Title: *Combinatorial Miller-Morita-Mumford classes and Witten Cycles*
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Abstract

The Miller-Morita-Mumford classes $\kappa_k$ are integer characteristic classes for oriented surface bundles given by taking the powers of the Euler class of the vertical tangent bundle and integrating along the fibers. This construction can be done combinatorially with interesting results.

First I will construct cocycles for the powers of the Euler class for the category of cyclically ordered sets by integrating a naturally defined curvature form on the geometric realization of this category. Then I represent (punctured) surfaces as "fat graphs" in the usual way. These are graphs with cyclic orderings of the half-edges incident to each vertex. Then, instead of integrating, I add the cyclic set cocycles obtained at each vertex multiplied by a suitable weight. This gives *combinatorial Miller-Morita-Mumford classes* $\beta_k \in H^{2k}(M_g^s, \mathbb{Z})$ where $M_g^s$ is the mapping class group of genus $g$ surfaces with $s \geq 1$ punctures.

It is easy to show that these combinatorial classes are dual to the *Witten cycles* $[W_k]$:

$$\beta_k = \frac{(k-2)k!}{(2k+2)!(2k-1)!}[W_k]^*$$

($\beta_2 = 0$ is exceptional.)

The *Witten conjecture* says that the Witten cycles $[W_k]$ are dual to the Miller-Morita-Mumford classes $\kappa_k$. Robert Penner proved this for $k = 1$. In the second half of this lecture I will explain how to prove the Witten conjecture for all $k \geq 1$ by modifying the above argument using the Framed Graph Theorem.

By Morse theory arguments we can see that the Miller-Morita-Mumford classes (minus a correction term $p_* e^k$) are given by adding up the cyclic set cocycles on the vertices of the framed fat graph. Using the dihedral symmetry ($D_{4k+4}$ action) of the Stasheff associahedron $A^{2k-1}$ we can see that

$$\kappa_k^0 = a_k [W_k]^*$$

for some nonzero rational number $a_k$ where $\kappa_k^0 = \kappa_k - p_* e^k$ is the "punctured Miller-Morita-Mumford class" given by subtracting the contribution of the 2-cells at the puncture, i.e., by subtracting the push-down $p_* e^k$ of the $k$-th power of the Euler class $e$ along the punctures. Stably, these correction terms are zero.

The lecture will begin with some elementary differential geometry on cyclically ordered sets, go into the geometric realization of categories and a short review of the theory of framed graphs/framed Morse functions/higher torsion as it pertains to this application.

Reference:

Robert C. Penner, *The Poincaré dual of the Weil-Petersson Kähler two-form*, Perspectives in Mathematical Physics, Int.Press, Cambridge, MA, 1994, 229-249.