CIRCLE-VALUED MORSE THEORY FOR COMPLEX HYPERPLANE ARRANGEMENTS

TOSHITAKE KOHNO AND A.V. PAJITNOV

ABSTRACT. Let $\mathcal{A}$ be an essential complex hyperplane arrangement in $\mathbb{C}^n$, and H denote the union of the hyperplanes. We develop the real-valued and circle-valued Morse theory on the space $M = \mathbb{C}^n \setminus H$ and prove, in particular, that M has the homotopy type of a space obtained from a finite n-dimensional CW complex fibered over a circle, by attaching $|\chi(M)|$ cells of dimension n. We compute the Novikov homology $\widehat{H}_*(M, \xi)$ for a large class of homomorphisms $\xi : \pi_1(M) \to \mathbb{R}$.

1. INTRODUCTION

Let Z be a complex analytic manifold, and $f : Z \to \mathbb{C}$ a holomorphic Morse function without zeros. It gives rise to a real-valued Morse function $|f| : Z \to \mathbb{R}$ and a circle-valued Morse function $f/|f| : Z \to S^1$. These two functions can be used to study the topology of Z. There are however numerous technical problems, and this approach works only in some rare particular cases. This paper is about one of such cases, namely the case when Z is the complement to a complex hyperplane arrangement in $\mathbb{C}^n$. The homology of such complements was extensively studied, see [3], [4]. The methods of the paper allow to obtain new results about the homology of the complement, in particular the Novikov homology, which can be viewed as homology with local coefficients.

Let $\xi_i : \mathbb{C}^n \to \mathbb{C}$ be non-constant affine functions ($1 \leqslant i \leqslant m$); put $H_i = \operatorname{Ker} \xi_i$. Denote by $\mathcal{A}$ the hyperplane arrangement $\{H_1, \ldots, H_m\}$ and put

$$H = \bigcup_i H_i, \qquad M(\mathcal{A}) = \mathbb{C}^n \setminus H.$$

We will abbreviate $M(\mathcal{A})$ to M. The *rank* of $\mathcal{A}$ is the maximal codimension of a non-empty intersection of some subfamily of $\mathcal{A}$. We say that $\mathcal{A}$ is *essential* if $\operatorname{rk} L = n$. In the sections 1 – 5 we will assume that L is essential.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a string of complex numbers. P. Orlik and H. Terao [5] proved that for α outside a closed algebraic subset of $\mathbb{C}^m$ the multivalued holomorphic function $\xi = \xi_1^{\alpha_1} \cdot \xi_2^{\alpha_2} \cdot \ldots \cdot \xi_m^{\alpha_m}$ has only non-degenerate critical points (see the works of K. Aomoto [2], and A. Varchenko [8] for partial results in this direction).

In this paper we work only with $\alpha \in \mathbb{R}^m$. It follows from the Orlik-Terao theorem, that there is an open dense subset $W \subset \mathbb{R}^m$ such that for $\alpha \in W$ the function ξ has only non-degenerate critical points. Consider a real-valued C^∞ function

$$f_\alpha(z) = \prod_i \left|\xi_i(z)\right|^{\alpha_i}, \qquad f_\alpha : \mathbb{C}^n \setminus H \to \mathbb{R}.$$

Lemma 1.1. *Let $\alpha \in W$. Then f_α is a Morse function. The index of every critical point of f_α equals n.*

Proof. Let $\omega_\alpha = \sum_i \alpha_i \frac{d\xi_i}{\xi_i}$. Then $f_\alpha(z) = \exp(\mathrm{Re} \int \omega_\alpha)$, therefore $\log f_\alpha(z)$ locally is the real part of a holomorphic Morse function. In general, if h is a holomorphic Morse function on an open subset of $\mathbb{C}^n$, then the real part of h is a real-valued Morse function, and the index of every critical point of this function equals n. Our assertion follows. $\square$

2. Main results

Let $\varepsilon > 0$ and put

(1) $$V = \{z \in \mathbb{C}^n \mid f_\alpha(z) = \varepsilon\}, \quad N = \{z \in \mathbb{C}^n \mid f_\alpha(z) \geqslant \varepsilon\}.$$

A vector $\alpha \in \mathbb{R}^m$ will be called *positive* if $\alpha_i > 0$ for all i. The set of all positive vectors is denoted by $\mathbb{R}^m_+$. The *rank* of the vector $\alpha \in \mathbb{R}^m$ is the dimension of the $\mathbb{Q}$-vector space generated by the components of α in $\mathbb{R}$. Recall that we denote $\mathbb{C}^n \setminus H$ by M.

Theorem 2.1. *Let α be any positive vector. Then for every $\varepsilon > 0$ small enough:*

1) The inclusion $N \subset M$ is a homotopy equivalence. The space $V = \partial N$ is a C^∞ manifold of dimension $2n - 1$.

2) The space N has the homotopy type of the space V with $|\chi(M)|$ cells of dimension n attached.

3) If α has rank 1, then V is fibered over a circle and the fiber has the homotopy type of a finite CW-complex of dimension $n - 1$.

To state the next theorem we recall the definition of the Novikov homology. Let G be a group, and $\mu : G \to \mathbb{R}$ a homomorphism. Put $G_C = \{g \in G \mid \mu(g) \geqslant C\}$. The Novikov completion $\widehat{\Lambda}_\mu$ of the group ring $\Lambda = \mathbb{Z}G$ with respect to the homomorphism $\mu : G \to \mathbb{R}$ is defined as follows (see the thesis of J.-Cl. Sikorav [7]):

$$\widehat{\Lambda}_\mu = \Big\{ \lambda = \sum_{g \in G} n_g g \;\Big|\; \text{where } \; n_g \in \mathbb{Z} \; \text{ and}$$

$$\operatorname{supp} \lambda \cap G_C \; \text{ is finite for every } \; C \Big\}.$$

Let X be a connected topological space and denote $\pi_1(X)$ by G. Let $\mu : G \to \mathbb{R}$ be a homomorphism. The Novikov homology $\widehat{H}_*(X, \mu)$ is by definition the homology of the chain complex

$$\widehat{\mathcal{S}}_*(\widetilde{X}) = \widehat{\Lambda}_\mu \underset{\Lambda}{\otimes} \mathcal{S}_*(\widetilde{X})$$

where $\mathcal{S}_*(\widetilde{X})$ is the singular chain complex of the universal covering of X.

Returning to the space $M = \mathbb{C}^n \setminus H$, observe that $H_1(M, \mathbb{Z})$ is a free abelian group of rank m generated by the meridians of the hyperplanes H_i. The elements of the dual basis in the group $H^1(M, \mathbb{Z})$ will be denoted by θ_i, where $1 \leqslant i \leqslant m$. For $\alpha \in \mathbb{R}^m$ denote by $\overline{\alpha} : \pi_1(M) \to \mathbb{R}$ the homomorphism $\sum_i \alpha_i \theta_i$.

Theorem 2.2. *For any positive α the Novikov homology $\widehat{H}_k(M, \overline{\alpha})$ vanishes for $k \neq n$ and is a free $\widehat{\Lambda}_{\overline{\alpha}}$-module of rank $|\chi(M)|$ if $k = n$.*

3. The gradient field in the neighbourhood of H

Let

$$v_\alpha(z) = \frac{\operatorname{grad} f_\alpha(z)}{f_\alpha(z)}.$$

Denote by u_j the gradient of the function $z \mapsto |\xi_j(z)|$. Then

$$v_\alpha(z) = \sum_{j=1}^m \alpha_j \frac{u_j(z)}{|\xi_j(z)|}.$$

For a linear form $\xi : \mathbb{C}^n \to \mathbb{C}$, $\xi(z) = a_1 z_1 + \ldots + a_m z_m$ the gradient of the function $|\xi(z)|$ is easy to compute:

$$\operatorname{grad} |\xi(z)| = \frac{\xi(z)}{|\xi(z)|} \cdot (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \tag{2}$$

(it follows in particular that the norm of this gradient is constant).

Lemma 3.1. *Assume that $\mathcal{A}$ is a central arrangement. Let $\Gamma \subset \mathbb{R}^m_+$ be a compact subset. Then there is $K > 0$ such that*

$$||v_\alpha(z)|| \geqslant K \sum_i \frac{1}{||\xi_i(z)||} \tag{3}$$

for every $z \in \mathbb{C}^n \setminus H$ and every $\alpha \in \Gamma$.

Proof. Observe that it suffices to prove the Lemma for the case when

$$\bigcap_j \mathrm{Ker}\ \xi_j = \{0\}. \tag{4}$$

Indeed, let $L = \cap_j \mathrm{Ker}\ \xi_j$. Then it follows from (2) that both sides of our inequality (3) are invariant with respect to translations by vectors in L, and it is sufficient to prove the formula for the vector fields $u_j|_{L^\perp}$.

Furthermore, the both sides of the inequality are homogeneous of degree -1, and it is sufficient to prove the inequality for $z \in \Sigma \setminus H$, where Σ stands for the sphere of radius 1 and center 0.

We will proceed by induction on m. Choose some $\varkappa > 0$, and for $i \neq j$ let

$$U_{i,j} = \Big\{ z \in \Sigma \ \Big|\ |\xi_i(z)| < \varkappa |\xi_i(z)| \Big\}.$$

These are open sets and it follows from the condition (4) that their union $U = \cup_{i,j} U_{i,j}$ covers the set $H \cap \Sigma$. We will now prove (3) for $z \in U_{i,j}$. To simplify the notation let us assume $i = 1, j = m$. Put

$$A_m(z) = v_\alpha(z), \quad A_{m-1}(z) = \sum_{j=1}^{m-1} \alpha_j \frac{u_j(z)}{|\xi_j(z)|},$$

$$B_m(z) = \sum_{i=1}^{m} \frac{1}{||\xi_i(z)||}, \quad B_{m-1}(z) = \sum_{i=1}^{m-1} \frac{1}{||\xi_i(z)||}.$$

By the induction assumption we have $||A_{m-1}(z)|| \geqslant DB_{m-1}(z)$, where D is some positive constant. An easy computation shows that for $z \in U_{i,j}$ we have

$$||A_m(z)|| \geqslant (D - \varkappa\alpha_m K_m - \varkappa D) B_m(z)$$

where $K_m = ||u_m(z)||$. Choosing $\varkappa$ sufficiently small we conclude that $||v_\alpha(z)|| \geqslant D' B_m(z)$ with some $D' > 0$ for every $\alpha \in \Gamma$ and $z \in U$.

The complement $\Sigma \setminus U$ is compact and the proof of the lemma will be over once we show that $v_\alpha(z) \neq 0$ for $z \in \Sigma \setminus U$. This is in turn obvious since

$$f_\alpha(\mu z) = \mu^{\alpha_1 + \ldots + \alpha_m} f_\alpha(z) \quad \text{for} \quad \mu \in \mathbb{R}_+,$$

therefore $f'_\alpha(z) \neq 0$ for every $z \notin H$, since all α_i are positive. The proof of Lemma 3.1 is now over. □

For a subset $X \subset \mathbb{C}^n$ and $\delta > 0$ let us denote by $X(\delta)$ the subset of all $z \in \mathbb{C}^n$ such that $d(z, X) \leqslant \delta$.

Proposition 3.2. *Let $\Gamma \subset \mathbb{R}^m_+$ be a compact subset. There is an open neighbourhood U of H, and numbers $A, B > 0$ such that*

1) *For some $\delta > 0$ the set $H(\delta)$ is in U.*
2) *For every $z \in U \setminus H$ and every $\alpha, \beta \in \Gamma$ we have*

$$||v_\alpha(z)|| \geqslant A, \tag{5}$$

$$||v_\alpha(z) - v_\beta(z)|| \leqslant B \cdot \max_i |\alpha_i - \beta_i| \cdot ||v_\beta(z)||. \tag{6}$$

Proof. For a multi-index $I = (i_1, \ldots, i_s)$ let us denote by H_I the intersection of the hyperplanes $H_{i_1}, \ldots, H_{i_s}$. Proceeding by induction on $\dim H_I$ we will construct for every I with $H_I \neq \varnothing$ a neighbourhood U_I of the subset H_I such that the properties 1) – 3) of the Proposition hold if we replace in the formulae H by H_I and U by U_I. Assume that this is done for every H_J with $\dim H_J \leqslant k - 1$; put

$$U_{k-1} = \bigcup_{\dim H_J \leqslant k-1} U_J.$$

Let I be a multi-index with $\dim H_I = k$. We will construct the neighbourhood U_I. We can assume that the multi-index I includes all the values of j such that $H_I \subset H_j$. To simplify the notation let us assume that $I = (1, 2, \ldots, r)$. Write

$$v_\alpha(z) = \sum_{j=1}^{r} \alpha_j \frac{u_j(z)}{|\xi_j(z)|} + \sum_{j=r+1}^{m} \alpha_j \frac{u_j(z)}{|\xi_j(z)|}. \tag{7}$$

Let $\mu > 0$, and consider the subset $U'_\mu = H_I(\mu) \setminus U_{k-1}$. For $z \in U'_\mu$ the second term of (7) is bounded (uniformly with respect to $\alpha \in \Gamma$). As for the first term, its norm converges to ∞ when $d(z, H_I) \to 0$ as it follows from Lemma 3.1, applied to the arrangement defined by a suitable translation of the hyperplanes $H_1, \ldots, H_r$. An easy computation shows now that for every $\mu > 0$ sufficiently small the inequalities (5) and (6) hold for $z \in U'_\mu$ and every $\alpha, \beta \in \Gamma$.

Put $U_I = U'_\mu \cup U_{k-1}$. The properties 2) and 3) for H_I and U_I are now easy to deduce, and the proof of Proposition 3.2 is complete. □

4. The homotopy type of M

In this section we prove the first two assertions of Theorem 2.1. Choose a neighbourhood U of H so that the conclusion of Proposition 3.2 holds. Observe that for $\varepsilon > 0$ small enough the set $f_\alpha^{-1}([0,\varepsilon])$ is in U, therefore ε is a regular level of f_α, and V is a submanifold of M of dimension $2n-1$. This proves the second part of the assertion 1).

To prove the first part, consider the normalized gradient

$$w_\alpha(z) = \frac{\mathrm{grad} f_\alpha(z)}{||\mathrm{grad} f_\alpha(z)||}.$$

It is clear that the trajectories of w_α are defined on $\mathbb{R}_+$.

We use the shift along the flow lines of w_α to construct the deformation retraction of M onto $N = f_\alpha^{-1}\big([\varepsilon, \infty[\big)$. If $\varepsilon > 0$ is sufficiently small then $M \setminus N \subset U$, and for every integral curve $\gamma(t)$ of w_α starting at a point $x \in M \setminus N$ we have

$$\frac{d}{dt} f_\alpha(\gamma(t)) = ||(\mathrm{grad} f_\alpha)(\gamma'(t))|| = f_\alpha(\gamma(t)) \cdot ||v_\alpha(\gamma(t))|| \geqslant A f_a(x)$$

(for every t such that $\gamma(t)$ is in the set U). Therefore this trajectory will reach $f_\alpha^{-1}(\varepsilon)$, and our deformation retraction is well-defined.

Moving forward to the assertion 2) let us first outline the proof. We are going to apply the Morse theory to the manifold N with boundary V. By the Orlik-Terao theorem we can choose a positive vector β close to α, so that f_β is a Morse function. This function is not constant on V, however it follows from Proposition 3.2 that for $\beta - \alpha$ small enough the gradient of f_β is still transversal to V and points inward N at any point of V. Thus we can apply the Morse theory to the Morse function $f_\beta \mid N$ and its gradient w_β and deduce that N is obtained from V by attaching several n-cells. The number of these n-cells equals the number of critical points of f_β, which is equal to $|\chi(M)|$ by Theorem 1.1 of [5].

Let us proceed to the details.

Proposition 4.1. *Let α be a positive vector. There is a neighbourhood U of H and $D > 0$ such that*

1) $H(\delta) \subset U$ for some $\delta > 0$,

2) $\big\langle v_\alpha(z), w_\beta(z)\big\rangle \geqslant D$ for every $z \in U$ and every positive vector β with $\max_i |\alpha_i - \beta_i|$ sufficiently small.

Proof. The neighbourhood U from the Proposition 3.2 will do. Indeed,

$$\Big|\big\langle v_\alpha(z) - v_\beta(z), w_\beta(z)\big\rangle\Big| \leqslant B\cdot \max_i |\alpha_i - \beta_i| \cdot ||v_\beta(z)||$$

On the other hand $\big\langle v_\beta(z), w_\beta(z)\big\rangle = ||v_\beta(z)||$ (since these vector fields are collinear), therefore

$$\big\langle v_\alpha(z), w_\beta(z)\big\rangle \geqslant (1 - B \max_i |\alpha_i - \beta_i|)\cdot ||v_\beta(z)||.$$

If $\alpha - \beta$ is small enough this is greater than a positive constant for $z \in U$ again by Proposition 3.2. □

The main result which guarantees the applicability of the Morse theory to our situation is the next theorem.

Theorem 4.2. *Let α be a positive vector. Let $\varepsilon > 0$ be sufficiently small so that $M \setminus N \subset U$. Let β be a positive vector sufficiently close to α so that the conclusion of the previous Proposition holds. Let $x \in N$. Denote by $\gamma(t)$ the trajectory of the vector field $-w_\beta$ starting at x.*

Then either $\gamma(t)$ converges to a critical point of f_β or it reaches the manifold $V = f_\alpha^{-1}(\varepsilon)$.

Proof. Choose $C > 0$ sufficiently large so that $f_\beta(x) < C$. Let

$$Y = f_\beta^{-1}([0, C]), \quad Y_0 = Y \cap f_\alpha^{-1}([\varepsilon, \infty[).$$

Let $p_1, \dots, p_N$ be the critical points of f_β and choose a neighbourhood R_i around each p_i. Put $R = \cup_i R_i$, and set $Y_1 = Y_0 \setminus R$. Then the function $f'_\alpha(z)(w_\beta(z)) = f_\alpha(z)\cdot \big\langle v_\alpha(z), w_\beta(z)\big\rangle$ is bounded away from 0 in Y_1. (Indeed, for $z \in U$ this is the subject of the previous Proposition. As for the set $Y \setminus \big(U \cup R\big)$, it is compact and the function in question is non-zero on it.) Then we can apply the same argument as in [6], page 95, Proposition 2.4, and the proof of the point 2) of our theorem is over. □

5. The Novikov homology of V and M

In this section we prove the assertion 3) of Theorem 2.1 and Theorem 2.2. We begin with a description of the homotopy type of $V = f_\alpha^{-1}(\varepsilon)$ for arbitrary positive vector α. Recall from the Section 2 the cohomology class $\bar\alpha \in H^1(M, \mathbb{R})$. By an abuse of notation we will denote the restriction of the class $\bar\alpha$ to N by the same letter $\bar\alpha$. Ther restriction of the class $\bar\alpha$ to V will be denoted by $\underline{\alpha}$. Recall the holomorphic 1-form $\omega_\alpha = \sum_j \alpha_j \frac{d\xi_j}{\xi_j}$ and denote its real and imaginary parts by $\mathfrak{R}$ and $\mathfrak{I}$ respectively. Then $\mathfrak{R} = \frac{df_\alpha}{f_\alpha}$ and the cohomology class of $\mathfrak{I}$ equals $2\pi\bar{\alpha}$. Let ι_α be the vector field dual to $\mathfrak{I}$. Since ω_α is a holomorphic form, we have

$$\frac{\mathrm{grad} f_\alpha(z)}{f_\alpha(z)} = i \cdot \iota_\alpha(z). \tag{8}$$

If $\varepsilon > 0$ is small enough so that V is contained in the neighbourhood U from Proposition 3.2 we have $||\iota_\alpha(z)|| \geqslant A$. Observe that $\iota_\alpha(z)$ is orthogonal to $\mathrm{grad} f_\alpha(z)$ and therefore tangent to $V = f^{-1}(\varepsilon)$. We deduce that the restriction to V of the 1-form $\mathfrak{I}$ does not vanish, and moreover, the norm of its dual vector field is bounded from below.

Proposition 5.1. *The Novikov homology $\widehat{H}_k(V, \underline{\alpha})$ vanishes for all k.*

Proof. Let $p : \widetilde{V} \to V$ be the universal covering of V. The closed 1-form $p^*(\mathfrak{I})$ is cohomologous to zero; let $p^*(\mathfrak{I}) = dF$, where $F : \widetilde{V} \to \mathbb{R}$ is a function without critical points. Denote by $\widetilde{V}_n$ the subset $F^{-1}\big(]-\infty, -n]\big)$. The chain complexes $C_*^{(n)} = \mathcal{S}_*(\widetilde{V}, \widetilde{V}_n)$ form an inverse system, and the Novikov homology $\widehat{H}_*(V, \underline{\alpha})$ is isomorphic to the homology of its inverse limit (see [7]). For every k we have an exact sequence

$$\lim{}^1 H_{k+1}\big(C_*^{(n)}\big) \to H_k\big(\varprojlim C_*^{(n)}\big) \to \varprojlim H_k\big(C_*^{(n)}\big).$$

The lift of the vector field ι_α to $\widetilde{V}$ will be denoted by the same letter ι_α. The standard argument using the shift diffeomorphism along the trajectories of $-\iota_\alpha$ shows that $H_k(C_*^{(n)}) = 0$ for every k; our Proposition follows. □

Consider now the case when α is of rank one, that is, all α_i are rational multiples of one real number. In this case the differential form $\mathfrak{I}$ is the differential of a map $g : V \to \mathbb{R}/a\mathbb{R} \approx S^1$ for some $a > 0$.

Proposition 5.2. *The map g is a fibration of V over S^1.*

Proof. The map g does not have critical points. Consider the vector field

$$y_\alpha(z) = \frac{\iota_\alpha(z)}{||\iota_\alpha(z)||^2}.$$

For $x \in V$ denote by $\gamma(x, t; y_\alpha)$ the y_α-trajectory starting at x. Since the norm of $\iota_\alpha(z)$ is bounded away from zero in V, the trajectory is defined on the whole of $\mathbb{R}$. We have also

$$\frac{d}{dt}\big(g(\gamma(x, t; y_\alpha)\big) = 1.$$

Pick any $\lambda \in S^1$ and let $V_0 = g^{-1}(\lambda)$. Denote by $\lambda' \in S^1$ the point opposite to λ. It is easy to check that for any $0 < \varkappa < \pi$ the map

$$(x,t) \mapsto \gamma(x,t;y_\alpha)$$

is a diffeomorphism

$$V_0 \times \left]\lambda - \pi, \lambda + \pi\right[\;\approx\; g^{-1}\big(S^1 \setminus \{\lambda'\}\big)$$

compatible with projections. Therefore g is a locally trivial fibration. □

It is clear that any fiber of g is locally the set of zeros of a holomorphic function, therefore it is a closed complex analytic submanifold of $\mathbb{C}^n$ and has a homotopy type of a finite CW-complex of dimension $\leqslant n-1$ (see [1]). The proof of Theorem 2.1 is now complete. □

Proof of Theorem 2.2. Let Λ be the group ring of the fundamental group of N. Let $\widehat{\Lambda}_{\bar\alpha}$ denote the Novikov completion of Λ with respect to $\bar\alpha$. Denote by $q : \widetilde{N} \to N$ the universal covering of N. Put $\bar V = q^{-1}(V)$.

We have the short exact sequence of free $\widehat{\Lambda}_{\bar\alpha}$-complexes

$$0 \to \widehat{\Lambda}_{\bar\alpha} \underset{\Lambda}{\otimes} \mathcal{S}_*(\bar V) \to \widehat{\Lambda}_{\bar\alpha} \underset{\Lambda}{\otimes} \mathcal{S}_*(\widetilde{N}) \to \widehat{\Lambda}_{\bar\alpha} \underset{\Lambda}{\otimes} \mathcal{S}_*(\widetilde{N}, \bar V) \to 0 \tag{9}$$

Observe that the inclusion $V \subset N$ induces a surjective homomorphism of fundamental groups. Therefore the space $\bar V$ is connected and the covering $\bar V \to V$ is a quotient of the universal covering $\widetilde{V} \to V$, so that

$$H_*\big(\widehat{\Lambda}_{\bar\alpha} \underset{\Lambda}{\otimes} \mathcal{S}_*(\bar V)\big) = H_*\Big(\widehat{\Lambda}_{\bar\alpha} \underset{\Lambda}{\otimes} \big(\widehat{\mathcal{L}}_{\underline{\alpha}} \underset{\mathcal{L}}{\otimes} \mathcal{S}_*(\widetilde{V})\big)\Big),$$

where $\mathcal{L}$ is the group ring of the fundamental group of V, and $\widehat{\mathcal{L}}_{\underline{\alpha}}$ is the Novikov completion of $\mathcal{L}$ with respect to $\underline{\alpha}$. By Proposition 5.1 the Novikov homology of V vanishes, and we deduce that the long exact sequence of homology modules, derived from the short exact sequence (9), splits into a sequence of isomorphisms

$$\widehat{H}_*(N, \bar\alpha) \approx H_*\Big(\widehat{\Lambda}_\xi \underset{\Lambda}{\otimes} \mathcal{S}_*(\widetilde{N}, \bar V)\Big).$$

Observe now that the homology of the couple $(\widetilde{N}, \bar V)$ is the free module over Λ of rank $|\chi(M)|$, concentrated in degree n. Theorem 2.2 follows. □

6. Non-essential arrangements

Let us consider the case of non-essential arrangements $\mathcal{A}$. As before we denote by m the number of hyperplanes in the family $\mathcal{A}$. Assume that rk $\mathcal{A} = l < n$. The function f_α is not a Morse function in this case. However the analog of Theorem 2.1 is easily obtained by reduction to the case of essential arrangements.

Denote by $\pi : \mathbb{C}^l \oplus \mathbb{C}^k \to \mathbb{C}^l$ the projection onto the first direct summand. Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^l$, defined by affine functions $\xi_i : \mathbb{C}^l \to \mathbb{C}$. The functions $\xi_i \circ \pi$ determine a hyperplane arrangement in $\mathbb{C}^{l+k}$ which will be called *k-suspension* of $\mathcal{A}$. It is not difficult to prove the next proposition.

Proposition 6.1. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$ of rank l. The $\mathcal{A}$ is linearly isomorphic to the $(n-l)$-suspension of an essential hyperplane arrangement $\mathcal{A}_0$ in $\mathbb{C}^l$.*

Put

$$H = \mathbb{C}^n \setminus \mathcal{M}(\mathcal{A}), \quad H_0 = \mathbb{C}^l \setminus \mathcal{M}(\mathcal{A}_0).$$

Then $\mathbb{C}^n \setminus H$ is diffeomorphic to $(\mathbb{C}^l \setminus H_0) \times \mathbb{C}^{n-l}$ and we obtain the following generalizations of the previous theorems.

Theorem 6.2. *There is a finite CW-complex Y of dimension $l-1$, fibered over a circle, such that the space $M = \mathbb{C}^n \setminus H$ is homotopy equivalent to a result of attaching to Y of $|\chi(M)|$ cells of dimension n.*

Corollary 6.3. *For every positive $\alpha \in \mathbb{R}^m$ the Novikov homology $\widehat{H}_k(M, \overline{\alpha})$ vanishes for every $k \neq l$ and is a free module of rank $|\chi(M)|$ for $k = l$.*

7. Acknowledgments

The work was supported by World Premier International Research Center Initiative (WPI Program), MEXT, Japan.

The work was finished during the second author's stay at the Graduate School of Mathematical Sciences, the University of Tokyo in the autumn of 2010. The second author is grateful to the Graduate School of Mathematical Sciences for warm hospitality.

References

[1] A. Andreotti and T. Frankel, *The Lefschetz theorem on hyperplane sections*, Ann. of Math. (3) **69** (1959), 713-717.

[2] K. Aomoto, *On vanishing of cohomology attached to certain many valued meromorphic functions*, J. Math. Soc. Japan (2) **27** (1975), 248-255.

[3] T. Kohno, *Homology of a local system on the complement of hyperplanes*, Proc. Japan Acad. **62** Ser. A (1986), 144–147.

[4] P. Orlik and H. Terao, Arrangements of Hyperplanes, Grundlehren der mathematischen Wissenschaften **300**, Springer-Verlag 1992.

[5] P. Orlik and H. Terao, *The number of critical points of a product of powers of linear functions*, Invent. Math. **120** (1995), 1 – 14.

[6] A. V. Pajitnov, Circle-valued Morse Theory, de Gruyter Studies in Mathematics **32**, Walter de Gruyter 2006.

[7] J.-Cl. Sikorav, *Points fixes de difféomorphismes symplectiques, intersections de sous-variétés lagrangiennes, et singularités de un-formes fermées*, Thèse de Doctorat d'Etat Es Sciences Mathématiques, Université Paris-Sud, Centre d'Orsay, 1987.

[8] A. Varchenko, *Critical points of the product of powers of linear functions and families of bases of singular vectors*, Compos. Math. **97** (1995), 385-401.

IPMU, GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO
E-mail address: kohno@ms.u-tokyo.ac.jp

LABORATOIRE MATHÉMATIQUES JEAN LERAY UMR 6629, UNIVERSITÉ DE NANTES, FACULTÉ DES SCIENCES, 2, RUE DE LA HOUSSINIÈRE, 44072, NANTES, CEDEX
E-mail address: pajitnov@math.univ-nantes.fr