Full strong exceptional collections of line bundles on global quotients of the projective plane

Ryo Ohkawa and Kazushi Ueda

Abstract

We show that the global quotient stack of the projective plane by the natural action of a finite subgroup G of $SL_3(\mathbb{C})$ has a full strong exceptional of line bundles if and only if G is abelian.

1 Introduction

Let $\mathcal{D}$ be a triangulated category over $\mathbb{C}$. An object E of $\mathcal{D}$ is *exceptional* if $\mathrm{Hom}(E, E) = \mathbb{C} \cdot \mathrm{id}_E$ and $\mathrm{Hom}(E, E[j]) = 0$ for any $j \neq 0$. A sequence $(E_1, \dots, E_n)$ of exceptional objects of $\mathcal{D}$ is an *exceptional collection* if the semiorthogonality $\mathrm{Hom}(E_i, E_j[k]) = 0$ for any $i > j$ and any $k \in \mathbb{Z}$ holds. An exceptional collection is *strong* if $\mathrm{Hom}(E_i, E_j[k]) = 0$ for any $i, j = 1, \dots, n$ and any $k \neq 0$. An exceptional collection is *full* if the smallest full triangulated subcategory of $\mathcal{D}$ containing E_i for all $i = 1, \dots, n$ coincides with $\mathcal{D}$.

Assume that $\mathcal{D}$ has a dg enhancement in the sense of Bondal and Kapranov [BK90]. This assumption is satisfied by derived categories of coherent sheaves on algebraic stacks. Under this assumption, Morita theory for derived categories [Bon89, Ric89] shows that if $\mathcal{D}$ has a full strong exceptional collection, then one has an equivalence of triangulated categories

$$\mathcal{D} \cong D^b \operatorname{mod} A$$

with the bounded derived category of finitely-generated modules over the total morphism algebra

$$A = \bigoplus_{i,j=1}^{n} \mathrm{Hom}(E_i, E_j).$$

The first example of a full strong exceptional collection is the sequence

$$(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n))$$

of line bundles on the projective space $\mathbb{P}^n$ discovered by Beilinson [Beĭ78]. Its *dual* collection

$$(\Omega^n_{\mathbb{P}^n}(n)[n], \Omega^{n-1}_{\mathbb{P}^n}(n-1)[n-1], \dots, \mathcal{O}_{\mathbb{P}^n})$$

gives another example of a full strong exceptional collection, consisting of shifts of vector bundles.

A semiorthogonal decomposition of a triangulated category is a decomposition of a triangulated category into simpler pieces, which is conjecturally related to the minimal model program in algebraic geometry when the triangulated category is the derived category of coherent sheaves on an algebraic variety [BO]. An exceptional collection corresponds to a semiorthogonal decomposition whose semiorthogonal summands are the derived category of vector spaces, and its existence imposes a strong condition on the structure of a triangulated category. Exceptional collections also have applications to gauge theory and string theory (cf. e.g. [HK09] and references therein). It is in general difficult to find an exceptional object in the derived category of coherent sheaves, let alone a full exceptional collection. It is conjectured that any homogeneous space G/P for a semisimple algebraic group G and its parabolic subgroup P has a full exceptional collection of vector bundles (cf. e.g. [KP, Section 1.1] for an overview).

Among full exceptional collections, full strong exceptional collections of line bundles are the 'nicest' ones. On the other hand, the combination of the strongness condition and the line bundle condition makes it hard to find examples of full strong exceptional collections of line bundles. King [Kin97, Conjecture 9.3] conjectured that a smooth complete toric variety has a full strong exceptional collection consisting of line bundles. This conjecture is shown to be false by Hille and Perling [HP06], who subsequently gave a necessary and sufficient condition for a smooth complete toric surface to have such a collection [HP]. Borisov and Hua [BH09] suggested to extend the conjecture to stacks, with an additional assumption that the toric stack be weak Fano, and proved it for toric Fano stacks of Picard number or dimension at most two. Efimov [Efi10] disproved this modified conjecture by showing the existence of a toric Fano manifold of Picard number three admitting no full exceptional collection of line bundles. On the other hand, Kawamata [Kaw06] shows that a smooth projective toric stack has a full exceptional collection consisting of sheaves.

In this paper, we study the problem of the existence of a full exceptional collection of line bundles on the quotient stack of the complex projective plane $\mathbb{P}^2$ by a finite subgroup of $SL_3(\mathbb{C})$. Our main result is the following:

Theorem 1.1. *The quotient stack $[\mathbb{P}^2/G]$ of the projective plane $\mathbb{P}^2$ by a finite subgroup G of $SL_3(\mathbb{C})$ has a full strong exceptional collection of line bundles if and only if G is an abelian group.*

This is in sharp contrast with the situation in dimension one:

Theorem 1.2 ([GL87, Proposition 4.1])**.** *Let X a smooth rational stack of dimension one. Then X has a full strong exceptional collection of line bundles.*

The quotient stack of $\mathbb{P}^1$ by any finite subgroup of $SL_2(\mathbb{C})$ is covered by Theorem 1.2. A natural generalization of Theorem 1.2 is the following:

Theorem 1.3 ([IU12])**.** *Let X be a smooth rational stack obtained from $\mathbb{P}^n$ by iterated root constructions along $n+1$ hyperplanes in general position. Then X has a full strong exceptional collection of line bundles.*

Here, the *root construction* is the operation introduced in [AGV08, Cad07] which produces a generic stabilizer along a divisor. A further generalization to arbitrary number of hyperplanes is announced by Herschend, Iyama, Minamoto and Oppermann. On the other hand, one has no full exceptional collection of line bundles when X is obtained from $\mathbb{P}^2$ by the root construction along a smooth conic:

Theorem 1.4. *Let X be the stack obtained from $\mathbb{P}^2$ by the root construction along a smooth conic. Then X does not admit a full exceptional collection of line bundles.*

A fine moduli interpretation of any smooth projective toric varieties in terms of quiver representations, which was one of the original motivations of King, is obtained by Craw and Smith [CS08]. This has been generalized to Mori dream spaces, toric stacks and rational orbifold stacks in [CW13], [Abd12] and [AU]. The quotient stack $[\mathbb{P}^2/G]$ gives an example of an MD-stack, which is introduced in [HM] as a generalization of Mori dream spaces. It is an interesting problem to find a fine moduli interpretation of the stack X appearing in Theorem 1.4 in terms of quiver representations.

Acknowledgement: This work has been initiated while the authors are visiting the Max Planck Institute for Mathematics in Bonn, whose hospitality and nice working environment is gratefully acknowledged. R. O. is supported by JSPS Grant-in-Aid for Young Scientists No. 25800016. K. U. is supported by JSPS Grant-in-Aid for Young Scientists No. 24740043.

2 Grassmannians

In this section, we give a proof of the following theorem, in order to illustrate the method of the proof of Theorem 1.1.

Theorem 2.1. *The Grassmannian $\operatorname{Gr}(r,n)$ of r-planes in n-space has a full exceptional collection of line bundles if and only if $r = 1$ or $r = n-1$.*

Proof. The 'if' part is proved by Beilinson [Beĭ78]. To show the 'only if' part, assume that $1 < r < n-1$. The universal subbundle $\mathcal{S}$ and the universal quotient bundle $\mathcal{Q}$ fits into the exact sequence

$$0 \to \mathcal{S} \to \mathcal{O}_{\operatorname{Gr}(r,n)}^{\oplus n} \to \mathcal{Q} \to 0,$$

which shows

$$\det \mathcal{S} \otimes \det \mathcal{Q} \cong \mathcal{O}_{\operatorname{Gr}(r,n)}.$$

The Picard group $\operatorname{Pic}\operatorname{Gr}(r,n)$ is the free abelian group generated by $\mathcal{O}_{\operatorname{Gr}(r,n)}(1) := \det \mathcal{Q} \cong \det \mathcal{S}^\vee$, which is an ample line bundle defining the Plücker embedding

$$\operatorname{Gr}(r,n) \hookrightarrow \mathbb{P}(\Lambda^r \boldsymbol{k}^n).$$

The tangent bundle is given by

$$\mathcal{T}_{\operatorname{Gr}(r,n)} \cong \mathcal{H}om_{\mathcal{O}_{\operatorname{Gr}(r,n)}}(\mathcal{S}, \mathcal{Q}),$$

and the canonical bundle is given by

$$\omega_{\operatorname{Gr}(r,n)} \cong \det \mathcal{T}^\vee \cong (\det \mathcal{S})^{\otimes(n-r)} \otimes (\det \mathcal{Q})^{\otimes(-r)} \cong \mathcal{O}_{\operatorname{Gr}(r,n)}(-n).$$

Set $N = \dim \operatorname{Gr}(r,n) = r(n-r)$. It follows that

$$\operatorname{Ext}^N(\mathcal{O}_{\operatorname{Gr}(r,n)}(k), \mathcal{O}_{\operatorname{Gr}(r,n)}) \cong H^0(\mathcal{O}_{\operatorname{Gr}(r,n)}(k-n)) = 0 \text{ if and only if } k < n,$$

so that the exceptional collection of line bundles with the maximal length is

$$\mathcal{O}_{\mathrm{Gr}(r,n)}, \mathcal{O}_{\mathrm{Gr}(r,n)}(1), \dots, \mathcal{O}_{\mathrm{Gr}(r,n)}(n-1).$$

This collection cannot be full since $\operatorname{rank} K(\mathrm{Gr}(r,n)) = \binom{n}{r} > n$. $\square$

On the other hand, $\mathrm{Gr}(r,n)$ has a full strong exceptional collection of vector bundles [Kap88].

3 The root stack along a smooth conic

We prove Theorem 1.4 in this section. Let $[\mathbb{A}^1/\mathbb{G}_m]$ be the quotient stack of the affine line $\mathbb{A}^1$ by the natural action of the multiplicative group $\mathbb{G}_m$. As a category fibered in groupoid, an object of $[\mathbb{A}^1/\mathbb{G}_m]$ over a scheme S is a pair $(\mathcal{L}, s)$ of a line bundle $\mathcal{L}$ on S and a section s of $\mathcal{L}$. For a positive integer r, let $\theta_r\colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism sending an object $(\mathcal{L}, s)$ to its tensor power $(\mathcal{L}^{\otimes r}, s^{\otimes r})$. Let X be an algebraic stack. Giving a morphism $\varphi\colon X \to [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to giving a pair $(\mathcal{L}, s)$ of a line bundle on X and a section. The root stack $\mathbb{X} = X_{\mathcal{L},s,r}$ is defined in [Cad07, AGV08] as the fiber product

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\pi_2} & [\mathbb{A}^1/\mathbb{G}_m] \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \theta_r} \\ X & \xrightarrow{\varphi} & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

Note that $\operatorname{Pic}[\mathbb{A}^1/\mathbb{G}_m] \cong \operatorname{Pic}^{\mathbb{G}_m} \mathbb{A}^1 \cong \operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ since any line bundle on $\mathbb{A}^1$ is trivial, and we write its generator as $\mathcal{O}_{[\mathbb{A}^1/\mathbb{G}_m]}(1)$. The pull-back morphisms $\pi_1^*\colon \operatorname{Pic} X \to \operatorname{Pic} \mathbb{X}$ and $\pi_2^*\colon \operatorname{Pic}[\mathbb{A}^1/\mathbb{G}_m] \to \operatorname{Pic} \mathbb{X}$ are injective, and $\operatorname{Pic} \mathbb{X}$ is generated by their images with one relation

$$\pi_1^* \mathcal{L} \cong \pi_2^* \mathcal{O}_{[\mathbb{A}^1/\mathbb{G}_m]}(r).$$

In this sense, the root construction is the operation of adding an r-th root $\pi_2^* \mathcal{O}_{[\mathbb{A}^1/\mathbb{G}_m]}(1)$ of $\pi_1^* \mathcal{L}$.

Now consider the case when X is the projective plane, $\mathcal{L} = \mathcal{O}_X(C)$ is the line bundle associated with a smooth conic C, and s is the canonical section of $\mathcal{O}_X(C)$. Any element of $\operatorname{Pic} \mathbb{X}$ can uniquely be written as $\mathcal{O}_{\mathbb{X}}\left(aH + b\sqrt[r]{C}\right)$ with $a \in \mathbb{Z}$ and $b \in \{0, 1, \dots, r-1\}$, where $\mathcal{O}_{\mathbb{X}}(H) = \pi_1^* \mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{O}_{\mathbb{X}}\left(\sqrt[r]{C}\right) = \pi_2^* \mathcal{O}_{[\mathbb{A}^1/\mathbb{G}_m]}(1)$. Let $L = (L_1, \dots, L_N)$ be an exceptional collection of line bundles. The canonical divisor is given by $K_{\mathbb{X}} = -H - \sqrt[r]{C}$, so that

$$H^2\left(\mathcal{O}_{\mathbb{X}}\left(-aH - b\sqrt[r]{C}\right)\right) = H^0\left(\mathcal{O}_{\mathbb{X}}\left((a-1)H + (b-1)\sqrt[r]{C}\right)\right).$$

This shows that $H^2\left(\mathcal{O}_{\mathbb{X}}\left(-aH - b\sqrt[r]{C}\right)\right) \neq 0$ if and only if $a \geq 1$ and $(a,b) \neq (1,0), (2,0)$. When this is the case, then one also has $H^0\left(\mathcal{O}_{\mathbb{X}}\left(aH + b\sqrt[r]{C}\right)\right) \neq 0$. Similarly, if $a \leq -3$,

$\cancel{(-3,0)}$	$\cancel{(-3,1)}$	$\cdots$	$\cancel{(-3,r-1)}$
$(-2,0)$	$(-2,1)$	$\cdots$	$(-2,r-1)$
$(-1,0)$	$(-1,1)$	$\cdots$	$(-1,r-1)$
$(0,0)$	$(0,1)$	$\cdots$	$(0,r-1)$
$(1,0)$	$\cancel{(1,1)}$	$\cdots$	$\cancel{(1,r-1)}$
$(2,0)$	$\cancel{(2,1)}$	$\cdots$	$\cancel{(2,r-1)}$
$\cancel{(3,0)}$	$\cancel{(3,1)}$	$\cdots$	$\cancel{(3,r-1)}$

Table 3.1: (a,b) such that $\mathcal{O}_{\mathbb{X}}\left(aH+b\sqrt[r]{C}\right)$ can be in L

then we have $H^2\left(\mathcal{O}_{\mathbb{X}}\left(aH+b\sqrt[r]{C}\right)\right)\neq 0$ and $H^0\left(\mathcal{O}_{\mathbb{X}}\left(-aH-b\sqrt[r]{C}\right)\right)\neq 0$. It follows that if $\mathcal{O}_{\mathbb{X}}\left(aH+b\sqrt[r]{C}\right)\in L$, then $\mathcal{O}_{\mathbb{X}}\left((a\pm a')H+(b\pm b')\sqrt[r]{C}\right)\notin L$ for any $a'\geq 1$, $b'=0,\dots,r-1$ and $(a',b')\neq(1,0),(2,0)$, or $a'\leq -3$. By tensoring a line bundle if necessary, one can assume that $\mathcal{O}_{\mathbb{X}}\in L$. Then the above condition shows that $(a,b)\in\mathbb{Z}\times\{0,\dots,r-1\}$ such that $\mathcal{O}_{\mathbb{X}}\left(aH+b\sqrt[r]{C}\right)$ can be in L are given by the pairs not crossed out in Table 3.1. Moreover, by setting $(a',b')=(1,1)$, one sees that (a,b) and $(a\pm 1,b\pm 1)$ can not simultaneously be in L. This immediately implies that the length of an exceptional collection of line bundles cannot exceed $r+2$. On the other hand, one has a fully faithful functor $\Phi\colon D^b\operatorname{coh}C\to D^b\operatorname{coh}\mathbb{X}$ and a semiorthogonal decomposition

$$D^b\operatorname{coh}\mathbb{X}=\left\langle\Phi(D^b\operatorname{coh}C)\otimes\mathcal{O}_{\mathbb{X}}\left((r-1)\sqrt[r]{C}\right),\cdots,\Phi(D^b\operatorname{coh}C)\otimes\mathcal{O}_{\mathbb{X}}\left(\sqrt[r]{C}\right),\pi_1^*(D^b\operatorname{coh}X)\right\rangle$$

by [IU, Theorem 1.5], so that the rank of the Grothendieck group of $\mathbb{X}$ is $2r+1$. This is strictly larger than $r+2$ since $r>1$, and Theorem 1.4 is proved.

4 Exceptional collection of line bundles on $[\mathbb{P}^2/G]$

Finite subgroups of $SL_3(\mathbb{C})$ are classified in [Bli17] and [MBD61]. The following list is taken from [YY93]:

(A) Diagonal abelian groups.

(B) A subgroup isomorphic to a finite subgroup G of $GL_2(\mathbb{C})$;

$$\left\{\begin{pmatrix}\alpha & 0 & 0\\ 0 & a & b\\ 0 & c & d\end{pmatrix}\,\middle|\, g=\begin{pmatrix}a & b\\ c & c\end{pmatrix}\in G\subset GL_2(\mathbb{C}),\ \alpha=(\det g)^{-1}\right\}.$$

(C) The group generated by (A) and $T=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$.

(D) A group generated by (C) and $Q = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}$ with $abc = -1$.

(E) The group of order 108 generated by $S = \mathrm{diag}(1, \omega, \omega^2)$, T, and $V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$ where $\omega = \exp(2\pi\sqrt{-1}/3)$.

(F) The group of order 216 generated by (E) and $\frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}$.

(G) The Hessian group of order 648 generated by (E) and $U = \mathrm{diag}(\varepsilon, \varepsilon, \varepsilon\omega)$ where $\varepsilon^3 = \omega^2$.

(H) The alternating group A_5 of order 60 generated by T, $\mathrm{diag}(1, -1, -1)$, and $\dfrac{1}{2} \begin{pmatrix} -1 & \mu_- & \mu_+ \\ \mu_- & \mu_+ & -1 \\ \mu_+ & -1 & \mu_- \end{pmatrix}$ where $\mu_\pm = \frac{1}{2}(-1 \pm \sqrt{5})$.

(I) The simple group of order 168 generated by T, $\mathrm{diag}(\beta, \beta^2, \beta^4)$, and $U = \frac{1}{\sqrt{-7}} \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$ where $\beta = \exp(2\pi\sqrt{-1}/7)$, $a = \beta^4 - \beta^3$, $b = \beta^2 - \beta^5$, and $c = \beta - \beta^6$.

(J) The group of order 180 generated by (H) and $W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}$.

(K) The group of order 504 generated by (I) and W.

(L) The simple group of order 1080 generated by (H) and $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & -\omega^2 & 0 \end{pmatrix}$.

We prove Theorem 1.1 in each cases. First note the following:

Lemma 4.1. *Let X be a stack containing a point with a non-abelian stabilizer. Then X does not have a full exceptional collection of line bundles.*

Proof. Let $\iota : B\Gamma \to X$ be a closed embedding of the quotient stack $B\Gamma = [\operatorname{Spec} \mathbb{C}/\Gamma]$ of a point by a non-abelian group Γ. Assume for a contradiction that X has a full exceptional collection $(L_1, \dots, L_n)$ of line bundles. Then $(\iota^* L_1, \dots, \iota^* L_n)$ generates $D^b \operatorname{coh} B\Gamma = D^b \operatorname{Rep}(\Gamma)$ as a triangulated category. This is impossible since $D^b \operatorname{Rep}(\Gamma)$ decomposes as the direct sum $\bigoplus_{\rho \in \operatorname{Irrep}(\Gamma)} D^b \operatorname{Rep}(\Gamma)_\rho$ of categories $D^b \operatorname{Rep}(\Gamma)_\rho$ equivalent to the derived categories of vector spaces, and the restriction $\iota^* L_i$ belongs to the subcategory $\bigoplus_{\rho \in \operatorname{Char}(\Gamma)} D^b \operatorname{Rep}(\Gamma)_\rho$ for any i. $\square$

The 'if' part of Theorem 1.1 is straightforward:

Theorem 4.2. *Let G be a finite abelian subgroup of $GL_{n+1}(\mathbb{C})$ acting naturally on $\mathbb{P}^n$. Then the quotient stack $[\mathbb{P}^n/G]$ has a full strong exceptional collection of line bundles*

Proof. The full strong exceptional collection

$$(\mathcal{O} \otimes \rho, \mathcal{O}(1) \otimes \rho, \dots, \mathcal{O}(n) \otimes \rho)_{\rho \in \mathrm{Irrep}(G)}$$

of vector bundles on $[\mathbb{P}^n/G]$ consists of line bundles if G is abelian. □

The 'only if' part is proved by a case-by-case analysis. Let G be a finite subgroup of $SL_3(\mathbb{C})$ and $X = [\mathbb{P}^2/G]$ be the quotient stack. Let further $S = \mathbb{C}[x, y, z]$ be the homogeneous coordinate ring of $\mathbb{P}^2$, and S^G be the invariant subring. By [HE71, Proposition 3 and page 1036], there exist algebraically-independent homogeneous elements $\theta_1, \theta_2, \theta_3 \in S^G$ such that S^G is a free $\mathbb{C}[\theta_1, \theta_2, \theta_3]$-module. Let $\eta_1, \dots, \eta_l$ be the free basis of S^G as a $\mathbb{C}[\theta_1, \theta_2, \theta_3]$-module, so that $S^G = \bigoplus_{i=1}^l \mathbb{C}[\theta_1, \theta_2, \theta_3]\eta_i$. Their degrees can be computed by GAP and Singular as in Table 4.1.

type	# Irrep(G)	# Char(G)	$(\deg \theta_i)_{i=1}^3$	$(\deg \eta_i)_{i=1}^l$
(E)	14	4	(12,6,6)	(21,12,9,0)
(F)	16	4	(12,9,6)	(24,12,0)
(G)	24	3	(18,12,9)	(36,18,0)
(H)	5	1	(10,6,2)	(15,0)
(I)	15	3	(15,6,6)	(24,12,0)
(L)	17	1	(30,12,6)	(45,0)

Table 4.1: Degrees of bases of the invariant rings

Theorem 4.3. *Let G be a finite non-abelian subgroup of $SL_3(\mathbb{C})$. Then the quotient stack $[\mathbb{P}^2/G]$ does not have a full exceptional collection of line bundles.*

Proof. Let $X = [\mathbb{P}^2/G]$ be the quotient stack of $\mathbb{P}^2$ by a finite non-abelian subgroup G of $SL_3(\mathbb{C})$. If G is of type (B), then the stabilizer of the point $[1 : 0 : 0] \in \mathbb{P}^2$ is the whole of G, which is non-abelian by the assumption. Hence X does not have a full exceptional collection of line bundles by Lemma 4.1.

If G is of type (C), then the polynomial xyz is G-invariant. Hence we have $H^0(\mathcal{O}_X(3l)) \neq 0$ for any non-negative integer l. For any four integers, their exits at least one pair (l, m) such that $l - m \equiv 0 \mod 3$. Thus any exceptional collection of line bundles on X must have size less than or equal to $3 \times \#\,\mathrm{Char}(G)$, which is smaller than rank $K(X) = 3 \times \#\,\mathrm{Irr}(G)$ since G is non-abelian. This shows that X does not admit a full exceptional collection of line bundles.

If G is of type (D), then consider the stabilizer Γ of the point $[1 : 0 : 0] \in \mathbb{P}^2$. This is the intersection of G with the parabolic subgroup P of $SL_3(\mathbb{C})$ consisting of block upper-triangular matrices of the form $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$. If Γ is non-abelian, then X does not have a full exceptional collection of line bundles by Lemma 4.1. Assume that Γ is abelian. Note that $Q \in \Gamma$, and a diagonal element $\operatorname{diag}(\alpha, \beta, \gamma)$ commutes with Q if and only if $\beta = \gamma$. Since $T^{-1} \cdot \operatorname{diag}(\alpha, \beta, \gamma) \cdot T = \operatorname{diag}(\gamma, \alpha, \beta)$, the diagonal subgroup of G must be trivial. Since $Q^2 = \operatorname{diag}(a^2, bc, bc)$, one has $a^2 = bc = 1$. Together with $abc = -1$, this gives $a = -1$ and $b = c = \pm 1$. Then one can set $a = b = c = -1$ by a choice of a coordinate, and G is isomorphic to the symmetric group S_3 of order 6. The group S_3 has three irreducible representations; the trivial representation ρ_0, the sign representation ρ_1, and the irreducible representation ρ_2 of dimension 2. The canonical bundle is given by $\mathcal{O}_X(-3) \otimes \rho_1$. One can easily see that the maximal length for exceptional collections of line bundles is 7, and a collection of length 7 is given by

$$(\mathcal{O}_X \otimes \rho_0, \mathcal{O}_X \otimes \rho_1, \mathcal{O}_X(1) \otimes \rho_0, \mathcal{O}_X(1) \otimes \rho_1, \mathcal{O}_X(2) \otimes \rho_0, \mathcal{O}_X(2) \otimes \rho_1, \mathcal{O}_X(3) \otimes \rho_0)$$

up to tensor by a line bundle. It is not full since $\operatorname{rank} K(X) = 9$.

Let G be the subgroup of type (E), and assume for a contradiction that there exists a full strong exceptional collection $L \subset \operatorname{Pic} X$ of line bundles. One must have

$$\# L = \operatorname{rank} K(X) = 3 \cdot \# \operatorname{Irrep}(G) = 42 \tag{4.1}$$

since L is full. For any $i, j \in \mathbb{Z}$, we have

$$\operatorname{Ext}^2(\mathcal{O}_X(i) \otimes \rho, \mathcal{O}_X(j) \otimes \rho) = H^0(\mathcal{O}_X(i - j - 3))$$

by the Serre duality. Table 4.1 shows

$$\bigoplus_{k=0}^{\infty} H^0(\mathcal{O}_X(k)) = S^G = \bigoplus_{i=1}^{4} \mathbb{C}[\theta_1, \theta_2, \theta_3]\eta_i, \tag{4.2}$$

where $\deg(\theta_1, \theta_2, \theta_3) = (12, 6, 6)$ and $\deg(\eta_1, \eta_2, \eta_3, \eta_4) = (21, 12, 9, 0)$. It follows that if either

(a) $i - j - 3 = 6k$ for $k \geq 0$, or

(b) $i - j = 6k$ for $k \geq 2$

is satisfied, then $\operatorname{Ext}^2(\mathcal{O}_X(i) \otimes \rho, \mathcal{O}_X(j) \otimes \rho)$ is non-zero. The assumption that L is strong implies that if both $\mathcal{O}_X(i) \otimes \rho$ and $\mathcal{O}_X(j) \otimes \rho$ are in L, then neither (a) nor (b) are satisfied. We put

$$L_\rho = \{ i \in \mathbb{Z} \mid \mathcal{O}_X(i) \otimes \rho \in L \}.$$

The condition (a) implies that $\# \{ [i] \in \mathbb{Z}/6\mathbb{Z} \mid i \in L_\rho \} \leq 3$, and the condition (b) implies that $\# L_\rho \leq 3 \times 2 = 6$. Hence $\# L \leq \# \operatorname{Char}(G) \cdot 6 = 24$, which contradicts (4.1).

The cases when G is of type (F), (G), (H), (I), or (L) are proved similarly. The non-vanishing conditions for $H^2(\mathcal{O}_X(-l))$ and the upper bound for $\# L_\rho$ is summarized in Table 4.2, and one can see that the upper bound is strictly smaller than the quotient $\operatorname{rank} K(X) / \# \operatorname{Char}(G)$ of the rank of the Grothendieck group G by the number of characters of G.

Let G be the group of type (J), and $H < G$ be the subgroup of type (H), so that $G = \langle H, W \rangle$ where $W = \operatorname{diag}(\omega, \omega, \omega)$. The corresponding quotient stacks will be denoted

type	non-vanishing condition for $H^2(\mathcal{O}_X(-l))$	upper bound for $\#L_\rho$
(E)	$l = 3$ or $3k + 9$ for $k \geq 0$	6
(F)	$l = 3$ or $3k + 9$ for $k \geq 0$	6
(G)	$l = 3, 12, 15, 21$ or $3k + 27$ for $k \geq 0$	9
(H)	$l = 2k + 3$ or $2k + 18$ for $k \geq 0$	3
(I)	$l = 6k + 3$ or $6k + 18$ for $k \geq 0$	9
(L)	$l = 6k + 3$ or $l = 6k + 48$ for $k \geq 0$	9

Table 4.2: Upper bounds for $\#L_\rho$

by $X = [\mathbb{P}^2/G]$ and $Y = [\mathbb{P}^2/H]$. Let $B\mathbb{G}_m = [\operatorname{Spec}\mathbb{C}/\mathbb{G}_m]$ be the classifying stack of the multiplicative group $\mathbb{G}_m$. As a category fibered in groupoid, an object of $B\mathbb{G}_m$ over a scheme S is a line bundle on S, so that giving a morphism from S to $B\mathbb{G}_m$ is equivalent to giving a line bundle on S. Let $\phi \colon Y \to B\mathbb{G}_m$ be the morphism defined by the line bundle $\mathcal{O}_Y(1)$, and $\vartheta_r \colon B\mathbb{G}_m \to B\mathbb{G}_m$ be the morphism sending an object $\mathcal{L}$ over S to the r-th tensor power $\mathcal{L}^{\otimes r}$. Since W acts by the cubic root of unity on the fiber of $\mathcal{O}_Y(1)$ and trivially on the base Y, the stack X is described as the fiber product

$$X = Y \,{}_\phi\!\times_{\vartheta_3} B\mathbb{G}_m.$$

See also [Cad07, Definition 2.2.6]. The abelian category of coherent sheaves on X is equivalent to the direct sum of 3 copies of the abelian category of coherent sheaves on Y by [IU, Theorem 1.5];

$$\operatorname{coh} X = \pi_1^*(\operatorname{coh} Y) \oplus \pi_1^*(\operatorname{coh} Y) \otimes \pi_2^*\mathcal{O}_{B\mathbb{G}_m}(1) \oplus \pi_1^*(\operatorname{coh} Y) \otimes \pi_2^*\mathcal{O}_{B\mathbb{G}_m}(2).$$

This implies that X has a full exceptional collection of line bundles if and only if Y has a full exceptional collection of line bundles. Similarly, the type (K) case can be reduced to the type (I) case. ☐

References

[Abd12] Tarig M. H. Abdelgadir, *Quivers of sections on toric orbifolds*, J. Algebra **349** (2012), 128–149. MR 2853630

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398. MR 2450211 (2009k:14108)

[AU] Tarig M. H. Abdelgadir and Kazushi Ueda, *Weighted projective lines as fine moduli spaces of quiver representations*, to appear in Communications in Algebra.

[Beĭ78] A. A. Beĭlinson, *Coherent sheaves on* $\mathbf{P}^n$ *and problems in linear algebra*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 68–69. MR MR509388 (80c:14010b)

[BH09] Lev Borisov and Zheng Hua, *On the conjecture of King for smooth toric Deligne-Mumford stacks*, Adv. Math. **221** (2009), no. 1, 277–301. MR MR2509327

[BK90] A. I. Bondal and M. M. Kapranov, *Enhanced triangulated categories*, Mat. Sb. **181** (1990), no. 5, 669–683. MR MR1055981 (91g:18010)

[Bli17] H. F. Blichfeldt, *Finite collineation groups*, The University of Chicago Press, 1917.

[BO] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, arXiv:alg-geom/9506012.

[Bon89] A. I. Bondal, *Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 25–44. MR MR992977 (90i:14017)

[Cad07] Charles Cadman, *Using stacks to impose tangency conditions on curves*, Amer. J. Math. **129** (2007), no. 2, 405–427. MR 2306040 (2008g:14016)

[CS08] Alastair Craw and Gregory G. Smith, *Projective toric varieties as fine moduli spaces of quiver representations*, Amer. J. Math. **130** (2008), no. 6, 1509–1534. MR MR2464026

[CW13] Alastair Craw and Dorothy Winn, *Mori dream spaces as fine moduli of quiver representations*, J. Pure Appl. Algebra **217** (2013), no. 1, 172–189. MR 2965915

[Efi10] Alexander I. Efimov, *Maximal lengths of exceptional collections of line bundles*, arXiv:1010.3755, 2010.

[GL87] Werner Geigle and Helmut Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), Lecture Notes in Math., vol. 1273, Springer, Berlin, 1987, pp. 265–297. MR MR915180 (89b:14049)

[HE71] M. Hochster and John A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058. MR 0302643 (46 #1787)

[HK09] Christopher P. Herzog and Robert L. Karp, *On the geometry of quiver gauge theories (stacking exceptional collections)*, Adv. Theor. Math. Phys. **13** (2009), no. 3, 599–636. MR 2610572 (2011d:14027)

[HM] Andreas Hochenegger and Elena Martinengo, *MD-stacks*, arxiv:1403.7984.

[HP] Lutz Hille and Markus Perling, *Exceptional sequences of invertible sheaves on rational surfaces*, arXiv:0810.1936.

[HP06] ———, *A counterexample to King's conjecture*, Compos. Math. **142** (2006), no. 6, 1507–1521. MR MR2278758 (2007h:14074)

[IU] Akira Ishii and Kazushi Ueda, *The special McKay correspondence and exceptional collection*, arXiv:1104.2381.

[IU12] ———, *A note on derived categories of Fermat varieties*, Derived categories in algebraic geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2012, pp. 103–110. MR 3050701

[Kap88] M. M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), no. 3, 479–508. MR MR939472 (89g:18018)

[Kaw06] Yujiro Kawamata, *Derived categories of toric varieties*, Michigan Math. J. **54** (2006), no. 3, 517–535. MR MR2280493 (2008d:14079)

[Kin97] Alastair King, *Tilting bundles on some rational surfaces*, preprint available at `http://www.maths.bath.ac.uk/~masadk/papers/`, 1997.

[KP] Alexander Kuznetsov and Alexander Polishchuk, *Exceptional collections on isotropic Grassmannians*, arXiv:1110.5607.

[MBD61] G. A. Miller, H. F. Blichfeldt, and L. E. Dickson, *Theory and applications of finite groups*, Dover Publications Inc., New York, 1961. MR 0123600 (23 #A925)

[Ric89] Jeremy Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456. MR MR1002456 (91b:18012)

[YY93] Stephen S.-T. Yau and Yung Yu, *Gorenstein quotient singularities in dimension three*, Mem. Amer. Math. Soc. **105** (1993), no. 505, viii+88. MR 1169227 (94b:14045)

Ryo Ohkawa

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan.

e-mail address : ohkawa@kurims.kyoto-u.ac.jp

Kazushi Ueda

Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan.

e-mail address : kazushi@math.sci.osaka-u.ac.jp