* \longrightarrow 0 \tag{2}$$

where π is the natural projection $t \longmapsto 0$. The induced long exact sequence in cohomology can be considered as an exact couple

$$\begin{array}{ccc} H^*\big(\mathcal{N}^*[[t]]\big) & \xrightarrow{\quad t \quad} & H^*\big(\mathcal{N}^*[[t]]\big) \\ & {\scriptstyle\delta}\nwarrow \quad \swarrow{\scriptstyle\pi_*} & \\ & H^*(\mathcal{N}^*) & \end{array} \tag{3}$$

Proposition 3.1. *The spectral sequence induced by the exact couple (3) depends only on the cohomology class of ξ.*

Proof. Let $\xi_1, \xi_2 \in A^1$ be cohomologous cocycles, $\xi_1 = \xi_2 + df$ with $f \in A^0$. Let $D_t = d + t\xi_1,\ D'_t = d + t\xi_2$ be the corresponding differentials. Multiplication by $e^{tf} \in A^0[[t]]$ determines an isomorphism $F : \mathcal{N}^*[[t]] \to \mathcal{N}^*[[t]]$, commuting with the differentials, namely, $F(D_t\omega) = D'_t(F(\omega))$. Thus the exact sequences (2) corresponding to ξ_1 and ξ_2 are isomorphic, as well as the exact couples (3) and their spectral sequences. □

Definition 3.2. Put $\alpha = [\xi]$. The spectral sequence associated to the exact couple (3) is called *deformation spectral sequence* and denoted by

$$\mathscr{E}_r^*(\mathcal{N}^*, \alpha) = \Big(D_r^*(\mathcal{N}^*, \alpha),\ E_r^*(\mathcal{N}^*, \alpha) \Big).$$

If the couple $(\mathcal{N}^*, \alpha)$ is clear from the context, we suppress it in the notation and write $\mathscr{E}_r^*$, respectively, D_r^*, E_r^*.

Denote by

$$L_\alpha : H^*(\mathcal{N}^*) \to H^*(\mathcal{N}^*)$$

the multiplication by α. It is clear that the first differential in the spectral sequence equals L_α and therefore

$$E_2^* = \text{Ker } L_\alpha / \text{Im } L_\alpha.$$

We are going to compute the higher differentials in this spectral sequence in terms of special Massey products. Let $a \in H^*(\mathcal{N}^*)$. An *r-chain starting from a* is a sequence of elements $\omega_1, \ldots, \omega_r \in \mathcal{N}^*$ such that

$$d\omega_1 = 0, \ \ [\omega_1] = a, \ \ d\omega_2 = \xi\omega_1, \ \ldots, \ d\omega_r = \xi\omega_{r-1}.$$

Denote by $MZ^m_{(r)}$ the subspace of all $a \in H^m(\mathcal{N}^*)$ such that there exists an r-chain starting from a. Thus

$$MZ^m_{(1)} = H^m(\mathcal{N}^*), \ \ MZ^m_{(2)} = \text{Ker } \Big(L_\alpha : H^m(\mathcal{N}^*) \to H^{m+1}(\mathcal{N}^*) \Big).$$

Denote by $MB^m_{(r)}$ the subspace of all $\beta \in H^m(\mathcal{N}^*)$ such that there exists an $(r-1)$-chain $(\omega_1, \ldots, \omega_{r-1})$ with $\xi\omega_{r-1}$ belonging to β. By definition

$$MB^m_{(1)} = 0, \ \ MB^m_{(2)} = \text{Im } \Big(L_\alpha : H^{m-1}(\mathcal{N}^*) \to H^m(\mathcal{N}^*) \Big).$$

It is clear that $MB^m_{(i)} \subset MZ^m_{(j)}$ for every i, j. Put

$$MH^m_{(r)} = MZ^m_{(r)} \Big/ MB^m_{(r)}.$$

In the next definition we omit the upper indices and write $MH_{(r)}, MZ_{(r)}$ etc. in order to simplify the notation.

Definition 3.3. Let $a \in H^*(\mathcal{N}^*)$, and $r \geqslant 1$. We say that the $(r+1)$-tuple Massey product $\langle \xi, \ldots, \xi, a \rangle$ is defined, if $a \in MZ_{(r)}$. In this case choose any r-chain $(\omega_1, \ldots, \omega_r)$ starting from a. The cohomology class of $\xi\omega_r$ is in $MZ_{(r)}$ (actually it is in $MZ_{(N)}$ for every N) and it is not difficult to show that it is well defined modulo $MB_{(r)}$.

The image of $\xi\omega_r$ in $MZ_{(r)}/MB_{(r)}$ is called the $(r+1)$-tuple Massey product of ξ and a:

$$\langle \xi, a \rangle_{(r+1)} = \Big\langle \underbrace{\xi, \ldots, \xi}_{r},\ a\ \Big\rangle \in MZ_{(r)} \Big/ MB_{(r)}.$$

Example 3.4. The double Massey product $\langle \xi, a\rangle_{(2)}$ equals ξa, the triple Massey product $\langle \xi, a\rangle_{(3)}$ equals the cohomology class of $\xi\omega_2$ where $d\omega_2 = \xi\omega_1$, and $[\omega_1] = a$, etc.

The correspondence $a \longmapsto \langle \xi, a\rangle_{(r+1)}$ gives rise to a well-defined homomorphism of degree 1

$$\Delta_r : MH_{(r)} \longrightarrow MH_{(r)}.$$

The next proposition is proved by an easy diagram chasing argument.

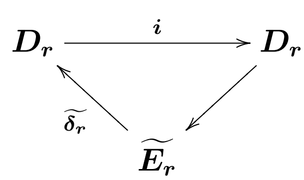
Proposition 3.5. *For any r we have $\Delta_r^2 = 0$, and the cohomology group $H^*(MH^*_{(r)}, \Delta_r)$ is isomorphic to $MH^*_{(r+1)}$.* □

Theorem 3.6. *For any $r \geqslant 2$ there is an isomorphism*

$$\phi : MH^*_{(r)} \xrightarrow{\approx} E^*_r$$

commuting with differentials.

Proof. Recall from Section 2 a spectral sequence $\widetilde{\mathscr{E}}^*_r$ isomorphic to $\mathscr{E}^*_r$. It is formed by exact couples



where $D_1 = H^*(\mathcal{N}^*)$, $D_r = \operatorname{Im}\Big(t^{r-1} : D \to D\Big)$, and $\widetilde{E}_r = Z_r/B_r$ (the modules Z_r, B_r are described in the definition 2.1).

Lemma 3.7. *1) $MZ_{(r)} = Z_r$, 2) $MB_{(r)} = B_r$.*

Proof. We will prove 1), the proof of 2) is similar. Let $\zeta \in H^*(\mathcal{N}^*)$ and z be a cocycle belonging to ζ. Then $\delta(\zeta)$ equals the cohomology class of $\xi z \in \mathcal{N}^*[[t]]$; and $\zeta \in Z_r$ if and only if there is $\mu \in \mathcal{N}^*[[t]]$ such that

$$\xi z - D_t\mu \in t^{r-1}\mathcal{N}^*[[t]]. \tag{4}$$

This condition is clearly equivalent to the existence of a sequence of elements $\mu_0, \mu_1, \ldots \in \mathcal{N}^*[[t]]$ such that

(5) $\quad \xi z = d\mu_0, \quad \text{and} \quad d\mu_i + \xi\mu_{i-1} = 0 \quad \text{for every} \quad 0 \leqslant i \leqslant r-2.$

The condition (5) is in turn equivalent to the existence of an r-chain starting from ζ. □

The Lemma implies that $\widetilde{E}^*_r \approx MH^*_{(r)}$ and it is not difficult to prove that this isomorphism is compatible with the boundary operators. □

In view of the Proposition 3.1 we obtain the next Corollary.

Corollary 3.8. *Let $\xi \in \mathcal{A}^1$ be a cocycle. The graded groups $MH^*_{(r)}$ defined above depend only on the cohomology class of ξ, which is denoted by α.* □

Therefore the differentials in the spectral sequence $\{\mathscr{E}^*_r\}$ are equal to the higher Massey products with the cohomology class of ξ. Observe that these Massey products, defined above, have smaller indeterminacy than the usual Massey products. The second term of the spectral sequence is described therefore in terms of multiplication by the cohomology class $\alpha = [\xi]$. It is convenient to give a general definition.

Definition 3.9. Let $\mathcal{K}^*$ be a differential graded algebra, and θ be an element of odd degree s. Denote by $L_\theta : \mathcal{K}^* \to \mathcal{K}^{*+s}$ the homomorphism of multiplication by θ. The quotient

$$\text{Ker } L_\theta \big/ \text{Im } L_\theta$$

is a graded module which is denoted by $\mathcal{H}^*(\mathcal{K}^*, \theta)$ and is called *θ-cohomology of $\mathcal{K}^*$*. The dimension of $\mathcal{H}^k(\mathcal{K}^*, \theta)$ is denoted by $\mathcal{B}_k(\mathcal{K}^*, \theta)$.

We have therefore

$$E^*_2(\mathcal{N}^*, \alpha) \approx \mathcal{H}^*(H^*(\mathcal{N}^*), \alpha).$$

Let us consider some examples, which will be important for the sequel.

Example 3.10. Let $\mathcal{N}^* = \mathcal{A}^*$. The homomorphism L_α is the multiplication by α in the cohomology $H^*(\mathcal{A}^*)$, and the differentials in the spectral sequence $\mathscr{E}_r$ are the higher Massey products induced by the ring structure in $H^*(\mathcal{A}^*)$.

Example 3.11. Let $\mathcal{A}^*$ be a DGA and $\eta \in \mathcal{A}^1$ be a cocycle. Endow the algebra $\mathcal{A}^*$ with the differential d_η defined by the following formula:

$$d_\eta(x) = dx + \eta x;$$

we obtain a DGM over $\mathcal{A}$, which we will denote by $\widetilde{\mathcal{A}}^*_\eta$. For an element $\alpha \in H^1(\mathcal{A}^*)$ we obtain a spectral sequence $\mathscr{E}^*_r$ with

$$E^*_1 = H^*(\widetilde{\mathcal{A}}^*_\eta); \quad E^*_2 = \mathcal{H}^*(H^*(\widetilde{\mathcal{A}}^*_\eta), \alpha).$$

Example 3.12. Let M be a connected C^∞ manifold, and E be an l-dimensional complex flat bundle over M. Denote by $\rho : \pi_1(M) \to \mathrm{GL}(l, \mathbb{C})$ the monodromy of E. Let $A^*(M)$ be the algebra of complex differential forms on M. The space $A^*(M, E)$ of the differential forms with coefficients in E is a DGM over $A^*(M)$; its cohomology is isomorphic to the cohomology $H^*(M, \rho)$ with local coefficients with respect to the representation ρ. For a de Rham cohomology class $\alpha \in H^1(M)$ we obtain therefore a spectral sequence $\mathscr{E}_r$ with

$$E^*_1 = H^*(M, \rho); \quad E^*_2 = \mathcal{H}^*(H^*(M, \rho), \alpha).$$

Now let us consider some cases when the spectral sequences constructed above, degenerate in its second term. Recall that a differential graded algebra $\mathcal{A}^*$ is called *formal* if it has the same minimal model as its cohomology algebra. Here is a useful characterization of minimal formal algebras.

Theorem 3.13. *(P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan,* [4], *Th. 4.1) Let $\mathcal{A}^*$ be a minimal algebra over a field of characteristic zero, generated (as a free graded-commutative algebra) in degree k by a vector space V_k; denote by $C_k \subset V_k$ the subspace of the closed generators. The algebra $\mathcal{A}^*$ is formal if and only if in each V_k there is a direct complement N_k to C_k in V_k, such that any cocycle in the ideal generated by $\oplus_k(N_k)$ is cohomologous to zero.*

This theorem leads to the proof of the well-known property that in formal algebras all Massey products vanish (see [4]). We will show that a similar result holds for the special Massey products (Example 3.10).

Theorem 3.14. *Let $\mathcal{A}^*$ be a formal differential algebra, $\alpha \in H^1(\mathcal{A}^*)$. Then the spectral sequence $\mathscr{E}^*_r(\mathcal{A}^*, \alpha)$ degenerates at its second term, and*

$$E^*_2(\mathcal{A}^*, \alpha) = \mathcal{H}^*(H^*(\mathcal{A}^*), \alpha).$$

Proof. It suffices to establish the property for the case of formal minimal algebras. Let $\mathcal{A}^*$ be a formal minimal algebra with the space of generators V_k in dimension k decomposed as $V_k = C_k \oplus N_k$, see 3.13. We will prove that $MZ_{(2)} = MZ_{(3)} = \ldots = MZ_{(r)}$ and $MB_{(2)} = MB_{(3)} = \ldots = MB_{(r)}$ for every $r \geqslant 2$.

Choose $\xi \in A^1$ representing $\alpha \in H^1(A^*)$. Let $a \in MZ_{(r)}$, and $(\omega_1, \dots, \omega_r)$ be an r-chain starting from a. so that

$$d\omega_1 = 0, \ \ [\omega_1] = a, \ \ d\omega_2 = \xi\omega_1, \ \dots, d\omega_r = \xi\omega_{r-1}.$$

Denote by $\Lambda(C_*)$ the algebra generated by the space of closed generators, so that $\mathcal{M} = \Lambda(C_*) \oplus I(N_*)$. Write $\omega_r = \omega_r^0 + \omega_r^1$ with $\omega_r^0 \in \Lambda(C_*)$, $\omega_r^1 \in I(N_*)$. Then $\xi\omega_{r-1} = d\omega_r^1$, and

$$d(\xi\omega_r^1) = d(\xi\omega_r) = \xi^2\omega_{r-1} = 0,$$

so that $\xi\omega_r^1$ is a cocycle belonging to $I(N_*)$. Therefore $\xi\omega_r^1 = d\omega_{r+1}$ for some $\omega_{r+1} \in I(N_*)$ and we obtain an $(r+1)$-chain starting from a, so that $a \in MZ_{(r+1)}$.

A similar argument shows that $MB_{(2)} = MB_{(r)}$ for every $r \geqslant 2$. Therefore the spectral sequence degenerates at its second term. □

The next proposition gives a sufficient condition for the degeneracy of the spectral sequence associated with a differential graded module over a DG-algebra $\mathcal{A}^*$. It will be used in Section 6 while studying the case of Kähler manifolds.

Definition 3.15. A DG-module $\mathcal{N}^*$ over a DG algebra $\mathcal{A}_*$ will be called *formal* if it is a direct summand of a formal DGA $\mathcal{B}^*$ over $\mathcal{A}_*$, that is,

$$\mathcal{B}^* = \mathcal{N}^* \oplus \mathcal{K}^*, \tag{6}$$

where both $\mathcal{N}^*$ and $\mathcal{K}^*$ are differential graded $\mathcal{A}_*$-submodules of $\mathcal{B}_*$.

Proposition 3.16. *Let $\mathcal{N}_*$ be a formal DG-module over $\mathcal{A}_*$, and $\alpha \in H^1(\mathcal{A}^*)$. Then the spectral sequence $\mathscr{E}_r^*(\mathcal{N}^*, \alpha)$ degenerates at its second term.*

Proof. The direct sum decomposition (6) implies that

$$\mathscr{E}_r^*(\mathcal{B}^*, \alpha) = \mathscr{E}_r^*(\mathcal{N}^*, \alpha) \oplus \mathscr{E}_r^*(\mathcal{K}^*, \alpha);$$

the spectral sequence $\mathscr{E}_r^*(\mathcal{B}^*, \alpha)$ degenerates by the previous proposition and the result follows. □

4. A spectral sequence converging to the twisted cohomology

Let X be a finite CW-complex, put $G = \pi_1(X)$. We endow the universal covering $\widetilde{X}$ with the natural left action of G, so that the cellular chain complex of $\widetilde{X}$ is a free finitely generated chain complex over $\mathbb{Z}G$. Let B be an integral domain and ρ be a left action of G on the free B-module B^l (or, equivalently, a representation $\rho : G \to \mathrm{GL}(l, B)$). The cohomology of the cochain complex

$$C^*(X, \rho) = \mathrm{Hom}_G\Big(C_*(\widetilde{X}), B^l\Big)$$

is a B-module called *the twisted cohomology of X with respect to ρ* and denoted by $H^*(X, \rho)$. Denote by $\{B\}$ the fraction field of B. The dimension over $\{B\}$ of the localization $H^k(X, \rho) \otimes \{B\}$ will be called the *k-th cohomological Betti number of X with respect to ρ* and denoted by $\beta^k(X, \rho)$.

Let us start with a given representation $\rho : G \to \mathrm{GL}(l, \mathbb{C})$. Pick a cohomology class $\alpha \in H^1(X, \mathbb{C})$ and consider the exponential deformation of ρ:

$$\gamma_t : G \to \mathrm{GL}(l, \mathbb{C}), \quad \gamma_t(g) = \rho(g) e^{t\langle\alpha, g\rangle} \quad (t \in \mathbb{C}). \tag{7}$$

Denote by $\mathcal{H}$ the ring of all entire holomorphic functions on $\mathbb{C}$ and let $\Lambda = \mathbb{C}[[t]]$; we have a natural inclusion $i : \mathcal{H} \hookrightarrow \Lambda$. The formula (7) defines a family of representations of G:

1) For a fixed $t \in \mathbb{C}$ a representation $\gamma_t : G \to \mathrm{GL}(l, \mathbb{C})$.
2) a representation $\bar{\gamma} : G \to \mathrm{GL}(l, \mathcal{H})$ (the *holomorphic exponential deformation* of ρ).
3) a representation $\widehat{\gamma} = i \circ \bar{\gamma} : G \to \mathrm{GL}(l, \Lambda)$ (the *formal exponential deformation* of ρ).

The inclusion i extends to the inclusion of the fields of fractions $\{\mathcal{H}\} \hookrightarrow \{\Lambda\}$, therefore

$$\beta^k(X, \bar{\gamma}) = \beta^k(X, \widehat{\gamma})$$

for every k.

Lemma 4.1. *For every k and t we have $\beta^k(X, \gamma_t) \geqslant \beta^k(X, \bar{\gamma})$. There is a subset $S \subset \mathbb{C}$ consisting of isolated points, such that*

$$\beta^k(X, \gamma_t) = \beta^k(X, \bar{\gamma}) \tag{8}$$

for every k and every $t \in \mathbb{C} \setminus S$.

Proof. Let n_k be the number of k-cells in X. The boundary operator ∂_k in $C_*(\widetilde{X})$ is represented by an $(n_{k-1} \times n_k)$-matrix with coefficients in $\mathbb{Z}G$. The chain complex

$$C^*(X, \bar{\gamma}) = \mathrm{Hom}_G\Big(C_*(\widetilde{X}),\ \mathcal{H}^l\Big)$$

computing the cohomology of X with coefficients in the local system defined by $\bar{\gamma}$, has $l \cdot n_k$ free generators in degree k; its boundary operator $\delta_{k+1} : C^k \to C^{k+1}$ is given by the formula

$$\delta_k = \bar\gamma(\partial^{\mathrm{T}}_{k+1}),$$

(we denote by M^T is the transpose of M). Denote by ρ_{k+1} the maximal rank of non-zero minors of this matrix. For $t \in \mathbb{C}$ denote by $\rho_{k+1}(t)$ the maximal rank of non-zero minors of the matrix $\bar\gamma\big(\partial^{\mathrm{T}}_{k+1}(t)\big)$. Then

$$\beta^k(X, \bar\gamma) = l \cdot n_k - \rho_k - \rho_{k+1},$$
$$\beta^k(X, \gamma_t) = l \cdot n_k - \rho_k(t) - \rho_{k+1}(t).$$

Observe that $\rho_k(t) \leqslant \rho_k$ for every t, and the set of $t \in \mathbb{C}$ where these inequalities are strict, consists of isolated points (since the minors of $\bar\gamma(\partial_k)$ are holomorphic functions of the variable $t \in \mathbb{C}$). The lemma follows. □

Thus

$$\beta^k(X, \widehat\gamma) = \beta^k(X, \bar\gamma) = \beta^k(X, \gamma_t) \qquad \text{for } \ t \notin S.$$

The exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{\ t\ } \Lambda \longrightarrow \mathbb{C} \longrightarrow 0$$

gives rise to a short exact sequence of complexes

$$0 \longrightarrow C^*(X, \widehat\gamma) \xrightarrow{\ t\ } C^*(X, \widehat\gamma) \longrightarrow C^*(X, \rho) \longrightarrow 0$$

inducing a long exact sequence in cohomology, which can be interpreted as an exact couple

$$\begin{array}{ccc} H^*\big(X, \widehat\gamma\big) & \xrightarrow{\quad\quad t \quad\quad} & H^*\big(X, \widehat\gamma\big) \\ & \nwarrow \qquad \swarrow & \\ & H^*(X, \rho) & \end{array} \tag{9}$$

Denote by $\mathscr{W}^*_r(X, \rho, \alpha) = (U^*_r, W^*_r)$ the induced spectral sequence.

Proposition 4.2. *Let X be a finite connected CW-complex. Then*

$$\dim_{\mathbb{C}} W^k_\infty = \beta^k(X, \widehat\gamma) = \beta^k(X, \gamma_t) \ \text{ for generic } t.$$

Proof. The ring Λ is principal, and the Λ-module $H^*\big(X, \widehat\gamma\big)$ is a finite direct sum of cyclic modules. Only the free summands survive to E_∞ and the contribution of each such summand to $\dim_{\mathbb{C}} W^k_\infty$ equals 1. □

The cohomology with local coefficients arises naturally in the de Rham theory. Later on we shall work also with the *homology* with local coefficients, which appears in the Morse-Novikov theory. We will now explain the relation between the two constructions.

An antihomomorphism $\theta : G \to GL(l, B)$ will be also called *a right representation*; it induces a structure of a right $\mathbb{Z}G$-module on B^l. The homology of the chain complex

$$C_*(X, \theta) = B^l \underset{\theta}{\otimes} C_*(\widetilde{X})$$

is a B-module called *the twisted homology of X with respect to θ* and denoted by $H_*(X, \theta)$. The dimension over $\{B\}$ of the localization $H_k(X, \theta) \otimes \{B\}$ will be called the *k-th homological Betti number of X with respect to θ* and denoted by $\beta_k(X, \theta)$.

To each representation $\rho : G \to GL(l, B)$ we associate the right representation $\rho^* : G \to GL(l, B)$, where the matrix $\rho^*(g)$ is the transpose of $\rho(g)$. The representation ρ^* will be called *conjugate* to ρ. Let E be the free finitely generated B-module endowed with the structure of a left $\mathbb{Z}G$-module via ρ, then the right representation ρ^* corresponds to the right $\mathbb{Z}G$-module $\mathrm{Hom}_B(E, B)$, which is denoted by E^*. Observe that $(\rho^*)^* = \rho$ and $(E^*)^* \approx E$.

Lemma 4.3. *Let $\rho : G \to GL(l, B)$ be a representation. Then there is a natural isomorphism*

$$\mathrm{Hom}_B\big(C_*(X, \rho^*), B\big) \approx C^*(X, \rho). \tag{10}$$

Proof. Put $R = \mathbb{Z}G$. For any R-module B there is a canonical isomorphism

$$\mathrm{Hom}_B\big(M \underset{R}{\otimes} B^l, B\big) \approx \mathrm{Hom}_R\big(M, \mathrm{Hom}_B(B^l, B)\big).$$

The lemma follows. □

The following Corollary is immediate.

Corollary 4.4.

$$\beta^k(X, \rho) = \beta_k(X, \rho^*). \tag{11}$$

Remark 4.5. Observe that for $l = 1$ every representation ρ is equal to its conjugate and therefore $\beta^k(X, \rho) = \beta_k(X, \rho)$ for 1-dimensional local systems.

5. Comparing the two spectral sequences

We are interested in the formal deformations of differential algebras related to topological spaces, mainly manifolds. Let M be a connected C^∞ manifold, denote $\pi_1(M)$ by G, and let $\rho : G \to \mathrm{GL}(l, \mathbb{C})$

be a representation. Let $\widetilde{M}$ be the universal covering of M and define a flat bundle E on M as follows:

$$E = (M \times \mathbb{C}^l)/\sim \quad \text{where } (gm, \xi) \sim (m, g^{-1}\xi) \ \text{ for } g \in G.$$

The module $\mathcal{N}^* = A^*(M, E)$ of E-valued C^∞ differential forms on M is a DGM over the algebra $A^*(M)$ of $\mathbb{C}$-valued C^∞ differential forms on M. The elements of $\mathcal{N}^*$ can be considered as G-equivariant differential forms on $\widetilde{M}$ with values in $\mathbb{C}^l$. Let $\alpha \in H^1(M, \mathbb{C})$. The corresponding exact couple

$$\begin{array}{ccc} H^*\Big(\mathcal{N}^*[[t]]\Big) & \xrightarrow{\quad t \quad} & H^*\Big(\mathcal{N}^*[[t]]\Big) \\ & \nwarrow \qquad \swarrow{\scriptstyle \pi_*} & \\ & H^*(\mathcal{N}^*) & \end{array} \tag{12}$$

gives rise to the deformation spectral sequence $\mathscr{E}_r^*(\mathcal{N}^*, \alpha)$ (see §3). Let $\widehat{\gamma}$ be the formal exponential deformation of ρ corresponding to the class α:

$$\widehat{\gamma}(g) = \rho(g)e^{t\langle \alpha, g\rangle} \in \mathrm{GL}(l, \Lambda), \quad \Lambda = \mathbb{C}[[t]].$$

We associate to this deformation the spectral sequence $\mathscr{W}^*(M, \rho, \alpha)$ (see §4).

Theorem 5.1. *The spectral sequences* $\mathscr{E}_r^*(\mathcal{N}^*, \alpha)$ *and* $\mathscr{W}_r^*(M, \rho, \alpha)$ *are isomorphic.*

Proof. The $A^*(M)$-module $A^*(M, E)[[t]]$ can be considered as the vector space of exterior differential forms on M with values in $E[[t]]$. Denote by $\mathcal{T}^*(M)$ the $A^*(M)$-submodule of $A^*(\widetilde{M}, \mathbb{C}^l[[t]])$, consisting of differential forms on $\widetilde{M}$ which are equivariant with respect to the representation $\widehat{\gamma}$. Choose a closed 1-form ξ within the cohomology class α. Let $F : \widetilde{M} \to \mathbb{C}$ be a C^∞ function such that $\pi^*\xi = dF$. The next lemma is obvious.

Lemma 5.2. *The homomorphism*

$$\Phi : A^*(M, E)[[t]] \longrightarrow \mathcal{T}^*(M)$$

defined by $\Phi(\omega) = e^{tF}\pi^*(\omega)$ *is an isomorphism of DG-modules over* $A^*(M)$. □

For a manifold N we denote by $\mathcal{S}_*(N)$ the graded group of singular C^∞-chains on N. The integration map determines a homomorphism

$$I : \mathcal{T}^*(M) \longrightarrow \mathrm{Hom}_G\Big(\mathcal{S}_*(\widetilde{M}), \mathbb{C}^l[[t]]\Big).$$

Here $\mathbb{C}^l[[t]]$ is endowed with the structure of a G-module via the representation $\widehat{\gamma} : G \to \mathrm{GL}(l, \Lambda)$.

Proposition 5.3. *The induced map in the cohomology groups*

$$I_* : H^*\big(\mathcal{T}^*(M)\big) \longrightarrow H^*(M, \widehat{\gamma})$$

is an isomorphism.

Proof. The argument follows the usual sheaf-theoretic proof of the de Rham theorem. Consider the sheaf $\mathcal{T}^k$ on M whose sections over an open subset $U \subset M$ are $\widehat{\gamma}$-equivariant k-forms on $\pi^{-1}(U)$ with values in $\mathbb{C}^l[[t]]$. Denote by $\mathcal{A}$ the sheaf on M, whose sections over U are $\widehat{\gamma}$-equivariant locally constant functions $\pi^{-1}(U) \to \mathbb{C}^l[[t]]$. We have an exact sequence of sheaves

$$\mathcal{R}_1 = \{0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{T}^0 \xrightarrow{d} \mathcal{T}^1 \longrightarrow \ldots\}. \tag{13}$$

Consider the sheaf $\mathcal{Z}^*$ of $\widehat{\gamma}$-equivariant singular cochains:

$$\mathcal{Z}^*(U) = \mathrm{Hom}_G\Big(\mathcal{S}_*(\pi^{-1}(U)), \mathbb{C}^l[[t]]\Big).$$

The sequence of sheaves

$$\mathcal{R}_2 = \{0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{Z}^0 \xrightarrow{\delta} \mathcal{Z}^1 \xrightarrow{\delta} \ldots\} \tag{14}$$

(where δ is the coboundary operator on singular cochains) is also exact, and we have two soft acyclic resolutions of the sheaf $\mathcal{A}$. The integration map I induces a homomorphism $\mathcal{R}_1 \to \mathcal{R}_2$ which equals identity on $\mathcal{A}$. The standard sheaf-theoretic result implies that the induced homomorphism in the cohomology groups of the complexes of global sections is an isomorphism (see for example [25], Corollary 3.14). □

The isomorphism

$$I_* \circ \Phi_* : H^*\Big(A^*(M, E)[[t]]\Big) \to H^*(M, \widehat{\gamma})$$

induces an isomorphism of the exact couple

$$\begin{array}{ccc} H^*\Big(A^*(M,E)[[t]]\Big) & \xrightarrow{\quad t \quad} & H^*\Big(A^*(M,E)[[t]]\Big) \\ & \nwarrow \qquad \swarrow_{\pi_*} & \\ & H^*(M,E) & \end{array} \tag{15}$$

to the exact couple

$$\begin{array}{ccc} H^*(M,\widehat{\gamma}) & \xrightarrow{\quad t \quad} & H^*(M,\widehat{\gamma}) \\ & \nwarrow \qquad \swarrow_{\pi_*} & \\ & H^*(M,E) & \end{array} \tag{16}$$

and therefore the isomorphism of the spectral sequences

$$\mathscr{E}_r^*(\mathcal{N}^*,\alpha) \approx \mathscr{W}_r^*(M,\rho,\alpha).$$

The proof of Theorem 5.1 is complete. □

Now let us proceed to general topological spaces. The rational homotopy theory of D. Sullivan (see [24], [4], [3]) associates to a connected topological space X a minimal algebra $\mathcal{M}^*(X)$ over $\mathbb{C}$, well defined up to isomorphism.

Let $\alpha \in H^1(X,\mathbb{C})$; we obtain a spectral sequence $\mathscr{E}_r^*(\mathcal{M}^*(X),\alpha)$. It is not difficult to see that for the case when X is a C^∞ manifold, this spectral sequence is isomorphic to the one considered in the previous section. Indeed, we have a homotopy equivalence $\phi : \mathcal{M}^*(X) \longrightarrow A^*(X)$; the induced homomorphism of the spectral sequences

$$\mathscr{E}_r^*(\mathcal{M}^*(X),\alpha) \to \mathscr{E}_r^*(A^*(X),\phi_*(\alpha))$$

is an isomorphism in the second term $\mathscr{E}_2$. Thus the two spectral sequences are isomorphic.

Theorem 5.4. *Let X be a finite CW complex. Then*

$$\mathscr{E}_r^*(\mathcal{M}^*(X),\alpha) \approx \mathscr{W}_r^*(X,\alpha).$$

Proof. Let $f : X \to M$ be a homotopy equivalence of X to a C^∞ manifold M (possibly non-compact). Denote by α' the $(f^{-1})^*$-image of α. The map f induces a homotopy equivalence $F : \mathcal{M}^*(X) \to A^*(M)$ of DGAs. We obtain isomorphisms of the spectral sequences:

$$\mathscr{E}_r^*(\mathcal{M}^*(X),\alpha) \approx \mathscr{E}_r^*\Big(A^*(M),(f^{-1})^*(\alpha)\Big)$$
$$\mathscr{W}_r^*(X,\alpha) \approx \mathscr{W}_r^*(M),\alpha).$$

Now apply Theorem 5.1 and the result follows. □

Remark 5.5. Theorem 5.4 can be generalized to the case of local coefficients so as to obtain a result similar to Theorem 5.1 in the setting of general topological spaces.

6. Strongly formal manifolds

The next theorem follows immediately from Theorems 5.4 and 3.14.

Theorem 6.1. *Let X be a finite connected CW-complex. Assume that X is formal. Then the spectral sequences*

$$\mathscr{E}_r^*(\mathcal{M}^*(X), \alpha) \approx \mathscr{W}^*(X, \alpha)$$

degenerate at their second term.

Thus the dimension of the homology of X with coefficients in a generic point of the exponential deformation of the trivial representation can be computed from the multiplicative structure of the ordinary homology. Namely, let $\alpha \in H^1(X, \mathbb{C})$ and let γ_t be the exponential deformation of the trivial representation:

$$\gamma_t(g) = e^{t\langle \alpha, g\rangle}.$$

Denote by $\beta^k(X, \gamma_{gen})$ the Betti number of X with coefficients in a generic point of the curve γ_t, and let L_α be the operation of multiplication by α in $H^k(X, \mathbb{C})$. The theorem above implies that

$$\beta^k(X, \gamma_{gen}) = \dim_{\mathbb{C}} \mathcal{H}^k(H^*(X, \mathbb{C}), \alpha).$$

The case of the spectral sequence $\mathscr{W}^*(X, \rho, \alpha)$ where $\rho : \pi_1(X) \to GL(l, \mathbb{C})$ is a non-trivial representation, is more complicated and to guarantee the degeneracy of the spectral sequence a stronger condition is necessary. Let us first consider the case of 1-dimensional representations ρ. Let M be a connected C^∞ manifold. Denote by G the fundamental group of M, let $Ch(G)$ be the group of homomorphisms $G \to \mathbb{C}^* = \mathrm{GL}(1, \mathbb{C})$. For a character $\rho \in Ch(G)$ denote by E_ρ the corresponding flat vector bundle over M. Put

$$\bar{A}^*(M) = \bigoplus_{\rho \in Ch(G)} A^*(M, E_\rho).$$

The pairing $E_\rho \otimes E_\eta \approx E_{\rho\eta}$ induces a natural structure of a differential graded algebra on the vector space $\bar{A}^*(M)$.

Definition 6.2. A C^∞ manifold M is *strongly formal* if the algebra $\bar{A}^*(M)$ is formal.

Theorem 6.3. *Let M be a strongly formal manifold, $\rho \in Ch(G)$ and $\alpha \in H^1(M, \mathbb{C})$. Then the spectral sequences*

$$\mathscr{E}_r^*(\mathcal{M}^*(M), \rho, \alpha) \approx \mathscr{W}_r^*(M, \rho, \alpha)$$

degenerate in their second term.

Proof. The DG-module $A^*(M, E_\rho)$ is formal; apply Proposition 3.16 and the proof is over. □

Denote by $b_k(M, \rho, \gamma_{gen})$ the k-th Betti number of M with coefficients in a generic point of the curve

$$\gamma_t(g) = \rho(g)e^{t\langle\alpha,g\rangle}.$$

Corollary 6.4. *Let M be a strongly formal manifold, $\rho \in Ch(G)$, and $\alpha \in H^1(M, \mathbb{C})$. Then*

$$b_k(M, \rho, \gamma_{gen}) = \mathcal{B}_k(H^*(M, \rho), \alpha).$$

An big class of examples of strongly formal spaces is formed by Kähler manifolds, as it follows from C. Simpson theory of Higgs bundles [23].

Theorem 6.5. *Any compact Kähler manifold is strongly formal.*

Proof. Let $\rho \in Ch(\pi_1(M))$. The flat bundle E_ρ has a unique structure of a harmonic Higgs bundle (see [1], [23]); the exterior differential D_ρ in the DG-module $A^*(M, E_\rho)$ writes therefore as $D_\rho = D'_\rho + D''_\rho$, and the natural homomorphisms of DG-modules

$$\begin{array}{ccc} & \Big(\text{Ker } (D'_\rho), D''_\rho\Big) & \\ \swarrow & & \searrow \\ \Big(A^*(M, E_\rho), D_\rho\Big) & & \Big(H^*_{DeRham}(M, E_\rho), 0\Big) \end{array}$$

induce isomorphisms in cohomology (see [23], Lemma 2.2 (Formality)). Denote the DG-module (Ker $(D'_\rho), D''_\rho$) by $K_\rho(M)$, and put

$$\mathcal{K}^*(M) = \bigoplus_{\rho \in Ch(G)} K_\rho(M).$$

The multiplicativity properties of Higgs bundles imply that $\mathcal{K}^*$ is a DG-algebra and we have the maps of DGAs:

$$\begin{array}{ccc} & \mathcal{K}^*(M) & \\ \swarrow & & \searrow \\ \bar{A}^*(M) & & \bigoplus_{\rho \in Ch(G)} H^*(M, E_\rho) \end{array}$$

both inducing isomorphisms in cohomology. The theorem follows. □

Remark 6.6. There are manifolds which are formal but not strongly formal. The example described below was indicated to the authors by H. Kasuya.

H. Sawai [22] constructed an 8-dimensional solvmanifold, having several remarkable properties. H. Sawai's construction starts with a 7-dimensional solvable Lie algebra $\mathfrak{g}$ generated by $A, X_1, X_2, X_3, Z_1, Z_2, Z_3$, with

$$[X_1, X_2] = Z_3, \quad [X_2, X_3] = Z_1, \quad [X_3, X_1] = Z_2,$$
$$[A, X_1] = -a_1 X_1, \quad [A, X_2] = -a_2 X_2, \quad [A, X_3] = -a_3 X_3,$$
$$[A, Z_1] = a_1 Z_1, \quad [A, Z_2] = a_2 Z_2, \quad [A, Z_3] = a_3 Z_3,$$

where a_1, a_2, a_3 are distinct real numbers. (This is a generalization of the Lie algebra constructed by Benson and Gordon in [2].) Denote by G the corresponding simply connected Lie group. H. Sawai proves that for some choice of a_1, a_2, a_3 there is a lattice Γ in $G \times \mathrm{R}$, and the quotient is a formal space, which has a symplectic structure and satisfies the hard Lefschetz property, but admits no Kähler structure.

The cohomology of $(G/\Gamma) \times S^1$ with local coefficients in one-dimensional local systems was studied by H. Kasuya in [11]. By the Mostow theorem [13] the computations can be carried out in the cohomology of the Lie algebra $\Gamma \times \mathrm{R}$ with coefficients in 1-dimensional modules. H. Kasuya gives an example of non-vanishing triple Massey product in the homology of $\Gamma \times \mathrm{R}$ with twisted coefficients. Thus M is a formal but not a strongly formal space.

H. Kasuya informed the authors that he has constructed a 4-dimensional solvmanifold which is formal but not strongly formal.

Conjecture 6.7. *For every $n \geqslant 2$ there exists a solvmanifold M, a character $\rho : \pi_1(M) \to \mathbb{C}^*$ and a cohomology class $\alpha \in H^1(M, \mathbb{R})$*

such that M is formal, but the differential d_n in the spectral sequence $\mathscr{W}_r^*(M,\rho,\alpha)$ is non-zero.

Consider now the case of representations of higher rank.

Proposition 6.8. *Let M be a compact Kähler manifold and $\rho : \pi_1(M) \to \mathrm{GL}(l,\mathbb{C})$ be a semi-simple representation. Then the differential graded $A^*(M)$-module $A^*(M,E_\rho)$ is formal.*

Proof. By [23], Theorem 1 there is a harmonic metric on the bundle E_ρ. The tensor powers of this metric provide harmonic metrics on the bundles $E_\rho^{\otimes n}$ for any $n \geqslant 1$. Put

$$\mathcal{L}_\rho^*(M) = \bigoplus_{n=0}^{\infty} A^*(M, E_\rho^{\otimes n})$$

(where $A^*(M,E_\rho^{\otimes 0}) = A^*(M)$ by convention). Then $\mathcal{L}_\rho^*(M)$ is a DG-algebra. The same argument as in the proof of Theorem 6.5 implies that this algebra is formal, and it remains to observe that $A^*(M,E_\rho)$ is a direct summand of $\mathcal{L}_\rho^*(M)$. □

7. Chain complexes over Laurent polynomial rings and their localizations

In this section we discuss the Betti numbers of complexes over Laurent polynomial rings in view of further applications to the Novikov Betti numbers and the homology with local coefficients.

Let T be an integral domain and $\{T\}$ be its fraction field. Let C_* be a finite free chain complex over T:

$$C_* = \{0 \longleftarrow C_0 \longleftarrow \ \ldots \ \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1} \longleftarrow \ \ldots\}$$

The tensor product of this chain complex with $\{T\}$ will be denoted by $\overline{C}_*$ and the boundary operator in $\overline{C}_*$ will be denoted by $\overline{\partial}_*$. The Betti number $b_k(\overline{C}_*)$ is equal to

$$\dim_{\{T\}} \overline{C}_k - \mathrm{rk}\overline{\partial}_k - \mathrm{rk}\overline{\partial}_{k+1} = \mathrm{rk}C_k - \mathrm{rk}\partial_k - \mathrm{rk}\partial_{k+1}$$

(where $\mathrm{rk}\partial_k$ stands for the maximal rank of a non-zero minor of the matrix of ∂_k).

Let $\phi : T \to U$ be a homomorphism to another integral domain. Denote by $b_k(C_*,\phi)$ the k-th Betti number of the tensor product $C_* \otimes_\phi \{U\}$. Then $b_k(C_*,\phi) \geqslant b_k(C_*)$. The inequality is strict if the ϕ-images of all the $\mathrm{rk}\partial_k$-minors of ∂_k vanish, or if the ϕ-images of all the $\mathrm{rk}\partial_{k+1}$-minors of ∂_{k+1} vanish. In general, for $q > 0$ the condition $b_k(C_*,\phi) \geqslant b_k(C_*) + q$ is equivalent to the existence of a number i, with $0 \leqslant i \leqslant q$ such that all the $(\mathrm{rk}\partial_k - i)$-minors of $\phi(\partial_k)$ vanish and all the $(\mathrm{rk}\partial_{k+1} - (q-i))$-minors of $\phi(\partial_{k+1})$ vanish.

We are interested in the case of Laurent polynomial rings. Let R be an integral domain and C_* be a finite free chain complex over $L_n = R[t_1^{\pm}, \ldots, t_n^{\pm}] = R[\mathbb{Z}^n]$. Let $p : \mathbb{Z}^n \to \mathbb{Z}^m$ be a group homomorphism. Extend it to a ring homomorphism $L_n \to L_m$ which will be denoted by the same letter p. Denote by $\mathcal{Q}_m$ is the field of fractions of L_m, that is, the field of the rational functions in m variables with coefficients in the fraction field of R. Form the chain complex $C_* \otimes_p \mathcal{Q}_m$, and denote by $b_k(C_*, p)$ the dimension of the vector space $H_k(C_* \otimes_p \mathcal{Q}_m)$ over $\mathcal{Q}_m$. Observe that if p is injective, then $b_k(C_*) = b_k(C_*, p)$. We will now study the dependance of $b_k(C_*, p)$ on p.

Definition 7.1. A subgroup $G \subset \mathbb{Z}^n$ is called *full* if it is a direct summand of $\mathbb{Z}^n$. We say that a homomorphism $p : \mathbb{Z}^n \to \mathbb{Z}^m$ is *subordinate* to a full subgroup $G \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ and we write $p \sqsubset G$, if all the coordinates of p are in G.

Remark 7.2. Let G be a full subgroup of $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$. Denote by K the subgroup of $\mathbb{Z}^n$ dual to G. Then $p \sqsubset G$ if and only if $p \mid K = 0$.

Theorem 7.3. *Let C_* be a finite free complex over L_n. Let $k \geqslant 0, q > 0$. Then there is a finite family of proper full subgroups $G_i \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that for $p \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$ the condition*

$$b_k(C_*, p) \geqslant b_k(C_*) + q$$

is equivalent to the following condition: $p \sqsubset G_i$ for some i.

Proof. Let us do the case $q = 1$, the general case is similar. Let $\mathscr{E}$ denote the set of all the $(\mathrm{rk}\partial_k)$-minors of the matrix $\partial_k : C_k \to C_{k-1}$, and all the $(\mathrm{rk}\partial_{k+1})$-minors of the matrix $\partial_{k+1} : C_{k+1}k \to C_k$. Let $\Delta \in \mathscr{E}$, write $\Delta = \sum_{g \in \mathbb{Z}^n} r_g \cdot g$ (where $r_g \in R$).

According to our previous observation it suffices to study the set Σ of all homomorphisms $p : \mathbb{Z}^n \to \mathbb{Z}^m$ such that $p(\Delta) = 0$. Let $\Gamma = \mathrm{supp}\,\Delta$, which is a finite subset of $\mathbb{Z}^n$. Any homomorphism $p : \mathbb{Z}^n \to \mathbb{Z}^m$ with $p(\Delta) = 0$ must be non-injective on Γ. To describe the set of all such homomorphisms let us say that a subdivision

$$\Gamma = \Gamma_1 \sqcup \ldots \sqcup \Gamma_N$$

is Δ-fitted, if for any j we have

$$\sum_{g_k \in \Gamma_j} r_g = 0.$$

For any $\boldsymbol{\Delta}$-fitted subdivision $\mathbb{S}$ consider the subgroup $L(\mathbb{S}) \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ consisting of all homomorphisms $h : \mathbb{Z}^n \to \mathbb{Z}$ such that $h \mid \Gamma_i$ is constant for every i. Then $L(\mathbb{S})$ is a full subgroup of $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$. Observe that $L(\mathbb{S})$ is a proper subgroup since every Γ_i contains at least two elements. It is clear that a homomorphism $p : \mathbb{Z}^n \to \mathbb{Z}^m$ belongs to Σ if and only if p is constant on each component Γ_i of a $\boldsymbol{\Delta}$-fitted subdivision of $\boldsymbol{\Delta}$, that is, $p \sqsubset L(\mathbb{S})$. □

8. The twisted Novikov Betti numbers

Let X be a finite connected CW complex, put $G = \pi_1(X)$. Let R be an integral domain, and $\eta : \pi_1(X) \to \mathrm{GL}(l, R)$ be a right representation (that is, η is an antihomomorphism of groups). Recall the definition of the twisted homological Betti numbers with coefficients in η:

$$\beta_k(X, \eta) = \dim_{\{R\}} H_k\Big(\{R\}^l \underset{\eta}{\otimes} C_*(\widetilde{X})\Big).$$

Starting with η we can construct several other representations of G. Let $n = \mathrm{rk} H_1(X, \mathbb{Z})$ and denote by π the projection $G \to H_1(G)/Tors \approx \mathbb{Z}^n$. Let $L_n = R[\mathbb{Z}^n]$ denote the ring of Laurent polynomials in n variables, then the group $\mathbb{Z}^n$ can be identified with the group of units $L_n^\times \subset \mathrm{GL}(1, L_n)$, and the homomorphism π can be condsidered as a representation $G \to \mathrm{GL}(1, L_n)$. Denote by $\langle\eta\rangle$ the tensor product of π and η, that is,

$$\langle\eta\rangle(g) = \pi(g) \cdot \eta(g) \in \mathrm{GL}(l, L_n).$$

Let $p : \mathbb{Z}^n \to \mathbb{Z}^m$ be a homomorphism. Similarly to the above we can consider the tensor product of representations η and $p \circ \pi : G \to \mathrm{GL}(1, L_m)$. This right representation will be denoted by $\langle\eta\rangle_p$. Observe that $\langle\eta\rangle_0 = \eta, \quad \langle\eta\rangle_{\mathrm{Id}} = \langle\eta\rangle$. It is not difficult to see that for every $p : \mathbb{Z}^n \to \mathbb{Z}^m$ we have

$$\beta_k(X, \eta) \geqslant \beta_k(X, \langle\eta\rangle_p) \geqslant \beta_k(X, \langle\eta\rangle). \tag{17}$$

Theorem 7.3 of the previous section implies the following.

Proposition 8.1. *Let $k \geqslant 0, q > 0$. Then there is a finite family of proper full subgroups $G_i \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that for $p \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$ the condition*

$$\beta_k(X, \langle\eta\rangle_p) \geqslant \beta_k(X, \langle\eta\rangle) + q$$

is equivalent to the following condition: $p \sqsubset G_i$ for some i.

Proceeding to the Novikov homology, let us first recall the definition of the Novikov ring. Let H be a free abelian group; denote $\mathbb{Z}H$

by $\mathcal{L}$. Let $\mu : H \to \mathbb{R}$ be a group homomorphism. The Novikov completion $\widehat{\mathcal{L}}_\mu$ of the ring $\mathcal{L}$ with respect to μ is defined as the set of all series of the form $\lambda = \sum_g n_g g$ (where $g \in H$ and $n_g \in R$) satisfying the following finiteness condition:

$$\widehat{\mathcal{L}}_\mu = \Big\{ \lambda \;\Big|\; \forall\, C \in \mathbb{R}, \;\; \text{the set } \; \operatorname{supp} \lambda \cap \mu^{-1}\big([C, \infty[\big) \text{ is finite } \Big\}.$$

In general the ring $\widehat{\mathcal{L}}_\mu$ is rather complicated, however if $R = \mathbb{Z}$ and μ is a monomorphism, this ring is Euclidean by a theorem of J.-Cl. Sikorav (see [18], Th. 1.4). If R is a field and μ is a monomorphism, this ring is a field.

Let $\alpha : G \to \mathbb{R}$ be a homomorphism. We can factor it as follows:

(18)
$$\begin{array}{ccccccc} G & \longrightarrow & H_1(X, \mathbb{Z})/Tors & \xrightarrow{\approx} & \mathbb{Z}^n & \longrightarrow & \mathbb{R} \\ & & & & {\scriptstyle p}\searrow & & \nearrow{\scriptstyle \widetilde{\alpha}} \\ & & & & & \mathbb{Z}^m & \end{array}$$

where p is an epimorphism and $\widetilde{\alpha}$ is a monomorphism. We will denote the Novikov completion of the ring $R[\mathbb{Z}^m]$ with respect to $\widetilde{\alpha}$ by $\widehat{L}_{m,\alpha}$. Denote by $\langle\langle \eta \rangle\rangle$ the composition

$$G \xrightarrow{\eta_p} \mathrm{GL}(l, L_m) \hookrightarrow \mathrm{GL}(l, \widehat{L}_{m,\alpha});$$

it is a right representation of G.

Definition 8.2. ([9], [19]) The twisted homology of X with respect to the representation $\langle\langle \eta \rangle\rangle$ is called *the η-twisted Novikov homology of X with respect to α*. The k-th homological Betti number of X with respect to $\langle\langle \eta \rangle\rangle$ will be denoted by $\widehat{b}^\eta_k(X, \alpha)$ and called *the twisted Novikov Betti number of X with respect to η*. Thus we have

$$\widehat{b}^\eta_k(X, \alpha) = \dim_{\{\widehat{L}_{m,\alpha}\}} H_k\Big(\{\widehat{L}_{m,\alpha}\}^l \underset{\langle\langle \eta \rangle\rangle}{\otimes} C_*(\widetilde{X}) \Big).$$

If $R = \mathbb{Z}$ so that $\widehat{L}_{m,\alpha}$ is a principal ideal domain, we denote by $\widehat{q}^\eta_k(X, \alpha)$ the torsion number of the module $H_k\big(\widehat{L}^l_{m,\alpha} \otimes C_*(\widetilde{X})\big)$.

The geometric reasons to consider these completions of the module $C_*(\widetilde{X})$ are as follows. If X is a compact manifold, and ω is a closed 1-form on M with non-degenerate zeroes, denote by α the period homomorphism $[\omega] = \alpha : H_1(M, \mathbb{Z}) \to \mathbb{R}$. The Morse-Novikov theory implies the following lower bounds on the number of zeroes of ω:

$$m_k(\omega) \geqslant \frac{1}{l}\Big(\widehat{b}^\eta_k(X, \alpha) + \widehat{q}^\eta_k(X, \alpha) + \widehat{q}^\eta_{k-1}(X, \alpha) \Big).$$

(Here the base ring R equals $\mathbb{Z}$.) Compared to the other versions of the Novikov homology, the twisted Novikov homology has the advantage of being computable, and at the same time to keep the information about the non-abelian structure of G and related invariants. In a recent work [8] S. Friedl and S. Vidussi proved that the twisted Novikov homology detects fibredness of knots in S^3. At present we will need only the simplest part of these invariants, namely, the twisted Novikov Betti numbers.

Proposition 8.3. *Let $\alpha : G \to \mathbb{R}$ be a homomorphism and $\eta : G \to GL(l, R)$ a right representation. Then for every k we have*

$$\beta_k(X, \langle\eta\rangle_p) = \widehat{b}_k^\eta(X, \alpha)$$

(where p is obtained from the diagram (18)).

Proof. The twisted Betti Novikov number in question equals the dimension of the module

$$H_k\big(L_m^l \underset{\langle\eta\rangle_p}{\otimes} C_*(\widetilde{X})\big) \otimes \{\widehat{L}_{m,\alpha}\}$$

over the field of fractions $\{\widehat{L}_{m,\alpha}\}$ of the Novikov ring.

The Betti number $b_k(X, \eta; p)$ is the dimension of the vector space

$$H_k\big(L_m^l \underset{\langle\eta\rangle_p}{\otimes} C_*(\widetilde{X})\big) \otimes \{L_m\}$$

over the field of fractions $\{L_m\}$. The inclusion $L_m \subset \widehat{L}_{m,\alpha}$ extends to an inclusion of fields $\{L_m\} \subset \{\widehat{L}_{m,\alpha}\}$ and the result follows. □

Definition 8.4. For a homomorphism $\alpha : G \to \mathbb{R}$ the subgroup $\alpha(G)$ is a free finitely generated abelian group; its rank is called *irrationality degree of* α and denoted $\mathrm{Irr}\alpha$. In particular $\mathrm{Irr}\alpha = 1$ if and only if α is a multiple of a homomorphism $G \to \mathbb{Z}$. We say that α is *maximally irrational* if $\mathrm{Irr}\alpha = \mathrm{rk} H_1(G)$.

Observe that the irrationality degree of α equals the number m from the diagram (18). If α is maximally irrational, then the homomorphism p in this diagram is an isomorphism. In this case the twisted Betti numbers $\widehat{b}_k^\eta(X, \alpha)$ do not depend on α.

Definition 8.5. The number $\widehat{b}_k^\eta(X, \alpha)$ where α is maximally irrational, will be denoted by $\widehat{b}_k^\eta(X)$.

The inequalities (17) together with the proposition 8.3 imply that

$$\widehat{b}_k^\eta(X, \alpha) \geqslant \widehat{b}_k^\eta(X).$$

Theorem 7.3 of the previous section implies the following.

Proposition 8.6. *Let $k \geqslant 0, q > 0$. Then there is a finite family of proper full subgroups $G_i \subset \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that the condition*

$$\widehat{b}^{\eta}_k(X, \alpha) \geqslant \widehat{b}^{\eta}_k(X) + q$$

is equivalent to the following condition: $\alpha \in \bigcup_i G_i \otimes \mathbb{R}$. □

Remark 8.7. The results of this section have natural cohomology analogs. Namely, given a representation $\rho : G \to GL(l, R)$ one defines the representations $\langle\rho\rangle$, $\langle\rho\rangle_p$, and the corresponding cohomological Betti numbers $\beta^k(X, \langle\rho\rangle)$ and $\beta^k(X, \langle\rho\rangle_p)$. One can define also the cohomological twisted Novikov numbers $\widehat{b}^k_\rho(X, \alpha)$ and $\widehat{b}^k_\rho(X)$. The following lemma follows from Lemma 4.3.

Lemma 8.8. *Let $\rho : G \to GL(l, R)$ be a representation; put $\eta = \rho^*$. We have*

$$\beta^k(X, \langle\rho\rangle) = \beta_k(X, \langle\eta\rangle), \quad \beta^k(X, \langle\rho\rangle_p) = \beta_k(X, \langle\eta\rangle_p),$$
$$\widehat{b}^{\eta}_k(X, \alpha) = \widehat{b}^k_\rho(X, \alpha), \quad \widehat{b}^{\eta}_k(X) = \widehat{b}^k_\rho(X). \qquad \square$$

9. Homology with local coefficients

Let us proceed now to the homology with local coefficients. Let $\eta : G \to \mathrm{GL}(l, \mathbb{C})$ be a right representation of G and $\alpha \in H^1(X, \mathbb{C})$. The cohomology class α can be considered as a homomorphism $\alpha : G \to \mathbb{C}$, which factors as follows

(19) $$\begin{array}{ccccccc} G & \longrightarrow & H_1(X,\mathbb{Z})/Tors & \xrightarrow{\approx} & \mathbb{Z}^n & \longrightarrow & \mathbb{R} \\ & & & & \searrow^{p} & & \nearrow_{\widetilde{\alpha}} \\ & & & & & \mathbb{Z}^m, & \end{array}$$

where p is an epimorphism and $\widetilde{\alpha}$ a monomorphism. Here $m = \mathrm{Irr}\alpha$; denote the coordinates of p by $p_i : \mathbb{Z}^m \to \mathbb{Z}$, $1 \leqslant i \leqslant m$. We have $\alpha = \sum_{i=1}^m \alpha_i p_i$, and the numbers $\alpha_i \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$. Recall the exponential and formal exponential deformations of η:

$$\gamma_t : G \to \mathrm{GL}(l, \mathbb{C}), \quad \gamma_t(g) = \eta(g) e^{t\langle\alpha, g\rangle} \in \mathbb{C} \quad (\text{where } t \in \mathbb{C});$$
$$\widehat{\gamma}(g) = \eta(g) e^{t\langle\alpha, g\rangle} \in \mathrm{GL}(l, \mathbb{C}[[t]]).$$

and the corresponding Betti numbers $\beta_k(X, \gamma_t)$, $\beta_k(X, \widehat{\gamma})$. We will need a simple lemma.

Lemma 9.1. *Let $\alpha_1, \ldots, \alpha_m$ be complex numbers, linearly independent over $\mathbb{Q}$. Then the power series $e^{\alpha_1 t}, \ldots, e^{\alpha_m t} \in \mathbb{C}[[t]]$ are algebraically independent.*

Proof. If $P \in \mathbb{C}[z_1, \ldots, z_m]$ is a polynomial such that $P(e^{\alpha_1 t}, \ldots, e^{\alpha_m t}) = 0$, write $P = \sum a_I t^I$ where the sum ranges over multiindices $I = (k_1, \ldots, k_m) \in \mathbb{N}^m$. Denote the string $(\alpha_1, \ldots, \alpha_m)$ by α. The series $\zeta = P(e^{\alpha_1 t}, \ldots, e^{\alpha_m t})$ is then a finite sum of exponential functions of the form $a_I e^{t\langle I, \alpha\rangle}$. Observe that $\langle I, \alpha\rangle \neq \langle J, \alpha\rangle$ if $I \neq J$, since α_i are linearly independent over $\mathbb{Q}$. Therefore ζ is a finite linear combination of exponential functions $e^{t\beta_I}$ with pairwise different β_I. Thus $\zeta = 0$ implies $a_I = 0$ for all I. □

Proposition 9.2. *There is a subset $S \subset \mathbb{C}$ consisiting of isolated points, such that for every $t \in \mathbb{C} \setminus S$ and every k we have*

1) $\beta_k(X, \gamma_t) = \beta_k(X, \widehat{\gamma}) = \beta_k(X, \langle\eta\rangle_p)$

2) If $\alpha \in H^1(X, \mathbb{R})$ then $\beta_k(X, \gamma_t) = \beta_k(X, \widehat{\gamma}) = \widehat{b}_k^\eta(X, \alpha)$.

Proof. The right representation $\widehat{\gamma}$ factors through $\mathbb{Z}^m$ as follows

$$G \xrightarrow{p\circ\pi} \mathbb{Z}^m \xrightarrow{\Gamma} \Big(\mathbb{C}[[t]]\Big)^*$$

Let $(e_1, \ldots, e_m)$ denote the canonical basis in $\mathbb{Z}^m$; then we have $\Gamma(e_i) = e^{t\alpha_i}$. The extension of Γ to a ring homomorphism $\mathbb{Z}[\mathbb{Z}^m] \to \mathbb{C}[[t]]$ is injective by the previous lemma, and therefore can be further extended to a homomorphism

$$\overline{\Gamma} : \{L_m\} \to \mathbb{C}((t))$$

of the fraction fields. Therefore

$$\dim_{\{L_m\}} H_k\Big(\{L_m\}^l \underset{\langle\eta\rangle_p}{\otimes} C_*(\widetilde{X})\Big) = \dim_{\mathbb{C}((t))} H_k\Big(\mathbb{C}((t))^l \underset{\Gamma\circ\eta\circ\pi}{\otimes} C_*(\widetilde{X})\Big)$$

and the point 1) is proved. The point 2) follows from 1) and Proposition 8.3. □

Remark 9.3. A particular case of this proposition corresponding to the vanishing Novikov Betti numbers was proved by S. Papadima and A. Suciu in [20].

The next proposition follows immediately.

Proposition 9.4. *Let $k \geqslant 0, q > 0$. Then there is a finite family of proper full subgroups $G_i \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ such that the condition*

$$\beta_k(X, \widehat{\gamma}) \geqslant \beta_k(X, \langle\eta\rangle) + q$$

is equivalent to the following condition: $\alpha \in \bigcup_i G_i \otimes \mathbb{C}$.

10. Strongly formal spaces II : the jump loci

Let X be a manifold, $\rho : \pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ a representation, and $\alpha \in H^1(X, \mathbb{C})$. In this section we assume that the spectral sequences $\mathscr{E}_r^*$ and $\mathscr{W}_r^*$ associated to the triple (X, ρ, α) degenerate at their second term, so that

$$E_r^*(X, \rho, \alpha) \approx W_r^*(X, \rho, \alpha) \approx \mathcal{H}^*\Big(H^*(X, \rho), \alpha\Big).$$

According to Section 6 this holds, in particular, when X is a Kähler manifold and ρ a semi-simple representation. Recall from Section 4 the exponential deformation γ_t and the formal exponential deformation $\widehat{\gamma}$ of the representation ρ.

Theorem 10.1. 1) $\beta^k(X, \widehat{\gamma}) = \beta^k(X, \gamma_{gen}) = \mathcal{B}^k(H^*(X, \rho), \alpha)$.
2) *If* $\alpha \in H^1(X, \mathbb{R})$ *then*

$$\widehat{b}_\rho^k(X, \alpha) = \beta^k(X, \widehat{\gamma}) = \mathcal{B}^k(H^*(X, \rho), \alpha).$$

Proof. The first point is a consequence of the degeneracy of the two spectral sequences above. To prove the second point, consider the right representation $\eta = \rho^*$, and the corresponding formal exponential deformation $\widehat{\gamma}^*$ of η. We have $\widehat{b}_\rho^k(X, \alpha) = \widehat{b}_k^\eta(X, \alpha)$ by Lemma 8.8. Further, $\widehat{b}_k^\eta(X, \alpha) = \beta_k(X, \widehat{\gamma}^*)$ by Proposition 9.2. Finally $\beta_k(X, \widehat{\gamma}^*) = \beta^k(X, \widehat{\gamma})$ by Corollary 4.4. The proof of the theorem is over. □

The proof of the following proposition, concerning the jump loci of the Betti numbers, is done on similar lines, using Proposition 9.4 and Proposition 8.1.

Proposition 10.2. *Let* $k \geqslant 0, q > 0$. *Then there is a finite family of proper full subgroups* $G_i \subset \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z})$ *such that each of the following conditions (20), (21)*

$$\beta^k(X, \widehat{\gamma}) \geqslant \beta^k(X, \langle\rho\rangle) + q \tag{20}$$

$$\mathcal{B}^k(H^*(X, \rho), \alpha) \geqslant \beta^k(X, \langle\rho\rangle) + q \tag{21}$$

is equivalent to the condition $\alpha \in \bigcup_i G_i \otimes \mathbb{C}$.

For $\alpha \in H^1(X, \mathbb{R})$ *each of the following conditions (22), (23)*

$$\widehat{b}_\rho^k(X, \alpha) \geqslant \beta^k(X, \langle\rho\rangle) + q; \tag{22}$$

$$\mathcal{B}^k(H^*(X, \rho), \alpha) \geqslant \beta^k(X, \langle\rho\rangle) + q \tag{23}$$

is equivalent to the condition $\alpha \in \bigcup_i G_i \otimes \mathbb{R}$. □

11. Acknowledgements

This work was supported by World Premier International Research Center Initiative (WPI Program), MEXT, Japan and by JSPS Grants-in-Aid for Scientific Research 23340014. A part of this work was accomplished while the second author was staying at Graduate School of Mathematical Sciences, the University of Tokyo, in 2011. The second author gratefully acknowledges the support by the Program for Leading Graduate Schools, MEXT, Japan, and thanks the Graduate School of Mathematical Sciences for warm hospitality.

The authors thank Hisashi Kasuya for several valuable discussions concerning his work and the subject of the present paper.

During the preparation of the paper, the authors became aware of the e-print [5] by A. Dimca and S. Papadima, on related subjects. Theorem 6.1 of the present paper is also proved by a completely different method in [5], Theorem E.

The second author thanks Fedor Bogomolov for many valuable discussions and for sharing his insight about several questions of algebraic geometry.

The authors thank the anonymous referees for the helpful comments and suggestions which lead to the improvement of the manuscript.

References

[1] D. Arapura, *Higgs line bundles, Green-Lazarsfeld sets, and maps of Kähler manifolds to curves,* Bull. Amer. Math. Soc. (N.S.), 26(2):310-314, 1992.

[2] C. Benson, C. S. Gordon, *Kähler structures on compact solvmanifolds*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 971–980.

[3] A. K. Bousfield, V. K. A. M. Guggenheim, *On PL de Rham theory and rational homotopy type,* Mem. AMS **179**, (1976).

[4] P. Deligne, Ph. Griffiths, J. Morgan, D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245 - 274.

[5] A. Dimca, S. Papadima, *Nonabelian cohomology jump loci from an analytic viewpoint*, arXiv:1206.3773.

[6] M. Farber, *Topology of closed 1-forms and their critical points*, Topology, **40**, (2001), 235 – 258.

[7] M. Farber, *Lusternik-Schnirelman theory for closed 1-forms*, Commentarii Mathematici Helvetici, **75**, (2000), 156 – 170.

[8] S. Friedl, S. Vidussi, *A vanishing theorem for twisted Alexander polynomials with applications to symplectic 4-manifolds*, arXiv:1205.2434

[9] H. Goda, A. Pajitnov, *Twisted Novikov homology and circle-valued Morse theory for knots and links*, Osaka Journal of Mathematics, **42** No. 3, (2005) 557 – 572.

[10] S.-T. Hu, *Homotopy Theory*, Academic Press, 1959.

[11] H. Kasuya, *Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems,* Journal of Differential Geometry, **93**, Number 2 (2013), 269-297.
[12] W. Massey, *Exact Couples in Algebraic Topology (Parts I and II)*, Ann. Math. **56** (1952), 363-396.
[13] G. D. Mostow, *Cohomology of topological groups and solvmanifolds* Ann. of Math. (2) **73** (1961), 20–48.
[14] S.P. Novikov, *Many-valued functions and functionals. An analogue of Morse theory*, Soviet Math. Dokl. **24** (1981), 222 – 226.
[15] S.P. Novikov, *Bloch homology, critical points of functions and closed 1-forms*, Soviet Math. Dokl. **33** (1986), 551 – 555.
[16] A. Pajitnov, *An analytic proof of the real part of the Novikov inequalities*, Soviet Math. Dokl.**35** (1987), 456 – 457.
[17] A. Pajitnov, *Proof of the Novikov's conjecture on homology with local coefficients over a field of finite characteristic*, Soviet Math. Dokl.**37** (1988), no. 3.
[18] A. Pajitnov, *On the sharpness of Novikov-type inequalities for manifolds with free abelian fundamental group*, Matem. Sbornik, **180** no. 11, (1989), 1486 – 1523. English translation: Math. USSR Sbornik, **68** (1991), 351 - 389.
[19] A. Pajitnov, *Novikov Homology, twisted Alexander polynomials and Thurston cones*, Algebra i Analiz, v. **18** N° 5 (2006), 173 – 209.
[20] S. Papadima, A. Suciu, *Bieri-Neumann-Strebel-Renz invariants and homology jumping loci*, Proceedings of the London Mathematical Society 100 (2010), no. 3, 795-834.
[21] S. Papadima, A. Suciu, *The spectral sequence of an equivariant chain complex and homology with local coefficients*, Trans. Amer. Math. Soc. **180** (2010), p. 2685–2721.
[22] H. Sawai, *A construction of lattices on certain solvable Lie groups*, Topology Appl. **154** (2007), no. 18, 3125–3134.
[23] C. Simpson, *Higgs bundles and local systems*, Publ. I.H.E.S. **75** (1992), 5-95.
[24] D. Sullivan, *Infinitesimal computations in topology*, Publ. I.H.E.S. **47** (1977), 269 - 331.
[25] R.O. Wells, *Differential Analysis on complex manifolds*, Prentice-Hall, 1973.

KAVLI IPMU (WPI), GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN
E-mail address: kohno@ms.u-tokyo.ac.jp

LABORATOIRE MATHÉMATIQUES JEAN LERAY UMR 6629, UNIVERSITÉ DE NANTES, FACULTÉ DES SCIENCES, 2, RUE DE LA HOUSSINIÈRE, 44072, NANTES, CEDEX
E-mail address: andrei.pajitnov@univ-nantes.fr