Persistent Age Distributions for An Age-Structured Two-Sex Population Model*

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Abstract

In this paper we formulate an age-structured two-sex population model which takes into account a monogamous marriage rule and the duration of marriage. We are mainly concerned with the existence of exponential solutions with a persistent age distribution. First we provide a semigroup method to deal with the time-evolution problem of our two-sex population model. Next, by constructing a fixed point mapping, we prove the existence of exponential solutions under homogeneity conditions.

Key words: Two-Sex Population Dynamics, Marriage Model, Exponential Solutions, Persistent Age Distributions, Fixed Point Theorem, Semigroups.

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1 Introduction

For a long time, Lotka's stable population theory has played a role as a central dogma in demographic analysis. Life table analysis and the stable population theory have long dominated demographer's thought. In fact, in spite of its simple formula, stable age distributions could often be observed as far as fertility is stable as in developing countries after the World War II or historical population in developed countries. The reader may refer to Hoppensteadt (1975), Keyfitz (1977) and Pollard (1973) for classical results of stable population theory. However, today those traditional tools are no longer suited to addressing many issues that concern demographers, biologists and social scientists, because the classical stable population theory is a one-sex model and so it disregards interaction between both sexes. Although demographers have long been interested in more realistic description of the human reproduction process, their efforts have not been very successful until the end of 70s. However in the 80s the rapid spread of the HIV/AIDS virus has revived people's concern to pair formation phenomena of humans, and so it has strongly promoted the recent developments of bisexual population models. The reader interested in demographic-epidemic two-sex models may refer to Castillo-Chavez *et al.* (1991a,b), Dietz and Hadelar (1988), Hadelar (1989a, 1989b, 1992b) and Waldstätter (1990).

In modern societies, the monogamous marriage system has been working to control human fertility. Recently ex-nuptial phenomena (cohabitation, child-bearing outside of the normal marital institution) have been spreading among the West and hence the traditional linkage of nuptiality and fertility is loosened in a legal sense. Nevertheless, it would be still true even in the future that most human fertility will be realized within stable unions, no matter whether they are legally sanctioned or not. Therefore, for example, if we once want to investigate the causes and effects of the recent fertility change in developed or developing countries, it is obvious that instead of the one-sex theory, we need to construct more fine reproduction theory that could take into account marriage behavior.

For age-independent marriage models, we already have had successful mathematical results (Castillo-Chavez and Huang 1995; Hadelar *et al.* 1988; Waldstätter, 1990; Yellin and Samuelson, 1974). However, up to now, very little is known for the age-dependent dynamics of a two-sex population, although over a quarter of a century have already passed since the work of Fredrickson (1971) that, as far as I know, would be the first mathematical formulation of a two-sex nonlinear age structured population model in continuous time. Subsequently Staroverov (1977) formulated an age-duration dependent two-sex model. Waldstätter (1990) has first provided an existence and uniqueness theorem for solutions of the age-duration dependent two-sex model by using the integral equation approach. Martcheva and Milner (1999) adopted the same kind of method and provided more refined results. Recently several authors have shown that the semigroup method could be a most powerful tool to investigate the so-

lutions of age-structured two-sex population dynamics (Prüss and Schappacher 1994b; Matsumoto *et al.* 1996; Iannelli and Martcheva 1997). For a discrete-time nonlinear two-sex model, Pollak (1986, 1987, 1990) has already shown some conditions under which the existence of persistent age-distributions can be proved.

In Inaba (1993) we have proposed a new formulation for age-structured two-sex population dynamics and proved the well-posedness by using a semigroup approach. It has also been shown that this type of formulation could be useful to model HIV/AIDS epidemics in a bisexual population (Inaba 1997). However, in the paper of 1993, in order to prove the existence of exponential solutions, we have adopted the assumption that monogamous marriage occurs only between never married individuals and only those first marriage couples produce children. Though the marriage system satisfying these assumptions are certainly a oversimplification of reality, depending on areas and times this type of marriage system is not necessarily unrealistic and it was an important starting point to approach more realistic but complex two-sex models. For example, in Japan, even now 99 per cent of newborns are produced by formally married couples and the percentage of remarriages to total marriages had been less than 10 percent until 1982. Since then it has begun to increase, but it is still below 12 percent for females and 13 percent for males in 1995. The mean age at remarriage for women is over 37 in 1989. Therefore so far we could expect that the contribution of remarried couples to total births in Japan would be very small and almost all newborns are produced in legitimate first marriages. That is, the Japanese marriage market in present days approximately satisfies the above conditions (Inaba 1995).

However it is also clear that in most western countries the first marriage assumption mentioned above does not hold and is far from reality. Hence in this paper we remove this restrictive assumption and again consider the existence problem of exponential solutions for the nonlinear age-structured two-sex population model. This problem is partly answered by Prüss and Schappacher (1994a) in a slightly different formulation under the assumption that the marriage function is harmonic:

$$2\kappa(a, b) \frac{s_m(t, a)s_f(t, b)}{\int_0^\infty s_m(t, a)da + \int_0^\infty s_f(t, b)db},$$

where s_m and s_f denote the age density of the single males and females, respectively. Our proof will not depend on such a special formula for the marriage function.

In what follows, first we develop a general formulation of the two-sex marriage model and prove its well-posedness by using the semigroup method. Next we will show the existence of exponential solutions under homogeneity conditions. Finally we propose another formulation of the marriage model and discuss some open problems which could be approached by this alternative formulation.

2 The monogamous marriage model

Let $p_m(t, a)$, $p_f(t, b)$ be the age-densities of single males at age a and time t and single females at age b and time t , and let $s(t, \tau; \zeta, \eta)$ be the density of *type* (ζ, η) couples at time t and marital duration τ , where *type* (ζ, η) of a couple means that the age at marriage of husband is ζ and that of wife is η . Let $\rho(t, \zeta, \eta)$ be the age-density of newly married couples with male of age ζ and female of age η at time t . In the following, we assume that the joint distribution ρ is produced by the *marriage function* Ψ from single male and single female distributions such that

$$\rho(t, \zeta, \eta) = \Psi(p_m(t, *), p_f(t, *))(\zeta, \eta).$$

Let $\mu_m(a)$ and $\mu_f(a)$ be the male and the female natural death rate at age a respectively. Let $\delta(\tau; \zeta, \eta)$ be the *divorce rate* at duration τ of *type* (ζ, η) couples, let $\beta(\tau; \zeta, \eta)$ be the *marital fertility rate* of *type* (ζ, η) couples at duration τ and let γ be the ratio of female newborn children to total newborns which is assumed to be constant.

Using the above notations, the dynamic model for human population with a monogamous marriage system is formulated as follows:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) p_m(t, a) = -\mu_m(a)p_m(t, a) - \int_0^\infty \rho(t, a, \eta) d\eta \quad (2.1)$$

$$+ \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; a - \tau, \eta)] s(t, \tau; a - \tau, \eta) d\eta d\tau,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) p_f(t, a) = -\mu_f(a)p_f(t, a) - \int_0^\infty \rho(t, \zeta, a) d\zeta \quad (2.2)$$

$$+ \int_0^a \int_0^\infty [\mu_m(\tau + \zeta) + \delta(\tau; \zeta, a - \tau)] s(t, \tau; \zeta, a - \tau) d\zeta d\tau,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) s(t, \tau; \zeta, \eta) = -[\mu_m(\tau + \zeta) + \mu_f(\tau + \eta) + \delta(\tau; \zeta, \eta)] s(t, \tau; \zeta, \eta), \quad (2.3)$$

$$p_m(t, 0) = (1 - \gamma) \int_0^\infty \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) s(t, \tau; \zeta, \eta) d\zeta d\eta d\tau \quad (2.4)$$

$$p_f(t, 0) = \gamma \int_0^\infty \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) s(t, \tau; \zeta, \eta) d\zeta d\eta d\tau, \quad (2.5)$$

$$s(t, 0; \zeta, \eta) = \rho(t, \zeta, \eta) = \Psi(p_m(t, *), p_f(t, *))(\zeta, \eta). \quad (2.6)$$

If we add initial data to system (2.1)-(2.6), we obtain the initial boundary value problem for the population vector (p_m, p_f, s) .

The problem what kind of marriage functions are suitable to describe the human pair formation process has been long discussed among demographers. In particular, McFarland (1972), Pollard (1975, 1977) and Keilman (1985) examined the axioms which must be satisfied by the marriage function. For example we could raise tentative axioms as follows, though those would not necessarily form a complete system exhausting all of the features for Ψ :

- [1] $\Psi(u, v) \geq 0$ for $(u, v) \geq 0$,
- [2] $\Psi(u, 0) = \Psi(0, v) = 0$,
- [3] $\int_0^\infty \int_0^\infty \Psi(u, v)(\zeta, \eta) d\zeta d\eta \leq \int_0^\infty \int_0^\infty \Psi(u', v')(\zeta, \eta) d\zeta d\eta$, if $(u, v) \leq (u', v')$,
- [4] $\frac{\partial \Psi(u, v)(\zeta, \eta)}{\partial u(\xi)} \leq 0$ for $\xi \neq \zeta$ and $\frac{\partial \Psi(u, v)(\zeta, \eta)}{\partial v(\xi)} \leq 0$ for $\xi \neq \eta$,
- [5] $\Psi(cu, cv) = c\Psi(u, v)$ for $c \in R_+$.

Axiom [1]-[2] are self-evident. Axiom [3] is also reasonable because it shows that the greater supply of unmarried male and unmarried female populations would produce the greater number of new couples. Axiom [4] (where the differential is used in symbolic sense) is known as the *competition condition*, which tells that the marriage chance of one age group is not increased by an increase in the supply of other age groups in the same sex. Axiom [5] is the *homogeneity condition*, which reflects the saturation effect in large population, that is, individuals have only a limited number of social contacts with other individuals.

A simple, but important example for Ψ satisfying the axioms [1]-[5] is given by the proportionate mixing function

$$\Psi(u, v)(\zeta, \eta) = \frac{\theta(\zeta, \eta)g(\zeta)u(\zeta)h(\eta)v(\eta)}{\int_0^\infty g(\zeta)u(\zeta)d\zeta + \int_0^\infty h(\eta)v(\eta)d\eta}, \quad (2.7)$$

where $g(\zeta)$ and $h(\eta)$ reflect the preference of respectively male and female in the marriage market, and $\theta(\zeta, \eta)$ denotes the *coefficient of nuptiality* between male aged ζ and female aged η . This form of the marriage function would follow if marriages were contracted via random interaction between unmarried males and unmarried females in a large population (Fredrickson, 1971).

Here we should remark that our simple marriage model does not take into account some important aspects of the human population reproduction process. For example it disregards the parity structure (*parity* means the number of children a woman has had) of the population. The parity status of woman is thought to be one of the most important determinants of controlled human fertility. Since the parity structure of remarriage couples would be so different from that of first marriage couples and the marital fertility function β can be

seen as an aggregated function with respect to the parity-specific fertility schedule, it is a simplification to apply the same $\beta(a; \zeta, \eta)$ to both types of couples. We also assume that there is no difference between the marital behavior of never married singles and that of singles who have already experienced marriage. This assumption, too, is clearly not realistic.

3 Semigroup approach

Here we adopt an abstract formulation of system (2.1)-(2.6) by which a semi-group method can be applied to prove the well-posedness of the system.

Let us define a population vector as $p(t, a) := (p_m(t, a), p_f(t, a), s(t, a; \zeta, \eta))^\tau$ (τ denotes the transpose of the vector). Then it takes a value in a positive cone of a Banach space $E := R \times R \times L^1(\Omega)$, where $\Omega := R_+ \times R_+$ is the parameter space of type of couples. Then it is natural to assume that the state space of the population vector is $X := L^1(R_+ : E)$ with the following norm:

$$\|p\|_X = \int_0^\infty |p_m(a)|da + \int_0^\infty |p_f(a)|da + 2 \int_0^\infty \int_0^\infty \int_0^\infty |s(a; \zeta, \eta)|d\zeta d\eta da,$$

since $\|p\|_X$ is the total size of the population. Next define a mapping F from X to E and a mapping G from X into X as follows:

$$\begin{aligned} G(p)(a) &:= \\ &\begin{pmatrix} -\mu_m(a)p_m(a) - \int_0^\infty \Psi(p_m, p_f)(a, \eta)d\eta + \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; a - \tau, \eta)]s(\tau; a - \tau, \eta)d\eta d\tau \\ -\mu_f(a)p_f(a) - \int_0^\infty \Psi(p_m, p_f)(\zeta, a)d\zeta + \int_0^a \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, a - \tau)]s(\tau; \eta, a - \tau)d\eta d\tau \\ -[\mu_m(a + \zeta) + \mu_f(a + \eta) + \delta(a; \zeta, \eta)]s(a; \zeta, \eta) \end{pmatrix}, \\ F(p) &:= \begin{pmatrix} (1 - \gamma) \int_0^\infty \int_0^\infty \int_0^\infty \beta(a; \zeta, \eta)s(a; \zeta, \eta)d\zeta d\eta da \\ \gamma \int_0^\infty \int_0^\infty \int_0^\infty \beta(a; \zeta, \eta)s(a; \zeta, \eta)d\zeta d\eta da \\ \Psi(p_m, p_f)(\zeta, \eta) \end{pmatrix}, \end{aligned}$$

where $p(a) := (p_m(a), p_f(a), s(a; \zeta, \eta))^\tau \in X$. Here we adopt the following technical assumption:

Assumption 3.1 (1) $\mu_m, \mu_f \in L_+^\infty(R_+)$, $\delta, \beta \in L_+^\infty(R_+; L_+^\infty(\Omega))$.

(2) The marriage function Ψ is a bounded operator from $Y_+ := L_+^1(R_+) \times L_+^1(R_+)$ to $L_+^1(\Omega)$, and it is locally Lipschitz continuous. That is, there exists an increasing function $L(r)$ such that $\|\Psi(p) - \Psi(p')\|_{L^1(\Omega)} \leq L(r)\|p - p'\|_{Y_+}$ for all $p, p' \in Y_+$ such that $\|p\|_{Y_+} \leq r, \|p'\|_{Y_+} \leq r$ where $\|\cdot\|_{Y_+}$ denotes the L^1 norm in Y_+ .

(3) For any $(u, v) \in Y_+$, there exists a number $K > 0$ such that

$$\int_0^\infty \Psi(u, v)(a, \eta)d\eta \leq Ku(a), \quad \int_0^\infty \Psi(u, v)(\zeta, a)d\zeta \leq Kv(a).$$

Note that in this section our argument depends on the Assumption 3.1, but it is not necessarily restricted by the axioms on the marriage function discussed in the previous section. Though we omit the proof, it is easy to see that the following holds:

Lemma 3.2 *Under Assumption 3.1, G is a bounded operator from L_+^1 to L_+^1 and F is a bounded operator from L_+^1 to E_+ .*

Then we can rewrite the system (2.1)-(2.6) as a general formula in age-dependent population dynamics:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)p(t, a) = G(p(t, *))(a), \quad t > 0, a > 0, \quad (3.1)$$

$$p(t, 0) = F(p(t, *)), \quad t > 0, \quad (3.2)$$

$$p(0, a) = \phi(a), \quad a > 0, \quad (3.3)$$

where $\phi \in L_+^1$ is the initial data.

Though there are different approaches to solving the problem (3.1)-(3.3), here we follow the method of Thieme (1990, 1991). Here we remark that for the case $E = \mathbb{R}^n$, a semigroup approach to the age-dependent population dynamics model (3.1)-(3.3) was systematically developed by Webb (1985). Since his work, age-dependent (or structured) population dynamics has stimulated the recent developments in semigroup theory and evolution equations (Metz and Diekmann, 1986). Tucker and Zimmerman (1988), Grabosch (1989) and Thieme (1991) treated the case in which E is a Banach space.

Let us introduce an extended state space Z as $Z := E \times X$ and its closed subspace Z_0 by $Z_0 := \{0\} \times X$. Define an operator $\mathcal{A}$ acting on Z such that

$$\mathcal{A}(0, \psi) := (-\psi(0), -\psi') \text{ for } (0, \psi) \in D(\mathcal{A}) := \{0\} \times D(A), \quad (3.4)$$

where A is a differential operator acting on X defined by

$$(A\psi)(a) := -\frac{d}{da}\psi(a), \quad D(A) = \{\psi \in L^1 : \psi \in W^{1,1}\}, \quad (3.5)$$

and $W^{1,1} := \{\psi \in X : \psi \text{ is absolutely continuous, almost everywhere differentiable and } \psi' \in L^1\}$. Then the operator A is densely defined in X . Let $C_0 := \{0\} \times X_+$. Define a bounded perturbation $\mathcal{B} : C_0 \rightarrow X$ as

$$\mathcal{B}(0, \psi) = (F(\psi), G(\psi)) \text{ for } (0, \psi) \in C_0. \quad (3.6)$$

Note that $\mathcal{B}$ is not necessarily a positive operator for our system. Using the above definitions, we can formally rewrite system (3.1)-(3.3) as an abstract semilinear Cauchy problem with non-densely defined operator on Z :

$$\frac{d}{dt}u(t) = \mathcal{A}u(t) + \mathcal{B}u(t), \quad u(0) = (0, \phi) \in C_0. \quad (3.7)$$

Using the above abstract setting, we can prove the existence of an integral solution (a weak solution) for the problem (3.1)-(3.3).

Because we are looking for a density of population, we are interested in solutions of (3.7) such that $u(t) \in C_0$, $t \geq 0$. It has been observed by Busenberg *et al.* (1991) that when looking for positive solutions, the following system (3.8) is equivalent to but more convenient than (3.7):

$$\frac{d}{dt}u(t) = \left(\mathcal{A} - \frac{1}{\epsilon}\right)u(t) + \frac{1}{\epsilon}(I + \epsilon\mathcal{B})u(t), \quad u(0) = (0, \phi) \in C_0, \quad (3.8)$$

if ϵ can be chosen so small that the operator $I + \epsilon\mathcal{B}$ maps C_0 into Z_+ (Z_+ denotes the positive cone of Z). In fact, if so, we could solve the variation of constants formula corresponding to (3.8) by positive iteration. In our case, this procedure works well. From Assumption 3.1, it is easy to prove the following lemma, though we omit the proof.

Lemma 3.3 *If we choose ϵ such that*

$$0 < \epsilon < \min\left(\frac{1}{\bar{\mu} + K}, \frac{1}{\bar{\mu} + K}, \frac{1}{2\bar{\mu} + \delta}\right),$$

then it follows that

$$(I + \epsilon\mathcal{B})(C_0) \subset Z_+. \quad (3.9)$$

Lemma 3.4 *$\mathcal{A}$ is a closed linear operator with non dense domain and the following holds:*

- (1) $\overline{D(\mathcal{A})} = Z_0$.
- (2) $\mathcal{A}$ satisfies the Hille-Yosida estimate such that for all $\lambda > 0$

$$\|(\lambda - \mathcal{A})^{-1}\|_Z \leq \frac{1}{\lambda}. \quad (3.10)$$

- (3) $(\lambda - \mathcal{A})^{-1}(X_+) \subset C_0$ for $\lambda > 0$.

Proof: Result (1) directly follows from the fact that $\overline{D(\mathcal{A})} = X$. Next let $(\lambda - \mathcal{A})(0, \psi) = (x, y)$. Then it is easily seen that

$$\psi(a) = e^{-\lambda a}x + \int_0^a e^{-\lambda(a-\sigma)}y(\sigma)d\sigma. \quad (3.11)$$

Hence we know that $(\lambda - \mathcal{A})^{-1}$ exists for all $\lambda > 0$. Using (3.11), it is an easy calculation to show that the estimate (3.10) holds. Result (3) follows immediately from (3.11). $\square$

Since the operator $\mathcal{A}$ is not densely defined, we cannot apply the classical Hille-Yosida theory to solve the ordinary differential equation (3.7) or (3.8) in the Banach space Z . Hence we seek a solution in a weak sense:

Definition 3.5 (a) A function $u(t) \in C^1(0, T; Z) \cap D(\mathcal{A})$ is called a classical solution of the Cauchy problem (3.7) if (3.7) is satisfied for all $t \in [0, T]$.

(b) $u(t) \in C(0, T; Z_0)$ is called an integral solution of (3.7) if the following conditions are satisfied:

$$(1) \quad \int_0^t u(s) ds \in D(\mathcal{A}) \text{ for all } t \in [0, T],$$

$$(2) \quad u(t) = u(0) + \mathcal{A} \int_0^t u(s) ds + \int_0^t \mathcal{B}u(s) ds.$$

Then it is known that the integral solution becomes a classical solution if $u(0) \in D(\mathcal{A})$, $\mathcal{A}u(0) + \mathcal{B}u(0) \in \overline{D(\mathcal{A})}$ (Thieme, 1990, Theorem 3.7). Therefore, in what follows we are mainly concerned with the integral solutions of (3.8). Define the part $\mathcal{A}_0$ of $\mathcal{A}$ in Z_0 as

$$\mathcal{A}_0 = \mathcal{A} \text{ on } D(\mathcal{A}_0) = \{(0, \psi) \in D(\mathcal{A}) : \mathcal{A}(0, \psi) \in Z_0\}.$$

Then the following holds:

Lemma 3.6 (1) $\overline{D(\mathcal{A}_0)} = Z_0$.

(2) $\mathcal{A}_0$ generates a strongly continuous semigroup $\mathcal{T}_0(t)$, $t \geq 0$ on Z_0 and

$$\mathcal{T}_0(C_0) \subset C_0.$$

Proof: It is clear from the definition that $(0, \psi) \in D(\mathcal{A}_0)$ if and only if $\psi \in W^{1,1}$ and $\psi(0) = 0$. Define a differential operator A_0 such that

$$(A_0\psi)(a) = -\frac{d}{da}\psi(a), \quad D(A_0) = \{\psi \in X : \psi \in W^{1,1}, \psi(0) = 0\}.$$

Then A_0 is a generator of translation semigroup $T_0(t)$ on X such that

$$(T_0(t)\psi)(a) = \begin{cases} \psi(a-t), & \text{if } a \geq t, \\ 0, & \text{if } a < t, \end{cases}$$

and it follows that $D(\mathcal{A}_0) = \{0\} \times D(A_0)$. Thus we know that $\mathcal{A}_0$ is densely defined on Z_0 and it generates a semigroup $\mathcal{T}_0(t)$ as $\mathcal{T}_0(t)(0, \psi) = (0, T_0(t)\psi)$ for $(0, \psi) \in Z_0$. $\square$

Using the semigroup $\mathcal{T}_0(t)$, $t \geq 0$, we can formulate an extended variation of constants formula for (3.8) (Thieme, 1990, 1991):

Proposition 3.7 A positive function $u(t) \in C(0, T; Z_0)$ is an integral solution for (3.8) if and only if $u(t)$ is the positive continuous solution of the variation of constants formula on Z_0

$$u(t) = e^{-\frac{1}{\varepsilon}t} \mathcal{T}_0(t)u(0) \tag{3.12}$$

$$+ \lim_{\lambda \rightarrow \infty} \frac{1}{\epsilon} \int_0^t e^{-\frac{1}{\epsilon}(t-s)} \mathcal{T}_0(t-s) \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} (I + \epsilon \mathcal{B}) u(s) ds.$$

Proof: First we assume that $u(t)$ is a positive integral solution of the problem (3.8):

$$u(t) = u(0) + \left(\mathcal{A} - \frac{1}{\epsilon} \right) \int_0^t u(s) ds + \frac{1}{\epsilon} \int_0^t (I + \epsilon \mathcal{B}) u(s) ds. \quad (3.13)$$

By applying the resolvent operator $\lambda(\lambda - (\mathcal{A} - \frac{1}{\epsilon}))^{-1}$, $\lambda > 0$ to both sides of (3.13), we obtain

$$\begin{aligned} u_\lambda(t) &= u_\lambda(0) + \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} \left(\mathcal{A} - \frac{1}{\epsilon} \right) \int_0^t u(s) ds \\ &\quad + \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} \frac{1}{\epsilon} \int_0^t (I + \epsilon \mathcal{B}) u(s) ds, \end{aligned} \quad (3.14)$$

where $u_\lambda(t) := \lambda(\lambda - (\mathcal{A} - \frac{1}{\epsilon}))^{-1} u(t)$. Then it is easily checked that $u_\lambda \in D(\mathcal{A}_0) \cap C_0$. Since $\lambda(\lambda - (\mathcal{A} - \frac{1}{\epsilon}))^{-1}(\mathcal{A} - \frac{1}{\epsilon})$ is bounded, it follows that

$$\begin{aligned} \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} \left(\mathcal{A} - \frac{1}{\epsilon} \right) \int_0^t u(s) ds &= \int_0^t \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} \left(\mathcal{A} - \frac{1}{\epsilon} \right) u(s) ds \\ &= \int_0^t \left(\mathcal{A}_0 - \frac{1}{\epsilon} \right) \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} u(s) ds = \int_0^t \left(\mathcal{A}_0 - \frac{1}{\epsilon} \right) u_\lambda(s) ds. \end{aligned}$$

From the continuity of $(I + \epsilon \mathcal{B})u(t)$, the integral and the resolvent operator can be interchanged in the third part of the right hand side of (3.14). Therefore we arrive at the following equation on $C_0 \cap D(\mathcal{A}_0)$:

$$\begin{aligned} u_\lambda(t) &= u_\lambda(0) + \int_0^t \left(\mathcal{A}_0 - \frac{1}{\epsilon} \right) u_\lambda(s) ds \\ &\quad + \frac{1}{\epsilon} \int_0^t \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} (I + \epsilon \mathcal{B}) u(s) ds. \end{aligned} \quad (3.15)$$

Then we know that $u_\lambda(t)$ is a classical solution of the Cauchy problem on Z_0 :

$$\frac{d}{dt} u_\lambda(t) = \left(\mathcal{A}_0 - \frac{1}{\epsilon} \right) u_\lambda(t) + \frac{1}{\epsilon} \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} (I + \epsilon \mathcal{B}) u(s), \quad (3.16)$$

$$u_\lambda(0) = \lambda(\lambda - \mathcal{A})^{-1} u(0).$$

Since $\mathcal{A}_0$ is a generator of $\mathcal{T}_0(t)$ in Z_0 , we can apply the classical variation of constants formula to (3.16):

$$u_\lambda(t) = e^{-\frac{1}{\epsilon}t} \mathcal{T}_0(t) u_\lambda(0) \quad (3.17)$$

$$+ \frac{1}{\epsilon} \int_0^t e^{-\frac{1}{\epsilon}(t-s)} \mathcal{T}_0(t-s) \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} (I + \epsilon \mathcal{B}) u(s) ds.$$

It follows from $u(t) \in Z_0$ that $\lambda(\lambda - (\mathcal{A} - \frac{1}{\epsilon}))^{-1} u(t) \rightarrow u(t)$ ($\lambda \rightarrow \infty$). Then we have the formula (3.12). Next we assume that $u(t)$ is the positive continuous solution of (3.12). Let us consider a linear inhomogeneous Cauchy problem on Z :

$$\frac{d}{dt} v(t) = \left(\mathcal{A} - \frac{1}{\epsilon} \right) v(t) + \frac{1}{\epsilon} (I + \epsilon \mathcal{B}) u(s), \quad v(0) = u(0). \quad (3.18)$$

Then $(\mathcal{A} - \frac{1}{\epsilon})$ satisfies the Hille-Yosida conditions with the exception of the density of its domain in Z . From the Da Prato-Sinestrari theorem, the above equation (3.18) has a unique integral solution $v(t) \in Z$ such that

$$v(t) = u(0) + \left(\mathcal{A} - \frac{1}{\epsilon} \right) \int_0^t v(s) ds + \frac{1}{\epsilon} \int_0^t (I + \epsilon \mathcal{B}) u(s) ds. \quad (3.19)$$

By applying the same argument as above, we obtain the expression

$$v(t) = e^{-\frac{1}{\epsilon}t} \mathcal{T}_0(t) u(0) \quad (3.20)$$

$$+ \lim_{\lambda \rightarrow \infty} \frac{1}{\epsilon} \int_0^t e^{-\frac{1}{\epsilon}(t-s)} \mathcal{T}_0(t-s) \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} (I + \epsilon \mathcal{B}) u(s) ds,$$

which shows that $u(t) = v(t)$ for all $t \in [0, T]$. Then $u(t)$ is the unique integral solution for (3.8). $\square$

From Proposition 3.7, it is sufficient to solve the extended variation of constants formula (3.12) to obtain the integral solution of (3.8). From the proof of Proposition 3.7, we know that the operator $\mathcal{H}$ on $C(0, T; C_0)$ given by

$$\mathcal{H}(u(t)) := e^{-\frac{1}{\epsilon}t} \mathcal{T}_0(t) u(0) \quad (3.21)$$

$$+ \lim_{\lambda \rightarrow \infty} \frac{1}{\epsilon} \int_0^t e^{-\frac{1}{\epsilon}(t-s)} \mathcal{T}_0(t-s) \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} (I + \epsilon \mathcal{B}) u(s) ds,$$

is well defined. Therefore, without any essential modification to the proof for the classical variation of constants formula, using the local Lipschitz continuity of the bounded perturbation $\mathcal{B}$ and applying the contraction mapping principle to $\mathcal{H}$, we can show the existence of the positive local solution for the extended variation of constants formula (3.12) (Pazy, 1983, Chapter 6). Moreover, using the fact that

$$\| \mathcal{T}_0(t) \| \leq 1, \quad \left\| \lambda \left(\lambda - \left(\mathcal{A} - \frac{1}{\epsilon} \right) \right)^{-1} \right\| \leq 1,$$

we have

$$\| \mathcal{H}(u(t)) \| \leq \| u(0) \| + \alpha \int_0^t \| u(s) \| ds, \quad (3.22)$$

where $\alpha := \frac{1}{\epsilon} \| (I + \epsilon \mathcal{B}) \|$. Hence we know that the fixed point of $\mathcal{H}$ grows at most exponentially as time evolves. That is, the local solution can be extended to the global solution. Then we conclude that the initial boundary value problem (3.1)-(3.3) has a unique global positive integral solution. This argument leads us to the following existence and uniqueness theorem.

Proposition 3.8 *The initial boundary problem (3.1)-(3.3) has a unique global positive integral solution.*

4 Exponential solutions for the homogeneous model

In the following, we concentrate our attention to the existence problem of exponentially growing persistent solutions of system (2.1)-(2.6). In the real world, it can be observed during some limited period and in certain areas that large scale human populations show approximately exponential growth with a stable (or quasi-stable) age structure, even though it is of course impossible for such a growth to persist forever. However, even in such cases in reality human populations must have been reproduced by any marriage system. Thus it is a most interesting question whether two-sex marriage models could allow exponentially growing solutions with a stable age distribution.

To consider the above question we here assume the following technical assumptions for the marriage function:

Assumption 4.1 *The marriage function Ψ is homogeneous of degree one. Moreover there exists a number ω such that divorced individuals do not reenter the marriage market if the duration of their last marriage is larger than ω . We choose ω larger than the upper bound of reproductive age, that is, $\beta(\tau; \zeta, \eta) = 0$ for $\tau > \omega$.*

This assumption is adopted in order to avoid mathematical difficulties related to infinite age intervals, but it would not be so unrealistic if we choose a sufficiently large number ω . In other words, we assume that the marriage market is composed of singles who have never experienced a previous marriage with duration larger than ω . In the following, only single individuals acting in the marriage market are simply called "singles", and it is assumed that marriage occurs only among such singles.

Now we look for the exponential solutions (persistent solutions) of system (2.1)-(2.6) as

$$p_m(t, a) = e^{\lambda t} x(a), \quad p_f(t, a) = e^{\lambda t} y(a), \quad s(t, \tau; \zeta, \eta) = e^{\lambda t} z(\tau; \zeta, \eta). \quad (4.1)$$

(x, y, z) in (4.1) represent the persistent age distributions with Malthusian parameter λ . Of course our concern is to seek for non-trivial exponential solutions, though a trivial persistent solution $(x, y, z) = (0, 0, 0)$ always exists.

In order to look for the solution as (4.1), first we use the following normalization:

$$u(t, a) = \frac{p_m(t, a)}{\|p_m(t)\|_{L^1} + \|p_f(t)\|_{L^1}}, \quad v(t, a) = \frac{p_f(t, a)}{\|p_m(t)\|_{L^1} + \|p_f(t)\|_{L^1}},$$

$$w(t, \tau; \zeta, \eta) = \frac{s(t, \tau; \zeta, \eta)}{\|p_m(t)\|_{L^1} + \|p_f(t)\|_{L^1}}.$$

Using Assumption 4.1, the system (2.1)-(2.6) can be written as a (u, v, w) -system:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) u(t, a) = -(\mu_m(a) + \lambda(t))u(t, a) - \int_0^\infty \Psi(u(t), v(t))(a, \eta) d\eta \quad (4.2)$$

$$+ \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; a - \tau, \eta)] H(\omega - \tau) w(t, \tau; a - \tau, \eta) d\eta d\tau,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) v(t, a) = -(\mu_f(a) + \lambda(t))v(t, a) - \int_0^\infty \Psi(u(t), v(t))(\eta, a) d\eta \quad (4.3)$$

$$+ \int_0^a \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, a - \tau)] H(\omega - \tau) w(t, \tau; \eta, a - \tau) d\eta d\tau,$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) w(t, \tau; \zeta, \eta) = -(\lambda(t) + \delta(\tau; \zeta, \eta) + \mu_m(\tau + \zeta) + \mu_f(\tau + \eta))w(t, \tau; \zeta, \eta), \quad (4.4)$$

$$u(t, 0) = (1 - \gamma) \int_0^\omega \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) w(t, \tau; \zeta, \eta) d\zeta d\eta d\tau, \quad (4.5)$$

$$v(t, 0) = \frac{\gamma}{1 - \gamma} u(t, 0), \quad (4.6)$$

$$w(t, 0, \zeta, \eta) = \Psi(u(t), v(t))(\zeta, \eta), \quad (4.7)$$

where $\lambda(t)$ is given by

$$\lambda(t) = \int_0^\omega \int_0^\infty \int_0^\infty (\beta(\tau; \zeta, \eta) + \mu_m(\tau + \zeta) + \mu_f(\tau + \eta) + 2\delta(\tau; \zeta, \eta)) w(t, \tau; \zeta, \eta) d\zeta d\eta d\tau \quad (4.8)$$

$$- \int_0^\infty \mu_m(a) u(t, a) da - \int_0^\infty \mu_f(a) v(t, a) da - 2 \int_0^\infty \int_0^\infty \Psi(u(t, *), v(t, *))(\zeta, \eta) d\zeta d\eta,$$

and $H(\tau)$ denotes the Heaviside unit function such that

$$H(\tau) = \begin{cases} 0, & \tau \leq 0 \\ 1, & \tau > 0 \end{cases}.$$

Then $\lambda(t)$ gives the growth rate of the size of single populations. If there exists an exponential solution $e^{\lambda t}(x, y, z)$, the following time-independent problem for (4.9)-(4.15) must have a solution (u, v, w) with $\|u\|_{L^1} + \|v\|_{L^1} = 1$:

$$u'(a) = -(\mu_m(a) + \lambda)u(a) - \int_0^\infty \Psi(u, v)(a, \eta) d\eta \quad (4.9)$$

$$+ \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; a - \tau, \eta)] H(\omega - \tau) w(\tau; a - \tau, \eta) d\eta d\tau,$$

$$v'(a) = -(\mu_f(a) + \lambda)v(a) - \int_0^\infty \Psi(u, v)(\eta, a) d\eta \quad (4.10)$$

$$+ \int_0^a \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, a - \tau)] H(\omega - \tau) w(\tau; \eta, a - \tau) d\eta d\tau,$$

$$\frac{\partial}{\partial \tau} w(\tau; \zeta, \eta) = -(\lambda + \delta(\tau; \zeta, \eta) + \mu_m(\tau + \zeta) + \mu_f(\tau + \eta)) w(\tau; \zeta, \eta), \quad (4.11)$$

$$u(0) = (1 - \gamma) \int_0^\omega \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) w(\tau; \zeta, \eta) d\zeta d\eta d\tau, \quad (4.12)$$

$$v(0) = \frac{\gamma}{1 - \gamma} u(t, 0), \quad (4.13)$$

$$w(0; \zeta, \eta) = \Psi(u, v)(\zeta, \eta), \quad (4.14)$$

$$\begin{aligned} \lambda = & \int_0^\omega \int_0^\infty \int_0^\infty (\beta(\tau; \zeta, \eta) + \mu_m(\tau + \zeta) + \mu_f(\tau + \eta) + 2\delta(\tau; \zeta, \eta)) w(\tau; \zeta, \eta) d\zeta d\eta d\tau \\ & - \int_0^\infty \mu_m(a) u(a) da - \int_0^\infty \mu_f(a) v(a) da - 2 \int_0^\infty \int_0^\infty \Psi(u, v)(\zeta, \eta) d\zeta d\eta, \end{aligned} \quad (4.15)$$

Conversely it is clear that if there exists a positive solution (u, v, w) of (4.9)-(4.15) with $\|u\|_{L^1} + \|v\|_{L^1} = 1$, $e^{\lambda t}(u, v, w)$ is an exponential solution of the original system (2.1)-(2.6).

It follows from (4.11) and (4.14) that

$$w(\tau; \zeta, \eta) = e^{-\lambda \tau} \Gamma(\tau; \zeta, \eta) \Psi(u, v), \quad (4.16)$$

where $\Gamma(\tau; \zeta, \eta)$ is the survival rate of type (ζ, η) pairs defined by

$$\Gamma(\tau; \zeta, \eta) = \frac{\ell_m(\zeta + \tau)}{\ell_m(\zeta)} \frac{\ell_f(\eta + \tau)}{\ell_f(\eta)} \exp \left(- \int_0^\tau \delta(\sigma; \zeta, \eta) d\sigma \right),$$

where $\ell_m(a)$, $\ell_f(a)$ are male and female survival rates given by

$$\ell_m(a) = \exp\left(-\int_0^a \mu_m(\sigma) d\sigma\right), \quad \ell_f(a) = \exp\left(-\int_0^a \mu_f(\sigma) d\sigma\right)$$

Using (4.16) we can rewrite the above (u, v, w) system into a (u, v, λ) system as follows:

$$u'(a) = -(\mu_m(a) + \lambda)u(a) - \int_0^\infty \Psi(u, v)(a, \eta) d\eta \quad (4.17)$$

$$+ \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; a - \tau, \eta)] H(\omega - \tau) e^{-\lambda\tau} \Gamma(\tau; a - \tau, \eta) \Psi(u, v)(a - \tau, \eta) d\eta d\tau,$$

$$v'(a) = -(\mu_f(a) + \lambda)v(a) - \int_0^\infty \Psi(u, v)(\eta, a) d\eta \quad (4.18)$$

$$+ \int_0^a \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, a - \tau)] H(\omega - \tau) e^{-\lambda\tau} \Gamma(\tau; \eta, a - \tau) \Psi(u, v)(\eta, a - \tau) d\eta d\tau,$$

$$u(0) = (1 - \gamma) \int_0^\omega \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) e^{-\lambda\tau} \Gamma(\tau; \zeta, \eta) \Psi(u, v)(\zeta, \eta) d\zeta d\eta d\tau, \quad (4.19)$$

$$v(0) = \frac{\gamma}{1 - \gamma} u(t, 0), \quad (4.20)$$

$$\lambda = \int_0^\omega \int_0^\infty \int_0^\infty (\beta(\tau; \zeta, \eta) + \mu_m(\tau + \zeta) + \mu_f(\tau + \eta) + 2\delta(\tau; \zeta, \eta)) \quad (4.21)$$

$$\times e^{-\lambda\tau} \Gamma(\tau; \zeta, \eta) \Psi(u, v)(\zeta, \eta) d\zeta d\eta d\tau$$

$$- \int_0^\infty \mu_m(a) u(a) da - \int_0^\infty \mu_f(a) v(a) da - 2 \int_0^\infty \int_0^\infty \Psi(u, v)(\zeta, \eta) d\zeta d\eta.$$

Note that in this new system, though w disappears, the parameter λ is not a functional of (u, v, w) but is acting as an independent variable.

Let us define the state space as

$$U_+ = \{(u, v) \in L_+^1(R_+) \times L_+^1(R_+) : \|u\|_{L^1} + \|v\|_{L^1} = 1\}.$$

Then if there exists a solution $(u^*, v^*, \lambda^*) \in U_+ \times R$ of the above system (4.17)-(4.21), an exponential solution of the original system is given by $\exp(\lambda^* t)(u^*, v^*, w^*)$ with $w^*(\tau; \zeta, \eta) = e^{-\lambda^* \tau} \Gamma(\tau; \zeta, \eta) \Psi(u^*, v^*)(\zeta, \eta)$.

For a small number $\epsilon > 0$, let us consider the following map $\Phi_\epsilon = (\Phi_\epsilon^1, \Phi_\epsilon^2, \Phi_\epsilon^3)$ defined on $L_+^1(R_+) \times L_+^1(R_+) \times R$:

$$\Phi_\epsilon^1(u, v, \lambda)(a) = (1 - \gamma) B(u, v, \lambda) e^{-\frac{1}{\epsilon} a} \quad (4.22)$$

$$\begin{aligned}
& + \frac{1}{\epsilon} \int_0^a e^{-\frac{1}{\epsilon}(a-s)} \left\{ u(s) - \epsilon(\mu_m(s) + \lambda)u(s) - \epsilon \int_0^\infty \Psi(u, v)(s, \eta) d\eta \right. \\
& + \epsilon \int_0^s \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; s - \tau, \eta)] H(\omega - \tau) e^{-\lambda\tau} \Gamma(\tau; s - \tau, \eta) \Psi(u, v)(s - \tau, \eta) d\eta d\tau \Big\} ds, \\
\Phi_\epsilon^2(u, v, \lambda)(a) &= \gamma B(u, v, \lambda) e^{-\frac{1}{\epsilon}a} \tag{4.23} \\
& + \frac{1}{\epsilon} \int_0^a e^{-\frac{1}{\epsilon}(a-s)} \left\{ v(s) - \epsilon(\mu_f(s) + \lambda)v(s) - \epsilon \int_0^\infty \Psi(u, v)(\eta, s) d\eta \right. \\
& + \epsilon \int_0^s \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, s - \tau)] H(\omega - \tau) e^{-\lambda\tau} \Gamma(\tau; \eta, s - \tau) \Psi(u, v)(\eta, s - \tau) d\eta d\tau \Big\} ds, \\
\Phi_\epsilon^3(u, v, \lambda) &= \int_0^\omega \int_0^\infty \int_0^\infty (\beta(a; \zeta, \eta) + \mu_m(a + \zeta) + \mu_f(a + \eta) + 2\delta(a; \zeta, \eta)) \tag{4.24} \\
& \times e^{-\lambda a} \Gamma(a; \zeta, \eta) \Psi(u, v)(\zeta, \eta) d\zeta d\eta da \\
& - \int_0^\infty \mu_m(a) u(a) da - \int_0^\infty \mu_f(a) v(a) da - 2 \int_0^\infty \int_0^\infty \Psi(u, v)(\zeta, \eta) d\zeta d\eta,
\end{aligned}$$

where the functional B is given by

$$B(u, v, \lambda) := \int_0^\omega \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) e^{-\lambda\tau} \Gamma(\tau; \zeta, \eta) \Psi(u, v)(\zeta, \eta) d\zeta d\eta d\tau. \tag{4.25}$$

If the above map Φ_ϵ is well defined, Φ_ϵ^1 and Φ_ϵ^2 are differentiable and it follows that

$$\begin{aligned}
\frac{d}{da} \Phi_\epsilon^1(a) &= -\frac{1}{\epsilon} \Phi_\epsilon^1(a) + \frac{1}{\epsilon} \left\{ u(a) - \epsilon(\mu_m(a) + \lambda)u(a) - \epsilon \int_0^\infty \Psi(u, v)(s, \eta) d\eta \right. \\
& + \epsilon \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; s - \tau, \eta)] H(\omega - \tau) e^{-\lambda\tau} \Gamma(\tau; s - \tau, \eta) \Psi(u, v)(s - \tau, \eta) d\eta d\tau \Big\}.
\end{aligned}$$

The same kind of equation holds for Φ_ϵ^2 . Then it can be seen that if Φ_ϵ has a fixed point $(u, v, \lambda) \in U_+ \times R$, it is no other than a solution of the (u, v, λ) system (4.17)-(4.21) for which we are looking. First we prove the following :

Lemma 4.2 *There exist numbers $\underline{\lambda}$ and $\bar{\lambda}$ such that*

$$\underline{\lambda} \leq \Phi_\epsilon^3(u, v, \lambda) \leq \bar{\lambda}, \quad \forall (u, v, \lambda) \in U_+ \times [\underline{\lambda}, \bar{\lambda}]. \tag{4.26}$$

Moreover if we choose $\epsilon > 0$ such that

$$0 < \epsilon < \frac{1}{\bar{\mu} + \bar{\lambda} + K}, \tag{4.27}$$

then Φ_ϵ is a continuous operator from $U_+ \times [\underline{\lambda}, \bar{\lambda}]$ into $L_+^1(R_+) \times L_+^1(R_+) \times [\underline{\lambda}, \bar{\lambda}]$.

Proof: From our Assumption 3.1, it follows that

$$\int_0^\infty \int_0^\infty \Psi(u, v)(\zeta, \eta) d\zeta d\eta \leq K \|u\|_{L^1} \leq K, \quad \text{if } (u, v) \in U_+.$$

Using the above inequality, for $(u, v, \lambda) \in U_+ \times R$, we have

$$\Phi_\epsilon^3(u, v, \lambda) \geq -\bar{\mu}(\|u\|_{L^1} + \|v\|_{L^1}) - 2K = -(\bar{\mu} + 2K),$$

where $\bar{\mu} = \sup_{a \geq 0} \{\mu_m(a), \mu_f(a)\}$. Then we can choose $\underline{\lambda}$ as $\underline{\lambda} = -(\bar{\mu} + 2K)$. Next observe that

$$\Phi_\epsilon^3(u, v, \lambda) \leq (\bar{\beta} + 2\bar{\mu} + 2\bar{\delta}) \int_0^\omega \int_0^\infty \int_0^\infty e^{-\lambda\tau} \Psi(u, v)(\zeta, \eta) d\zeta d\eta d\tau,$$

where $\bar{\beta} = \sup \beta(\tau; \zeta, \eta)$, $\bar{\delta} = \sup \delta(\tau; \zeta, \eta)$. Therefore for any $(u, v, \lambda) \in U_+ \times [\underline{\lambda}, \infty)$ we arrive at

$$\Phi_\epsilon^3(u, v, \lambda) \leq K(\bar{\beta} + 2\bar{\mu} + 2\bar{\delta}) \int_0^\omega e^{-\lambda\tau} d\tau \leq K(\bar{\beta} + 2\bar{\mu} + 2\bar{\delta}) \frac{1 - e^{-\lambda\omega}}{\lambda}.$$

Thus if we define $\bar{\lambda}$ as

$$\bar{\lambda} = K(\bar{\beta} + 2\bar{\mu} + 2\bar{\delta}) \frac{e^{(\bar{\mu} + 2K)\omega} - 1}{\bar{\mu} + 2K},$$

then we can conclude that $\Phi_\epsilon^3(u, v, \lambda) \in [\underline{\lambda}, \bar{\lambda}]$ for all $(u, v, \lambda) \in U_+ \times [\underline{\lambda}, \bar{\lambda}]$.

Subsequently observe that if $(u, v, \lambda) \in U_+ \times [\underline{\lambda}, \bar{\lambda}]$, it follows that

$$u(a) - \epsilon(\mu_m(a) + \lambda)u(a) - \epsilon \int_0^\infty \Psi(u, v)(a, \eta) d\eta \geq u(a)(1 - \epsilon(\bar{\mu} + \bar{\lambda}) - \epsilon K).$$

The same kind of inequality holds for v , so if we choose ϵ in advance so small that (4.27) is satisfied, then we have $\Phi_\epsilon^1(u, v, \lambda) \geq 0$ and $\Phi_\epsilon^2(u, v, \lambda) \geq 0$.

Finally for $(u, v, \lambda), (u', v', \lambda') \in U_+ \times [\underline{\lambda}, \bar{\lambda}]$, it follows from Assumption 3.1 that the following estimate holds:

$$\begin{aligned} & \|\Phi_\epsilon^1(u, v, \lambda) - \Phi_\epsilon^1(u', v', \lambda')\|_{L^1} \\ & \leq (1 + \epsilon(\bar{\mu} + \bar{\lambda}))\|u - u'\|_{L^1} + \epsilon L(1)\{\|u - u'\|_{L^1} + \|v - v'\|_{L^1}\} \\ & \quad + \epsilon((1 - \gamma)\bar{\beta} + \bar{\mu} + \bar{\delta}) \frac{K\omega^2 e^{-\lambda\omega}}{2} |\lambda - \lambda'| + \epsilon \frac{1 - e^{-\lambda\omega}}{\lambda} \{\|u - u'\|_{L^1} + \|v - v'\|_{L^1}\}. \end{aligned}$$

Since the same kind of estimate can be established for Φ_ϵ^2 and Φ_ϵ^3 , we can conclude that Φ_ϵ is a continuous operator from $U_+ \times [\underline{\lambda}, \bar{\lambda}]$ into $L_+^1(R_+) \times L_+^1(R_+) \times [\underline{\lambda}, \bar{\lambda}]$. $\square$

Lemma 4.3 *Let $(u, v, \lambda) \in U_+ \times [\underline{\lambda}, \bar{\lambda}]$ and (4.27) be satisfied. Then the following relation holds:*

$$\|\Phi_\epsilon^1(u, v, \lambda)\|_{L^1} + \|\Phi_\epsilon^2(u, v, \lambda)\|_{L^1} = (1 - \epsilon\lambda)(\|u\|_{L^1} + \|v\|_{L^1}) + \epsilon\Phi_\epsilon^3(u, v, \lambda). \quad (4.28)$$

Proof: Since $\Phi_\epsilon^1(a) = \Phi_\epsilon^1(u, v, \lambda)(a)$ and $\Phi_\epsilon^2(a) = \Phi_\epsilon^2(u, v, \lambda)(a)$ are differentiable, we obtain

$$\begin{aligned} \frac{d}{da}\Phi_\epsilon^1(a) + \frac{1}{\epsilon}\Phi_\epsilon^1(a) &= \frac{1}{\epsilon} \left[u(a) - \epsilon(\mu_m(a) + \lambda)u(a) - \epsilon \int_0^\infty \Psi(u, v)(a, \eta) d\eta \right. \\ &\quad \left. + \epsilon \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; a - \tau, \eta)] H(\omega - \tau) e^{-\lambda\tau} \Gamma(\tau; a - \tau, \eta) \Psi(u, v)(a - \tau, \eta) d\eta d\tau \right], \\ \frac{d}{da}\Phi_\epsilon^2(a) + \frac{1}{\epsilon}\Phi_\epsilon^2(a) &= \frac{1}{\epsilon} \left[v(a) - \epsilon(\mu_f(a) + \lambda)v(a) - \epsilon \int_0^\infty \Psi(u, v)(\eta, a) d\eta \right. \\ &\quad \left. + \epsilon \int_0^a \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, a - \tau)] H(\omega - \tau) e^{-\lambda\tau} \Gamma(\tau; \eta, a - \tau) \Psi(u, v)(\eta, a - \tau) d\eta d\tau \right]. \end{aligned}$$

Integrating from zero to infinity both sides of the above equations and thanks to the positivity of $\Phi_\epsilon^{(i)}$ ($i = 1, 2$), we have

$$\begin{aligned} -\Phi_\epsilon^1(0) + \frac{1}{\epsilon}\|\Phi_\epsilon^1\|_{L^1} &= \left(\frac{1}{\epsilon} - \lambda\right) \|u\|_{L^1} - \int_0^\infty \mu_m(a)u(a)da - \int_0^\infty \int_0^\infty \Psi(u, v)(a, \eta) d\eta da \\ &\quad + \int_0^\infty \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; a - \tau, \eta)] H(\omega - \tau) e^{-\lambda\tau} \Gamma(\tau; a - \tau, \eta) \Psi(u, v)(a - \tau, \eta) d\eta d\tau da, \\ -\Phi_\epsilon^2(0) + \frac{1}{\epsilon}\|\Phi_\epsilon^2\|_{L^1} &= \left(\frac{1}{\epsilon} - \lambda\right) \|v\|_{L^1} - \int_0^\infty \mu_f(a)v(a)da - \int_0^\infty \int_0^\infty \Psi(u, v)(\eta, a) d\eta da \\ &\quad + \int_0^\infty \int_0^a \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, a - \tau)] H(\omega - \tau) e^{-\lambda\tau} \Gamma(\tau; \eta, a - \tau) \Psi(u, v)(\eta, a - \tau) d\eta d\tau da, \end{aligned}$$

where we are using the fact that $\Phi_\epsilon^1(\infty) = \Phi_\epsilon^2(\infty) = 0$. By adding term to term and changing the order of integrals, we obtain

$$\begin{aligned} & - \int_0^\omega \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) e^{-\lambda\tau} \Gamma(\tau; \zeta, \eta) \Psi(u, v)(\zeta, \eta) d\zeta d\eta d\tau + \frac{1}{\epsilon}(\|\Phi_\epsilon^1\|_{L^1} + \|\Phi_\epsilon^2\|_{L^1}) \\ &= \left(\frac{1}{\epsilon} - \lambda\right) (\|u\|_{L^1} + \|v\|_{L^1}) - \int_0^\infty \mu_m(a)u(a)da - \int_0^\infty \mu_f(a)v(a)da \\ &\quad - 2 \int_0^\infty \int_0^\infty \Psi(u, v)(\zeta, \eta) d\zeta d\eta \end{aligned}$$

$$+ \int_0^\omega \int_0^\infty \int_0^\infty (\mu_m(\tau + \zeta) + \mu_f(\tau + \zeta) + 2\delta(\tau; \zeta, \eta)) e^{-\lambda\tau} \Gamma(\tau; \zeta, \eta) \Psi(u, v)(\zeta, \eta) d\zeta d\eta d\tau.$$

It follows from the definition of Φ_ϵ^3 that the relation (4.28) holds. $\square$

With the above preparations, we can prove the following existence theorem:

Proposition 4.4 *Suppose that Assumption 4.1 holds. Then for system (2.1)-(2.6), there exists at least one non-trivial exponential solution.*

Proof: From the above arguments, it is sufficient to show that Φ_ϵ has a fixed point in $U_+ \times [\underline{\lambda}, \bar{\lambda}]$. Let us define a new operator $\mathcal{F}_\epsilon$ from $U_+ \times [\underline{\lambda}, \bar{\lambda}]$ by

$$\begin{aligned} \mathcal{F}_\epsilon : (u, v, \lambda) \\ \rightarrow \left(\frac{\Phi_\epsilon^1(u, v, \lambda)}{\|\Phi_\epsilon^1(u, v, \lambda)\|_{L^1} + \|\Phi_\epsilon^2(u, v, \lambda)\|_{L^1}}, \frac{\Phi_\epsilon^2(u, v, \lambda)}{\|\Phi_\epsilon^1(u, v, \lambda)\|_{L^1} + \|\Phi_\epsilon^2(u, v, \lambda)\|_{L^1}}, \Phi_\epsilon^3(u, v, \lambda) \right). \end{aligned}$$

From Lemma 4.3, it follows that if $(u, v, \lambda) \in U_+ \times [\underline{\lambda}, \bar{\lambda}]$, then

$$\|\Phi_\epsilon^1(u, v, \lambda)\|_{L^1} + \|\Phi_\epsilon^2(u, v, \lambda)\|_{L^1} \geq 1 - \epsilon(\bar{\lambda} - \underline{\lambda}).$$

Therefore if we choose ϵ such that

$$0 < \epsilon < \min \left(\frac{1}{\bar{\lambda} - \underline{\lambda}}, \frac{1}{\bar{\mu} + \bar{\lambda} + K} \right),$$

then $\mathcal{F}_\epsilon$ is well-defined and it becomes a continuous operator from $U_+ \times [\underline{\lambda}, \bar{\lambda}]$ into itself. Moreover it is easily seen that $\mathcal{F}_\epsilon$ is a completely continuous operator on a bounded convex closed set $U_+ \times [\underline{\lambda}, \bar{\lambda}]$, because it is a sum of a one-dimensional operator and a Volterra integral operator. From the well-known Schauder's fixed point principle, we can conclude that $\mathcal{F}_\epsilon$ has a fixed point $(u^*, v^*, \lambda^*) \in U_+ \times [\underline{\lambda}, \bar{\lambda}]$. From Lemma 4.3, it follows that

$$\|\Phi_\epsilon^1(u^*, v^*, \lambda^*)\|_{L^1} + \|\Phi_\epsilon^2(u^*, v^*, \lambda^*)\|_{L^1} = (1 - \epsilon\lambda^*) + \epsilon\lambda^* = 1.$$

Then we know that in fact (u^*, v^*, λ^*) is a fixed point of Φ_ϵ itself. This completes our proof. $\square$

5 Integro-differential equation approach

Although the proof given in the previous section is a very general one in the sense that we do not need to assume any specific form of marriage function, it has a shortcoming that we could not get any information about the location of the Malthusian parameter. For example, could we define the basic reproduction number for the two-sex system and relate it to the Malthusian parameter

? Furthermore we do not know even conditions for existence of stationary solutions. Here we propose another approach to show the existence of exponential solutions, which is applied to less general situation but it would allow us to get some ideas for problems mentioned above.

From (2.3) and (2.6), we obtain

$$s(t, \tau; \zeta, \eta) = \Gamma(\tau; \zeta, \eta) s(t - \tau, 0; \zeta, \eta) = \Gamma(\tau; \zeta, \eta) \Psi(p_m(t - \tau), p_f(t - \tau))(\zeta, \eta).$$

Then the monogamous marriage model (2.1)-(2.6) can be rewritten as the following integro-differential equation system for male and female single populations:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) p_m(t, a) &= -\mu_m(a) p_m(t, a) - \int_0^\infty \Psi(p_m(t), p_f(t))(a, \eta) d\eta \quad (5.1) \\ &+ \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; a - \tau, \eta)] \Gamma(\tau; a - \tau, \eta) \Psi(p_m(t - \tau), p_f(t - \tau))(a - \tau, \eta) d\eta d\tau, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) p_f(t, a) &= -\mu_f(a) p_f(t, a) - \int_0^\infty \Psi(p_m(t), p_f(t))(\eta, a) d\eta \quad (5.2) \\ &+ \int_0^a \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, a - \tau)] \Gamma(\tau; \eta, a - \tau) \Psi(p_m(t - \tau), p_f(t - \tau))(\eta, a - \tau) d\eta d\tau, \end{aligned}$$

$$p_m(t, 0) = (1 - \gamma) \int_0^\infty \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) \Gamma(\tau; \zeta, \eta) \Psi(p_m(t - \tau), p_f(t - \tau))(\zeta, \eta) d\zeta d\eta d\tau \quad (5.3)$$

$$p_f(t, 0) = \gamma \int_0^\infty \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) \Gamma(\tau; \zeta, \eta) \Psi(p_m(t - \tau), p_f(t - \tau))(\zeta, \eta) d\zeta d\eta d\tau, \quad (5.4)$$

where we assume that the initial data $p_m(t, a), p_f(t, a), (t, a) \in (-\infty, 0) \times R_+$ is given.

If we substitute $(e^{\lambda t} u(a), e^{\lambda t} v(a))$ into the above system, we obtain the following system:

$$u'(a) = -(\mu_m(a) + \lambda) u(a) - \int_0^\infty \Psi(u, v)(a, \eta) d\eta \quad (5.5)$$

$$+ \int_0^a \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; a - \tau, \eta)] e^{-\lambda \tau} \Gamma(\tau; a - \tau, \eta) \Psi(u, v)(a - \tau, \eta) d\eta d\tau,$$

$$v'(a) = -(\mu_f(a) + \lambda) v(a) - \int_0^\infty \Psi(u, v)(a, \eta) d\eta \quad (5.6)$$

$$+ \int_0^a \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, a - \tau)] e^{-\lambda \tau} \Gamma(\tau; \eta, a - \tau) \Psi(u, v)(\eta, a - \tau) d\eta d\tau,$$

$$u(0) = (1 - \gamma) \int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda \tau} \beta(\tau; \zeta, \eta) \Gamma(\tau; \zeta, \eta) \Psi(u, v)(\zeta, \eta) d\zeta d\eta d\tau, \quad (5.7)$$

$$v(0) = \frac{\gamma}{1-\gamma}u(0), \quad (5.8)$$

To seek the solution (u, v, λ) satisfying the above relations, first for a fixed λ we solve the system (5.1)-(5.6) with a constant boundary condition as

$$u(0) = 1 - \gamma, \quad v(0) = \gamma. \quad (5.9)$$

Proposition 5.1 *If $\lambda > -\mu$, the system (5.5)-(5.6) with constant boundary condition (5.9) has at least one positive solution in $L^1(0, \infty) \times L^1(0, \infty)$.*

Proof: For a sufficiently large parameter $\alpha > K$, let us consider a positive mapping as

$$\begin{aligned} \mathcal{F}_1^\lambda(u, v)(a) &:= (1 - \gamma)\ell_m(a)e^{-(\alpha+\lambda)a} + \int_0^a e^{-(\alpha+\lambda)(a-s)} \frac{\ell_m(a)}{\ell_m(s)} \left[\alpha \left(u(s) - \frac{1}{\alpha} \int_0^\infty \Psi(u, v)(s, \eta) d\eta \right) \right. \\ &\quad \left. + \int_0^s \int_0^\infty [\mu_f(\tau + \eta) + \delta(\tau; s - \tau, \eta)] e^{-\lambda\tau} \Gamma(\tau; s - \tau, \eta) \Psi(u, v)(s - \tau, \eta) d\eta d\tau \right] ds, \\ \mathcal{F}_2^\lambda(u, v)(a) &:= \gamma \ell_f(a) e^{-(\alpha+\lambda)a} + \int_0^a e^{-(\alpha+\lambda)(a-s)} \frac{\ell_f(a)}{\ell_f(s)} \left[\alpha \left(v(s) - \frac{1}{\alpha} \int_0^\infty \Psi(u, v)(\eta, s) d\eta \right) \right. \\ &\quad \left. + \int_0^s \int_0^\infty [\mu_m(\tau + \eta) + \delta(\tau; \eta, s - \tau)] e^{-\lambda\tau} \Gamma(\tau; \eta, s - \tau) \Psi(u, v)(\eta, s - \tau) d\eta d\tau \right] ds, \end{aligned}$$

It is easily seen that if the map

$$\mathcal{F}^\lambda : \begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{F}_1^\lambda(u, v) \\ \mathcal{F}_2^\lambda(u, v) \end{pmatrix}$$

has a fixed point in $L_+^1 \times L_+^1$, it is no other than the solution for which we are looking and it is independent of the choice of α .

Observe that

$$\|\mathcal{F}_1^\lambda(u, v)\|_{L^1} \leq J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= (1 - \gamma) \int_0^\infty \ell_m(a) e^{-(\alpha+\lambda)a} da, \\ J_2 &= \int_0^\infty \int_0^a e^{-(\alpha+\lambda)(a-s)} \frac{\ell_m(a)}{\ell_m(s)} \left(\alpha u(s) - \int_0^\infty \Psi(u, v)(s, \eta) d\eta \right) ds da, \\ J_3 &= \int_0^\infty da \int_0^a ds e^{-(\alpha+\lambda)(a-s)} \frac{\ell_m(a)}{\ell_m(s)} \\ &\quad \times \int_0^s d\tau \int_0^\infty d\eta [\mu_f(\tau + \eta) + \delta(\tau; s - \tau, \eta)] e^{-\lambda\tau} \Gamma(\tau; s - \tau, \eta) \Psi(u, v)(s - \tau, \eta). \end{aligned}$$

Then it follows that

$$\begin{aligned}
J_1 &\leq (1 - \gamma) \int_0^\infty e^{-(\underline{\mu} + \alpha + \lambda)a} da = \frac{1 - \gamma}{\underline{\mu} + \alpha + \lambda}, \\
J_2 &\leq \int_0^\infty \int_0^a e^{-(\underline{\mu} + \alpha + \lambda)(a-s)} \left(\alpha u(s) - \int_0^\infty \Psi(u, v)(s, \eta) d\eta \right) ds da \\
&= \int_0^\infty ds \int_s^\infty e^{-(\underline{\mu} + \alpha + \lambda)(a-s)} \left(\alpha u(s) - \int_0^\infty \Psi(u, v)(s, \eta) d\eta \right) ds da \\
&= \frac{\alpha \|u\|_{L^1} - \|\Psi\|_{L^1(\Omega)}}{\underline{\mu} + \alpha + \lambda} \\
J_3 &\leq \int_0^\infty ds \int_s^\infty da e^{-(\underline{\mu} + \alpha + \lambda)(a-s)} \\
&\quad \times \int_0^s d\tau \int_0^\infty d\eta [\mu_f(\tau + \eta) + \delta(\tau; s - \tau, \eta)] e^{-\lambda\tau} \Gamma(\tau; s - \tau, \eta) \Psi(u, v)(s - \tau, \eta) \\
&\leq \frac{1}{\underline{\mu} + \alpha + \lambda} \int_0^\infty ds \int_0^s d\tau \\
&\quad \times \int_0^\infty d\eta [\mu_f(\tau + \eta) + \delta(\tau; s - \tau, \eta)] e^{-\lambda\tau} \Gamma(\tau; s - \tau, \eta) \Psi(u, v)(s - \tau, \eta) \\
&= \frac{1}{\underline{\mu} + \alpha + \lambda} \int_0^\infty d\eta \int_0^\infty ds \\
&\quad \times \int_0^s dz [\mu_f(s - z + \eta) + \delta(s - z; z, \eta)] e^{-\lambda(s-z)} \Gamma(s - z; z, \eta) \Psi(u, v)(z, \eta) \\
&= \frac{1}{\underline{\mu} + \alpha + \lambda} \int_0^\infty d\eta \int_0^\infty ds \\
&\quad \times \int_0^s dz [-\partial_s \Gamma(s - z; z, \eta) - \mu_m(s - z + \eta) \Gamma(s - z; z, \eta)] e^{-\lambda(s-z)} \Psi(u, v)(z, \eta) \\
&= \frac{1}{\underline{\mu} + \alpha + \lambda} \int_0^\infty d\eta \int_0^\infty dz \\
&\quad \times \int_z^\infty ds [-\partial_s \Gamma(s - z; z, \eta) - \mu_m(s - z + \eta) \Gamma(s - z; z, \eta)] e^{-\lambda(s-z)} \Psi(u, v)(z, \eta) \\
&= \frac{1}{\underline{\mu} + \alpha + \lambda} \int_0^\infty d\eta \int_0^\infty dz
\end{aligned}$$

$$\begin{aligned}
& \times \left(1 - \int_z^\infty \mu_m(s - z + \eta) \Gamma(s - z; z, \eta) ds \right) e^{-\lambda(s-z)} \Psi(u, v)(z, \eta) \\
& \leq \frac{\|\Psi(u, v)\|_{L^1(\Omega)}}{\underline{\mu} + \alpha + \lambda}
\end{aligned}$$

Therefore we arrive at

$$\|\mathcal{F}_1^\lambda(u, v)\| \leq J_1 + J_2 + J_3 \leq \frac{1 - \gamma + \alpha \|u\|_{L^1}}{\underline{\mu} + \alpha + \lambda}$$

Then we obtain

$$\|\mathcal{F}_1^\lambda(u, v)\|_{L^1} + \|\mathcal{F}_2^\lambda(u, v)\|_{L^1} \leq \frac{1 + \alpha(\|u\|_{L^1} + \|v\|_{L^1})}{\underline{\mu} + \alpha + \lambda}. \quad (5.10)$$

Let us define a closed convex set B_λ as

$$B_\lambda = \{(u, v) \in L_+^1(R_+) \times L_+^1(R_+) : \|u\|_{L^1} + \|v\|_{L^1} \leq \frac{1}{\lambda + \underline{\mu}}\}$$

It follows from (5.10) that

$$\mathcal{F}^\lambda(B_\lambda) \subset B_\lambda.$$

Therefore we can again apply Schauder's fixed point principle to conclude that for a fixed $\lambda > -\underline{\mu}$ the mapping $\mathcal{F}^\lambda$ has at least one fixed point in B_λ , which is not the origin because $\mathcal{F}_j^\lambda(0, 0) \neq 0$. This completes our proof. $\square$

Let (u_λ, v_λ) be a positive fixed point of $\mathcal{F}^\lambda$. Then it is easy to see that if (u_λ, v_λ) satisfies the following equation

$$\int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda\tau} \beta(\tau; \zeta, \eta) \Gamma(\tau; \zeta, \eta) \Psi(u_\lambda, v_\lambda)(\zeta, \eta) d\zeta d\eta d\tau = 1, \quad (5.11)$$

then $(e^{\lambda t} u_\lambda(a), e^{\lambda t} v_\lambda(a))$ satisfies (5.5)-(5.8) and hence it is no other than the exponential solution of the system (5.1)-(5.4). In particular, we know that

Corollary 5.2 *There exists a stationary solution if and only if*

$$\int_0^\infty \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) \Gamma(\tau; \zeta, \eta) \Psi(u_0, v_0)(\zeta, \eta) d\zeta d\eta d\tau = 1. \quad (5.12)$$

The equation (5.11) is the nonlinear analogue of Lotka's characteristic equation, so the basic reproduction number in the stationary state could be defined as

$$R_0 = \int_0^\infty \int_0^\infty \int_0^\infty \beta(\tau; \zeta, \eta) \Gamma(\tau; \zeta, \eta) \Psi(u_0, v_0)(\zeta, \eta) d\zeta d\eta d\tau. \quad (5.13)$$

Demographically speaking, $\frac{u_0(a)}{u_0(0)}$ and $\frac{v_0(a)}{v_0(0)}$ can be seen as the survival rates of males and females in the single status at age a under a constant flow of newborns. In fact we have a relation as

$$\begin{aligned} u_0(a) &= (1 - \gamma) - \int_0^a \mu_m(\sigma) u_0(\sigma) d\sigma - \int_0^\infty d\eta \int_0^a \Gamma(a - \zeta; \zeta, \eta) \Psi(u_0, v_0)(\zeta, \eta) d\zeta \\ &\quad - \int_0^\infty d\eta \int_0^a \int_\zeta^a \mu_m(\xi) \Gamma(\xi - \zeta; \zeta, \eta) d\xi \Psi(u_0, v_0)(\zeta, \eta) d\zeta, \end{aligned}$$

which shows that the number of single males at age a equals the number of newborn males minus the number of deaths in the single status until age a minus the number of married males at age a minus the number of deaths in the marital status until age a .

From the above observations, we would naturally be led to, for example, a question whether the system (5.1)-(5.4) has at least one exponential solution with a Malthusian parameter $\lambda_0 > 0$ if $R_0 > 1$. The answer would be yes if there exists a solution (u_λ, v_λ) continuously depending on the parameter $\lambda \in [0, \infty)$. In fact if so, a function $F(\lambda)$ defined by

$$F(\lambda) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda\tau} \beta(\tau; \zeta, \eta) \Gamma(\tau; \zeta, \eta) \Psi(u_\lambda, v_\lambda)(\zeta, \eta) d\zeta d\eta d\tau$$

is a positive continuous function of λ . Since $F(0) = R_0 > 1$ and $\lim_{\lambda \rightarrow +\infty} F(\lambda) = 0$, there exists at least one $\lambda_0 > 0$ such that $F(\lambda_0) = 1$. Therefore we could conclude that $(e^{\lambda_0 t} u_{\lambda_0}(a), e^{\lambda_0 t} v_{\lambda_0}(a))$ is an exponential solution of the integro-differential equation system (5.1)-(5.4) with positive Malthusian parameter. It would be an interesting open problem whether there exists such a global continuous branch (u_λ, v_λ) and the basic reproduction number R_0 could play a role as threshold value.

6 Discussion

In this paper, we have established a semigroup approach to the two-sex non-linear age-structured population model and proved an existence theorem for exponential solutions in the homogeneous system. Certainly this is only a first step in the mathematical analysis for two-sex population dynamics. Since the two-sex population dynamics is difficult to analyze and very little is known of its properties, there remains a lot of work to be done in this field in the future. Since the analytical approach to two-sex models is so hard, numerical studies would become a useful support to understand their properties (Arbogast and Milner, 1989; Milner, 1988; Milner and Rabbio, 1992; Martcheva and Milner 1996b).

To determine the number of exponential solutions and to show their stability would be most important problems in the next step. In particular, the reader

interested in global behavior of the homogeneous system may refer to Haderler *et al.* (1988), Haderler and Ngoma (1990), Haderler (1992a) and Webb (1993). Of course, the existence and stability of steady states for nonhomogeneous models should be also considered (see Castillo-Chavez and Huang 1995).

From the standpoint of demography, another interesting problem is to provide a more substantial mathematical description for the mechanism of the marriage market. Intuitively it would be difficult to believe that a simple mathematical function Ψ such as harmonic type could well describe the real marriage behavior of humans. In fact Keyfitz (1972) and Martcheva and Milner (1996) reported that the fit of the harmonic type marriage function to real data is not necessarily better than other simple functions, even though it is preferable from a theoretical point of view. The axioms for the marriage function give no more than formal restrictions to the marriage function. We would have to learn from socio-economic, substantive observations if we once want to go beyond the simple, mechanistic stage of the problem and to construct a general mathematical theory of human population reproduction.

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