HIGHER HOLONOMY MAPS FOR HYPERPLANE ARRANGEMENTS

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Abstract. We develop a method to construct representations of the homotopy 2-groupoid of a manifold as a 2-category by means of K.-T. Chen's formal homology connections. As an application we describe 2-holonomy maps for hyperplane arrangements and discuss representations of the category of braid cobordisms.

1. Introduction

The purpose of this article is to give a systematic treatment of representations of the homotopy 2-groupoid of a manifold as a 2-category by means of K.-T. Chen's formal homology connections. The 2-categories play an important role in higher gauge theory (see Baez and Huerta [1]). In particular, the 2-holonomy maps have been investigated in the framework of 2-connections. The notion of formal homology connections was developed by K.-T. Chen in the theory of iterated integrals of differential forms in order to describe the homology group of the loop space of a manifold M by the chain complex formed by the tensor algebra of the homology group of M (see [3], [4] and [5]).

We apply such method to the complement of complex hyperplane arrangements. In this case because of the formality of the space the formal homology connection can be described by quadratic derivations. In particular, we discuss in details the 2-flatness condition in the case of the configuration space of ordered distinct points in the complex plane. We describe categorified infinitesimal pure braid relations in this setting. It is an important problem to construct a categorification of the Knizhnik-Zamolodchikov (KZ) connections. There is a work by Cirio and Martins [8] on the categorification of the KZ connections by means of 2-Yang-Baxter operators for $sl_2(\mathbf{C})$. In this paper we give a universal expression of 2-holonomy maps based on the formal homology connections. One of our motivations is to apply such methods to braided surfaces in 4-space studied by Carter, Kamada and Saito (see [2], [10]). We discuss an application of 2-holonomy maps to a construction of representations of the 2-category of braid cobordisms.

The paper is organized in the following way. In Section 2 we briefly review K.-T. Chen's iterated integrals and their basic properties. In particular, we recall the formula for the composition of plots. In Section 3 we describe the notion of formal homology connections. In particular, we explain the notion of 2-connections and 2-curvatures in this framework. In Section 4 we give a general method to construct representations of homotopy 2-groupoids by means of the formal homology connection. We also describe the notion of crossed modules in this setting. In Section 5 we apply the above method to the complement of complex hyperplane arrangement. In particular, we describe 2-flatness condition for braid arrangements. We discuss an application to a representation of the category of braid cobordisms.

2. Preliminaries on K.-T. Chen's iterated integrals

First, we briefly recall the notion of iterated integrals of differential forms due to K.-T. Chen. We refer the reader to [3], [4] and [5] for details. Let M be a smooth manifold and $\omega_1, \cdots, \omega_k$ be differential forms on M. We fix two points $\mathbf{x}_0$ and $\mathbf{x}_1$ in M and consider the space of piecewise smooth paths $\gamma : [0,1] \to M$ with $\gamma(0) = \mathbf{x}_0$ and $\gamma(1) = \mathbf{x}_1$. We denote by $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ the above space of paths. In particular, in the case $\mathbf{x}_0 = \mathbf{x}_1$ the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ is called the based loop space of M and is denoted by $\Omega_{\mathbf{x}_0} M$. In the following we suppose that the differential forms $\omega_1, \cdots, \omega_k$ are of positive degrees. We denote by

$$p_j : \underbrace{M \times \cdots \times M}_{k} \longrightarrow M, \ 1 \leq j \leq k$$

the projection to the j-th factor and set

$$\omega_1 \times \cdots \times \omega_k = p_1^* \omega_1 \wedge \cdots \wedge p_k^* \omega_k.$$

We consider the simplex

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbf{R}^k \ ; \ 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

and the evaluation map

$$\varphi : \Delta_k \times \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1) \to \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$. The iterated integral of $\omega_1, \cdots, \omega_k$ is defined as

$$\int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

where the expression

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

is the integration along the fiber with respect to the projection

$$p : \Delta_k \times \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1) \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1).$$

The above iterated integral is considered as a differential form on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ with degree $q_1 + \cdots + q_k - k$, where we set $q_j = \deg \omega_j$. To justify differential forms on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ we use the notion of plots. A plot $\alpha : U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ is a family of piecewise linear paths smoothly parametrized by a compact convex set U in a finite dimensional Euclidean space. Given a plot α we denote the corresponding iterated integral

$$\left(\int \omega_1 \cdots \omega_k \right)_\alpha$$

as a differential form on U obtained by pulling back the iterated integral $\int \omega_1 \cdots \omega_k$ by the plot α. Namely, the above expression stands for

$$\int_{\Delta_k} ((\mathrm{id} \times \alpha) \circ \varphi)^* (\omega_1 \times \cdots \times \omega_k)$$

where we consider the integration along the fiber with respect to the projection $\Delta_k \times U \to U$.

We denote by $\Omega^*(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1))$ the set of such differential forms on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ obtained as iterated integrals of differential forms of positive degrees on M. In particular, in the case $\omega_1, \cdots, \omega_k$ are 1-forms, the iterated integral $\int \omega_1 \cdots \omega_k$ is a function on the path space and its value on a path $\gamma : [0,1] \to M$ is the iterated line integral

$$\int_\gamma \omega_1 \cdots \omega_k = \int_{\Delta_k} f_1(t_1) \cdots f_k(t_k)\ dt_1 \cdots dt_k$$

where $\gamma^* \omega_j = f_j(t)\ dt$, $1 \leq j \leq k$.

We take an extra point $\mathbf{x}_2$ in M and consider the plots

$$\alpha : U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad \beta : U \longrightarrow \mathcal{P}(M; \mathbf{x}_1, \mathbf{x}_2).$$

The composition of the plots α and β

$$\alpha\beta :\ U \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_2)$$

is defined by

$$\alpha\beta(x)(t) = \begin{cases} \alpha(x)(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(x)(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $x \in U$. As is shown by K.-T. Chen, we have the following rule for the composition of plots.

Proposition 2.1. *The relation*

$$\left(\int \omega_1 \cdots \omega_k \right)_{\alpha\beta} = \sum_{0 \leq i \leq k} \left(\int \omega_1 \cdots \omega_i \right)_\alpha \wedge \left(\int \omega_{i+1} \cdots \omega_k \right)_\beta$$

holds

For a path α we define its inverse path α^{-1} by

$$\alpha^{-1}(t) = \alpha(1-t).$$

For the composition $\alpha\alpha^{-1}$ we have

$$\left(\int \omega_1 \cdots \omega_i \right)_{\alpha\alpha^{-1}} = 0.$$

As a differential form on the path space $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ we have the following.

Proposition 2.2. *For the iterated integral* $\int \omega_1 \cdots \omega_k$ *we have*

$$\begin{aligned} & d \int \omega_1 \cdots \omega_k \\ = & \sum_{j=1}^k (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j\ \omega_{j+1} \cdots \omega_k \\ & + \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1} (\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k \end{aligned}$$

where we put $\nu_j = \deg \omega_1 + \cdots + \deg \omega_j - j$ *for* $j \geq 1$ *and* $\nu_0 = 0$.

Thus we obtain the complex $\Omega^*(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1))$ with the differential

$$d : \Omega^q(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)) \longrightarrow \Omega^{q+1}(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1))$$

explicitly geven as in Proposition 2.2.

3. Formal homology connections

Let M be a smooth manifold. We put

$$H_+(M) = \bigoplus_{q>0} H_q(M;\mathbf{R})$$

and consider the tensor algebra

$$TH_+(M) = \bigoplus_{k\geq 0} \left(\bigotimes^k H_+(M) \right).$$

In the following we suppose that $\dim H_+(M)$ is finite. We denote by $\Omega^*(M)$ the algebra of differential forms on M and consider the tensor product $\Omega^*(M)\otimes TH_+(M)$. We suppose that the differential d acts trivially on $TH_+(M;\mathbf{R})$. Namely, we set

$$d(\omega\otimes X) = d\omega\otimes X,\ \omega\in\Omega^*(M),\ X\in TH_+(M;\mathbf{R}).$$

When $H_+(M)$ has a basis $X_1,\cdots,X_m$, $\Omega^*(M)\otimes TH_+(M)$ is identified with the ring of non-commutative polynomials

$$\Omega^*(M)[X_1,\cdots,X_m]$$

over $\Omega^*(M)$. We assign the degree of $X_i\in H_{p_i}(M)$ as

$$\deg X_i = p_i - 1$$

by shifting the degree by 1. For the product of homogeneous elements we define the degree of $X_{i_1}\cdots X_{i_k}$ as

$$\deg X_{i_1}\cdots X_{i_k} = \sum_{p=1}^k \deg X_{i_p}.$$

In this way we regard $TH_+(M)$ as a graded algebra. For homogeneous elements X,Y in $\Omega^*(M)[X_1,\cdots,X_m]$ we define the graded Lie bracket by

$$[X,Y] = XY - (-1)^{pq}YX$$

where $\deg X = p$ and $\deg Y = q$.

We consider the augmentation map

$$\epsilon : \mathbf{R}[X_1,\cdots,X_m] \longrightarrow \mathbf{R}$$

defined by $\varepsilon(X_k)=0$, $1\leq k\leq m$. We denote by J the kernel of the augmentation map ϵ, which is the 2-sided ideal of $TH_+(M)$ generated by $X_1,\cdots,X_m$. We consider the completion of $TH_+(M)$ with respect to the powers of the augmentation ideal, which is denoted by $T\widehat{H_+(M)}$. The tensor product $\Omega^*(M)\otimes T\widehat{H_+(M)}$ is identified with the ring of non-commutative formal power series

$$\Omega^*(M)\langle\langle X_1,\cdots,X_m\rangle\rangle$$

over $\Omega^*(M)$. We denote by $T\widehat{H_+(M)}_q$ the degree q part of $T\widehat{H_+(M)}$ with respect to the above degrees.

For a differential form ω we define the parity operator ε as $\varepsilon(\omega)=\omega$ when ω is of even degree and $\varepsilon(\omega)=-\omega$ when ω is of odd degree. For $\omega\otimes X\in\Omega^*(M)\otimes T\widehat{H_+(M)}$ we set $\varepsilon(\omega\otimes X)=\omega\otimes X$ if ω is a differential form of even degree and $\varepsilon(\omega\otimes X) = -\omega\otimes X$ if ω is a differential form of odd degree. Extending the above map linearly we obtain the operator ε on $\Omega^*(M)\otimes T\widehat{H_+(M)}$.

We extend naturally the wedge product and iterated integrals on $\Omega^*(M)\otimes T\widehat{H_+(M)}$. This means that we define as

$$(\omega \otimes X) \wedge (\varphi \otimes Y) = (\omega \wedge \varphi) \otimes XY,$$
$$\int (\varphi_1 \otimes Z_1) \cdots (\varphi_k \otimes Z_k) = \left(\int \varphi_1 \cdots \varphi_k \right) \otimes Z_1 \cdots Z_k.$$

Here the right hand side of the second equation is considered as an element of

$$\Omega^*(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)) \otimes T\widehat{H_+(M)}.$$

We say that a linear map

$$\delta : T\widehat{H_+(M)}_q \longrightarrow T\widehat{H_+(M)}_{q-1}$$

is a derivation of degree -1 if it satisfies the Leibniz rule

$$\delta(uv) = (\delta u)v + (-1)^{\deg u} u(\delta v).$$

According to K.-T. Chen a formal homology connection

$$\omega \in \Omega^*(M) \otimes T\widehat{H_+(M)}$$

is by definition an expression written as

$$\omega = \sum_{i=1}^{m} \omega_i X_i + \cdots + \sum_{i_1 \cdots i_k} \omega_{i_1 \cdots i_k} X_{i_1} \cdots X_{i_k} + \cdots$$

with differential forms of positive degrees $\omega_{i_1 \cdots i_k}$ satisfying the following properties.

- $[\omega_i]$, $1 \leq i \leq m$, is a dual basis of X_i, $1 \leq i \leq m$.
- $\delta\omega + d\omega - \varepsilon(\omega) \wedge \omega = 0$.
- $\deg \omega_{i_1 \cdots i_k} = \deg X_{i_1} \cdots X_{i_k} + 1$
- δ is a derivation of degree -1.
- $\delta X_j \in \widehat{J}^2$ where $\widehat{J}$ is the augmentation ideal of $T\widehat{H_+(M)}$.

For a formal homology connection ω we define the generalized curvature κ by

$$\kappa = d\omega - \varepsilon(\omega) \wedge \omega.$$

From the above conditions it can be shown that $\delta \circ \delta = 0$ and $(T\widehat{H_+(M)}, \delta)$ forms a complex. The formal homology connection can be written in the sum

$$\omega = \omega^{(1)} + \omega^{(2)} + \cdots + \omega^{(p)} + \cdots$$

with the p-form part

$$\omega^{(p)} \in \Omega^p(M) \otimes T\widehat{H_+(M)}_{p-1}.$$

The 2-form part of κ is written as

$$\kappa^{(2)} = d\omega^{(1)} + \omega^{(1)} \wedge \omega^{(1)}$$

which coincides with the usual curvature form for $\omega^{(1)}$. From the equation $\delta\omega + \kappa = 0$ we have the equation

$$\delta\omega^{(2)} + d\omega^{(1)} + \omega^{(1)} \wedge \omega^{(1)} = 0.$$

Let us consider the 3-form part of κ given by

$$\kappa^{(3)} = d\omega^{(2)} - \omega^{(1)} \wedge \omega^{(2)} + \omega^{(2)} \wedge \omega^{(1)}.$$

We call $\kappa^{(3)}$ the 2-curvature of the pair $\omega^{(1)}$ and $\omega^{(2)}$. We have the equation

$$\delta\omega^{(3)} + d\omega^{(2)} - \omega^{(1)} \wedge \omega^{(2)} + \omega^{(2)} \wedge \omega^{(1)} = 0.$$

Although the formal homology connection is not uniquely determined, we can construct it inductively starting from the initial term $\sum_{i=1}^{m} \omega_i X_i$. Here are some examples. For details about the examples (1) and (3) we refer the reader to [15].

Examples : (1) Let $T = S^1 \times S^1$ be the 2-dimensional torus. The de Rham cohomology $H^*(T)$ has a basis represented by $\omega_1, \omega_2, \omega_1 \wedge \omega_2$ and we put X_1, X_2, Y its dual basis of the homology. The formal homology connection is given as

$$\omega = \omega_1 \otimes X_1 + \omega_2 \otimes X_2 + (\omega_1 \wedge \omega_2) \otimes Y$$

with the derivation defined by

$$\delta(X_1) = 0,\ \delta(X_2) = 0,\ \delta(Y) = -[X_1, X_2].$$

(2) Let $\mathbf{C}P^n$ denote the complex n-dimensional projective space. and τ the Kähler form. For $k = 0, 1, \cdots, n$ the cohomology group $H_{2k}(\mathbf{C}P^n; \mathbf{R})$ is isomorphic to $\mathbf{R}$ and has a basis $[\tau^k]$. Let X_k denote the dual basis of $[\tau^k]$ in the homology group $H_{2k}(\mathbf{C}P^n; \mathbf{R})$. We put

$$\omega = \tau \otimes X_1 + \tau^2 \otimes X_2 + \cdots + \tau^n \otimes X_n$$

Then we have

$$\kappa = -\omega \wedge \omega = -\sum_{i,j \geq 1} \tau^{i+j} \otimes X_i X_j.$$

By defining

$$\begin{aligned} \delta X_1 &= 0 \\ \delta X_k &= \sum_{1 \leq i \leq k-1} X_i X_{k-i}, \quad 2 \leq k \leq n, \end{aligned}$$

we get the condition $\delta\omega + \kappa = 0$. The above (ω, δ) is a formal homology connection for $\mathbf{C}P^n$.

(3) Let G be the unipotent Lie group consisting of the matrices

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbf{R}$$

and $G_{\mathbf{Z}}$ its subgroup consisting of the above matrices with $x, y, z \in \mathbf{Z}$. We denote by M the quotient space of G by the left action of $G_{\mathbf{Z}}$. We see that M has a structure of a compact smooth 3-dimensional manifold. We put

$$\omega_1 = dx,\ \omega_2 = dy,\ \omega_{12} = -xdy + dz.$$

We observe that $H^1(M)$ has a basis represented by ω_1, ω_2 and $H^2(M)$ has a basis represented by $\omega_1 \wedge \omega_{12}, \omega_2 \wedge \omega_{12}$. These are typical examples of non-trivial Massey product. We denote by $X_1, X_2 \in H_1(M)$ the dual basis of $[\omega_1], [\omega_2]$ and by $Y_1, Y_2 \in H_2(M)$ the dual basis of $[\omega_1 \wedge \omega_{12}], [\omega_2 \wedge \omega_{12}]$. We obtain that the derivation δ is given by

$$\delta(X_1) = 0,\ \delta(X_2) = 0,\ \delta(Y_1) = [[X_1, X_2], X_1],\ \delta(Y_2) = [[X_1, X_2], X_2].$$

In the above examples (1) and (2) the derivations δ are quadratic, which reflects the fact that the corresponding spaces are formal. On the other hand in the example

(3) there are non-trivial Massey products and the derivations are not quadratic. We recall celebrated theorem of Deligne, Griffiths, Morgan and Sullivan [9] that a compact Kähler manifold is formal. Consequently the derivation, for the formal homology connection is quadratic in this case.

For the formal homology connection ω we define its transport by

$$T = 1 + \sum_{k=1}^{\infty} \int \underbrace{\omega \cdots \omega}_{k}$$

which is considered to be an element of

$$\Omega^*(\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)) \otimes \widehat{TH_+(M)}.$$

The following proposition plays a key role for the construction of higher holonomy maps. For the proof we refer the reader to [15].

Proposition 3.1. *Given a formal homology connection (ω, δ) for a manifold M the transport T satisfies $dT = \delta T$.*

4. Path groupoids, 2-path groupoids and their representations

In this section we recall the notion of path groupoids and 2-path groupoids. We refer the reader to [1] for more details including the notion of 2-categories. Let us recall that a groupoid is a category such that all the morphisms are invertible. In particular, a groupoid with one object is nothing but a group. For a smooth manifold M we define the path groupoid $\mathcal{P}_1(M)$. For this purpose we introduce the notion of a thin homotopy. We take $\mathbf{x}_0, \mathbf{x}_1 \in M$ and let γ_0 and γ_1 be piecewise smooth paths $\gamma_i : [0,1] \to M$, $i = 0, 1$, such that $\gamma_i(0) = \mathbf{x}_0$ and $\gamma_i(1) = \mathbf{x}_1$. We shall say that the paths γ_0 and γ_1 are thin homotopic if there exists a piecewise smooth map $H : [0,1]^2 \to M$ with

$$\begin{aligned} &H(t, 0) = \gamma_0(t),\ H(t, 1) = \gamma_1(t),\ 0 \le t \le 1 \\ &H(0, s) = \mathbf{x}_0,\ H(1, s) = \mathbf{x}_1,\ 0 \le s \le 1. \end{aligned}$$

satisfying rank $dH_p \le 1$ for any $p \in [0,1]^2$ such that dH_p is defined.

The path groupoid $\mathcal{P}_1(M)$ is a category whose objects are points in M and whose morphisms are piecewise smooth paths between points up to a thin homotopy. Namely, for $\mathbf{x}_0, \mathbf{x}_1 \in M$ the set of morphisms between them is

$$\mathrm{Hom}(\mathbf{x}_0, \mathbf{x}_1) = \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)/\sim$$

where the paths $\gamma_0, \gamma_1 \in \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ satisfy the equivalence relation $\gamma_0 \sim \gamma_1$ if and only if the one is obtained from the other by a thin homotopy. We see that $\mathcal{P}_1(M)$ has a structure of a groupoid since there is an associativity and each morphism has its inverse.

Now we explain the notion of 2-categories. In general, a 2-category consists of objects, 1-morphisms and 2-morphisms, which are morphisms between morphisms. There are two kinds of compositions for 2-morphisms, horizontal compositions and vertical compositions and there are several coherency conditions among them. We do not give here a full definition of a 2-category.

The path 2-groupoid $\mathcal{P}_2(M)$ is a 2-category defined as follows. The objects are points in M and the 1-morphisms are piecewise smooth paths between points up to a thin homotopy. To define the 2-morphisms we consider a disc given by a piecewise smooth function $F : [0,1]^2 \to M$ spanning 2 paths γ_0 and γ_1 connecting $\mathbf{x}_0$ and $\mathbf{x}_1$. Namely, we consider a piecewise smooth function $F : [0,1]^2 \to M$ with

$$F(t,0) = \gamma_0(t),\ F(t,1) = \gamma_1(t),\ 0 \le t \le 1,$$
$$F(0,s) = \mathbf{x}_0,\ F(1,s) = \mathbf{x}_1,\ 0 \le s \le 1.$$

Let F_0 and F_1 be piecewise smooth discs $F_i : [0,1]^2 \to M,\ i = 0,1$, spanning the paths γ_0 and γ_1. We shall say that F_0 and F_1 are thin homotopic if there exists a family of discs $F_r : [0,1]^2 \to M,\ 0 \le r \le 1$, spanning the 2 paths γ_0 and γ_1 such that the following conditions (1) and (2) are satisfied. We put $G(t,s,r) = F_r(t,s)$.

(1) The function $G(t,s,r)$ is piecewise smooth.
(2) $\operatorname{rank} dG_p \le 2$ for any $p \in [0,1]^3$ such that dG_p is defined.

A 2-morphism between the paths γ_0 and γ_1 is a piecewise smooth disc $F : [0,1]^2 \to M$ spanning the paths γ_0 and γ_1 considered up to thin homotopy. Putting $c(s)(t) = F(t,s)$, we obtain a family of paths

$$c : [0,1] \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1).$$

We represent a 2-morphism beween the paths γ_0 and γ_1 as such family of paths.

Let γ_0, γ_1 and γ_2 be piecewise smooth paths connecting $\mathbf{x}_0$ and $\mathbf{x}_1$. For a 2-morphism c_1 between γ_0 and γ_1 and a 2-morphism c_2 between γ_1 and γ_2 we define their vertical composition $c_2 \cdot c_1$ by the family of paths given by

$$(c_2 \cdot c_1)(s)(t) = \begin{cases} c_1(2s)(t),\ 0 \le s \le \frac{1}{2} \\ c_2(2s-1)(t),\ \frac{1}{2} \le s \le 1 \end{cases}$$

as depicted in Figure 1.

For the 2-morphisms c_1 and c_2 respectively represented by the plots

$$\alpha_1 : I \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad \alpha_2 : I \longrightarrow \mathcal{P}(M; \mathbf{x}_1, \mathbf{x}_2).$$

We define their horizontal composition $c_2 \circ c_1$ by the composition of the plots $\alpha_2\alpha_1$.

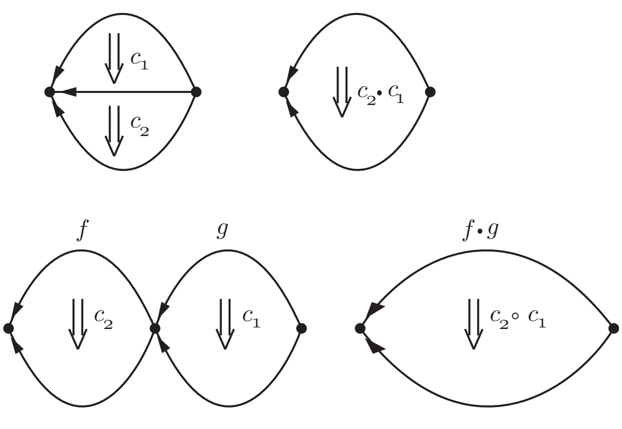

Figure 1. vertical and horizontal compositions

We consider the homotopy equivalence classes of paths in the groupoid $\mathcal{P}_1(M)$ fixing the endpoints. We denote the set of such equivalence classes by $\Pi_1(M)$. We see that $\Pi_1(M)$ has a structure of a groupoid and call it the homotopy path groupoid of M. In a similar way, we define the homotopy 2-groupoid $\Pi_2(M)$ by taking the homotopy equivalence classes of discs fixing the boundary in the path 2-groupoid $\mathcal{P}_2(M)$.

Now we construct a representation of the path groupoid $\mathcal{P}_1(M)$ by means of the iterated integrals of a formal homology connection. Let ω be a formal homology connection for M with the derivation δ. First, we consider the 1-form part of ω which is denoted by $\omega^{(1)}$. For a piecewise smooth path γ in M the holonomy of the connection $\omega^{(1)}$ is given the transport as

$$Hol(\gamma) = \langle T, \gamma \rangle = 1 + \sum_{k=1}^{\infty} \int_\gamma \underbrace{\omega^{(1)} \cdots \omega^{(1)}}_{k}$$

which is an element of $T\widehat{H_+(M)}_0$. Let us notice that the iterated integrals are independent of a thin homotopy of a path and that the above holonomy is well-defined. For the composition of paths we have

$$Hol(\alpha\beta) = Hol(\alpha)Hol(\beta)$$

by Proposition 2.1. Moreover, the relation

$$Hol(\alpha^{-1}) = Hol(\alpha)^{-1}$$

holds. Therefore, we obtain a representation of the path groupoid

$$Hol : \mathcal{P}_1(M) \longrightarrow T\widehat{H_+(M)}_0.$$

We denote by $T\widehat{H_+(M)}_0^\times$ the group of invertible elements in $T\widehat{H_+(M)}_0$. The above Hol is considered to be a functor from the path groupoid $\mathcal{P}_1(M)$ to the group $T\widehat{H_+(M)}_0^\times$. Let us construct a representation of the homotopy path groupoid $\Pi_1(M)$. We consider $T\widehat{H_+(M)}_1$ as a 2-sided module over $T\widehat{H_+(M)}_0$. Let $\mathcal{I}_0$ denote the 2-sided ideal of $T\widehat{H_+(M)}_1$ generated by the image of the derivation

$$\delta : T\widehat{H_+(M)}_1 \longrightarrow T\widehat{H_+(M)}_0.$$

We define the category $\mathcal{H}_M(1)$ as follows. The set of objects of $\mathcal{H}_M(1)$ consists of one point and the set of morphisms consists of invertible elements in $T\widehat{H_+(M)}_0/\mathcal{I}_0$.

Proposition 4.1. *The above holonomy map induces a well-defined functor.*

$$Hol : \Pi_1(M) \longrightarrow \mathcal{H}_M(1)$$

Proof. We consider Hol as a function in $\gamma \in \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$. Then we have

$$d\, Hol(\gamma) = \langle dT, \gamma \rangle = \langle \delta T, \gamma \rangle$$

by Proposition 3.1. Hence we have $d\, Hol(\gamma) = 0$ in $T\widehat{H_+(M)}_0/\mathcal{I}_0$. This shows that if $\gamma_0, \gamma_1 \in \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ are connected by a homotopy fixing the endpoints $\mathbf{x}_0, \mathbf{x}_1$, then we have $Hol(\gamma_0) = Hol(\gamma_1)$ in $T\widehat{H_+(M)}_0/\mathcal{I}_0$. This completes the proof. $\square$

By fixing a base point $\mathbf{x}_0 \in M$ we have the holonomy map

$$Hol : \pi_1(M, \mathbf{x}_0) \longrightarrow T\widehat{H_+(M)}_0/\mathcal{I}_0.$$

One of the main results due to K.-T. Chen is that the holonomy map induces an isomorphism

$$\mathbf{R}\widehat{\pi_1(M, \mathbf{x}_0)} \cong T\widehat{H_+(M)}_0/\mathcal{I}_0$$

where $\mathbf{R}\widehat{\pi_1(M, \mathbf{x}_0)}$ is the completion of the group ring $\mathbf{R}\pi_1(M, \mathbf{x}_0)$ with respect to the powers of the augmentation ideal. The algebra $\mathbf{R}\widehat{\pi_1(M, \mathbf{x}_0)}$ is called the Malcev completion of the fundamental group $\pi_1(M, \mathbf{x}_0)$.

We consider a piecewise smooth 1-parameter family of paths

$$c : [0, 1] \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1),$$

as in the definition of the path 2-groupoid $\mathcal{P}_2(M)$. We regard c as a 1-chain of $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$. For the formal homology connection we consider the transport

$$T = 1 + \sum_{k=1}^{\infty} \int \underbrace{\omega \cdots \omega}_{k}.$$

We regard its pullback $c^*T = T_c$ an element of $\Omega^*(I) \otimes T\widehat{H_+(M)}$. We denote by $\langle T, c\rangle$ the integration of the 1-form part of c^*T over the unit interval I. We define the 2-holonomy

$$Hol_2 : \mathcal{P}_2(M) \longrightarrow T\widehat{H_+(M)}_1$$

by $Hol_2(c) = \langle T, c\rangle$. The symbol $\langle T, c\rangle$ stands for the integration of the 1-form part of T on the 1-chain c.

For the vertical composition of the 1-morphisms

$$\alpha : I \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad \beta : I \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$$

we have

$$Hol_2(\alpha \cdot \beta) = Hol_2(\alpha) + Hol_2(\beta)$$

since the left hand side is considered to be the integration of over the some of the 1-chains represented by α and β. The horizontal composition of the 1-morphisms

$$c_1 : I \longrightarrow \mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1), \quad c_2 : I \longrightarrow \mathcal{P}(M; \mathbf{x}_1, \mathbf{x}_2)$$

the 2-holonomy is expressed as

$$Hol_2(c_2 \circ c_1) = \int T_{c_2} \wedge T_{c_1}$$

by means of Proposition 2.1. We obtain that the 2-holonomy map Hol_2 gives a representation of the path 2-groupoid $\mathcal{P}_2(M)$.

Theorem 4.1. *The above 2-holonomy map gives a representation of the homotopy 2-groupoid*

$$Hol_2 : \Pi_2(M) \longrightarrow T\widehat{H_+(M)}_1/\mathcal{I}_1$$

where $\mathcal{I}_1$ is the ideal generated by the image of the derivation

$$\delta : T\widehat{H_+(M)}_2 \longrightarrow T\widehat{H_+(M)}_1$$

Moreover, The 2-holonomy map satisfies

$$\delta Hol_2(c) = Hol(\gamma_1) - Hol(\gamma_0)$$

where c is a 2-morphism between γ_0 and γ_1.

Proof. As is shown in the above argument we have a representation of the path 2-groupoid given by

$$Hol_2 : \mathcal{P}_2(M) \longrightarrow \widehat{TH_+(M)}_1.$$

Suppose that for paths γ_0 and γ_1 in $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ piecewise smooth discs $F_j : [0,1]^2 \to M$, $j = 1, 2$ with

$$\begin{aligned} F_j(t,0) = \gamma_0(t), \ F_j(t,1) = \gamma_1(t) \\ F_j(0,s) = \mathbf{x}_0, \ F_j(1,s) = \mathbf{x}_1, \end{aligned}$$

are connected by a piecewise smooth homotopy preserving the above boundary conditions. This gives homologous 1-chains c_1 and c_2 in $\mathcal{P}(M; \mathbf{x}_0, \mathbf{x}_1)$ and there is a 2-chain y such that $c_1 - c_2 = \partial y$. We have

$$Hol_2(c_1) - Hol_2(c_2) = Hol_2(\partial y)$$

which is by definition $\langle T, \partial y \rangle$. By the Stokes theorem we have

$$\langle T, \partial y \rangle = \langle dT, y \rangle.$$

On the other hand we have $dT = \delta T$ by Proposition 3.1. This shows that $Hol_2(c_1) = Hol_2(c_2)$ in $\widehat{TH_+(M)}_1/\mathcal{I}_1$ and the 2-holonomy map from the homotopy 2-groupoid $\Pi_2(M)$ is well-defined. The equality $\delta Hol_2(c) = Hol(\gamma_1) - Hol(\gamma_0)$ follows from Proposition 3.1 and the Stokes theorem. We refer the reader to [15] for details of this part. □

We define the category $\mathcal{H}_M(2)$ as follows. The objects of $\mathcal{H}_M(2)$ consist of invertible elements in $\widehat{TH_+(M)}_0$. For invertible elements g_1 and g_2 in $\widehat{TH_+(M)}_0$ the set of morphisms from g_1 to g_2 is defined by

$$\mathrm{Hom}(g_1, g_2) = \{ v \in \widehat{TH_+(M)}_1/\mathcal{I}_1 \mid \delta(v) = g_2 - g_1 \}.$$

The above theorem shows that Hol_2 can be considered to be a functor sending a 2-morpshim in the homotopy 2-groupoid $\Pi_2(M)$ to a morphism in the category $\mathcal{H}_M(2)$. We regard $\widehat{TH_+(M)}_1/\mathcal{I}_1$ as a 2-sided module over $\widehat{TH_+(M)}_0$. The derivation δ has the compatibility

$$\delta(X \cdot v) = X \cdot \delta(v), \quad \delta(v \cdot X) = \delta(v) \cdot X$$

for $X \in \widehat{TH_+(M)}_0$ and $v \in \widehat{TH_+(M)}_1/\mathcal{I}_1$. In this sense the pair $\widehat{TH_+(M)}_1/\mathcal{I}_1$ and $\widehat{TH_+(M)}_0$ together with the derivation

$$\delta : \widehat{TH_+(M)}_1/\mathcal{I}_1 \longrightarrow \widehat{TH_+(M)}_0$$

has a structure of a crossed module.

5. 2-holonomy for the complement of hyperplane arrangements

We start by recalling basic facts on hyperplane arrangements. Let

$$\mathcal{A} = \{H_1, \cdots, H_\ell\}$$

be a collection of finite number of complex hyperplanes in $\mathbf{C}^n$. We call $\mathcal{A}$ a hyperplane arrangement. Let f_j, $1 \le j \le \ell$, be linear forms dining the hyperplanes H_j. We consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$$

and denote by $\Omega^*(M(\mathcal{A}))$ the algebra of differential forms on $M(\mathcal{A})$ with values in $\mathbf{C}$. The Orlik-Solomon algebra $OS(\mathcal{A})$ is the subalgebra of $\Omega^*(M(\mathcal{A}))$ generated by the logarithmic forms $\omega_j = d\log f_j$, $1 \leq j \leq \ell$. We refer the reader to [18] for basic properties of the Orlik-Solomon algebra. The fundamental fact is that the inclusion map

$$i : OS(\mathcal{A}) \longrightarrow \Omega^*(M(\mathcal{A}))$$

induces an isomorpshim of cohomology, where the differential on $OS(\mathcal{A})$ is trivial. In particular, we have an isomorpshims of algebras

$$OS(\mathcal{A}) \cong H^*(M(\mathcal{A}); \mathbf{C}).$$

A formal homology connection for $M(\mathcal{A})$ is described as follows. Let $\{Z_j\}$ be a basis of $H_+(M(\mathcal{A}); \mathbf{C})$ and $\{\varphi_j\}$ be its basis in the Orlik-Solomon algebra $OS(\mathcal{A})$. We define the derivation $\delta : T\widehat{H_+(M(\mathcal{A}))}_p \longrightarrow T\widehat{H_+(M(\mathcal{A}))}_{p-1}$ as the dual of the wedge product. More explicitly, when the wedge product is given by

$$\varepsilon(\varphi_i) \wedge \varphi_j = \sum_k c_{ij}^k \varphi_k$$

the derivation δ is defined as

$$\delta Z_k = \sum_{i,j} c_{ij}^k Z_i Z_j.$$

We obtain the condition

$$\varepsilon(\omega) \wedge \omega = \delta(\omega)$$

by defining the derivation δ in the above way. Therefore, we have the following theorem, which reflects the formality of $M(\mathcal{A})$ (see [14] for details).

Theorem 5.1. *For a complex hyperplane arrangement $\mathcal{A}$ a formal homology connection for its complement $M(\mathcal{A})$ is given by*

$$\omega = \sum_{j=1}^m \varphi_j \otimes Z_j$$

with the derivation $\delta : T\widehat{H_+(M(\mathcal{A}))}_p \longrightarrow T\widehat{H_+(M(\mathcal{A}))}_{p-1}$ defined as the dual of the wedge product.

Let us consider the derivation

$$\delta : T\widehat{H_+(M(\mathcal{A}))}_1 \longrightarrow T\widehat{H_+(M(\mathcal{A}))}_0$$

and the ideal $\mathcal{I}_0$ generated by the image of δ. The ideal $\mathcal{I}_0$ has generators

$$[X_{j_p}, X_{j_1} + \cdots + X_{j_k}], \quad 1 \leq p < k,$$

for the maximal family of hyperplanes $\{H_{j_1}, \cdots, H_{j_k}\}$ such that

$$\mathrm{codim}_{\mathbf{C}}(H_{j_1} \cap \cdots \cap H_{j_k}) = 2.$$

The primitive part of $T\widehat{H_+(M(\mathcal{A}))}$ is call the holonomy Lie algebra for the arrangement $\mathcal{A}$. As is shown in [11] the holonomy Lie algebra is isomorphic to the nilpotent completion of the fundamental group $\pi_1(M(\mathcal{A}))$ over $\mathbf{C}$.

We consider the configuration space of ordered distinct n points in the complex plane $\mathbf{C}$ given as

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}.$$

The configuration space X_n is the complement of the union of big diagonal hyperplanes H_{ij} defined by $z_i = z_j$ in $\mathbf{C}^n$ for $1 \leq i < j \leq n$. By considering the action of the symmetric group $\mathfrak{S}_n$ by the permutation of coordinates, we consider the quotient space

$$Y_n = X_n/\mathfrak{S}_n.$$

There is a covering map

$$\pi : X_n \longrightarrow Y_n.$$

The fundamental group $\pi_1(Y_n)$ is the braid group of n strings denoted by B_n and $\pi_1(X_n)$ is the pure braid group of n strings denoted by P_n.

We set

$$\omega_{ij} = d\log(z_i - z_j), \quad 1 \leq i < j \leq n.$$

Then the Orlik-Solomon algebra $OS(X_n)$ is generated by ω_{ij}, $1 \leq i < j \leq n$, with the Arnold relations

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{ij} = 0, \quad 1 \leq i < j < k \leq n. \tag{5.1}$$

It is known that the degree q part of $OS(X_n)$ has a basis represented by

$$\omega_{i_1 j_1} \wedge \cdots \wedge \omega_{i_q j_q}, \quad j_1 < \cdots < j_q.$$

This is called the normal form of a basis of $OS(X_n)$. We denote by $X_{i_1 j_1, \cdots, i_q j_q}$ its dual basis of the homology $H_q(X_n)$. The formal homology connection is given by

$$\omega = \sum_{j_1 < \cdots < j_q, 1 \leq q \leq n} (\omega_{i_1 j_1} \wedge \cdots \wedge \omega_{i_q j_q}) \otimes X_{i_1 j_1, \cdots, i_q j_q}. \tag{5.2}$$

Example: Let us describe the case $n = 4$, which will play an important role in the following computation. In this case we have

$$\dim H^1(X_4) = 6, \ \dim H^2(X_4) = 11, \ \dim H^3(X_4) = 6.$$

Let us denote by A^q the degree q part of the Orlik-Solomon algebra $OS(X_4)$. Then A^1 has a basis ω_{ij}, $1 \leq i < j \leq 4$, and A^2 has a basis $\omega_{ij} \wedge \omega_{k\ell}$, $i < j, k < \ell, j < \ell$. A basis of A^3 is given by

$$\omega_{12} \wedge \omega_{i3} \wedge \omega_{j4}, \quad 1 \leq i \leq 2, \ 1 \leq j \leq 3$$

To describe the derivation

$$\delta : \widehat{TH_+(X_4)}_2 \longrightarrow \widehat{TH_+(X_4)}_1$$

we need to determine the product structure $\wedge : A^1 \times A^2 \longrightarrow A^3$. In order to write down this product in terms of the above basis we use the Arnold relation (5.1) in the form

$$\omega_{ik} \wedge \omega_{jk} = \omega_{ij} \wedge \omega_{jk} - \omega_{ij} \wedge \omega_{ik}, \ i < j < k$$

successively. For example, we have

$$\begin{aligned}
\omega_{14} \wedge (\omega_{23} \wedge \omega_{34}) =& \omega_{13} \wedge \omega_{23} \wedge \omega_{34} - \omega_{13} \wedge \omega_{23} \wedge \omega_{14} \\
=& \omega_{12} \wedge \omega_{23} \wedge \omega_{34} - \omega_{12} \wedge \omega_{13} \wedge \omega_{34} - \omega_{12} \wedge \omega_{23} \wedge \omega_{14} + \omega_{12} \wedge \omega_{13} \wedge \omega_{14}.
\end{aligned}$$

The 1-form part of the formal homology connection is

$$\omega^{(1)} = \sum_{i<j} \omega_{ij} \otimes X_{ij}$$

where X_{ij}, $1 \leq i < j \leq n$, is a basis of $H_1(X_n; \mathbf{C})$ corresponding to the hyperplanes H_{ij}. The representation of the path groupoid described in the previous section is give as

$$Hol : \mathcal{P}_1(X_n) \longrightarrow \mathbf{C}\langle\langle X_{ij} \rangle\rangle$$

where $\mathbf{C}\langle\langle X_{ij} \rangle\rangle$ is the ring of non-cummutative formal power series with indeterminates X_{ij}, $1 \leq i < j \leq n$. Let us describe the derivation

$$\delta : T\widehat{H_+(X_n)}_1 \longrightarrow T\widehat{H_+(X_n)}_0.$$

For i, j, k such that $1 \leq i < j < k \leq n$ we have

$$\delta X_{ij,ik} = [X_{ik}, X_{ij} + X_{jk}], \quad \delta X_{ij,jk} = [X_{jk}, X_{ij} + X_{ik}]$$

and for distinct i, j, k, ℓ we have

$$\delta X_{ij,k\ell} = -[X_{ij}, X_{k\ell}].$$

Hence the ideal $\mathcal{I}_0$ is generated by the infinitesimal pure braid relations:

$$\begin{aligned} &[X_{ik}, X_{ij} + X_{jk}], \ [X_{ij} + X_{ik}, X_{jk}] \quad (i < j < k), \\ &[X_{ij}, X_{k\ell}], \quad (i, j, k, \ell \ \text{ distinct}). \end{aligned}$$

We obtain the representation of the homotopy path groupoid

$$Hol : \Pi_1(X_n) \longrightarrow \mathbf{C}\langle\langle X_{ij} \rangle\rangle / \mathcal{I}_0.$$

In particular, we obtain the holonomy map

$$Hol : P_n \longrightarrow \mathbf{C}\langle\langle X_{ij} \rangle\rangle / \mathcal{I}_0$$

which is a prototype of the Kontsevich integral [16] for knots and gives a universal finite type invariants for pure braids (see [12], [13] and [6]).

Here we explain a relation to the Knizhnik-Zamolodchikov (KZ) connection. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\{I_\mu\}$ an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. We set $\Omega = \sum_\mu I_\mu \otimes I_\mu$. Let $r_i : \mathfrak{g} \to End(V_i)$, $1 \leq i \leq n$, be representations of the Lie algebra $\mathfrak{g}$. We define Ω_{ij} to be the action of Ω on the i-th and j-th components of the tensor product $V_1 \otimes \cdots \otimes V_n$ by means of the above representations r_i, $1 \leq i \leq n$. We put

$$\omega = \frac{1}{\kappa} \sum_{i<j} \Omega_{ij} d \log(z_i - z_j),$$

where κ is a non-zero complex parameter. Then we have a representation of the algebra

$$\rho : \mathbf{C}\langle\langle X_{ij} \rangle\rangle / \mathcal{I}_0 \longrightarrow \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$$

by defining $\rho(X_{ij}) = \frac{1}{\kappa}\Omega_{ij}$. The above 1-form ω defines a flat connection for a trivial vector bundle over the configuration space X_n with fiber $V_1 \otimes \cdots \otimes V_n$. This is called the KZ connection. The holonomy map of the KZ connection gives linear representation of the pure braid group P_n, wich was studied in [12].

We deal with the 1-form part and the 2-form part

$$\omega^{(1)} = \sum_{i<j} \omega_{ij} \otimes X_{ij}, \quad \omega^{(2)} = \sum_{j_1<j_2} (\omega_{i_1j_1} \wedge \omega_{i_2j_2}) \otimes X_{i_1j_1,i_2j_2}.$$

From the condition $\delta\omega + \kappa = 0$ for the formal homology connection we obtain the equation

$$\delta\omega^{(2)} + \omega^{(1)} \wedge \omega^{(1)} = 0.$$

Let us consider the derivation

$$\delta : T\widehat{H_+(X_n)}_2 \longrightarrow T\widehat{H_+(X_n)}_1.$$

We express the 3-form part of the generalized curvature κ by the normal form of a basis of $OS(X_n)$ as

$$\omega^{(1)} \wedge \omega^{(2)} - \omega^{(2)} \wedge \omega^{(1)} = \sum_{j_1<j_2<j_3} (\omega_{i_1j_1} \wedge \omega_{i_2j_2} \wedge \omega_{i_3j_3}) \otimes Z_{i_1j_1,i_2j_2,i_3j_3}$$

Then we have

$$\delta(X_{i_1j_1,i_2j_2,i_3j_3}) = -Z_{i_1j_1,i_2j_2,i_3j_3}$$

and the ideal $\mathcal{I}_1$ is generated by $Z_{i_1j_1,i_2j_2,i_3j_3}$, which are expressed by Lie brackets of X_{ij} and $X_{i_1j_1,i_2j_2}$. Let us describe explicitly these generators. First, we have

$$[X_{k\ell}, X_{i_1j_1,i_2j_2}] \text{ for } \{k,\ell\} \cap \{i_1,j_1,i_2,j_2\} = \emptyset,\ k<\ell, i_1<j_1, i_2<j_2, j_1<j_2 \tag{5.3}$$

For i,j,k,ℓ such that $1 \leq i < j < k < \ell \leq n$ we have the following generators.

$$\begin{aligned}&[X_{i\ell}, X_{ij,ik} + X_{jk,k\ell}] + [X_{ij} + X_{jk} + X_{j\ell}, X_{ik,i\ell}] - [X_{ik} + X_{k\ell}, X_{ij,i\ell}] \\ &- [X_{ik} + X_{k\ell}, X_{jk,i\ell}] + [X_{ik}, X_{ij,k\ell} - X_{ik,j\ell}]\end{aligned} \tag{5.4}$$

$$\begin{aligned}&[X_{j\ell}, X_{ij,jk} + X_{ik,k\ell}] + [X_{ij} + X_{ik} + X_{i\ell}, X_{jk,k\ell}] - [X_{jk} + X_{k\ell}, X_{ij,j\ell}] \\ &+ [X_{jk} + X_{j\ell} + X_{k\ell}, X_{ik,j\ell}] + [X_{j\ell}, X_{ij,k\ell} - X_{jk,i\ell}]\end{aligned} \tag{5.5}$$

$$\begin{aligned}&- [X_{jk}, X_{ij,i\ell} + X_{ik,i\ell}] + [X_{i\ell}, X_{jk,j\ell} + X_{jk,k\ell} - X_{ij,jk}] \\ &+ [X_{ij} + X_{ik} + X_{j\ell} + X_{k\ell}, X_{jk,i\ell}]\end{aligned} \tag{5.6}$$

$$\begin{aligned}&- [X_{ik}, X_{ij,i\ell} + X_{jk,j\ell}] + [X_{j\ell}, X_{ik,i\ell} + X_{ik,k\ell} - X_{ij,ik}] \\ &+ [X_{ij} + X_{i\ell} + X_{jk} + X_{k\ell}, X_{ik,j\ell}]\end{aligned} \tag{5.7}$$

$$\begin{aligned}&[X_{k\ell}, X_{ij,ik} + X_{ij,i\ell}] + [X_{ij} + X_{jk} + X_{j\ell}, X_{ik,k\ell}] - [X_{ik} + X_{i\ell}, X_{jk,k\ell}] \\ &- [X_{i\ell}, X_{ij,k\ell}] + [X_{k\ell}, -X_{ik,j\ell} + X_{jk,i\ell}]\end{aligned} \tag{5.8}$$

$$\begin{aligned}&[X_{k\ell}, X_{ij,jk} + X_{ij,j\ell}] + [X_{ij} + X_{ik} + X_{i\ell}, X_{jk,k\ell}] - [X_{jk} + X_{j\ell}, X_{ik,k\ell}] \\ &- [X_{jk} + X_{j\ell}, X_{ij,k\ell}] + [X_{k\ell}, X_{ik,j\ell} - X_{jk,i\ell}]\end{aligned} \tag{5.9}$$

Let us take a representation of the pair $T\widehat{H_+(X_n)}_1/\mathcal{I}_1$ and $T\widehat{H_+(X_n)}_0$ together with the derivation

$$\delta : T\widehat{H_+(M)}_1/\mathcal{I}_1 \longrightarrow T\widehat{H_+(M)}_0$$

as a crossed module. Namely, we consider the following. There is a representation

$$\rho : T\widehat{H_+(X_n)}_0 \longrightarrow \mathrm{End}(V)$$

as a complete algebra with some vector space V and a representation

$$\rho' : T\widehat{H_+(X_n)}_1/\mathcal{I}_1 \longrightarrow \mathrm{End}(W)$$

as 2-sided modules, which means that there are right and left action of $\mathrm{End}(V)$ on $\mathrm{End}(W)$ compatible with ρ and ρ'. We also assume that there is a linear map

$$\tau : \mathrm{End}(W) \longrightarrow \mathrm{End}(V)$$

such that the condition $\rho \circ \delta = \tau \circ \rho'$ holds. For $X \in \widehat{TH_+(X_n)}_0$ and $Y \in \widehat{TH_+(X_n)}_1$ we have

$$\rho'([X, Y]) = \rho(X) \cdot \rho'(Y) - \rho'(Y) \cdot \rho(X)$$

which is denoted by $[\rho(X), \rho'(Y)]$.

We set $\rho(X_{ij}) = T_{ij}$ and $\rho'(X_{i_1j_1,i_2j_2}) = W_{i_1j_1,i_2j_2}$. Let us consider the 1-form part and the 2-form part

$$\omega^{(1)} = \sum_{i<j} \omega_{ij} \otimes X_{ij}, \quad \omega^{(2)} = \sum_{j_1<j_2} (\omega_{i_1j_1} \wedge \omega_{i_2j_2}) \otimes X_{i_1j_1,i_2j_2}.$$

of the formal homology connection. Then the 1-form

$$A = \sum_{i<j} \omega_{ij} T_{ij}$$

with values in $\mathrm{End}(V)$ and the 2-form

$$B = \sum_{j_1<j_2} \omega_{i_1j_1} \wedge \omega_{i_2j_2} \; W_{i_1j_1,i_2j_2}$$

with values in $\mathrm{End}(W)$ define a so-called 2-connection. They satisfy the the equation

$$\tau(A) + B \wedge B = 0.$$

The pair A and B satisfy the 2-flatness condition in the sense that the 2-curvature vanishes. Namely, we have

$$A \wedge B - B \wedge A = 0$$

where we use the right and left actions of $\mathrm{End}(V)$ on $\mathrm{End}(W)$ in the above equation. This 2-flatness condition holds since we consider the representation ρ' modulo the ideal $\mathcal{I}_1$. This corresponds to the Lie bracket relations between T_{ij} and $W_{i_1j_1,i_2j_2}$ derived from the equations (5.4)–(5.9).

Let us consider a representation ρ' satisfying

$$\rho'(X_{i_1j_1,i_2j_2}) = 0$$

if $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$. In this case the 2-form B is written as

$$B = \sum_{i<j<k} (\omega_{ij} \wedge \omega_{ik} \; W_{ij,ik} + \omega_{ij} \wedge \omega_{jk} \; W_{ij,jk})$$

and the 2-flatness condition can be reduced to the following equations.

$$\begin{aligned}
&[T_{i\ell}, W_{ij,ik} + W_{jk,k\ell}] + [T_{ij} + T_{jk} + T_{j\ell}, W_{ik,i\ell}] - [T_{ik} + T_{k\ell}, W_{ij,i\ell}] = 0 \\
&[T_{j\ell}, W_{ij,jk} + W_{ik,k\ell}] + [T_{ij} + T_{ik} + T_{i\ell}, W_{jk,k\ell}] - [T_{jk} + T_{k\ell}, W_{ij,j\ell}] = 0 \\
&[T_{jk}, W_{ij,i\ell} + W_{ik,i\ell}] - [T_{i\ell}, W_{jk,j\ell} + W_{jk,k\ell} - W_{ij,jk}] = 0 \\
&[T_{ik}, W_{ij,i\ell} + W_{jk,j\ell}] - [T_{j\ell}, W_{ik,i\ell} + W_{ik,k\ell} - W_{ij,ik}] = 0 \\
&[T_{k\ell}, W_{ij,ik} + W_{ij,i\ell}] + [T_{ij} + T_{jk} + T_{j\ell}, W_{ik,k\ell}] - [T_{ik} + T_{i\ell}, W_{jk,k\ell}] = 0 \\
&[T_{k\ell}, W_{ij,jk} + W_{ij,j\ell}] + [T_{ij} + T_{ik} + T_{i\ell}, W_{jk,k\ell}] - [T_{jk} + T_{j\ell}, W_{ik,k\ell}] = 0
\end{aligned}$$

This recovers the 2-flatness condition described by Cirio and Martins in [7]. In [8] they investigated the categorification of the KZ connection by means of 2-Yang-Baxter operator for $sl_2(\mathbf{C})$ with $\mathfrak{S}_n$ symmetry.

It is shown in [14] that the complex $(\widehat{TH_+(X_n)}_*, \delta)$ is acyclic. This means that the homology group $H_q(\widehat{TH_+(X_n)}_*) = 0$ if $q \neq 0$. We have

$$H_0(\widehat{TH_+(X_n)}_*) \cong \widehat{TH_+(X_n)}_0/\mathcal{I}_0.$$

It is well-known that the configuration space X_n is a $K(\pi, 1)$ space and, in particular, we have $\pi_2(X_n) = 0$. Although, it is worthwhile to study the 2-holonomy map for a representation of the category of braid cobordisms. First, we describe the notion of the category of braid cobordisms. Let us recall that a braid is an embedding of a 1-manifold which is a disjoint union of closed intervals into $\mathbf{C} \times [0, 1]$ so that the projection onto $[0, 1]$ has no critical points, and the boundary of the 1-manifold is mapped to $2n$ points

$$(1, 0), (2, 0), \cdots, (n, 0), (1, 1), (2, 1), \cdots, (n, 1) \in \mathbf{C} \times [0, 1].$$

The isotopy classes of braids fixing the boundary form the braid group B_n. A braid cobordism between braids g and h is a compact surface S with boundary and corners, smoothly and properly embedded in $\mathbf{C} \times [0, 1]^2$, such that the following conditions are satisfied.

(1) The boundary of S is the union of 1-manifolds

$$\begin{aligned}
&S \cap (\mathbf{C} \times [0, 1] \times \{0\}) = g,\\
&S \cap (\mathbf{C} \times [0, 1] \times \{1\}) = h,\\
&S \cap (\mathbf{C} \times \{0\} \times [0, 1]) = \{1, 2, \cdots, n\} \times \{0\} \times [0, 1],\\
&S \cap (\mathbf{C} \times \{1\} \times [0, 1]) = \{1, 2, \cdots, n\} \times \{1\} \times [0, 1].
\end{aligned}$$

(2) The projection of S onto $[0, 1]^2$ is a branched covering with simple branch points only.

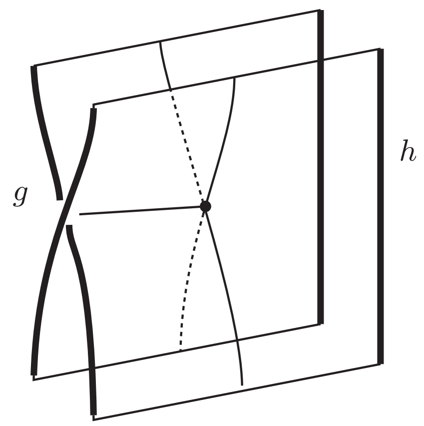

FIGURE 2. braid cobordism

An example of a braid cobordism is depicted in Figure 2. We shall say that two braid cobordisms S and S' between the braids g and h are equivalent if there is an isotopy through braid cobordism between S and S' relative to the boundary. Let $\mathcal{B}C_n$ be the category whose objects are braids with n-strands and whose morphisms are the equivalence classes of braid cobordisms. We define the composition of morphisms in the following way. Let S_1 be a braid cobordism between g and h and S_2 be a braid cobordism between h and k. We define the composition S_2S_1 as the braid cobordism given by the concatination of S_1 and S_2 along their common boundary h. Let us notice that in $\mathcal{B}C_n$ two isotopic braids are isomorphic, but not equal. We equip $\mathcal{B}C_n$ with a monoidal structure in the following way. Let S_1 be a braid cobordism from g_1 to h_1 and S_2 be a braid cobordism from g_2 to h_2. We define a braid cobordism $S_2 \circ S_1$ from g_1g_2 to h_1h_2 by identifying the top portion of ∂S_1 and the bottom portion of ∂S_2. This monoidal structure is not strictly associative since the braids $g_3(g_2g_1)$ and $(g_3g_2)g_1$ are isotopic but not equal. A braid cobordism is also called a braided surface (see [2] and [10]).

A braid cobordism can be described diagramatically by Carter and Saito's braid movie which is a one-parameter deformation family of singular braids with double points (see [2]). To construct a representation of $\mathcal{B}C_n$ based on the 2-holonomy map for the configuration space X_n we consider the integration of the transport T along the braid movie associated with a braid cobordism as shown in Figure 2. A divergence for a braid with double points for such iterated integrals can be described by a method similar to the one used by Le and Murakami [17]. Let us start with a simple case. For a small positive number ε we consider the path γ given as the segment connecting from 1 to ε on the real line. We consider the 1-form $\omega = \frac{dt}{t}$. For a parameter λ we consider the transport

$$T = 1 + \lambda \int_\gamma \omega + \cdots + \lambda^k \int_\gamma \underbrace{\omega \cdots \omega}_{k} + \cdots$$

We have $T = \varepsilon^\lambda = e^{\lambda \log \varepsilon}$. This shows the asymptotic behavior of the above transport when ε tends to 0. In particular, we have

$$\int_\gamma \underbrace{\omega \cdots \omega}_{k} = \frac{1}{k!}(\log \varepsilon)^k. \tag{5.10}$$

We take n distinct points $p_1, \cdots, p_n$ with coordinate functions $z_1, \cdots, z_n$. We consider the open set U in X_n defined by $|z_1| < 1$, $|z_2| < 1$ and $|z_j| > 1$ for $j \geq 3$. We put $|z_1 - z_2| = \varepsilon$ and consider the situation when ε is sufficiently small. In this case the equation (5.10) can be applied to estimate the asymptotics of the 2-holonomy map when ε tends to 0 and we can regularize the 2-holonomy maps along braid movies.

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