SYMMETRIES OF HYPERBOLIC SPATIAL GRAPHS IN 3-MANIFOLDS

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Dedicated to Professor Yukio Matsumoto on his 70th birthday

ABSTRACT. We consider symmetries of spatial graphs in compact 3-manifolds described by smooth finite group actions. We first present a method for constructing an infinite family of hyperbolic spatial graphs with given symmetry. Next, we apply this method to the study of links in 3-manifolds which can be regarded as systems of rotation axes in closed hyperbolic 3-manifolds obtained by Dehn surgeries.

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1. INTRODUCTION

Let $\mathscr{G}_e$ be the set of finite graphs with no vertex of degree less than two each of whose components has Euler characteristic e. The set of spatial embeddings of graphs in $\mathscr{G}_e$ into M is denoted by $\mathscr{G}_e(M)$. We refer to a finite group generated by self-diffeomorphisms of M as a smooth finite group action on M.

Myers proved in [8] that every closed connected 3-manifold contains infinitely many hyperbolic links up to ambient isotopy. Myers' result is generalized in [4] to the case of hyperbolic spatial graphs in closed orientable 3-manifolds. It is a natural question to ask whether this result extends to the case of symmetric hyperbolic spatial graphs. In this talk, we first present a method for constructing hyperbolic spatial graphs setwise invariant under a finite group action to prove the following theorem (see [4]).

Theorem 1.1. *Let G be a smooth finite group action on a compact connected 3-manifold M. For any integer $e \leq 0$, there are infinitely many ambient isotopy classes of setwise G-invariant hyperbolic spatial graphs in $\mathscr{G}_e(M)$.*

As an application, we show the following theorem (see [5]).

Theorem 1.2. *Let M be a closed orientable 3-manifold, $\{p_1, p_2, \ldots\}$ a sequence of integers greater than one. For any link L in M, there exists an infinite sequence $\{\mathscr{L}_1, \mathscr{L}_2, \ldots\}$ of framed links in $M - L$ such that the Dehn surgery of M along each $\mathscr{L}_i$ yields a complete hyperbolic 3-manifold M_i of finite volume which admits an orientation-preserving smooth finite cyclic group action of order p_i with fixed point set L, and that any two of the M_i's are not diffeomorphic.*

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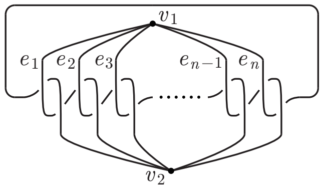

FIGURE 1. Hyperbolic spatial graphs in S^3

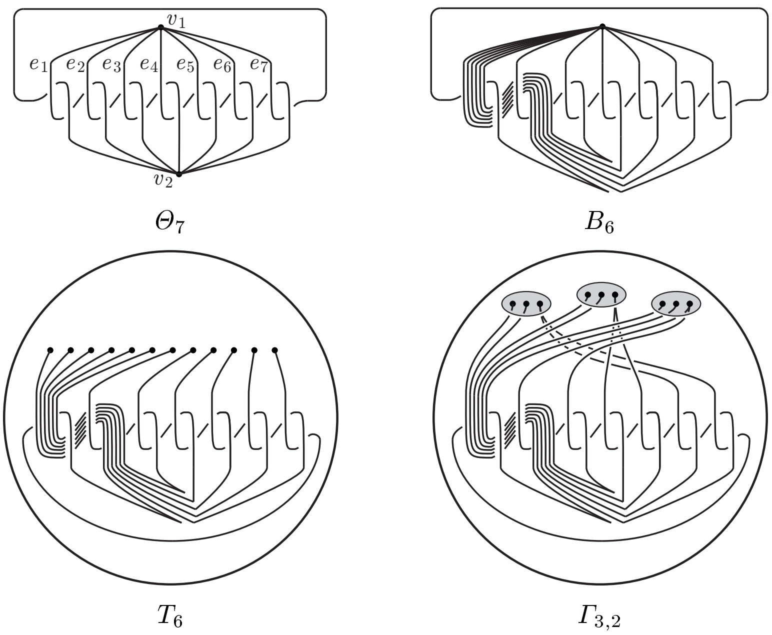

FIGURE 2. Construction of $\Gamma_{3,2}$

2. Hyperbolic spatial graphs with given symmetries

The basic idea of this talk is to use the spatial graph Θ_n in S^3 illustrated in Figure 1, where n is an integer greater than two. Paoluzzi and Zimmermann proved in [9] that the exterior $E(\Theta_n)$ of Θ_n is a compact hyperbolic 3-manifold with totally geodesic boundary. Denote by B_{n-1} the bouquet of $n-1$ circles obtained from Θ_n by sliding each of the edges $e_3, \ldots, e_n$ along e_1 so that one of the endpoints goes from v_2 to v_1. For example, Θ_7 and B_6 are illustrated in Figure 2.

Let n be a positive integer, and Δ_k, where $k \geq 4$, a closed 3-ball associated with a system $\mathscr{F}(\Delta_k)$ of k disjoint disks marked on the boundary. Note that $T_{kn} = (E(v_1), B_{kn} \cap E(v_1))$ is a kn-string tangle. Take an orientation-preserving diffeomorphism $f\colon E(v_1) \to \Delta_k$. By sliding the arcs, one can assume that each arc has its endpoints on the same disk in $\mathscr{F}(\Delta_k)$, and that each disk meets n arcs. By sliding further, one can assume that on each disk n endpoints, one for each arc, are collapsed to one single point creating a system $\Gamma_{k,n}$ of k disjoint copies of a star with n edges. For example, T_6 and $\Gamma_{3,2}$ are illustrated in Figure 2.

Let (Δ, Γ) and (Δ', Γ') be copies of $(\Delta_k, \Gamma_{k,n})$ and $(\Delta_\ell, \Gamma_{\ell,n})$ for some k and ℓ. For a disk F in $\mathscr{F}(\Delta)$ and a disk F' in $\mathscr{F}(\Delta')$, suppose that an orientation-reversing diffeomorphism $\varphi\colon F \to F'$ takes $\Gamma \cap F$ to $\Gamma' \cap F'$. Glue Δ and Δ' along F and F' by φ. Denote by $\Gamma \cup_\varphi \Gamma'$ the result of gluing Γ and Γ' by this operation. Then $\Gamma \cup_\varphi \Gamma'$ has a component of Euler characteristic $1-n$. The following proposition implies that $E(\Gamma \cup_\varphi \Gamma')$ is a complete hyperbolic 3-manifold of finite volume.

Proposition 2.1 (Myers [8]). *Let M be a compact connected 3-manifold with non-empty boundary, and F a compact connected proper surface of negative Euler characteristic in M bounded by essential loops on ∂M. Then M is a compact, irreducible, atoroidal, anannular 3-manifold with incompressible boundary if so is each piece obtained by splitting M along F.*

Suppose that a 3-manifold M is equipped with a triangulation K with 3-simplices $\sigma_1, \dots, \sigma_n$. Since a regular neighborhood $\mathscr{N}$ of the 1-skeleton is a handlebody, K is modified to a cell decomposition of M, called a *polyhedral decomposition* of M induced from K, whose 3-cells are the 0-handles and 1-handles of $\mathscr{N}$, and $\sigma_i \cap E(M_1)$ for $1 \leq i \leq n$. In this decomposition, at most three 3-cells meet along each 1-cell, and at most four 3-cells meet at each 0-cell. Therefore, the union of any subcollection of 3-cells is a 3-dimensional submanifold of M in contrast to the case of a triangulation of M possibly containing a pair of 3-simplices which intersect in a 0- or 1-simplex.

Let P be a 3-dimensional polyhedral cell complex with 3-cells $\rho_1, \dots, \rho_n$. The number of the faces of each ρ_i is denoted by $f(i)$. Let $e \leq 0$ be an integer, and $\varphi_i\colon \Delta_{f(i)} \to \rho_i$ an orientation-preserving diffeomorphism such that it takes the precisely one disk in $\mathscr{F}(\Delta_{f(i)})$ into each face of ρ_i, and that $\varphi_i(\Gamma_{f(i),1-e}) \cap F_{i,j} = \varphi_j(\Gamma_{f(j),1-e}) \cap F_{i,j}$ holds for each pair of ρ_i and ρ_j with a common face $F_{i,j}$. We denote by Γ^P_{1-e} the union $\bigcup_{i=1}^n \varphi_i(\Gamma_{f(i),1-e})$.

Let M be a compact 3-manifold. The *closed Haken number* of M, denoted by $\overline{h}(M)$, is the maximal number of disjoint, incompressible, pairwise non-parallel, closed surfaces that can be embedded in M (see [6]).

Lemma 2.2. *Let γ_1 and γ_2 be hyperbolic spatial graphs in distinct closed 3-manifolds X_1 and X_2 with k-valent vertices v_1 and v_2, where $k \geq 3$, respectively. Let γ be the vertex connected sum of γ_1 and γ_2 at v_1 and v_2. Then $\overline{h}(E(\gamma)) > \overline{h}(E(\gamma_1)) + \overline{h}(E(\gamma_2))$.*

Let $\Lambda_{k,i}$ be the hyperbolic spatial graph obtained from i copies of Θ_{2k} by repeating the vertex connected sum operation $i-1$ times. Denote by $\Gamma^P_{1-e,i}$ the result of the vertex connected sum operation with $\Lambda_{k,i}$ at each vertex of Γ^P_{1-e}. Lemma 2.2 implies that the sequence $\{\overline{h}(\Gamma^P_{1-e,1}), \overline{h}(\Gamma^P_{1-e,2}), \overline{h}(\Gamma^P_{1-e,3}), \dots\}$ is strictly increasing.

Lemma 2.3. *Let F be a polygon, and v an interior point or a vertex of F. Let M be a prism with base F. Suppose that M is equipped with the cell decomposition P induced from the stellar subdivision of F at v, in which the 3-cells are triangular prisms. Then $E(\Gamma^P_{1-e,i})$ is a compact, irreducible, atoroidal, anannular 3-manifold with incompressible boundary.*

Lemma 2.4. *Let M be a convex polyhedron, and v an interior point or a vertex of M. Suppose that M is equipped with a stellar subdivision P at v. Then $E(\Gamma^P_{1-e,i})$ is a compact, irreducible, atoroidal, anannular 3-manifold with incompressible boundary.*

Outline of proof of Theorem 1.1. A triangulation K of the quotient orbifold M/G is constructed using a G-invariant triangulation of M (see [2, Lemma 3.1]) so that the singular locus $\mathscr{S}$ is a subcomplex of K. Denote by P the polyhedral decomposition of M/G induced from K. Modify P by subdividing the 3-cells intersecting $\mathscr{S}$ as stated in Lemmas 2.3 and 2.4. Deform $\Gamma^P_{1-e,i}$ so that each component attaching $\partial M/G$ is modified to a bouquet of $1-e$ circles by sliding its $2-e$ vertices to one point along the 2-cell on which they lies. Then we obtain a G-invariant spatial graph $\overline{\Gamma}^P_{1-e,i}$ in $\mathscr{G}_e(M)$. The hyperbolicity of $\overline{\Gamma}^P_{1-e,i}$ follows by applying Proposition 2.1 to each step of the construction of $E(\overline{\Gamma}^P_{1-e,i})$ by gluing the pieces given by the polyhedral decomposition of M. $\square$

3. Hyperbolic rotations about links in 3-manifolds

From now on, Σ_n for an integer $n \geq 0$ will denote a closed orientable 3-manifold which is a 3-sphere if $n = 0$, and otherwise the connected sum of n copies of $S^2 \times S^1$. Moreover, we will refer to a finite cyclic group action G on Σ_n as a *standard cyclic action* of order $p \geq 2$ if it is induced from a finite cyclic group action on S^3 generated by a rotation about a trivial knot K of period p via the 0-surgeries along n disjoint setwise invariant meridians of K.

Let γ be a spatial graph in a closed 3-manifold X. Suppose that a sphere S in X intersects γ in k points avoiding the vertices and splits X into two manifolds X_1 and X_2. By collapsing each sphere ∂X_i to a point, $\gamma \cap X_i$ is deformed to a spatial graph γ_i in a closed 3-manifold $\overline{X}_i$. We say that γ is a *vertex connected sum* of γ_1 and γ_2 at v_1 and v_2. Note that the exterior $E(\gamma)$ is obtained by gluing $E(\gamma_1)$ and $E(\gamma_2)$ along compact planar surfaces of Euler characteristic $2-k$ on the boundaries.

Proposition 3.1. *Let G be a standard cyclic action on Σ_n, where $n \geq 0$, of order $p \geq 2$. For any positive integer μ, there exists a setwise G-invariant hyperbolic link L_μ in Σ_n disjoint from the fixed point set of G such that $\overline{h}(E(L_\mu)) > \mu$ holds.*

Outline of proof. Suppose that the polyhedral decomposition P of Σ_n/G and the spatial graph $\Gamma^P_{k,i}$ in Σ_n/G are as stated in the proof of Theorem 1.1. Modify $\Gamma^P_{k,i}$ by performing the vertex connected sum operation with B_k at each vertex. Then we obtain a setwise G-invariant link $\overline{\Gamma}^P_{k,i}$ in Σ_n disjoint from the fixed point set of G. The hyperbolicity of $\overline{\Gamma}^P_{k,i}$ follows inductively from Proposition 2.1. Since the sequence $\{\overline{h}(\overline{\Gamma}^P_{k,1}), \overline{h}(\overline{\Gamma}^P_{k,2}), \ldots\}$ is strictly increasing, $\overline{h}(\overline{\Gamma}^P_{k,i}) > \mu$ holds for some i. $\square$

Lemma 3.2. *Let M be a closed orientable 3-manifold. For any n-component link L in M and for any integer $p \geq 2$, there exists a framed link $\mathscr{L}$ in $M - L$ such that Dehn surgery of M along $\mathscr{L}$ yields Σ_{n-1} which admits a standard cyclic action of order p with fixed point set L.*

Outline of proof. By a Dehn surgery on a framed link $\mathscr{L}'$ in $M - L$, we see L as a link in S^3 (see [7, 11]). Trivialize L by the ± 1-Dehn surgeries corresponding to crossing changes of its diagram, as illustrated in Figure 3. The components $L_0, \ldots, L_{n-1}$ of L respectively bound disjoint disks $\Delta_0, \ldots, \Delta_{n-1}$ in S^3. Take disjoint disks $D_1, \ldots, D_{n-1}$ in S^3 such that each D_i is a band sum of meridian disks of $N(L_0)$ and $N(L_i)$ along a band in $E(L)$. The open solid torus $S^3 - L_0$ admits an S^1-bundle structure with fibers $\partial D_1, \ldots, \partial D_{n-1}$ such that each ∂D_i has a fibered regular neighborhood V_i containing Δ_i in its interior. Let G be the finite cyclic group action on S^3 of order p generated by a rotation about L_0 setwise preserving each fiber of $S^3 - L_0$. The 0-surgeries along $\partial D_1, \cdots, \partial D_{n-1}$, which yields Σ_{n-1}, takes each V_i

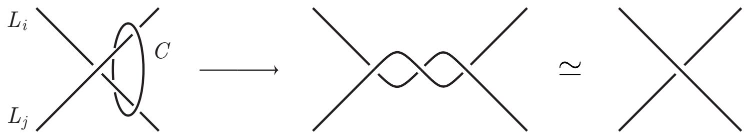

FIGURE 3. Dehn surgery realizing crossing change

to a solid torus with the core L_i. Then the action of G on $S^3 - \mathrm{int} \bigcup_{i=1}^{n-1} V_i$ extends to a required action on Σ_{n-1}. □

Let X be a compact orientable hyperbolic 3-manifold with a torus boundary component T. For a slope α on T, denote by $X(\alpha)$ the 3-manifold obtained by Dehn filling X along α. Thurston's hyperbolic Dehn surgery theorem [10] and the result of Bachman, Derby, Talbot and Sedgwick [1] lead to the following proposition.

Proposition 3.3. *Let X be a compact orientable hyperbolic 3-manifold with a torus boundary component T. Then there are infinitely many slopes δ on T such that $X(\delta)$ is a hyperbolic 3-manifold with the closed Haken number $\overline{h}(X(\gamma)) = \overline{h}(X)$.*

Outline of proof of Theorem 1.2. Assume by Lemma 3.2 that G is a standard cyclic action on Σ_{n-1} with fixed point set L. Let $\{\mu_1, \mu_2, \ldots\}$ be a strictly increasing sequence of positive integers defined by $\mu_1 = 1$ and $\mu_{i+1} = \overline{h}(E(L_{\mu_i}))$, where L_{μ_i} is a link given by Proposition 3.1. Proposition 3.3 implies that some equivariant Dehn surgeries along L_{μ_i} yields a closed hyperbolic 3-manifold $\overline{\Sigma}_i$ with $\overline{h}(\overline{\Sigma}_i) = \mu_{i+1}$ in which L is the fixed point set of the action induced from G. □

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