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Effects of Age Shift on the Tempo and Quantum of Non-Repeatable Events

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Effects of age shift on the tempo and quantum of non-repeatable demographic events are examined. The purpose is to develop a period index theory based on the survival model and to provide a mathematically consistent interpretation of Bongaarts and Feeney’s tempo adjustment arguments. The survival model for non-repeatable events is introduced. In the time-inhomogeneous case, three types of period survival models are considered. McKendrick equation is used to formulate the risk population dynamics. The tempo and quantum indices for three period survival models are computed when the period age shift occurs for the hazard, the incidence, and the survival rates. Bongaarts and Feeney’s tempo adjustment arguments are consistently based on the scenario of the period age shift on the survival rate, and they give translation formulae between period indices without referring to cohort. Traditional demographic translation formulae between cohort and period indices are reviewed to clarify differences between cohort- and period-oriented translation procedures.

Keywords: age shift; non-repeatable event; quantum; survival model; tempo

1. NON-REPEATABLE EVENTS AND THE SURVIVAL MODEL

1.1. The Basic Model

A demographic event is called non-repeatable if the population at risk is composed of individuals who have not yet experienced the event. A demographic event is called repeatable if the population at risk is not affected by the occurrence of the event (Keilman, 1994). Repeatable events can occur in the same population again and again without change of the risk population. Events could produce any qualitative change in the population, so it seems that there are few repeatable
events in a strict sense. For example, first marriage and first birth must be considered as non-repeatable events, because first marriage occurs in the population at risk composed of never married individuals and first birth happens to individuals who have never experienced childbearing. Though death is a non-repeatable event, it is special because the population at risk is the whole population.

The occurrence of a non-repeatable event in a cohort is described by the survival model. Let \( X \) be a random variable describing age at a non-repeatable event. Let \( F(a) \) be the distribution function, \( f(a) \) the probability density function of \( X \) and \( Pr(A) \) the probability that the event \( A \) occurs, then the hazard \( \lambda(a) \) for the event is defined as

\[
\lambda(a) = \lim_{h \to 0} \frac{Pr(a \leq X < a + h \mid a \leq X)}{h} = \lim_{h \to 0} \frac{Pr(a \leq X < a + h)}{Pr(a \leq X)h}
\]

(1.1)

where the survival rate \( \Lambda(a) := 1 - F(a) \) is the probability that the event does not occur until age \( a \). If the event of interest is death, \( \lambda(a) \) and \( \Lambda(a) \) are written as \( \mu(a) \) and \( \ell(a) \) respectively, \( \mu(a) \) is called the force of mortality, and \( \ell(a) \) the life table survival rate. Therefore we use notations \( \mu \) and \( \ell \) instead of \( \lambda \) and \( \Lambda \) when death is the non-repeatable event.

From definition Eq. (1.1), we have

\[
\frac{d\Lambda(a)}{da} = -\lambda(a)\Lambda(a).
\]

(1.2)

Then we obtain

\[
\Lambda(a) = \exp\left(-\int_0^a \lambda(\sigma)d\sigma\right).
\]

(1.3)

Using the survival rate, the incidence rate of this event is expressed as

\[
\phi(a) = \lambda(a)\Lambda(a),
\]

(1.4)

which is the probability density that the event occurs at age \( a \).

Once the incidence rate is observed, the hazard function \( \lambda(a) \) is calculated as

\[
\lambda(a) = \frac{\phi(a)}{1 - \int_0^a \phi(\sigma)d\sigma} = \frac{\phi(a)}{\Lambda(a)},
\]

(1.5)

where \( dF(0)/da = \phi(a) \) and \( F'(0) = 0 \) are used.

The total sum of incidence rate is called the quantum for the event. The quantum \( Q \) is the lifelong average number of occurrences of the non-repeatable event given by
\[
Q := \int_0^\omega \phi(a)da = 1 - \Lambda(\omega) = 1 - \exp\left(-\int_0^\omega \lambda(\sigma)d\sigma\right),
\tag{1.6}
\]

where \(\omega\) is an upper bound of age of occurrence of the event. We assume that \(\lambda(a) = 0\) for \(a > \omega\). For a non-repeatable event such as death, its quantum equals unity, \(\Lambda(\omega) = 0\) and \(\int_0^\omega \lambda(\sigma)d\sigma = \infty\). For first birth, \(Q\) is the probability that an individual has at least one child, or the average expected number of the first child that an individual has. If the event is not necessarily experienced by everybody, \(\Lambda(\omega) > 0\) is the proportion of individuals in the cohort who never experience the event.

Let \(A\) be the average age at which the event occurs in the cohort:
\[
A := \frac{\int_0^\omega a\lambda(a)\Lambda(a)da}{\int_0^\omega \lambda(a)\Lambda(a)da} = \frac{1}{Q}\left(-\omega\Lambda(\omega) + \int_0^\omega \Lambda(a)da\right). \tag{1.7}
\]

\(A\) is the average sojourn time until the occurrence of the event, and it is called tempo for the event. When all individuals of the cohort necessarily experience the event, \(Q = 1\), \(\Lambda(\omega) = 0\) and
\[
A = \int_0^\omega \Lambda(a)da. \tag{1.8}
\]

In particular, if the event of interest is death, \(A\) is the life expectancy, denoted by \(e_0\).

To simplify the calculation, we sometime use the unlimited upper bound \(\omega = \infty\) of age of occurrence. However if \(Q < 1\), we must use Eq. (1.7) to calculate the average age instead of Eq. (1.8) with \(\omega = \infty\), even though the age interval is \([0, \infty)\).

\subsection*{1.2. Age Shift}

The effect of tempo change on life cycle parameters has been an important topic in demography (Ryder, 1956, 1964, 1980a, 1980b; Butz and Ward, 1979; Suzuki, 2002). The adequate tool to discuss this topic is the age shift in parameter functions because it expresses the most simple perturbation of the timing of age-specific parameters. Let \(f(a)\) be an age-specific schedule of a life cycle event. The age shift of the life cycle schedule is expressed by the function \(f(a - h)\), where the domain of \(f\) is extended such that \(f(a) = 0\) for \(a < 0\). If \(h > 0\), the schedule of the occurrence of the event is shifted toward older age, while it is shifted toward younger age if \(h < 0\).

In the survival model, we consider the age shift perturbation for the hazard rate, the incidence rate, and the survival rate.
Assume that the standard hazard rate $\lambda_0(a)$ is shifted to a new schedule $\lambda(a) = \lambda_0(a - h)$. From
\[
\frac{d\Lambda(a)}{da} = -\lambda_0(a - h)\Lambda(a),
\]
we obtain
\[
\Lambda(a) = \exp\left(- \int_0^a \lambda_0(\sigma - h)d\sigma \right) = \exp\left(- \int_0^{a-h} \lambda_0(\sigma)d\sigma \right) = \Lambda_0(a - h),
\]
(1.9)
where $\Lambda_0$ denotes the survival rate associated with the standard hazard rate:
\[
\Lambda_0(a) := \exp\left(- \int_0^a \lambda_0(\sigma)d\sigma \right).
\]
(1.10)

Then the incidence rate is
\[
\phi(a) = \lambda(a)\Lambda(a) = \lambda_0(a - h)\Lambda_0(a - h) = \phi_0(a - h),
\]
(1.11)
where $\phi_0(a)$ denotes the incidence by the standard hazard rate. The age shift of the hazard rate leads to the same kind of age shift in the incidence and the survival rates.

If the incidence rate is shifted as $\phi(a) = \phi_0(a - h)$, it follows from Eq. (1.5) that
\[
\lambda(a) = \frac{\phi_0(a - h)}{1 - \int_0^a \phi_0(\sigma - h)d\sigma} = \frac{\phi_0(a - h)}{1 - \int_0^{a-h} \phi_0(\sigma)d\sigma} = \lambda_0(a - h).
\]
(1.12)
The hazard rate has the same age shift. Moreover,
\[
\Lambda(a) = \frac{\phi(a)}{\lambda(a)} = \frac{\phi_0(a - h)}{\lambda_0(a - h)} = \Lambda_0(a - h),
\]
(1.13)
so the same age shift occurs in the survival rate.

Finally, if the standard survival rate $\Lambda_0(a)$ is shifted as $\Lambda(a) = \Lambda_0(a - h)$, then
\[
\frac{d\Lambda(a)}{da} = \frac{d\Lambda_0(a - h)}{da} = -\lambda_0(a - h)\Lambda_0(a - h) = -\phi_0(a - h),
\]
(1.14)
which shows that the incidence and the hazard rates have the same age shift.

In summary, for the time-independent survival model, the age shift has the same effect for hazard, survival, and incidence rates. We shall see that this conclusion holds true also for the time-inhomogeneous cohort survival model if the age shift occurs along the life line, but it does not hold for the time-inhomogeneous period survival model.
2. THE TIME-INHOMOGENEOUS SURVIVAL MODELS

We introduce time into the survival model for non-repeatable events and examine the change of demographic indices under the age shift.

2.1. The Cohort Model

Let $\lambda(t, a)$ be the hazard rate for a life cycle event at time $t$ and age $a$ and $\Lambda(a + T, a)$ the survival rate, which is the proportion of individuals born at time $T$ and have not yet experienced the event at time $a + T$ and age $a$. Then we have

$$\frac{d\Lambda(a + T, a)}{da} = -\lambda(a + T, a)\Lambda(a + T, a), \quad \Lambda(T, 0) = 1. \quad (2.1)$$

By solving Eq. (2.1), we obtain

$$\Lambda(a + T, a) = \exp\left(-\int_0^a \lambda(\sigma + T, \sigma)d\sigma\right). \quad (2.2)$$

Inserting $T = t - a$ into Eq. (2.2),

$$\Lambda(t, a) = \exp\left(-\int_0^a \lambda(t - a + \sigma, \sigma)d\sigma\right), \quad (2.3)$$

which implies that the event at time $t$ and age $a$ occurs to individuals who were born at time $t - a$. The corresponding incidence is

$$\phi(t, a) = \dot{\lambda}(t, a)\Lambda(t, a) = \dot{\lambda}(t, a)\exp\left(-\int_0^a \lambda(t - a + \sigma, \sigma)d\sigma\right). \quad (2.4)$$

The quantum of the cohort born at time $T$ is

$$Q(T) : = \int_0^\omega \phi(a + T, a)da$$

$$= 1 - \Lambda(\omega + T, \omega) = 1 - \exp\left(-\int_0^\omega \lambda(\sigma + T, \sigma)d\sigma\right). \quad (2.5)$$

The average age of the occurrence of the event, or the tempo index, $A(T)$ for the cohort born at time $T$ is

$$A(T) : = \frac{\int_0^\omega a\dot{\lambda}(a + T, a)\Lambda(a + T, a)da}{\int_0^\omega \dot{\lambda}(a + T, a)\Lambda(a + T, a)da}$$

$$= \frac{1}{Q(T)} \left[-\omega\Lambda(T + \omega, \omega) + \int_0^\omega \Lambda(a + T, a)da\right]. \quad (2.6)$$
The incidence rate at age $a$ for individuals born at time $T$, denoted by $\phi(a + T, a)$, is

$$\phi(a + T, a) = \lambda(a + T, a) \exp \left( - \int_0^a \lambda(\sigma + T, \sigma) d\sigma \right). \tag{2.7}$$

Integrating Eq. (2.7) from 0 to $a$, it follows that

$$\int_0^a \phi(\sigma + T, \sigma) d\sigma = 1 - \exp \left( - \int_0^a \lambda(\sigma + T, \sigma) d\sigma \right). \tag{2.8}$$

Applying Eq. (2.8) to Eq. (2.7),

$$\lambda(a + T, a) = \frac{\phi(a + T, a)}{1 - \int_0^a \phi(\sigma + T, \sigma) d\sigma}, \tag{2.9}$$

which gives an estimation for the hazard rate. Inserting $T = t - a$ into Eq. (2.9), we obtain the hazard rate at time $t$:

$$\lambda(t, a) = \frac{\phi(t, a)}{1 - \int_0^a \phi(t - a + \sigma, \sigma) d\sigma} = \frac{\phi(t, a)}{\Lambda(t, a)} \tag{2.10}$$

If we have data for the incidence rate $\phi(t, a)$ or the survival rate $\Lambda(t, a)$, we can estimate the hazard rate $\lambda(t, a)$.

In particular, if the event of interest is death, instead of $\lambda(t, a)$ and $\Lambda(t, a)$, we use symbols $\mu(t, a)$ (the force of mortality) and $\ell(t, a)$. The cohort life table survival rate for death is given as

$$\ell(t, a) = \exp \left( - \int_0^a \mu(t - a + \sigma, \sigma) d\sigma \right), \tag{2.11}$$

which is the proportion of individuals born at time $t - a$ who are alive at time $t$ and age $a$.

### 2.2. The Period Model

In order to apply the cohort survival model to real data, we need to observe the life cycle event from the beginning of the cohort to the terminal age of occurrence of the event. If the life cycle event of interest is death, we have to observe the cohort for more than 80 years. The cohort index calculated by the cohort model summarizes the occurrence of the event which already concerns a real cohort.

To calculate the demographic index summarizing the present situation of occurrence of life cycle events, we apply the survival model
to the hypothetical cohort; that is, we interpret the occurrence of the event at each age at a given time (period data) as if it had occurred in a hypothetical cohort. The survival model applied to the period data is called the period survival model.

If the occurrence of the event depends only on age and not on time, the cohort observation and the period observation would lead to the same result. But the real population is made of various cohorts whose life cycle indices change with time, so that the demographic indices calculated by the cohort model and those calculated by the period model are likely to differ from each other. The former corresponds to the real cohort process and is a summary for the past data; the latter summarizes the occurrence of the event at an observation time.

The period survival model can be divided into three types according to the method of estimating the hazard rate. I call these types as A, B, and C.

2.2.1. Period Model A Based on the Hazard Rate

Assume that the hazard rate \( \lambda(t, a) \) is known at time \( t \). For example, if we have the incidence rate \( \phi(t, a) \) for \( t \leq t \), or the knowledge of \( \phi(t, a) \) and \( \Lambda(t, a) \), we apply Eq. (2.10) to obtain the hazard rate \( \lambda(t, a) \) at time \( t \).

If we use \( \lambda(t, a) \) as the hazard rate for a hypothetical cohort, the period survival rate based on a given hazard rate, denoted by \( \Lambda_p^a(t, a) \), at time \( t \) is

\[
\Lambda_p^a(t, a) := \exp \left( - \int_0^a \lambda(t, \sigma) \, d\sigma \right). 
\]  

Then the incidence rate is given by

\[
\phi_p^a(t, a) := \lambda(t, a) \Lambda_p^a(t, a),
\]

and the period quantum based on the model A, denoted by \( Q_p^a(t) \), is

\[
Q_p^a(t) := \int_0^\omega \phi_p^a(t, a) \, da = 1 - \exp \left( - \int_0^\omega \lambda(t, \sigma) \, d\sigma \right) = 1 - \Lambda_p^a(t, \omega).
\]  

Let \( A_p^a(t) \) be the period average age of the occurrence of the event. We have

\[
A_p^a(t) := \frac{\int_0^\omega a \phi_p^a(t, a) \, da}{\int_0^\omega \phi_p^a(t, a) \, da} = \frac{\int_0^\omega a \lambda(t, a) \exp \left( - \int_0^a \lambda(t, \sigma) \, d\sigma \right) \, da}{\int_0^\omega \lambda(t, a) \exp \left( - \int_0^a \lambda(t, \sigma) \, d\sigma \right) \, da}.
\]
If the event of interest is death, instead of \( k(t; a) \), \( k_p(t; a) \) and \( \Lambda_p(t; a) \), we use symbols \( \mu(t; a) \), \( \ell_p(t; a) \) and \( e_p(t; a) \) to denote the force of mortality, the period life table survival rate, and the period life expectancy at time \( t \). Then \( Q_p(t) = 1 \) and
\[
\ell_p(t; a) = \exp \left( -\int_0^a \mu(t; \sigma)d\sigma \right),
\]
(2.16)
\[
e_p(t) = \int_0^\infty \ell_p(t; a)da,
\]
where we use the convention that \( \omega = \infty \).

The period model A is the period life table model for life cycle events. If the event of interest is first marriage, \( k_p(t; a) \) is the proportion of never married people at age \( a \) in the marital status life table at period \( t \).

Though in general we need either the incidence data until time \( t \) or the incidence at time \( t \) and the survival rate at time \( t \) to estimate the hazard rate at time \( t \) from Eq. (2.10), it is sufficient to know the incidence at time \( t \) and the population at time \( t \) to compute the force of mortality \( \mu(t; a) \), because all individuals are at risk of death. This is the reason why we can calculate the life table functions only from period data.

### 2.2.2. Period Model B Based on the Incidence Rate

Assume that the incidence rate \( \phi(t; a) \) is known but there is no data for the population at risk, that is, we do not have the survival rate at time \( t \). Then we cannot estimate the true hazard \( \lambda(t; a) \). Instead, we compute the implicit hazard \( \lambda_p^b(t; a) \) from the incidence rate \( \phi(t; a) \) at time \( t \) as:
\[
\lambda_p^b(t; a) := \frac{\phi(t; a)}{1 - \int_0^a \phi(t; \sigma)d\sigma}.
\]
(2.17)

By using the implicit hazard, the survival rate is defined as
\[
\Lambda_p^b(t; a) := \exp \left( -\int_0^a \lambda_p^b(t, \sigma)d\sigma \right) = 1 - \int_0^a \phi(t, \sigma)d\sigma.
\]
(2.18)
The incidence rate is \( \phi(t; a) \) itself and
\[
\lambda_p^b(t; a)\Lambda_p^b(t; a) = \phi(t; a) = \lambda(t; a)\Lambda(t; a).
\]
(2.19)
The quantum of this period model B is
\[
Q_p^b(t) := \int_0^\omega \phi(t; a)da,
\]
(2.20)
and the average age of the occurrence of events is given by

$$A_p^b(t) := \frac{\int_0^\infty a \phi(t,a) da}{\int_0^\infty \phi(t,a) da}. \quad (2.21)$$

In this model B, the quantum $Q_p^b(t)$ and the tempo $A_p^b(t)$ can be calculated only from the data of the incidence rate $\phi(t,a)$; we do not need to know the implicit hazard $\lambda_p^b(t,a)$. However the hazard rate is not $\lambda(t,a)$ but $\lambda_p^b(t,a)$, even though the incidence rate $\phi$ is the same. To calculate the quantum $Q_p^b(t)$ from the period incidence rate by Eq. (2.20) implies that we apply the period model B using the implicit hazard rate.

For example, this interpretation is usually adopted when the event is childbearing and the incidence rate is the age-specific birth rate by birth order. The total fertility rate (TFR) is calculated as the sum of age-specific birth rates at a given time. If we compute by birth order, we obtain the parity-specific total fertility rate, which is the quantum $Q_p^b(t)$ based on the period survival model B with the implicit hazard. If the event of interest is death, the tempo index $e_p^b(t) (= A_p^b(t))$ is called the standardized mean age at death (SMAD) (Bongaarts and Feeney, 2003).

However, the sum of incidence rates at time $t$ does not necessarily satisfy the condition for total probability $\int_0^\infty \phi(t,a) da \leq 1$, because the incidence rate $\phi(t,a)$ at each age $a$ covers different cohorts. The nonnegative hazard rate is not necessarily obtained from Eq. (2.17), so that we cannot apply the survival model to the period incidence data.

Even if the denominator of Eq. (2.17) is positive and we can calculate $\lambda_p^b(t,a)$, the denominator of Eq. (2.17) is an estimate of the survival rate at age $a$, which is different from the real cohort survival rate $\Lambda(t,a)$. Under the assumption of model B, the proportion of population at risk at time $t$ and age $a$ is not estimated correctly.

The period total fertility rate by birth order, which is the period total sum of age-specific birth rates, is calculated from the implicit hazard rates, so it is distorted because the proportion of population at risk is not estimated correctly. Instead of $Q_p^b(t)$ corresponding to the TFR, Yamaguchi and Beppu (2004) recommended to use the quantum $Q_p^a(t)$ computed from period model A, because it has no logical shortcoming and $Q_p^a$ gives a better approximation to the cohort TFR under the cohort age shift (see Proposition 5.2). The period quantum $Q_p^a(t)$ has the advantage that it summarizes the real data of occurrence of the event at a given time.
2.2.3. Period Model C Based on the Survival Rate

Consider the case when we know only the survival rate \( \Lambda(t, a) \) at time \( t \), for example in the absence of vital statistics and when only census data are available. We can interpret \( \Lambda(t, a) \) as if it had given a survival rate for a hypothetical cohort, so that the implicit hazard, denoted by \( \dot{\gamma}^c(t, a) \), is defined as

\[
\Lambda(t, a) = \exp \left( - \int_0^a \dot{\gamma}^c(t, \sigma) d\sigma \right).
\] (2.22)

We have

\[
\dot{\gamma}^c(t, a) := - \frac{1}{\Lambda(t, a)} \frac{\partial \Lambda(t, a)}{\partial a}.
\] (2.23)

This interpretation is inadequate if \( \Lambda(t, a) \) is not monotone decreasing with respect to age \( a \).

Then the period incidence rate for period model C, denoted by \( \phi^c_p(t, a) \), is calculated as

\[
\phi^c_p(t, a) = \dot{\gamma}^c(t, a) \Lambda(t, a).
\] (2.24)

Moreover the quantum in this model is

\[
Q^c_p(t) := \int_0^\omega \phi^c_p(t, a) da = 1 - \Lambda(t, \omega),
\] (2.25)

but it is directly observed from \( \Lambda(t, \omega) \). The average age of the occurrence of the event is

\[
A^c_p(t) := \frac{\int_0^\omega a \phi^c_p(t, a) da}{\int_0^\omega \phi^c_p(t, a) da} = \frac{1}{1 - \Lambda(t, \omega)} \left[ -\omega \Lambda(t, \omega) + \int_0^\omega \Lambda(t, a) da \right].
\] (2.26)

As the right hand sides of Eq. (2.25)–(2.26) are calculated from the survival rate, for calculation purpose we do not use the implicit hazard \( \dot{\gamma}^c(t, a) \), but it is needed to interpret \( Q^c_p(t) \) as quantum and \( A^c_p(t) \) as tempo under the framework of the survival model.

For example, if the life cycle event described by the survival model is first marriage, \( A^c_p(t) \) is the average age of first marriage calculated by the period data of never married population, which is called sin-gulate mean age of marriage (SMAM). Moreover, if the event of interest is death, we use the symbol \( e^c_p(t) \) instead of \( A^c_p(t) \). As \( \Lambda(t, a) = \ell(t, a) \) and \( \ell(t, \infty) = 0 \), it follows that

\[
e^c_p(t) = \int_0^\infty \ell(t, a) da,
\] (2.27)
which is the cross-sectional average length of life (CAL) introduced by Brouard (1986) (see also Guillot, 2003).

3. MCKENDRICK EQUATION AND ITS APPLICATIONS

3.1. Age-Specific Growth Rate and Period Quantum

From Eq. (2.3), the survival rate satisfies the McKendrick equation with unit boundary value:

$$\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) \Lambda(t, a) = -\dot{\lambda}(t, a)\Lambda(t, a) = -\phi(t, a), \quad \Lambda(t, 0) = 1. \tag{3.1}$$

Let us introduce the age-specific growth rate of the survival rate as

$$\rho(t, a) := \frac{1}{\Lambda(t, a)} \frac{\partial \Lambda(t, a)}{\partial t}. \tag{3.2}$$

Then McKendrick equation (3.1) is also written as

$$\frac{\partial \Lambda(t, a)}{\partial a} = - (\rho(t, a) + \dot{\lambda}(t, a))\Lambda(t, a), \tag{3.3}$$

whose solution under the condition $\Lambda(t, 0) = 1$ is

$$\Lambda(t, a) = \exp \left( - \int_0^a [\rho(t, \sigma) + \dot{\lambda}(t, \sigma)]d\sigma \right) = \exp \left( - \int_0^a \rho(t, \sigma)d\sigma \right) \Lambda_p^a(t, a). \tag{3.4}$$

This means that the cohort survival rate at time $t$ is turned into the period (model A) survival rate $\Lambda_p^a(t, a)$ by using the age-specific growth rate. By comparing Eq. (2.23) and (3.3),

$$\rho(t, a) = \dot{\lambda}_p^c(t, a) - \dot{\lambda}(t, a), \tag{3.5}$$

which means that the cohort hazard rate is transformed into the period hazard rate based on period model C by using the age-specific growth rate.

Differentiating Eq. (2.3) with respect to age:

$$\frac{\partial \Lambda(t, a)}{\partial a} = \left( -\dot{\lambda}(t, a) - \int_0^a \frac{\partial \dot{\lambda}}{\partial a} (t - a + \sigma, \sigma) d\sigma \right) \Lambda(t, a). \tag{3.6}$$

As

$$\frac{\partial \dot{\lambda}}{\partial a} (t - a + \sigma, \sigma) = - \frac{\partial \dot{\lambda}}{\partial t} (t - a + \sigma, \sigma), \tag{3.7}$$
by using Eq. (3.3), (3.6), and (3.7), we obtain
\[ \rho(t,a) = - \int_0^a \frac{\partial \lambda}{\partial t}(t - \sigma, \sigma) d\sigma. \] (3.8)

Then

**Proposition 3.1.** If the hazard rate \( \lambda(t,a) \) decreases at each age with time \( \partial \lambda / \partial t \leq 0 \), then \( \rho \geq 0 \) and \( A_p^c(t,a) \geq \lambda(t,a) \). Conversely if \( \lambda(t,a) \) increases at each age with time \( \partial \lambda / \partial t \geq 0 \), then \( \rho \leq 0 \) and \( A_p^c(t,a) \leq \lambda(t,a) \).

If \( A(t,\omega) = 0 \), from Eq. (3.4), \( A_p^c(t,\omega) = 0 \) as long as \( \int_0^\omega \rho(t,\sigma)d\sigma < +\infty \). Then \( Q_p^u(t) = Q_p^l(t) = 1 \) and
\[ A_p^c(t) = - \int_0^\omega A_p^c(t,a) da, \]
\[ A_p^c(t) = \int_0^\omega A_p^c(t,a) da = \int_0^\omega A_p^c(t,a) \exp\left(-\int_0^a \rho(t,\sigma)d\sigma\right) da. \] (3.9)

We conclude that \( A_p^c(t) \geq A_p^c(t) \) if \( \rho \geq 0 \), while \( A_p^c(t) \geq A_p^c(t) \) if \( \rho \leq 0 \).

**Proposition 3.2.** Assume that \( A(t,\omega) = 0 \). If the hazard rate \( \lambda(t,a) \) decreases at each age with time \( \partial \lambda / \partial t \leq 0 \), then \( A_p^c(t) \leq A_p^c(t) \). If \( \lambda(t,a) \) increases at each age with time \( \partial \lambda / \partial t \geq 0 \), then \( A_p^c(t) \geq A_p^c(t) \).

If the event of interest is death, Proposition 3.2 tells us that the life expectancy based on the period life table is greater [smaller] than CAL if the force of mortality decreases [increases] with time. This fact was pointed out by Bongaarts and Feeney (2002, 2003) under the proportionality assumption. Finkelstein (2005) has shown that this phenomenon holds true without the proportionality assumption.

Integrating both sides of McKendrick Eq. (3.1),
\[ - \int_0^\omega \phi(t,a) da = \int_0^\omega \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) A(t,a) da \]
\[ = \frac{d}{dt} \int_0^\omega A(t,a) da + A(t,\omega) - 1. \] (3.10)

By using Eq. (2.26) and (3.10), we obtain

**Proposition 3.3.** Between two period quanta \( Q_p^u(t) \) and \( Q_p^c(t) \),
\[ \frac{d}{dt} \int_0^\omega A(t,a) da = \frac{d}{dt} (Q_p^c(t)A_p^c(t) + \omega A(t,\omega)) = Q_p^c(t) - Q_p^b(t). \] (3.11)

In particular, if \( A(t,\omega) = 0 \), then \( Q_p^c(t) = 1 \) and
\[ Q_p^b(t) = 1 - \frac{dA_p^c(t)}{dt}. \] (3.12)
Eq. (3.12) was shown by Guillot (2003) when the event of interest is death. In this case, $Q_p^b$ is called the period total mortality rate (TMR) (Guillot, 2006). Hirosima (2005) pointed out that Eq. (3.12) holds true for any non-repeatable event and called it the “total vital rates theorem.” Even for death for which the quantum should be one, Eq. (3.12) tells that the period quantum based on period model B could become greater or smaller than one.

### 3.2. The Proportionality Assumption

Consider the case when $\lambda(t,a)/\lambda_p^c(t,a)$ is age-independent. This assumption is called the “proportionality assumption” by Bongaarts and Feeney (2003), in which they insisted that the proportionality assumption holds true for real data.

If

$$p(t) := \frac{\lambda(t,a)}{\lambda_p^c(t,a)} \tag{3.13}$$

is an age-independent constant proportion, the sign of $\lambda_p^c(t,a)$ is constant over age at time $t$, so that the survival rate $\Lambda(t,a)$ must be a monotone function with respect to age $a$ at each time $t$. Conversely, if $\Lambda(t,a)$ is not monotone at time $t$, the proportionality assumption does not hold at time $t$.

From Eq. (3.13) and (2.24), $\phi(t,a) = p(t)\phi_p^c(t,a)$. By integration,

$$Q_p^b(t) = \int_0^\omega \phi(t,a)da = p(t) \int_0^\omega \phi_p^c(t,a)da = p(t)Q_p^c(t), \tag{3.14}$$

hence

$$p(t) = \frac{\lambda(t,a)}{\lambda_p^c(t,a)} = \frac{Q_p^b(t)}{Q_p^c(t)}. \tag{3.15}$$

Moreover, as

$$A_p^c(t) = \frac{\int_0^\omega a\phi_p^c(t,a)da}{\int_0^\omega \phi_p^c(t,a)da} = \frac{\int_0^\omega a\phi(t,a)da}{\int_0^\omega \phi(t,a)da} = A_p^b(t), \tag{3.16}$$

we have $A_p^c(t) = A_p^b(t)$. In summary,

**Proposition 3.4.** Under the proportionality assumption, the tempo index based on period model B equals the tempo index based on period model C. In particular, CAL = SMAD.
For life cycle events such as death where the final survival rate 
\( \Lambda(t, \omega) \) is zero, \( Q_p^c(t) = 1 \) and under the proportionality assumption

\[
p(t) = Q_p^b(t) = 1 - \frac{dA_p^c(t)}{dt}.
\] (3.17)

Bongaarts and Feeney (2003) obtained Eq. (3.17) without noting that 
\( p(t) = Q_b^c(t) \). From Eq. (3.12) and (3.15),

\[
\lambda(t, a) = Q_p^b(t) \lambda_p^c(t, a) = \left(1 - \frac{dA_p^c(t)}{dt}\right) \lambda_p^c(t, a),
\] (3.18)

For the force of mortality, the transformation formula (3.18) was 

From Eq. (3.18), we state that if 
\( A_p^c(t) \) is increasing \( (dA_p^c(t)/dt > 0) \), then 
\( \lambda(t, a) < \lambda_p^c(t, a) \) and \( A_p^c(t) = A_b^c(t) < A_p^a(t) \).

Instead of symbol \( A \), we denote the tempo index for death as \( e \), 
though the meanings of subscripts are the same. If we let 
\( \delta := de_p^c/\delta t \) and assume that \( \delta \) is sufficiently small, from Eq. (3.18):

\[
CAL(t) = \int_0^\infty \exp \left(- \int_0^a \frac{\mu(t, \sigma)}{1 - \delta} \, d\sigma\right) \, da
\]
\[
\approx \int_0^\infty \exp \left(- (1 + \delta) \int_0^a \mu(t, \sigma) \, d\sigma\right) \, da
\]
\[
\approx \int_0^\infty \ell_p^a(t, a) \left(1 - \delta \int_0^a \mu(t, \sigma) \, d\sigma\right) \, da.
\] (3.19)

As the period life expectancy is given by

\[
e_p^a(t) = \int_0^\infty \ell_p^a(t, a) \, da,
\] (3.20)

then

\[
\frac{CAL(t) - e_p^a(t)}{e_p^a(t)} \approx - \frac{de_p^c(t)}{dt} H(t),
\] (3.21)

where \( H \) denotes the period life table entropy defined by

\[
H(t) := - \frac{1}{e_p^a(t)} \int_0^\infty \ell_p^a(t, a) \ln \ell_p^a(t, a) \, da
\] (3.22)

and introduced by Keyfitz (1977). It gives the elasticity of the life span 
to a uniform change of death rate. If the death rate is improved one 
percent uniformly, the life span is prolonged \( 100 \times H \) percent.

**Proposition 3.5.** Assume that \( \Lambda(t, \omega) = 0 \). Under the proportionality 
assumption, if the tempo (the average age of occurrence of the event)
based on model C increases [decreases], the tempo index based on model A is greater [less] than the tempo index based on model B or model C. If the event of interest is death, the difference between those tempo indices is proportional to the speed of change of CAL or of SMAD and to the period life table entropy.

Bongaarts and Feeney (2003) claimed that when the proportionality assumption holds true, if the force of mortality is decreasing with time, the period life expectancy $e_p(t)$ is biased by the tempo distortion, which overestimates the real life expectancy, so CAL ($= e_p^c(t)$) or SMAD ($= e_p^b(t)$) would be more appropriate tempo indices to summarize the real situation of death. We will return to this point in section 4.5.

4. EFFECTS OF PERIOD AGE SHIFT

We consider the effects of the period age shift of event schedule in the time-inhomogeneous survival model on tempo and quantum. For the time-inhomogeneous survival model, the period age shift occurring at the level of hazard, incidence and survival rates have different effects. The period age shift at the three levels occurring for three period survival models yields nine combinations of assumptions and models. We now examine the effects of the period age shift at each level. The effect of the cohort age shift is considered in the next section.

4.1. Period Age Shift on Hazard Rate

Assume that the hazard rate has shifted from the standard schedule $\lambda_0(a)$ to a new schedule $\lambda_0(a - F(t))$ at time $t$. We assume that the age shift is occurring in a finite time interval, so that there exists a time $T > 0$ such that $F(t) = 0$ for $t \leq 0$ and $F(t) = F(T)$ for $t \geq T$. The standard survival rate and the standard incidence are given by

$$\Lambda_0(a) = \exp \left( - \int_0^a \lambda_0(\sigma)d\sigma \right), \quad \phi_0(a) = \lambda_0(a)\Lambda_0(a). \quad (4.1)$$

For period model A, we observe that

$$\Lambda_p^a(t, a) = \exp \left( - \int_0^a \lambda_0(\sigma - F(t))d\sigma \right) = \exp \left( - \int_{F(t)}^{a-F(t)} \lambda_0(\sigma)d\sigma \right) = \Lambda_0(a - F(t)), \quad (4.2)$$

with the convention that $\int_{-F(t)}^{0} \lambda_0(a)da = 0$. We have

$$\phi_p^a(t, a) = \lambda_0(t - F(t))\Lambda_0(t - F(t)) = \phi_0(a - F(t)). \quad (4.3)$$
For period model A, the perturbed schedules of survival rate and of incidence are given by the same age shift of the standard schedule. If we define the standard tempo and quantum by

\[
Q_0 := \int_0^\infty \phi_0(a)da = 1 - \Lambda_0(\omega),
\]

\[
A_0 := \frac{\int_0^\infty a\phi_0(a)da}{\int_0^\infty \phi_0(a)da} = \frac{1}{1 - \Lambda_0(\omega)} \left[ -\omega \Lambda_0(\omega) + \int_0^\infty \Lambda_0(a)da \right],
\]

then

\[
Q^a_p(t) = Q_0, \quad A^a_p(t) = F(t) + A_0.
\] (4.5)

In integrals of Eq. (4.4), the upper bound \( \omega \) indicates the maximum age of occurrence of the event. It changes to \( \omega + F(t) \) corresponding to the age shift of the integrand.

Consider period model B. Under the age-shift of the hazard rate, the survival rate satisfies the McKendrick equation:

\[
\frac{\partial \Lambda(t,a)}{\partial t} + \frac{\partial \Lambda(t,a)}{\partial a} = -\lambda_0(a - F(t))\Lambda(t,a). \tag{4.6}
\]

By solving Eq. (4.6) with the boundary condition \( \Lambda(t,0) = 1 \),

\[
\Lambda(t,a) = \exp \left( -\int_0^a \lambda_0(\sigma - F(t - a + \sigma))d\sigma \right). \tag{4.7}
\]

The incidence rate \( \phi(t,a) \) is calculated as

\[
\phi(t,a) = \lambda_0(a - F(t))\Lambda(t,a), \tag{4.8}
\]

and the quantum index is

\[
Q^b_p(t) = \int_0^\infty \phi(t,a)da = \int_0^\infty \lambda_0(a - F(t))\Lambda(t,a)da. \tag{4.9}
\]

The tempo index \( A^b_p(t) \) based on period model B is given by Eq. (2.21) using Eq. (4.8).

Finally consider period model C. From Eq. (4.7),

\[
Q^c_p(t) = 1 - \Lambda(t,\omega) = 1 - \exp \left( -\int_0^\omega \lambda_0(\sigma - F(t - a + \sigma))d\sigma \right), \tag{4.10}
\]

and the tempo \( A^c_p(t) \) is computed from Eq. (2.26). By summarizing Eq. (4.5), (4.9), and (4.10), we conclude that:

**Proposition 4.1.** If the period age shift occurs on the hazard rate, period tempi and quanta for three period models are calculated as
\[ Q_p^a(t) = Q_0 = 1 - \Lambda_0(\omega), \]
\[ A_p^a(t) = F(t) + A_0, \]
\[ Q_p^b(t) = \int_0^\omega \lambda_0(a - F(t)) \Lambda(t, a) da, \]
\[ A_p^b(t) = \frac{1}{Q_p^b(t)} \left( -\omega \Lambda_p^b(t, \omega) + \int_0^\omega \Lambda_p^b(t, a) da \right), \]
\[ Q_p^c(t) = 1 - \Lambda(t, \omega), \]
\[ A_p^c(t) = \frac{1}{1 - \Lambda(t, \omega)} \left( -\omega \Lambda(t, \omega) + \int_0^\omega \Lambda(t, a) da \right), \]

where \( \Lambda(t, a) \) is given by Eq. (4.7) and \( \Lambda_p^b(t, a) \) by Eq. (2.18) and (4.8) as
\[ \Lambda_p^b(t, a) = 1 - \int_0^a \lambda_0(\sigma - F(t)) \Lambda(t, \sigma) d\sigma. \]  

Under the period age shift on the hazard rate, the survival rate and the incidence rate cannot be expressed as age shift of the standard schedule. For period models B and C, there is no simple transformation formula as observed in period model A between tempo and quantum indices of the standard schedule and the perturbed schedules.

To calculate period tempo and quantum using Eq. (4.11) and (4.12), we need to specify the shift \( F(t) \). Consider the most simple case that \( F(t) \) is given by a linear function \( F(t) = kt \) with a shift speed \( k \). By simple calculation,
\[ \Lambda(t, a) = \exp \left( -\int_0^a \lambda_0(\sigma - F(t - a + \sigma)) d\sigma \right) \]
\[ = \exp \left( -\frac{1}{1 - k} \int_0^{a-kt} \lambda_0(\sigma) d\sigma \right) = \Lambda_0(a_0 - kt)^{\frac{1}{1-k}}, \]  

where we assume that the integral \( \int_0^{-k(t-a)} \lambda_0(\sigma) d\sigma \) is zero.

If \( Q_0 = 1 \), the quantum based on period model B is
\[ Q_p^b(t) = \int_0^\omega \phi(t, a) da = \int_0^\omega \lambda_0(a - kt) e^{-\frac{1}{1-k} \int_0^{a-kt} \lambda_0(\sigma) d\sigma} da \]
\[ = \left[ -(1 - k) e^{-\frac{1}{1-k} \int_0^{a-kt} \lambda_0(\sigma) d\sigma} \right]_0^\omega = 1 - k, \]  

where we assume that \( \int_0^{-kt} \lambda_0(a) da = \infty \) so as to make the quantum invariant. From Proposition 3.3, the righthand side of Eq. (4.14)
equals $1 - dA_p(t)/dt$. By direct calculation, $dA_p(t)/dt = k$. In fact,

$$
\dot{\lambda}_p(t,a) = \frac{\dot{\lambda}_0(a - kt)}{1 - k},
$$

$$
\phi_p(t,a) = \frac{\dot{\lambda}_0(a - kt)}{1 - k}\exp\left(-\int_0^{a-kt} \frac{\dot{\lambda}_0(\sigma)}{1 - k}d\sigma\right). \quad (4.15)
$$

As $\int_0^\infty \dot{\lambda}_0(a)da = \infty$,

$$
A_p(t) = \int_0^\infty a\phi_p(t,a)da = \int_0^\infty \Lambda_0(a)^{1/k}da + kt, \quad (4.16)
$$

which shows that the tempo of period model C also increases or decreases with constant speed $k$.

### 4.2. Period Age Shift on Incidence Rate

Subsequently, if the incidence rate has shifted from the standard schedule $\dot{\lambda}_0(a)$ to a new schedule $\dot{\lambda}_0(a - F(t))$ at time $t$ by the period age shift $F(t)$, the survival rate satisfies the McKendrick equation:

$$
\frac{\partial \Lambda(t,a)}{\partial t} + \frac{\partial \Lambda(t,a)}{\partial a} = -\phi_0(a - F(t)) = -\dot{\lambda}(t,a)\Lambda(t,a). \quad (4.17)
$$

The unknown hazard rate $\dot{\lambda}(t,a)$ is calculated by Eq. (2.10) as

$$
\dot{\lambda}(t,a) = \frac{\phi_0(a - F(t))}{1 - \int_0^{a} \phi_0(\sigma - F(t - a + \sigma))d\sigma}, \quad (4.18)
$$

and the survival rate is

$$
\Lambda(t,a) = 1 - \int_0^{a} \phi_0(\sigma - F(t - a + \sigma))d\sigma. \quad (4.19)
$$

As the standard schedules of survival and hazard rates are given by

$$
\Lambda_0(a) = 1 - \int_0^{a} \phi_0(\sigma)d\sigma, \quad \dot{\lambda}_0(a) = \frac{\phi_0(a)}{1 - \int_0^{a} \phi_0(\sigma)d\sigma}, \quad (4.20)
$$

then $\Lambda(t,a)$ and $\dot{\lambda}(t,a)$ cannot be expressed by the age shift of the standard schedules.

The quantum and tempo for period model A are obtained from Eq. (2.14) and (2.15) using Eq. (4.18). As the incidence rate is $\phi(t,a) = \phi_0(a - F(t))$, for period model B, from Eq. (2.20) and (2.21):

$$
Q_b^p(t) = 1 - \Lambda_0(\omega), \quad A_b^p(t) = A_0 + F(t). \quad (4.21)
$$

For period model C,

$$
Q_c^p(t) = 1 - \int_0^{\omega} \phi_0(\sigma - F(t - a + \sigma))d\sigma, \quad (4.22)
$$
and $A_c^p(t)$ is calculated from Eq. (4.19) and (2.26). We conclude that:

**Proposition 4.2.** If the period age shift occurs on the incidence rate, period tempi and quanta for three period models are calculated as

\[
\begin{align*}
Q_a^p(t) &= 1 - \exp\left(-\int_0^\omega \lambda(t, \sigma) d\sigma\right), \\
A_a^p(t) &= \int_0^\omega a\lambda(t, a) \exp\left(-\int_0^\omega \lambda(t, \sigma) d\sigma\right) da, \\
Q_b^p(t) &= Q_0 = 1 - \Lambda_0(\omega), \\
A_b^p(t) &= A_0 + F(t), \\
Q_c^p(t) &= 1 - \int_0^\omega \phi_0(\sigma - F(t - a + \sigma)) d\sigma, \\
A_c^p(t) &= \frac{1}{1 - \Lambda(t, \omega)} \left[-\omega\Lambda(t, \omega) + \int_0^\omega \Lambda(t, a) da\right],
\end{align*}
\]

where $\lambda(t, a)$ and $\Lambda(t, a)$ are given by Eq. (4.18) and (4.19).

### 4.3. Period Age Shift on Survival Rate

Consider the case when the survival rate has shifted from the standard schedule $\Lambda_0(a)$ to a new schedule $\Lambda_0(a - F(t))$ at time $t$ by the period age shift $F(t)$. The survival rate $\Lambda(t, a) = \Lambda_0(a - F(t))$ satisfies the McKendrick equation:

\[
\frac{\partial\Lambda(t, a)}{\partial t} + \frac{\partial\Lambda(t, a)}{\partial a} = - (1 - F'(t)) \lambda_0(a - F(t)) \Lambda(t, a)
\]  

where $F'(t) = dF(t)/dt$ is the speed of shift. The hazard and the incidence rate are

\[
\begin{align*}
\lambda(t, a) &= (1 - F'(t)) \lambda_0(a - F(t)), \\
\phi(t, a) &= (1 - F'(t)) \lambda_0(a - F(t)) \Lambda_0(a - F(t)) \\
&= (1 - F'(t)) \phi_0(a - F(t)).
\end{align*}
\]

The hazard and the incidence rates are given by age-shift and proportional transformation. Eq. (4.25) is meaningful if and only if $F'(t) < 1$, which means that the shift cannot outpace ageing.

Quantum and tempo for period model A can be obtained by inserting Eq. (4.25) into Eq. (2.14) and (2.15). For period model B, its quantum
and tempo are obtained by applying Eq. (4.26) to Eq. (2.20) and (2.21). For period model C, from Eq. (2.23) we know that

\[
\begin{align*}
\dot{\gamma}_c^c(t,a) &= \dot{\gamma}_0(a - F(t)), \\
\dot{\phi}_c^c(t,a) &= \dot{\gamma}_c^c(t,a)\Lambda(t,a) = \dot{\gamma}_0(a - F(t)).
\end{align*}
\]

(4.27)

For period model C, the incidence rate is given by the same age shift \( F(t) \) of the standard schedule. The quantum is invariant and the tempo is changed into \( A_0 + F(t) \). From Eq. (4.26)–(4.27), as \( \dot{\phi}_c^c(t,a) \) is proportional to \( \dot{\phi}_c^c(t,a) \), tempo indices of period model B and C are identical. We conclude that:

**Proposition 4.3.** If the period age shift occurs on the survival rate, period tempi and quanta for the three period models are calculated as

\[
\begin{align*}
Q_p^a(t) &= 1 - \Lambda_0(\omega)^{1-F(t)}, \\
A_p^a(t) &= F(t) + \frac{1}{Q_p^a(t)} \left( -\omega\Lambda_0(\omega)^{1-F(t)} + \int_0^\infty \Lambda_0(\omega)^{1-F(t)} d\omega \right), \\
Q_p^b(t) &= (1-F(t))Q_0 = (1-F(t))(1-\Lambda_0(\omega)), \\
A_p^b(t) &= F(t) + A_0, \\
Q_p^c(t) &= Q_0 = 1 - \Lambda_0(\omega), \\
A_p^c(t) &= F(t) + A_0.
\end{align*}
\]

(4.28)

In particular, \( A_p^b(t) = A_p^c(t) \).

### 4.4. Age Shift and the Proportionality Assumption

Bongaarts and Feeney (2003) claimed that the proportionality assumption is equivalent to the assumption of the period age shift of the survival rate \( \Lambda(t,a) \). Here we prove this statement. We assume that proportionality is satisfied on a finite time interval \((0,T)\), which is stronger than proportionality valid at a given time. Bongaarts and Feeney (2003) have shown that for a real human population, the proportionality assumption was satisfied at an observed period.

Assume that the proportionality assumption \( \dot{\lambda}(t,a) = p(t)\dot{\gamma}_p(t,a) \) holds true for a time interval \( t \in (0,T) \). From Eq. (3.5),

\[
\rho(t,a) = \dot{\lambda}(t,a) \left( \frac{1}{p(t)} - 1 \right),
\]

(4.29)

and from Eq. (3.2) and (3.3),
\[
\frac{1}{\Lambda(t,a)} \frac{\partial \Lambda(t,a)}{\partial t} = \lambda(t,a) \frac{1 - p(t)}{p(t)},
\]
(4.30)

\[
\frac{1}{\Lambda(t,a)} \frac{\partial \Lambda(t,a)}{\partial a} = - \frac{1}{p(t)} \lambda(t,a).
\]

Hence

\[
\frac{\partial \Lambda(t,a)}{\partial t} + (1 - p(t)) \frac{\partial \Lambda(t,a)}{\partial a} = 0.
\]
(4.31)

Define \( F(t) = F(0) + \int_0^t (1 - p(x)) dx, \ t \in (0, T) \) and let \( y = a - F(t) \). By differentiating \( \Lambda(t,a) = \Lambda(t,y + F(t)) \) with respect to \( t \),

\[
\frac{\partial}{\partial t} \Lambda(t,y + F(t)) = \frac{\partial \Lambda}{\partial t}(t,y + F(t)) + \frac{\partial \Lambda}{\partial a}(t,y + F(t)) \frac{dF(t)}{dt} = 0.
\]
(4.32)

As \( \Lambda(t,y + F(t)) \) is a function of only \( y \), there exists a function \( \Lambda_0 \) such that \( \Lambda(t,y + F(t)) = \Lambda_0(y) \). Using \( y = a - F(t) \),

\[
\Lambda(t,a) = \Lambda_0(a - F(t)),
\]
(4.33)

which shows that \( \Lambda(t,a) \) is given by the age shift \( F(t) \) of the standard schedule \( \Lambda_0(a) \).

Conversely, if Eq. (4.33) holds true, \( \Lambda \) satisfies Eq. (4.24) and the hazard rate is calculated as

\[
\lambda(t,a) = - \frac{1}{\Lambda(t,a)} \left( \frac{\partial \Lambda}{\partial t} + \frac{\partial \Lambda}{\partial a} \right) = (1 - F'(t)) \lambda_0(a - F(t)),
\]
(4.34)

where

\[
\lambda_0(a) := - \frac{1}{\Lambda_0(a)} \frac{d \Lambda_0(a)}{da}.
\]
(4.35)

Moreover

\[
\lambda^c_p(t,a) = - \frac{1}{\Lambda_0(a - F(t))} \frac{d \Lambda_0(a - F(t))}{da} = \lambda_0(a - F(t)).
\]
(4.36)

If \( p(t) = 1 - F'(t) \), then

\[
\lambda(t,a) = (1 - F'(t)) \lambda^c_p(t,a) = p(t) \lambda^c_p(t,a),
\]
(4.37)

which means that the proportionality assumption holds true. In summary, the proportionality assumption for a finite time interval is equivalent to the assumption of the period age shift on the survival rate.
4.5. Perturbation Scenario by Bongaarts and Feeney

We discuss the perturbation scenario by Bongaarts and Feeney (1998, 2002, 2003, 2006), which is formulated as period age shift on the survival rate.\(^1\)

First consider that the event of interest is death. Assume that the period survival rate is shifted from its standard \(s_0(a)\) to a new schedule \(s_0(a - F(t))\). As shown in subsection 4.4, the proportionality assumption holds true. If the age shift occurs during the time interval \(0 < t < T\) and there is no shift for \(t \leq 0\) and \(t \geq T\), the survival rate is changing as

\[
l(t, a) = \begin{cases} 
  s_0(a), & t \leq 0 \\
  s_0(a - F(t)), & 0 < t < T \\
  s_0(a - F(T)), & T \leq t,
\end{cases}
\]

where \(F\) is a piecewise continuous function, \(\omega = \infty\) and we adopt the convention that \(s_0(a) = 1\) and \(\mu_0(a) = 0\) for \(a < 0\), \(\mu_0(a)\) is the unper- turbed force of mortality. As the proportionality assumption holds true for the time interval \((0, T)\), the hazard rate for \(0 < t < T\) is calculated from Eq. (4.37) as

\[
\mu(t, a) = (1 - F'(t))\mu_0(a - F(t)).
\]

In the time domain \(t \leq 0\) and \(T \leq t\), as the hazard rate is fixed, the tempo indices of the three period models are identical:

\[
e^{a}_{p}(t) = \text{SMAD}(t) = \text{CAL}(t) = \int_{0}^{\infty} s_0(a)da, \quad t \leq 0,
\]

\[
e^{a}_{p}(t) = \text{SMAD}(t) = \text{CAL}(t) = F(T) + \int_{0}^{\infty} s_0(a)da, \quad t \geq T,
\]

where \(\text{SMAD}(t) [\text{CAL}(t)]\) denotes \(\text{SMAD} [\text{CAL}]\) at time \(t\).

For the time interval \(0 < t < T\),

\[
\text{SMAD}(t) = \text{CAL}(t) = F(t) + \int_{0}^{\infty} s_0(a)da.
\]

From the expression of \(A^{a}_{p}(t)\) in Eq. (4.28), we obtain

\[
e^{a}_{p}(t) = F(t) + \int_{0}^{\infty} s_0(a)^{1-F(t)}da.
\]

We know that \(\text{CAL}(t)\) and \(\text{SMAD}(t)\) are changing smoothly from \(e^{a}_{p}(0)\)

\(^1\)Early criticism to their argument and reply are in Bongaarts and Feeney (2000), Kim and Schoen (2000), van Imhoff and Keilman (2000), and van Imhoff (2001).
to $e_p^a(T) = e_p^a(0) + F(T)$ as time evolves. However, the difference between CAL($t$) and $e_p^a(t)$ is

$$\Delta(t) := e_p^a(t) - \text{CAL}(t) = e_p^a(t) - \text{SMAD}(t)$$

$$= \int_0^\infty (\ell_0(a)^{1-F(t)} - \ell_0(a))da,$$  \hspace{1cm} (4.43)

which could be very large if the shift speed $F'(t)$ is close to unity. This difference was designated as tempo bias in period life expectancy by Bongaarts and Feeney (2002, 2003).

However, as I mentioned in subsection 4.1–4.3, this difference is due to the difference in model assumptions. To call it a bias is misleading, because from the beginning, the period indices are defined from a hypothetical cohort, so there is no real index outside models.

From Proposition 3.4 and Eq. (3.17),

$$F'(t) = 1 - p(t) = \frac{de_p^b}{dt} = \frac{de_p^c}{dt} = 1 - Q_p^b(t).$$  \hspace{1cm} (4.44)

Bongaarts and Feeney (2003) define a hazard rate from which the tempo bias is eliminated as

$$\mu^a(t,a) := \frac{\mu(t,a)}{1 - \frac{de_p^a}{dt}} = \frac{\mu(t,a)}{Q_p^b(t)},$$  \hspace{1cm} (4.45)

where $\mu(t,a)$ is the observed force of mortality and $Q_p^b(t)$ is the total mortality rate. They state that the average age at death

$$\int_0^\infty \exp \left( - \int_0^a \frac{\mu(t,\sigma)}{1 - \frac{de_p^c}{dt}} d\sigma \right) da,$$  \hspace{1cm} (4.46)

is the tempo-adjusted life expectancy. From Eq. (4.27) and (4.39) we know that

$$\mu^a(t,a) = \mu_0(a - F(t)) = \mu_p^c(t,a),$$  \hspace{1cm} (4.47)

then Eq. (4.46) gives CAL by using the hazard rate $\mu_p^c(t,a)$ based on period model C. If the survival rate $\ell(t,a)$ is available, we calculate CAL directly by integrating $\ell(t,a)$. If $\ell(t,a)$ is unknown, we use $\mu^a$ as an estimate of the hazard rate $\mu_p^c(t,a)$.

Given a life table expectancy, CAL or SMAD depends on the availability of real data. Except when the whole population is at risk, it is difficult to obtain the hazard rate. When the hazard rate can be calculated, the survival and the incidence rates are also available. Then period model A is the most restrictive from the point of view of data
availability. Period model A can always be applied to real data without contradiction, but period model B and period model C cannot be applied consistently if the implicit hazard takes a negative value.

If the period age shift occurs on the survival rate, CAL and SMAD are changing smoothly, but the life table expectancy $e_p^a(t)$ changes rapidly and can take abnormal values if the shift speed is high. In such a case, it is inappropriate to interpret $e_p^a(t)$ as a realistic life expectancy of the population, and CAL or SMAD could give a better approximation to the real life expectancy. However, as Bongaarts and Feeney showed, the excess of $e_p^a(t)$ over CAL and SMAD in real data is not extraordinary under a gradual decrease of the death rate. In a realistic situation, the few years of difference between $e_p^a(t)$ and CAL (or SMAD) reflects only the fact that $e_p^a(t)$ depends only on current mortality while CAL and SMAD depend also on the past history of mortality. The sensitivity of $e_p^a(t)$ to the current change of mortality testifies better for mortality variation.

Consider the scenario of fertility. From Eq. (4.26), we know that the period age shift on the survival rate can be expressed as the period transformation of the incidence rate as

$$\phi(t, a) = \begin{cases} 
\phi_0(a), & t \leq 0 \\
(1 - F'(t))\phi_0(a - F(t)), & 0 < t < T, \\
\phi_0(a - F(T)), & T \leq t.
\end{cases} \quad (4.48)$$

The quantum of the (unperturbed) standard schedule $\phi_0(a) = \lambda_0(a)\Lambda_0(a)$, which would be observed if there were no period age shift on survival rate, is

$$Q_0 = \int_0^\infty \phi_0(a)da$$

$$= \frac{1}{1 - F'(t)}\int_0^\infty \phi(t, a)da = \frac{Q_p^b(t)}{1 - F'(t)}, \quad t \in (0, T). \quad (4.49)$$

If we consider first birth as the event of interest, Eq. (4.49) gives the tempo-adjusted total fertility rate introduced by Bongaarts and Feeney (1998), which is a counterfactual measure of what the period TFR would be if there were no period age shift on survival rate. Inaba (2003) pointed out that Bongaarts and Feeney (1998) implicitly use the scenario of Eq. (4.48) to induce the formula (4.49). Our point is that the special scenario of Eq. (4.48) is derived from the underlying assumption of period age shift on survival rate.

In summary, from the point of view of the survival model, Bongaarts and Feeney always use a scenario of period age shift on survival rate, and their argument is mathematically consistent. As far as
the proportionality assumption holds true, their translation formula is useful to reveal tempo effect in period demographic indices. However, the adjustment of fertility in Eq. (4.49) returns the quantum which would be observed if there were no age shift, while the adjusted life expectancy in Eq. (4.46) does not provide the life expectancy of the baseline mortality schedule, which would be observed if there were no age shift.

Remark 1. Cohort interpretation: assume that the scenario of Eq. (4.48) covers a sufficiently long period and the hazard rate $\left(1-F'(t)\right)\lambda_0(\alpha - F(t))$ is applied to any age class of a cohort. The cumulative hazard rate of a cohort born at time $T$ is

$$\int_0^\infty \lambda(T + \sigma, \sigma) d\sigma = \int_0^\infty (1-F'(T + \sigma))\lambda_0(\sigma - F(T + \sigma))d\sigma
= \int_0^\infty \lambda_0(z)dz, \quad (4.50)$$

where we changed the variables as $\sigma - F(T + \sigma) = z$, and assumed that $\lambda_0(z) = 0$ for $z < 0$. If $\lambda_0(a) = 0$ for $a > \omega$, from Eq. (2.5):

$$Q(T) = 1 - \exp\left(-\int_0^\omega \lambda_0(\sigma)d\sigma\right) = 1 - \Lambda_0(\omega), \quad (4.51)$$

which equals the quantum of the standard schedule observed before or after the occurrence of the age shift. Then the cohort quantum equals the quantum of the standard schedule.

From this result, Zeng and Land (2001) and Rodríguez (2006) derived a cohort interpretation for the Bongaarts–Feeney adjusted total fertility rate. The adjusted total fertility rate tells how many children the future cohort would have if the period age shift on the survival rate Eq. (4.48) would continue for a sufficiently long period, and the hazard rate $\left(1-F'(t)\right)\lambda_0(\alpha - F(t))$ would be applied to any age class of the cohort.

Similarly, if the scenario of Eq. (4.38) covers a sufficiently long period and the hazard rate $\ell_0(\alpha - F(t))$ is applied to any age class of a cohort, the cohort tempo, denoted by $e_c(T)$, is calculated as

$$e_c(T) = \int_0^\infty \ell_0(\sigma - F(\sigma + T))d\sigma. \quad (4.52)$$

If we assume that $F(t) = kt$, then

$$e_c(T) = \frac{1}{1-k} \left(kT + e_p^a(0)\right) = \frac{\text{CAL}(T)}{1-k}, \quad (4.53)$$
which shows that the synthetic cohort interpretation for the Bongaarts–Feeney scenario for death gives a new index $e_c(T)$, which is greater than CAL and SMAD and different from $e_p^a(t)$.

As shown by Goldstein (2006) and Rodriguez (2006), Eq. (4.53) implies

$$\text{CAL}(T + e_c(T)) = e_c(T),$$

(4.54)

because $\text{CAL}(t) = \text{CAL}(0) + kt$. If we choose $T$ as $T = t - e_c(T)$, the lagged cohort life expectancy $e_c(t - e_c(T))$ at time $t$, introduced by Bongaarts (2005), equals $\text{CAL}(t)$.

\textbf{Remark 2. Role of the Gompertz law:} The period age shift on the survival rate for death is produced by a uniform change of mortality, which is a reason why the period age shift on the survival rate is observed in reality.

If mortality is Gompertz $\mu_0(a) = \mu_0(0)e^{ba}$, with $b$ a positive constant, decreasing mortality at young age is negligible, and period age shift occurs on the survival rate, then from Eq. (4.37):

$$\mu(t, a) = p(t)\mu_0(a - F(t)) = p(t)\mu_0(0)e^{b(a - F(t))}.$$  

(4.55)

Introduce a function $h$ by

$$h(t) := -\frac{\ln p(t)}{b},$$

(4.56)

then $p(t) = e^{-bh(t)}$ and

$$\mu(t, a) = \mu_0(0)e^{b(a - F(t) - h(t))} = \mu_p^c(t, a - h(t))$$

$$= e^{-b(F(t) + h(t))} \times \mu_0(a),$$

(4.57)

Under the Gompertz law, the proportion $p(t)$ is expressed as an additional shift $h(t)$ in the standard schedule, so that the perturbed hazard rate $\mu(t, a)$ can be given only by the age shift of the standard schedule. Then

$$e_p^a(t) = \int_0^\infty \exp\left(- \int_0^a \mu(t, \sigma)d\sigma\right) = \int_0^\infty \exp\left(- \int_0^a \mu_p^c(t, \sigma - h(t))d\sigma\right)$$

$$= \int_{-h(t)}^\infty \exp\left(- \int_0^a \mu_p^c(t, \sigma)d\sigma\right) = h(t) + e_p^c(t),$$

(4.58)

where we use the adoption that $\mu_0(a) = 0$ for $a < 0$. Hence

$$e_p^a(t) - e_p^c(t) = h(t) = -\frac{\ln p(t)}{b} = -\frac{1}{b}\ln\left(1 - \frac{de_p^c(t)}{dt}\right),$$

(4.59)
which is the estimate of the difference between the period life table expectancy and CAL given by Bongaarts and Feeney (2002).

Conversely assume that the force of mortality is changing uniformly, so there is a positive function \( k(t) \) of time \( t \) such that \( \mu(t, a) = k(t)\mu_0(a) \). Under the Gompertz law \( \mu_0(a) = \mu_0(0)e^{ba} \), this mortality is produced by the period age shift on the survival rate if and only if there exists a shift function \( F(t) \) such that

\[
k(t) = (1 - F'(t))e^{-bF(t)}.
\]

This differential equation (4.60) is solved as

\[
F(t) = -\frac{1}{b} \ln \left( e^{-bt-bF(0)} + \int_0^t be^{-b(t-s)}k(s)ds \right).
\]

Therefore, any proportional change in the exponential mortality is expressed by the period age shift on the survival rate. If mortality is improved uniformly in reality, we observe a period age shift of the survival rate.

Consider a uniform proportional change of mortality brought by a new treatment prolonging life.\(^2\) Let \( \pi(a)(0 \leq \pi \leq 1) \) be the treatment rate which is applied at the time of occurrence of death. Let \( \mu_0(a) \) be the natural death rate for the non-treated population. The treatment occurs repeatedly whenever death occurs. Let \( \ell_n(a) \) be the survival rate after the \( n \)-th treatment, then

\[
\frac{d\ell_0(a)}{da} = -\mu_0(a)\ell_0(a),
\]

\[
\frac{d\ell_n(a)}{da} = \pi\mu_0(a)\ell_{n-1}(a) - \mu_0(a)\ell_n(a), \quad (n \geq 1)
\]

where \( \ell_0 \) denotes the survival rate for individuals who have never received the treatment. The initial data is given by \( \ell_0(0) = 1 \) and \( \ell_n(0) = 0 \) \( (n \geq 1) \). By solving Eq. (4.62) iteratively:

\[
\ell_0(a) = \exp \left( -\int_0^a \mu_0(\sigma)d\sigma \right),
\]

\[
\ell_n(a) = \ell_0(a)^n (-\pi \ln \ell_0(a))! n!.
\]

\(^2\)The model (4.62) was developed by Mitra (1979), Vaupel (2002, 2005) and Le Bras (2005).
Let $\ell(a)$ be the survival rate for the total population. Then
\[
\ell(a) = \sum_{n=0}^{\infty} \ell_n(a) = \ell_0(a) \sum_{n=0}^{\infty} \frac{(-\pi \ln \ell_0(a))^n}{n!}
= \ell_0(a) \exp(-\pi \ln \ell_0(a)) = \ell_0(a)^{1-\pi}.
\] (4.64)

The force of mortality for the total population is given by
\[
\mu(a) = -\frac{1}{\ell(a)} \frac{d\ell(a)}{da} = (1-\pi)\mu_0(a).
\] (4.65)

This equation shows that the uniform proportional change of mortality is obtained by iterative treatments.

5. TRANSLATION FORMULAE BETWEEN COHORT AND PERIOD INDICES

Since Ryder (1956, 1964), the purpose of translation theory in demography is to obtain cohort indices from period indices. The idea is that as demographic events occur for real cohorts, demographic indices observed for cohorts are more important than corresponding period indices, which only reflect cohort indices. In order to contrast Bongaarts and Feeney’s period-oriented idea of translation with the traditional cohort-oriented point of view, we review translation formulae between cohort and period indices focusing on the effect of age shift.

5.1. General Quantum Translation Formulae

First consider a general quantum translation formula between period and cohort (Foster, 1990). Decompose the incidence $\phi(t,a)$ at time $t$ and age $a$ as
\[
\phi(a + T, a) = Q(T)\psi_c(a; T),
\] (5.1)

where
\[
\psi_c(a; T) := \frac{\phi(a + T, a)}{Q(T)};
\] (5.2)
denotes the age-specific incidence rate of a cohort born at time $T$ normalized along the life line.

For the period quantum
\[
Q_p^b(t) = \int_0^\infty \phi(t,a)da,
\] (5.3)
a general relationship between the period and the cohort quantum is

$$Q_p^b(t) = \int_0^\infty Q(t - a) \psi_c(a; t - a) da. \quad (5.4)$$

Let $\psi_p(t, a)$ be the normalized pattern of the period incidence rate. Then

$$\phi(t, a) = Q_p^b(t) \psi_p(t, a), \quad (5.5)$$

and

$$Q(T) = \int_0^\infty \phi(T + a, a) da = \int_0^\infty Q_p^b(T + a) \psi_p(T + a, a) da, \quad (5.6)$$

which shows that the cohort quantum is an average period quantum. From Eq. (5.4), the period quantum is decomposed as

$$Q_p^b(t) = \int_0^\infty \psi_c(a; t - a) da \times \int_0^\infty w(t, a) Q(t - a) da, \quad (5.7)$$

where

$$w(t, a) := \frac{\psi_c(a; t - a)}{\int_0^\infty \psi_c(\sigma; t - \sigma) d\sigma}, \quad (5.8)$$

is the age-specific incidence pattern normalized again cross-sectionally at time $t$.

The first part of Eq. (5.7) is called the timing index, denoted by $TI(t)$ (Butz and Ward, 1979; Ryder, 1980b; Otani, 1993):

$$TI(t) := \int_0^\infty \psi_c(a; t - a) da. \quad (5.9)$$

The second part of Eq. (5.7) is called the average cohort quantum, denoted by $AQ(t)$. It is the weighted mean of the cohort quantum:

$$AQ(t) := \int_0^\infty w(t, a) Q(t - a) da. \quad (5.10)$$

$AQ(t)$ is the weighted harmonic mean of the cohort quantum:

$$AQ(t) = \frac{1}{\int_0^\infty \phi_0(t, \sigma) d\sigma}, \quad (5.11)$$

where the weight $\phi_0(t, a)$ is the normalized incidence rate at time $t$ defined by

$$\phi_0(t, a) := \frac{\phi(t, a)}{\int_0^\infty \phi(t, \sigma) d\sigma}. \quad (5.12)$$
The decomposition formula by Butz and Ward (1979) and Ryder (1980b) is

\[ Q^b_p(t) = A Q(t) \times TI(t). \tag{5.13} \]

If the cohort incidence rate \( \phi(a + T, a) \) is given by a linear combination as

\[ \phi(a + T, a) = \phi_0(a) + \phi_1(a) T, \tag{5.14} \]

where \( \phi_0 \) and \( \phi_1 \) are given functions of age, we have

\[ Q(t) = a_0 + a_1 t \]

\[ Q^b_p(t + A(t)) = Q(t) + a_1 A(t) - \int_0^\infty a \phi_1(a) da, \tag{5.15} \]

where

\[ a_0 := Q(0) = \int_0^\infty \phi_0(a) da, \quad a_1 := \int_0^\infty \phi_1(a) da. \tag{5.16} \]

Let \( A(t) \) be the average age of the occurrence of the event given by

\[ A(t) = \frac{1}{Q(t)} \int_0^\infty a \phi(a + t, a) da. \tag{5.17} \]

Then

\[ \frac{dA(t)}{dt} = - \frac{1}{Q(t)} \left( a_1 A(t) - \int_0^\infty \sigma \phi_1(\sigma) d\sigma \right). \tag{5.18} \]

Comparing Eq. (5.15) and (5.18),

\[ Q^b_p(t + A(t)) = \left( 1 - \frac{dA(t)}{dt} \right) Q(t), \tag{5.19} \]

which is Ryder’s translation formula (Ryder, 1964). However, there is no simple relation between period and cohort tempo indices under the linear assumption of Eq. (5.14).

### 5.2. Quantum Translation Under Age Shift

Consider the case when the age shift occurs along the cohort at a constant speed keeping the quantum constant:

\[ \phi(a + T, a) = Q \psi_0(a - kT), \tag{5.20} \]

where \( Q \) is the constant cohort quantum, \( \psi_0(a) \) is the normalized standard schedule of the incidence rate and \( k \) is the constant speed.
Define \( \psi_0(a) = 0 \) for \( a < 0 \). For the age-time domain \( t - a > 0 \), we have
\[
\phi(t, a) = Q \psi_0(a - k(t - a)) = Q \psi_0((1 + k)a - kt)
\]
\[
= Q \psi_0 \left( (1 + k) \left( a - \frac{k}{1 + k} t \right) \right), \quad t - a > 0.
\]
(5.21)

At time \( t > 0 \), the period incidence is given by the age-shift of a function \( Q \psi_0((1 + k)a) \) with constant speed \( k/(1 + k) \). The period quantum is
\[
Q^b_p(t) = \int_0^\infty \phi(t, a) \, da = \frac{Q}{1 + k} \int_0^t \psi_0(a) \, da + \int_t^\infty \phi(t, a) \, da,
\]
(5.22)
where we take \((0, \infty)\) as the integral interval to avoid that the interval is changing as the age-shift goes by. We know that for \( t > \omega \), the period quantum becomes constant:
\[
Q^b_p = \frac{Q}{1 + k}.
\]
(5.23)

From the point of view of the decomposition formula (5.13), \( 1/(1 + k) \) is the timing index.\textsuperscript{3}

Conversely, assume that the period incidence has a constant quantum \( Q^b_p \), that its schedule has a standard pattern \( \phi_0(a) \) and that it is age-shifted with a constant speed \( k^* \). Then the period incidence is \( \phi(t, a) = Q^b_p \phi_0(a - k^* t) \) at time \( t \). Hence
\[
\phi(a + T, a) = Q^b_p \phi_0((a - k^*(a + T))) = Q^b_p \phi_0 \left( (1 - k^*) \left( a - \frac{k^*}{1 - k^*} \right) \right),
\]
(5.26)
so we observe that the incidence distribution \( Q^b_p \phi_0((1 - k^*)a) \) is age-shifted with a constant speed \( k^*/(1 - k^*) \) along the cohort. The cohort quantum is
\[
Q = \int_0^\infty \phi(a + T, a) \, da = \frac{Q^b_p}{1 - k^*} \int_0^\infty \phi_0(a) \, da,
\]
(5.27)

\textsuperscript{3}If \( k(t - a) \) is sufficiently small and we use an approximation by the Taylor series
\[
\phi(t, a) \approx Q[\psi_0(a) - k(t - a)\psi'_0(a)],
\]
then from the same calculation as the Ryder formula (5.19), we have another translation formula
\[
Q^b_p = (1 - k)Q,
\]
(5.25)
which coincides with Eq. (5.23) if we neglect the second order term of \( k \).
which is a constant:

\[ Q = \frac{Q_p^b}{1 - k^*}. \tag{5.28} \]

In summary, the cohort age-shift of the incidence with a constant speed leads the period age-shift with a constant speed. Conversely, the period age-shift of the incidence with a constant speed leads the cohort age-shift with a constant speed. Then the cohort shift speed \( k \) is greater than the period shift speed \( k^*/C_3 \). The equation

\[ k = k^*/C_3 \]

is referred to as the period-cohort tempo equation (Zeng and Land, 2002). The age pattern of the cohort incidence is different from the period pattern. If \( k > 0 \), the incidence is ageing, the period quantum is smaller than the cohort quantum. If \( k < 0 \), the period quantum is larger than the cohort quantum.

If the event of interest is non-repeatable and described by the survival model, and if the cohort age-shift is given by Eq. (5.20), the hazard rate is estimated by Eq. (2.10) as

\[ \lambda(t, a) = \frac{Q\psi_0(a - k(t - a))}{1 - Q\int_0^a \psi_0(\sigma - k(t - a))d\sigma}. \tag{5.29} \]

By changing the variable in the denominator,

\[ \int_0^a \psi_0(\sigma - k(t - a))d\sigma = \int_{-k(t-a)}^{(1+k)a-kt} \psi_0(z)dz = \int_0^{(1+k)a-kt} \psi_0(z)dz, \tag{5.30} \]

where we assume that \( t - a > 0 \) for all \( a \) in the support of \( \psi_0 \). Then

\[ \lambda(t, a) = -\frac{1}{1+k} \frac{d}{da} \ln \left( 1 - Q\int_0^{(1+k)a-kt} \psi_0(z)dz \right). \tag{5.31} \]

Integrating Eq. (5.31),

\[ (1+k) \int_0^\infty \lambda(t, a)da = -\ln \left( 1 - Q\int_0^{\infty} \psi_0(z)dz \right) = -\ln(1 - Q), \tag{5.32} \]

and

\[ \left[ e^{-\int_0^\infty \lambda(t,a)da} \right]^{1+k} = (1 - Q_p)^{1+k} = 1 - Q. \tag{5.33} \]

Then we arrive at the translation formula by Keilman (1994):

**Proposition 5.1.** Let \( k \) be the constant speed of the cohort age-shift and let \( k^* = k/(1 + k) \) be the speed of the period age-shift. Then
\[ Q_p^a = 1 - (1 - Q)^{1/k} = 1 - (1 - Q)^{1-k}, \]
\[ Q = 1 - (1 - Q_p^a)^{1+k} = 1 - (1 - Q_p^a)^{1+k}. \]  

(5.34)

We decompose the difference between the period quantum \( Q_p^b \) and the cohort quantum as

\[ Q_p^b - Q = (Q_p^b - Q_p^a) + (Q_p^a - Q) =: \Delta_1 + \Delta_2. \]  

(5.35)

As

\[ \Delta_2 = Q_p^a - Q = (1 - Q) - (1 - Q)^{1/k}, \]  

(5.36)

we know that if the tempo is delayed \( (k > 0) \), then \( \Delta_2 < 0 \), while if the tempo is advanced \( (k < 0) \), then \( \Delta_2 > 0 \). From Eq. (5.23) and (5.34),

\[ \Delta_1 = Q_p^b - Q_p^a = \frac{Q}{1+k} - 1 + (1 - Q)^{1/k}, \]  

(5.37)

where the right hand side is a decreasing function of \( Q \) if \( k > 0 \), while it is an increasing function of \( Q \) if \( k < 0 \). Then if the tempo is delayed by age shift, \( \Delta_1 < 0 \), while if the tempo is advanced, \( \Delta_1 > 0 \).

If the incidence schedule is delayed by the age-shift with constant speed, then \( Q > Q_p^a > Q_p^b \), while if the schedule is advanced, then \( Q > Q_p^a > Q_p^b \). We arrive at the conclusion by Yamaguchi and Beppu (2004):

**Proposition 5.2.** If the incidence schedule is delayed or advanced by the age-shift with constant speed, the period quantum based on model A is closer to the cohort quantum than the period quantum based on model B.

### 5.3. Tempo Translation Under Age Shift

Under assumption (5.20), the average age at which the event occurs for the cohort born at time \( T \), denoted by \( A(T) \), is

\[ A(T) = \int_0^\infty a\psi_0(a-kT)da = kT + A(0). \]  

(5.38)

The average age or the tempo increases or decreases proportionally as the birth time of the cohort.

The period tempo based on model B is

\[ A_p^b(t) = \frac{\int_0^\infty a\phi(t,a)da}{\int_0^\infty \phi(t,a)da}. \]  

(5.39)
Then
\[
\int_0^\infty \phi(t,a)da = Q \int_0^\infty \phi_0(a - k(t-a))da = Q \int_0^\infty \phi_0(a) \frac{da}{1+k}, \tag{5.40}
\]
\[
\int_0^\infty a\phi(t,a)da = Q \int_0^\infty a\phi_0(a - k(t-a))da
\]
\[
= Q \int_0^\infty a + kt \phi_0(a) \frac{da}{1+k} \]
\[
= \frac{Q}{(1+k)^2} \left( \int_0^\infty a\phi_0(a)da + kt \int_0^\infty \phi_0(a)da \right), \tag{5.41}
\]
where we assume that all cohorts receive the same age-shift in Eq. (5.20) for the age interval where the period incidence is not zero. Therefore,
\[
A^b_p(t) = \frac{1}{1+k}(kt + A(0)). \tag{5.42}
\]
Eliminating the constant term \(A(0)\) from Eq. (5.38) and (5.42), we arrive at a tempo translation formula based on the cohort age shift:
\[
A(T) = k(T - t) + (1+k)A^b_p(t). \tag{5.43}
\]
Let \(T^*\) be the birth year of a cohort such that the period tempo \(A^b_p(t)\) observed at time \(t\) equals the tempo of the cohort born at time \(T^*\). Let \(A^b_p(t) = A(T^*)\) in Eq. (5.43) to solve for \(T^*\). Then
\[
T^* = t - A^b_p(t), \tag{5.44}
\]
which shows that if the age-shift occurs at a constant speed, the period tempo \(A^b_p(t)\) at time \(t\) equals the cohort tempo of the cohort born at \(t - A^b_p(t)\).

Conversely, assume that the change on the cohort is induced by the period change. The period incidence \(\phi(t,a)\) uses the normalized standard schedule \(\psi_0(a)\) as
\[
\phi(t,a) = Q^b_p\psi_0(a - k^*t), \tag{5.45}
\]
where \(k^*\) is the period shift speed. Then
\[
A^b_p(t) = A^b_p(0) + kt, \tag{5.46}
\]
and
\[
A(T) = \frac{\int_0^\infty a\phi(a + T,a)da}{\int_0^\infty \phi(a + T,a)da} = \frac{\int_0^\infty a\psi_0(a - k^*(a + T))da}{\int_0^\infty \psi_0(a - k^*(a + T))da}. \tag{5.47}
\]
We have

\[
\int_0^\infty \psi_0(a - k^*(a + T))da = \frac{1}{1 - k^*} \int_0^\infty \psi_0(a)da, \tag{5.48}
\]

\[
\int_0^\infty a\psi_0(a - k^*(a + T))da = \frac{1}{(1 - k^*)^2} \left( \int_0^\infty a\psi_0(a)da + k^*T \int_0^\infty \psi_0(a)da \right), \tag{5.49}
\]

and so

\[
A(T) = \frac{1}{1 - k^*} \left( A^b_p(0) + k^*T \right). \tag{5.50}
\]

By eliminating the constant term, we obtain a tempo transformation formula:

\[
A(T) = \frac{1}{1 - k^*} \left( A^b_p(t) + k^*(T - t) \right). \tag{5.51}
\]

If we transform the cohort shift speed \( k \) and the period shift speed \( k^* \) by using \( k^* = k/(1 + k) \), we know that Eq. (5.43) and (5.51) are the same transformation formulae. In summary,

**Proposition 5.3.** Assume that the incidence rate is age-shifted with a constant speed. \( A^b_p(t) \) and \( A(T) \) are mutually translated by (5.43) or (5.51). In particular,

\[
A(t - A^b_p(t)) = A^b_p(t). \tag{5.52}
\]

Eq. (5.52) is also written as

\[
A^b_p(T^* + A(T^*)) = A(T^*), \tag{5.53}
\]

where \( T^* = t - A^b_p(t) \) denotes the birth time of the cohort. If the event of interest is death and the age shift in Eq. (5.20) on incidence rate along the cohort is produced by the period age shift on the survival rate, the period tempo \( A^b_p(t) \) (SMAD) equals CAL and Eq. (5.53) implies that CAL at time \( T^* + A(T^*) \) equals the life expectancy of the cohort born at time \( T^* \), so we again arrive at Eq. (4.54).

6. CONCLUSION: PERIOD VERSUS COHORT

If parameters are time-dependent, there is no unique choice of the indices expressing tempo and quantum of life-cycle events. However,
we have two different points of view, cohort and period. The basic purpose of demographic translation formulae by Ryder was to compute cohort from period indices, because the tempo effect was originally considered as the effect of timing changes of the cohort schedule on the period schedule. The event schedule observed along a cohort corresponds to a real biological process such as aging, so it has a clear mathematical pattern. The period schedule appears as a synthetic result of cohort schedules.

In the short term, socio-economic environmental change can dominate the period event schedule for a while. For example, in Japan, from 1965 to 1966 the TFR dropped from 2.14 to 1.58, then rose again to 2.27 in 1967, because of a hino-e-uma superstition against childbearing in 1966. It is also possible to view the cohort schedule as a superposition of period schedules. Bongaarts and Feeney have developed a demographic translation theory between period indices without referring to any real cohort, which reflects the idea that the life-cycle event schedule is determined at each period.

Though the period-oriented translation theory is mathematically consistent, the assumptions of period survival models are not necessarily satisfied by demographic data. Even when they are applied to real data without contradiction, we ought to use a hypothetical cohort to interpret the period indices. Those hypothetical indices are not experienced by individuals, so that we cannot say which index is distorted in comparison with other period indices. The reference index, with respect to which we say that an index is distorted, should be calculated from the cohort model.

In the period model, it is often unclear what kind of change at the individual level perturbs the parameters. For example, as seen in section 4.5, thanks to the Gompertz law, the uniform improvement of individual mortality can induce a period age shift on the survival rate. If the event of interest is birth, however, it is unclear what kind of fertility change at individual level produces the period age shift on the survival rate. When we adopt a period perturbation scenario, we ought to check what kind of underlying process at the individual level produces the perturbation scenario at the population level.

REFERENCES


