BAR COMPLEX, CONFIGURATION SPACES AND FINITE TYPE INVARIANTS FOR BRAIDS

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ABSTRACT. We show that the bar complex of the configuration space of ordered distinct points in the complex plane is acyclic. The 0-dimensional cohomology of this bar complex is identified with the space of finite type invariants for braids. We construct a universal holonomy homomorpshim from the braid group to the space of horizontal chord diagrams over **Q**, which provides finite type invariants for braids with values in **Q**.

1. INTRODUCTION

The purpose of this paper is to present some refinements of the author's work on configuration spaces and finite type invariants for braid groups developed in [9] and [10].

Let M_n be the configuration space of ordered distinct n points in the complex plane $\mathbf{C}$ and A be the Orlik-Solomon algebra, which is the subalgebra of the algebra of differential forms on M_n generated by the logarithmic forms

$$\omega_{ij} = \frac{1}{2\pi\sqrt{-1}}\, d\log(z_i - z_j), \quad 1 \leq i < j \leq n.$$

Let $B^*(A)$ be the bar complex of the Orlik-Solomon algebra. There is an acyclicity of the bar complex $B^*(A)$. Namely, the vanishing of the cohomology $H^j(B^*(A)) = 0$ holds for $j \neq 0$. The 0-dimensional cohomology $H^0(B^*(A))$ is identified with the space of finite type invariants for the pure braid group. It turns out that this space is isomorphic to the dual of the space of horizontal chord diagrams with n vertical strands. This isomorphism is derived from a universal holonomy map defined by Chen's iterated integrals from the braid group to the completion of the space of horizontal chord diagrams. This is a prototype of the Kontsevich integral for knots appearing in [12]. In the preceding articles [9], [10], we described this holonomy map over $\mathbf{C}$ since it is defined by the iterated integral of the above logarithmic forms. In this article we focus on invariants derived from a universal holonomy homomorphism of braid groups defined by means of a rational Drinfel'd associator and we shall show the above isomorphisms over $\mathbf{Q}$.

The paper is organized in the following way. In Section 2 we recall basic facts about the bar complex of the Orlik-Solomon algebra for a hyperplane arrangement based on [11]. In Section 3 we show the acyclicity of the bar complex for the configuration space M_n. Section 4 is devoted to a relation between the 0-dimensional cohomology of the above bar complex and the space of finite type invariants for braids. In Section 5 we construct a universal holonomy map over $\mathbf{Q}$ by means of a rational Drinfel'd associator. In Section 6 we describe some consequence of the Cohen-Gitler theorem on the structure of the loop space of the configutation space of points in $\mathbf{R}^m$, $m \geq 3$, in relation with the above results for finite type invariants.

2. Bar complex of the Orlik-Solomon algebra

This section is devoted to preliminaries on hyperplane arrangements and some properties of the bar complex of the Orlik-Solomon algebra. We refer the reader to [11] for a more detailed description.

Let $\{H_j\}$, $1 \leq j \leq r$, be a family of complex hyperplanes in $\mathbf{C}^m$ and f_j be a linear form defining the hyperplane H_j. We consider the logarithmic differential forms

$$\omega_j = \frac{1}{2\pi\sqrt{-1}} \frac{df_j}{f_j}, \quad 1 \leq j \leq r.$$

Let M be the complement of the hyperplanes

$$M = \mathbf{C}^m \setminus \bigcup_{1 \leq j \leq r} H_j$$

and we denote by $\mathcal{E}^*(M)$ the de Rham complex of differential forms with values in $\mathbf{C}$. The Orlik-Solomon algebra A is the $\mathbf{Z}$ subalgebra of $\mathcal{E}^*(M)$ generated by the logarithmic forms ω_j, $1 \leq j \leq r$. The algebra A is isomorphic to the cohomology ring $H^*(M; \mathbf{Z})$. We shall say that a subset $\{H_{i_1}, \cdots H_{i_p}\}$ of the set of hyperplanes $\{H_1, \cdots H_r\}$ is dependent if the condition

$$\mathrm{codim}_{\mathbf{C}}[H_{i_1} \cap \cdots \cap H_{i_p}] < p$$

is satisfied. It is shown by Orlik-Solomon [14] that the algebra A is isomorphic to the exterior algebra generated by e_j, $1 \leq j \leq r$, with relations

$$\sum_{s=1}^{p} (-1)^{s-1} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_s} \wedge \cdots \wedge e_{i_p} = 0$$

for any dependent family $\{H_{i_1}, \cdots H_{i_p}\}$.

Following Chen [3] we briefly recall the definition of the bar complex of the de Rham complex $\mathcal{E}^*(M)$. We denote by $\mathcal{E}^{*-1}(M)$ the differential graded algebra whose degree j part is given by $\mathcal{E}^{j+1}(M)$ if $j > 0$ and by $\mathcal{E}^1(M)/d\mathcal{E}^0(M)$ if $j = 0$. The tensor algebra $T\mathcal{E}^{*-1}(M)$ is equipped with the structure of a graded algebra. We set

$$B^{-k,p}(M) = \left[\bigotimes^{k} \mathcal{E}^{*-1}(M) \right]^{p-k}$$

where the right hand side stands for the degree $p - k$ part.

For a differential q form φ we set $J\varphi = (-1)^q \varphi$. The differential $d' : B^{-k,p}(M) \longrightarrow B^{-k,p+1}(M)$ is defined by

$$d'(\varphi_1 \otimes \cdots \otimes \varphi_k) = \sum_{i=1}^{k} (-1)^i \; J\varphi_1 \otimes \cdots \otimes J\varphi_{i-1} \otimes d\varphi_i \otimes \varphi_{i+1} \otimes \cdots \otimes \varphi_k$$

and the differential $d'' : B^{-k,p}(M) \longrightarrow B^{-k+1,p}(M)$ is defined by

$$\begin{aligned} & d''(\varphi_1 \otimes \cdots \otimes \varphi_k) \\ & \quad = \sum_{i=1}^{k} (-1)^{i-1} \; J\varphi_1 \otimes \cdots \otimes J\varphi_{i-1} \otimes [(J\varphi_i) \wedge \varphi_{i+1}] \otimes \varphi_{i+2} \otimes \cdots \otimes \varphi_k. \end{aligned}$$

With the above differentials d' and d'' the direct sum $\oplus_{k,p} B^{-k,p}(M)$ has a structure of a double complex. The associated total complex is denoted by $B^*(M)$ and is called the bar complex of the de Rham complex of M. Restricting the above construction to the Orlik-Solomon algebra A, we obtain the double complex $\oplus_{k,p} B^{-k,p}(A)$ whose associated total complex $B^*(A)$ is called the bar complex of the Orlik-Solomon algebra. Let us notice that $d' = 0$ on $B^*(A)$. There is an isomorphism of algebras

$$H^*(B^*(A)) \otimes \mathbf{C} \cong H^*(B^*(M)).$$

We denote by $H_{*-1}(M; \mathbf{Z})$ the graded module whose degree j part is $H_{j+1}(M; \mathbf{Z})$ if $j \geq 0$ and is defined to be 0 if $j < 0$. Let us consider the tensor algebra $TH_{*-1}(M; \mathbf{Z})$ with the structure of the graded algebra. Let 1 and $\omega_1, \cdots, \omega_m$ be a basis of the Orlik-Solomon algebra A as a $\mathbf{Z}$ module and $X_1, \cdots, X_m$ be its dual basis of $H_{*-1}(M; \mathbf{Z})$.

A derivation δ on $TH_{*-1}(M; \mathbf{Z})$ is a $\mathbf{Z}$ linear endomorphism of degree -1 with $\delta \circ \delta = 0$ such that

$$\delta(uv) = (\delta u)v + (-1)^{\deg u} u(\delta v)$$

for any homogeneous elements $u, v \in TH_{*-1}(M; \mathbf{Z})$. We define a derivation δ on $TH_{*-1}(M; \mathbf{Z})$ in the following way. When the wedge product on the Orlik-Solomon algebra is written as $\omega_i \wedge \omega_j = \sum_k c_{ij}^k \omega_k, c_{ij}^k \in \mathbf{Z}, 1 \leq i < j \leq m$, we define δ by

$$\delta X_k = -\sum_k (-1)^{p_i} c_{ij}^k [X_i, X_j]$$

if $\deg X_k > 0$ and by $\delta X_k = 0$ if $\deg X_k = 0$. Here the Lie bracket is taken in a graded sense and $\deg X_i = p_i - 1$. It can be shown that the bar complex $B^*(A)$ and the complex $TH_{*-1}(M; \mathbf{Z})$ with the derivation δ are dual to each other.

3. Acyclicity of the bar complex

For a space X we denote by $\mathrm{Conf}_n(X)$ the configuration space of ordered distinct n points in X defined by

$$\mathrm{Conf}_n(X) = \{(x_1, \cdots x_n) \in X^n \ ; \ x_i \neq x_j \text{ if } i \neq j\}.$$

In this section we deal with the configuration space $\mathrm{Conf}_n(\mathbf{C})$, which is also denoted by M_n.

The following vanishing holds for the cohomology of the bar complex of the Orlik-Solomon algebra for the configuration space M_n.

Theorem 3.1. *Let A be the Orlik-Solomon algebra for M_n. Then, we have*

$$H^j(B^*(A)) \cong 0, \quad j \neq 0.$$

Proof. Let $p : M_{n+1} \longrightarrow M_n$ be the projection map on the last n coordinates and F be its fiber. It has a structure of a fiber bundle and the fundamental group of the base space acts trivially on the homology of a fiber. The only non-trivial differential in the Serre spectral sequence is

$$d^2 : E^2_{p+1,0} \longrightarrow E^2_{p-1,1}.$$

There is a long exact sequence

$$\begin{aligned} &\longrightarrow H_{p+1}(M_{n+1}) \longrightarrow H_{p+1}(M_n) \cong E^2_{p+1,0} \longrightarrow E^2_{p-1,1} \\ &\longrightarrow H_p(M_{n+1}) \longrightarrow \cdots \end{aligned}$$

Since the fibration $p : M_{n+1} \longrightarrow M_n$ admits a section $\sigma : M_n \longrightarrow M_{n+1}$ such that $p \circ \sigma = id$ we obtain an exact sequence

$$0 \longrightarrow H_{p-1}(M_n) \otimes H_1(F) \longrightarrow H_p(M_{n+1}) \longrightarrow H_p(M_n) \longrightarrow 0.$$

It follows that there is an isomorphism

$$H_*(M_{n+1}) \cong H_*(M_n) \otimes H_*(F).$$

Since $H_{*-1}(F)$ has only degree 0 elements we have $H_j(TH_{*-1}(F)) = 0$ for $j \neq 0$. Now by an inductive argument it can be shown that $H_j(TH_{*-1}(M_{n+1})) = 0$ for $j \neq 0$. Hence by duality we have $H^j(B^*(A)) \cong 0, j \neq 0$. This completes the proof. □

The above vanishing theorem can be generalized in a similar way to fiber-type arrangements in the sense of Falk and Randell [8]. This type of acyclicity of the bar complex was first noticed by Aomoto [1] in the case of an example of a fiber-type arrangement.

4. Finite type invariants

Let $\mathbf{K}$ be a field. We recall the notion of finite type invariants for braids. First, let us consider the diagram of a singular braid with finitely many transverse double points as shown in Figure 1. We replace the double points $p_1, \cdots, p_k$ by positive or negative crossings according as $\epsilon_j = \pm 1$ and we denote by $\beta_{\epsilon_1 \cdots \epsilon_k}$ the obtained braid. For a function $v : B_n \longrightarrow \mathbf{K}$ we define its extension on singular braids with transverse double points by

$$\tilde{v}(\beta) = \sum_{\epsilon_j = \pm 1, 1 \leq j \leq k} \epsilon_1 \cdots \epsilon_k \; v(\beta_{\epsilon_1 \cdots \epsilon_k}). \tag{4.1}$$

There is an increasing sequence of singular braids

$$B_n \subset S_1(B_n) \subset \cdots \subset S_k(B_n) \subset \cdots$$

where $S_k(B_n)$ is the set of singular n-braids with at most k transverse double points. We shall say that v is of finite type of order k if and only if its extension $\tilde{v}$ vanishes on $S_m(B_n)$ for $m > k$. We denote by $V_k(B_n)_{\mathbf{K}}$ the space of order k invariants for B_n with values in K. There is an increasing sequence of vector spaces.

$$V_0(B_n)_{\mathbf{K}} \subset V_1(B_n)_{\mathbf{K}} \subset \cdots \subset V_k(B_n)_{\mathbf{K}} \subset \cdots$$

We set $V(B_n)_{\mathbf{K}} = \bigcup_{k \geq 0} V_k(B_n)_{\mathbf{K}}$ and call it the space of finite type invariants for B_n with values in $\mathbf{K}$. In a similar way we define $V(P_n)_{\mathbf{K}}$, the space of finite type invariants for the pure braid group P_n with values in $\mathbf{K}$.

For the bar complex there is a filtration defined by

$$\mathcal{F}^{-k} B^*(A) = \bigoplus_{q \leq k} B^{-q,p}(A), \quad k = 0, 1, 2, \cdots$$

This induces a filtration $\mathcal{F}^{-k} H^0(B^*(A))$, $k \geq 0$, on the cohomology of the bar complex.

For 1-forms $\varphi_1, \cdots \varphi_k$ and a loop γ in M_n we define the iterated integral by

$$\int_\gamma \varphi_1 \cdots \varphi_k = \int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} f_1(t_1) \cdots f_k(t_k) \; dt_1 \cdots dt_k$$

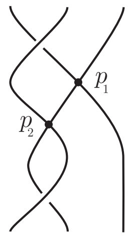

FIGURE 1. A singluar braid with double points

where $f_j(t)dt$ is the pull back $\gamma^*\varphi_k$. The above iterated integral defines a map

$$\iota : H^0(B^*(A)) \longrightarrow Hom(\mathbf{Z}P_n, \mathbf{K}).$$

Theorem 4.1. *Let A be the Orlik-Solomon algebra for the configuration space $M_n = \mathrm{Conf}_n(\mathbf{C})$. The iterated integral map ι gives the isomopshisms*

$$\mathcal{F}^{-k}H^0(B^*(A)) \otimes \mathbf{K} \cong V_k(P_n)_{\mathbf{K}},$$
$$H^0(B^*(A)) \otimes \mathbf{K} \cong V(P_n)_{\mathbf{K}}.$$

Proof. The iterated integral map induces

$$\iota : \mathcal{F}^{-k}H^0(B^*(A)) \longrightarrow Hom(\mathbf{Z}P_n/J^{k+1}, \mathbf{K})$$

where J is the augumentation ideal of the group ring $\mathbf{Z}P_n$. By a theorem of Chen [3] for fundamental groups we see that the above map ι is an isomorphism. It follows from the definition of order k invariants that there is an isomorphim

$$Hom(\mathbf{Z}P_n/J^{k+1}, \mathbf{K}) \cong V_k(P_n)_{\mathbf{K}}.$$

This completes the proof. □

5. Drinfel'd associator and holonomy of braid groups

We denote by $\mathcal{A}_n$ the algebra over $\mathbf{Z}$ generated by X_{ij}, $1 \leq i \neq j \leq n$, with the relations :

$$X_{ij} = X_{ji} \tag{5.1}$$
$$[X_{ik}, X_{ij} + X_{jk}] = 0 \quad i, j, k \text{ distinct}, \tag{5.2}$$
$$[X_{ij}, X_{k\ell}] = 0 \quad i, j, k, \ell \text{ distinct}. \tag{5.3}$$

We assign to each X_{ij} degree 1 and put the structure of a graded algebra on $\mathcal{A}_n$. We denote by $\mathcal{A}_{n,k}$ the degree k part of $\mathcal{A}_n$. A basis of $\mathcal{A}_{n,k}$ is represented by a chord diagram with n vertical strands and k horizontal chords. The relation (5.2) is shown graphically as in Figure 2. There is a direct sum decomposition $\mathcal{A}_n = \oplus_{k\geq 0}\mathcal{A}_{n,k}$. The algebra $\widehat{\mathcal{A}}_n$ is defined to be the direct product

$$\widehat{\mathcal{A}}_n = \prod_{k\geq 0} \mathcal{A}_{n,k}.$$

The algebra $\mathcal{A}_n$ has a natural structure of a graded Hopf algebra and is called the algebra of horizontal chord diagrams on n vertical strands.

Let $\mathbf{Z}S_n$ denote the group algebra of the symmetric group S_n over $\mathbf{Z}$. We define the semi-direct product $\mathcal{A}_n \rtimes \mathbf{Z}S_n$ by the relation

$$X_{ij} \cdot \sigma = \sigma \cdot X_{\sigma(i)\sigma(j)}$$

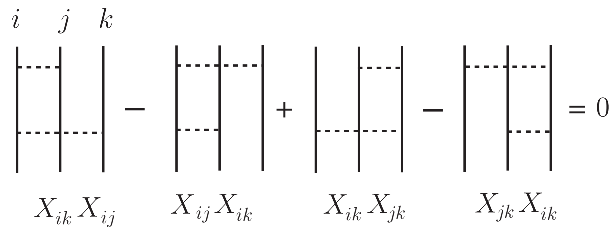

FIGURE 2. 4 term relation

for $\sigma \in S_n$. We equip $\mathcal{A}_n \rtimes \mathbf{Z}S_n$ the structure of a graded algebra so that $\deg X_{ij} = 1$ and $\deg x = 0$ for $x \in \mathbf{Z}S_n$

We define the operation of doubling the i-th vertical strand

$$\Delta_i : \mathcal{A}_n \longrightarrow \mathcal{A}_{n+1}$$

by the correspondence

$$\begin{aligned} X_{i,j} &\mapsto X_{i,j+1} + X_{i+1,j+1}, \quad i < j \\ X_{k,i} &\mapsto X_{k,i} + X_{k,i+1}, \quad k < i \\ X_{p,q} &\mapsto X_{p,q}, \quad p < q < i \\ X_{p,q} &\mapsto X_{p+1,q+1}, \quad i < p < q. \end{aligned}$$

This map is extended in a natural way to

$$\Delta_i : \mathcal{A}_n \rtimes \mathbf{Z}S_n \longrightarrow \mathcal{A}_{n+1} \rtimes \mathbf{Z}S_n.$$

The map $\varepsilon_i : \mathcal{A}_n \longrightarrow \mathcal{A}_{n-1}$, $1 \leq i \leq n$, is defined by setting $\varepsilon_i(X)$ to be represented by the chord diagram obtained by deleting the i-th vertical strand if there is no horizontal chord on the i-th vertical strand in $X \in \mathcal{A}_n$ and to be 0 otherwise.

The symbol $t_{ij} \in S_n$ stands for the permutation of i-th and j-th letters. The element $R \in (\widehat{\mathcal{A}}_2 \otimes \mathbf{Q}) \rtimes \mathbf{Z}S_2$ is defined by

$$R = t_{12} \exp\left(\frac{1}{2} X_{12}\right).$$

A Drinfel'd associator Φ is an element of $\widehat{\mathcal{A}}_3 \otimes \mathbf{C}$ satisfying the following properties.

- (strong invertibility)
$$\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1$$
- (skew symmetry)
$$\Phi^{-1} = t_{13} \cdot \Phi \cdot t_{13}$$
- (pentagon relation)
$$(\Phi \otimes id) \cdot (\Delta_2 \Phi) \cdot (id \otimes \Phi) = (\Delta_1 \Phi) \cdot (\Delta_3 \Phi) \quad \text{in} \quad \widehat{\mathcal{A}}_4 \otimes \mathbf{C}.$$
- (hexagon relation)
$$\Phi \cdot (\Delta_2 R) \cdot \Phi = (R \otimes id) \cdot \Phi \cdot (id \otimes R)$$

The original Drinfel'd associator was introduced in [6] for the purpose of describing the monodromy representation of the KZ equation. It is an element in the ring of non-commutative formal power series $\mathbf{C}[[X,Y]]$ describing a relation of the solutions $G_0(z)$ and $G_1(z)$ of the differential equation

$$G'(z) = \left(\frac{X}{z} + \frac{Y}{z-1}\right) G(z) \tag{5.4}$$

with the asymptotic behavior

$$G_0(z) \sim z^X, \quad z \longrightarrow 0$$
$$G_1(z) \sim (1-z)^Y, \quad z \longrightarrow 1.$$

We set

$$G_0(z) = G_1(z)\Phi_{KZ}(X,Y)$$

and it can be shown that $\Phi_{KZ}(X_{12}, X_{23})$ satisfies the above properties for an associator.

In [7] Drinfel'd shows that there exists an associator with coefficients in $\mathbf{Q}$. Bar-Natan [2] gave an algorithm to construct such rational Drinfel'd associator. An explicit rational associator up to degree 4 terms is of the form

$$\begin{aligned}\Phi(X,Y) = 1 &- \frac{\zeta(2)}{(2\pi i)^2}[X,Y] \\ &- \frac{\zeta(4)}{(2\pi i)^4}[X,[X,[X,Y]]] - \frac{\zeta(4)}{(2\pi i)^4}[Y,[Y,[X,Y]]] \\ &- \frac{\zeta(3,1)}{(2\pi i)^4}[X,[Y,[X,Y]]] + \frac{1}{2}\frac{\zeta(2)^2}{(2\pi i)^4}[X,Y]^2 + \cdots\end{aligned}$$

with $X = X_{12}$, $Y = X_{23}$, where $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$ and $\zeta(3,1) = \pi^4/360$.

We set

$$R_{j,j+1} = t_{j,j+1} \exp\left(\frac{1}{2} X_{j,j+1}\right).$$

We adapt a construction of the universal Vassiliev-Kontsevich invariant for tangles due to Le and Murakami [13] to the case of braids and obtain the following theorem.

Theorem 5.1. *For the generators σ_j, $1 \le j \le n-1$, of the braid group B_n we put*

$$\Theta(\sigma_j) = \Phi_j \cdot R_{j,j+1} \cdot \Phi_j^{-1}, \quad 1 \le j \le n-1.$$

Here Φ_j is defined by means of a rational Drinfel'd associator by the formulae

$$\Phi_j = \Phi\left(\sum_{i=1}^{j-1} X_{ij},\ X_{j,j+1}\right), \quad j > 1$$

and $\Phi_1 = 1$. Then, Θ defines an injective homomorphism

$$\Theta : B_n \longrightarrow (\widehat{\mathcal{A}}_n \otimes \mathbf{Q}) \rtimes \mathbf{Z}S_n.$$

Proof. The fact that Θ is a homomorphism is directly derived by computation based on pentagon and hexagon relations in the definition of a Drinfel'd associator. Now the kernel of

$$\Theta_k : \mathbf{Z}P_n \longrightarrow \bigoplus_{\ell \le k} \mathcal{A}_{n,\ell} \otimes \mathbf{Q}$$

is equal to J^{k+1}, where J is the augmentation ideal $\mathbf{Z}P_n$. It is known that the pure braid group P_n is residually torsion free nilpotent. Namely, we have

$$\bigcap_{k\geq 0} J^{k+1} = \{0\}.$$

We refer the reader to [8] and [9] for this fact. The injectivity of Θ follows immediately. This completes the proof. □

The map Θ is called a universal holonomy homomorphism of the braid group over $\mathbf{Q}$. The expressions of Θ and Φ_j in the above theorem come from partial compactifications of the configuration space M_n with normal crossing divisors.

Theorem 5.2. *We have the following isomorpshisms for finite type invariants over the field of rational numbers.*

$$V_k(P_n)_{\mathbf{Q}}/V_{k-1}(P_n)_{\mathbf{Q}} \cong \mathrm{Hom}(\mathcal{A}_{n,k}, \mathbf{Q}),$$
$$V_k(B_n)_{\mathbf{Q}}/V_{k-1}(B_n)_{\mathbf{Q}} \cong \mathrm{Hom}(\mathcal{A}_{n,k} \rtimes \mathbf{Z}S_n, \mathbf{Q}).$$

Proof. For an element X in $\mathcal{A}_{n,k}$ we contract its horizontal chords to obtain a diagram β for a singular pure braid with transverse double points. Given $v \in V_k(P_n)_{\mathbf{Q}}$ we define $w(v)(\beta)$ by

$$w(v)(\beta) = \sum_{\epsilon_j=\pm 1, 1\leq j\leq k} \epsilon_1 \cdots \epsilon_k \; v(\beta_{\epsilon_1\cdots\epsilon_k})$$

as in the formula (4.1). This defines a homomorphism $w : V_k(P_n)_{\mathbf{Q}} \longrightarrow \mathrm{Hom}(\mathcal{A}_{n,k}, \mathbf{Q})$. It follows from the definition of order k invriants that $\mathrm{Ker}\, w = V_{k-1}(P_n)_{\mathbf{Q}}$. The homomorphsim w induces a map

$$\widehat{w} : V(P_n)_{\mathbf{Q}} \longrightarrow \mathrm{Hom}(\widehat{\mathcal{A}}_n, \mathbf{Q}).$$

An element in $\mathrm{Hom}(\widehat{\mathcal{A}}_n, \mathbf{Q})$ is called a weight system for a horizontal chord diagram. Given such weight system α and $\gamma \in P_n$ we define $v(\gamma)$ to be the rational number obtained by applying α to $\Theta(\gamma)$ in the previous theorem. This construction gives an inverse of the map $\widehat{w}$ and it follows that there is an isomorphism $V(P_n)_{\mathbf{Q}} \cong \mathrm{Hom}(\widehat{\mathcal{A}}_n, \mathbf{Q})$, which induces the isomorphism $V_k(P_n)_{\mathbf{Q}}/V_{k-1}(P_n)_{\mathbf{Q}} \cong \mathrm{Hom}(\mathcal{A}_{n,k}, \mathbf{Q})$. This isomorpshim is extended to $V_k(B_n)_{\mathbf{Q}}/V_{k-1}(B_n)_{\mathbf{Q}} \cong \mathrm{Hom}(\mathcal{A}_{n,k} \rtimes \mathbf{Z}S_n, \mathbf{Q})$. □

6. Configuration spaces and the rescaling isomorphism

We recall the following theorem due to Cohen and Gitler [4] describing the structure of the homology of the based loop space of the configuration space $\Omega\,\mathrm{Conf}_n(\mathbf{R}^m)$ as an algebra.

Theorem 6.1 (Cohen-Gitler [4]). *The homology $H_*(\Omega\,\mathrm{Conf}_n(\mathbf{R}^m))$, $m \geq 3$, is isomorphic to the algebra generated by degree $m-2$ elements X_{ij}, $1 \leq i \neq j \leq n$, with $X_{ij} = (-1)^{m-2}X_{ji}$ and the relations (5.2) and (5.3), where the Lie bracket is taken in a graded sense.*

We describe some consequence of the above theorem and a relation to finite type invariants for the braid group. Let $\widehat{T}H_{*-1}(M;\mathbf{R})$ be the completed tensor algebra of $H_{*-1}(M;\mathbf{R})$. Chen's formal connection ω is by definition an element $\omega \in \mathcal{E}^*(M) \otimes \widehat{T}H_{*-1}(M;\mathbf{R})$ and a derivation ∂ on $\widehat{T}H_{*-1}(M;\mathbf{R})$ such that

$$\partial\omega + d\omega - J\omega \wedge \omega = 0.$$

We shall say that the formal connection is quadratic if the derivation ∂ is written in the form

$$\partial X_k = \sum_{i,j} c_{ij}^k X_i X_j$$

with the structure constants c_{ij}^k of the cohomology ring $H^*(M; \mathbf{R})$.

Theorem 6.2. *The configuration space* $\mathrm{Conf}_n(\mathbf{R}^m)$*,* $m \geq 2$*, has a quadratic formal connection in the sense of Chen, and is therefore formal.*

Proof. In the case $m = 2$ we may take

$$\omega = \sum_{1 \leq i < j \leq n} \omega_{ij} \otimes X_{ij}$$

and $\partial = \delta$ where X_{ij} is considered to be an elements of $H_1(M_n; \mathbf{R})$ dual to ω_{ij} and δ is the derivation introduced in Section 2. This shows that the formal connection is quadratic in this case.

In the case $m \geq 3$ the configuration space $\mathrm{Conf}_n(\mathbf{R}^m)$ is simply connected and by a theorem of Chen [3] the homology of the loop space $\Omega\,\mathrm{Conf}_n(\mathbf{R}^m)$ is isomorphic to the homology of the complex $(TH_{*-1}(M; \mathbf{R}), \partial)$. Then by the Cohen-Gitler theorem we can conclude that the formal homology connection is quadratic. It is known that the formality in the sense of Sullivan is equivalent to the fact that Chen's formal connection is quadratic. Thus we obtain the statement of the theorem. □

We denote by $\mathcal{A}_n[\ell]$ the graded algebra whose degree p part is given by

$$\mathcal{A}_n[\ell]_p = \begin{cases} \mathcal{A}_{n,k}, & p = 2\ell k \\ 0, & \text{otherwise} \end{cases}$$

and call it the ℓ rescaling of the algebra of horizontal chord diagrams $\mathcal{A}_n$. The relation between the homology of the loop space of the configuration space and the algebra of chord diagram can be formulated in the following way.

Theorem 6.3. *There is a rescaling isomorphism of Hopf algebras*

$$H_*(\Omega\,\mathrm{Conf}_n(\mathbf{C}^{\ell+1}); \mathbf{Z}) \cong \mathcal{A}_n[\ell].$$

This type of rescaling theorem was discussed systematically in [15] in relation with the rescaling in the level of cohomology rings. A similar rescaling isomorphism for the orbit configuration space for the action on the upper half plane of a discrete subgroup of $PSL(2, \mathbf{R})$ in [5].

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