Local Sobolev-Bergman kernels of strictly pseudoconvex domains

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Introduction

This article grew out of an attempt to understand analytic aspect of Fefferman's invariant theory [F3] of the Bergman kernel on the diagonal of $\Omega\times\Omega$ for strictly pseudoconvex domains Ω in $\mathbb{C}^n$ with smooth (C^∞ or real analytic) boundary. The framework of his invariant theory applies equally to the Szegö kernel if the surface element on $\partial\Omega$ is appropriately chosen, while the Szegö kernel is regarded as the reproducing kernel of a Hilbert space of holomorphic functions in Ω which belong to the L^2 Sobolev space of order $1/2$. This fact is our starting point. For each $s\in\mathbb{R}$, we first globally define the Sobolev-Bergman kernel K^s of order $s/2$ to be the reproducing kernel of the Hilbert space $H^{s/2}(\Omega)$ of holomorphic functions which belong to the L^2 Sobolev space of order $s/2$, where the inner product is specified arbitrarily.

In order to put the Sobolev-Bergman kernel K^s in the invariant theory, it is necessary to assume that K^s has two crucial properties which are satisfied by the Bergman kernel $K^{\mathrm{B}}=K^{\mathrm{B}}_\Omega$ and the (invariantly defined) Szegö kernel $K^{\mathrm{S}}=K^{\mathrm{S}}_\Omega$. The first one is the transformation law of weight $w\in\mathbb{Z}$ under biholomorphic mappings $\Phi\colon\Omega_1\to\Omega_2$

$$K_{\Omega_1}=(K_{\Omega_2}\circ\Phi)\,|\det\Phi'|^{2w/(n+1)} \tag{0.1}$$

for a kernel (or a domain functional) $K=K_\Omega$, where $\det\Phi'$ denotes the holomorphic Jacobian of Φ. If we write $w=\mathrm{w}^{\mathrm{TL}}(K)$ for w in (0.1), then $\mathrm{w}^{\mathrm{TL}}(K^{\mathrm{B}})=n+1$ and $\mathrm{w}^{\mathrm{TL}}(K^{\mathrm{S}})=n$. We require the inner product of $H^{s/2}(\Omega)$ to satisfy

$$\mathrm{w}^{\mathrm{TL}}(K^s)=w(s)\quad\text{with}\quad w(s)=n+1-s\in\mathbb{Z},$$

and say that such K^s is weakly invariant. However, we don't know how to define such an inner product for $s>0$, except for the Szegö kernel case $s=1$. So far, we could have defined weakly invariant Sobolev-Bergman kernels $K^s=K^s_\Omega$ only for $s=1$ and $s\le 0$ real (see Section 1). This is a motivation to abandon the global definition via the inner product and consider the local kernels regarded as singularities (i.e. kernels modulo smooth error) near a boundary point of reference.

The second crucial property satisfied by the Bergman kernel and the Szegö kernel is that the singularity is simple holonomic, that is,

$$K = \frac{\varphi}{r^w} + \psi \log r \quad (w > 0), \quad K = \varphi\, r^{-w} \log r \quad (w \le 0) \tag{0.2}$$

with $w \in \mathbb{Z}$, where r is a (smooth) defining function of $\partial\Omega$ such that $r > 0$ in Ω, and φ, ψ are smooth functions on $\overline{\Omega}$ (near $\partial\Omega$) such that φ does not vanish on $\partial\Omega$. We have $w = \mathrm{w}^{\mathrm{TL}}(K)$ for w in (0.2) if $K = K^{\mathrm{B}}$, K^{S}. Furthermore, the singularities of K^{B} and K^{S} are localizable to a neighborhood of a reference boundary point. In fact, these are obtained by patching locally defined singularities along the boundary $\partial\Omega$. We thus require $w = w(s)$ in defining local Sobolev-Bergman kernels $K^s = K^s_{\mathrm{loc}}$ with simple holonomic singularity. If in addition K^s is weakly invariant, we say that K^s is strongly invariant. This property is necessary in discussing the invariant theory of K^s.

In order to define local Sobolev-Bergman kernels, we first assume for simplicity that the (local) defining function r of $\partial\Omega$ is real analytic, so that we may write $r = r(z, \overline{z})$. We then use Kashiwara's characterization of the local Bergman kernel $K^{\mathrm{B}} = K^{\mathrm{B}}_{\mathrm{loc}}(z, \overline{z})$. Kashiwara [Kas] wrote down a system of microdifferential equations characterizing K^{B} up to a constant multiple by using another system satisfied by $\log r$. According to Boutet de Monvel [BM1]–[BM3], one can in fact define a transformation $\log r \mapsto K^{\mathrm{B}}$, where the singularity $\log r$ represents the domain Ω locally. In other words, K^{B} is a local domain functional via $\log r$. On the other hand, Sato's hyperfunction theory asserts that any simple holonomic singularity $\widehat{K}$, with respect to r which is fixed, is written as $\widehat{K} = \mathbf{A} \log r$, where $\mathbf{A} = \mathbf{A}[\widehat{K}]$ is a specific linear transformation (a microdifferential operator of finite order) which is holomorphic in z. Then $K = \mathbf{A}^{*-1} K^{\mathrm{B}}$ is again a simple holonomic singularity, where $\mathbf{A}^*$ denotes the formal adjoint of $\mathbf{A}$ defined formally by integration by parts without taking the complex conjugate. The mapping $\widehat{K} \mapsto K$ is consistent with Kashiwara's transformation $\log r \mapsto K^{\mathrm{B}}$, and the Szegö kernel $K = K^{\mathrm{S}}$ is obtained by choosing $\widehat{K}$ to be a constant multiple of $1/r$ with an appropriate choice of r which defines $\partial\Omega$ locally. Taking account of this fact, we first define in Subsection 2.1 the local Sobolev-Bergman kernels $K^s = K^s_{\mathrm{loc}}$ with respect to any (local) defining function r by taking

$$\widehat{K^s} = r^{-s} \quad (s > 0), \qquad \widehat{K^s} = r^{-s} \log r \quad (s \le 0)$$

for $s \in \mathbb{Z}$, where normalization constants are ignored. We then define in Subsection 2.3 the (invariant) local Sobolev-Bergman kernel $K^s_{\mathrm{loc}} = K^s_{\Omega}$

for each $s \in \mathbb{Z}$ as a local domain functional by requiring that the defining functions $r = r_\Omega$ are so chosen that $K^s_{\rm loc}$ is strongly invariant. Here, the word "strongly" can be omitted, because the strong invariance of $K^s_{\rm loc}$ is reduced to the weak one, the (local) transformation law. In case $\partial\Omega$ is not real analytic but C^∞, the local Sobolev-Bergman kernels are regarded as formal singularities (see Section 3).

As we prove in Subsection 2.2, the invariance of $K^s = K^s_{\rm loc}$ is equivalent to that of $\widehat{K^s}$, which obviously comes from the transformation law for the defining function $r = r_\Omega$ as a local domain functional. However, the situation is somewhat complicated because the transformation law for r holds only approximately. In [F2], Fefferman constructed r such that $\mathrm{w}^{\rm TL}(r) = -1$ modulo $O(r^{n+2})$. This error estimate is optimal (Theorem 2). Consequently, the local Sobolev-Bergman kernel $K^s_{\rm loc}$ which by definition is invariant exists if and only if $0 \le s \le n+1$ (Theorem 1). These two theorems are the main results of this paper stated in Subsection 2.3. Theorem 1 suggests that, for $0 > s \in \mathbb{Z}$, weakly invariant Sobolev-Bergman kernels K^s which are globally defined do not have simple holonomic singularities, though we don't know anything about the singularities in this case.

We emphasize that the invariance of $K^s_{\rm loc}$ for $0 \le s \le n+1$ holds without error, though that of the best possible r is approximate with error of $O(r^{n+2})$. More precisely, the invariance of $K^s_{\rm loc}$ follows from that of r modulo $O(r^{s+1})$ for $0 \le s \le n+1$.

For the local Sobolev-Bergman kernel K^s with $0 \le s \le n+1$, we can apply Fefferman's invariant theory to get an approximately invariant asymptotic expansion similar to those for $K^{\rm B}$ and $K^{\rm S}$. Though there are some technical difficulties to be examined such as the polynomial dependence on Moser's normal form coefficients $A = (A^\ell_{\alpha\overline{\beta}})$, we can verify these by inspecting the construction (see Section 4). In fact, the polynomial dependence on $A = (A^\ell_{\alpha\overline{\beta}})$ is taken into account in the definition of $K^s_{\rm loc}$. All abstract results as in Fefferman [F3] and Bailey-Eastwood-Graham [BEG] for $K^{\rm B}$ are evidently valid as well for K^s, whereas explicit results for K^s such as the determination of universal constants in Graham [G1] and [HKN1], [HKN2] for $K^{\rm B}$ and $K^{\rm S}$ are obtained by computer-aided calculation. These results are stated in Sections 4 with the method of computation explained in Appendix B.

The first author has recently obtained in [Hi] an invariant asymptotic expansion of the Bergman kernel without error via a special family of defining functions r of $\partial\Omega$, where the family is parametrized formally by $C^\infty(\partial\Omega)$ and the transformation law is made to hold within the family. The method applies in getting similar expansions of the Szegö kernel and the local Sobolev-Bergman kernels in the present paper as well. Though the present paper discusses Fefferman's approach so that the best possible defining functions r has the ambiguity $O(r^{n+2})$, the proof of the optimality of this error estimate (i.e. Theorem 2) is done here by using the theory in [Hi] (see Section 5).

1. Globally defined Sobolev-Bergman kernels

Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. For $s \in \mathbb{R}$, we denote by $H^{s/2}_{\mathrm{top}}(\Omega)$ the topological vector space consisting of holomorphic functions in Ω which are contained in the L^2 Sobolev space of order $s/2$. When an inner product $(\,\cdot\,,\,\cdot\,)_{s/2}$ is specified, we write $H^{s/2}_{\mathrm{top}}(\Omega)$ as $H^{s/2}(\Omega)$. Then $H^{s/2}(\Omega)$ is a Hilbert space which admits the reproducing kernel $K^s(z,\overline{w})$ for $z, w \in \Omega$ defined by

$$K^s(z,\overline{w}) = \sum_j h_j(z)\,\overline{h_j(w)},$$

where $\{h_j\}_j$ is an arbitrary complete orthonormal system of $H^{s/2}(\Omega)$. We set $K^s(z) = K^s(z,\overline{z})$.

Definition 1.1. The reproducing kernel $K^s(z,\overline{w})$, or rather $K^s(z)$, is called the *Sobolev-Bergman kernel associated with* $H^{s/2}(\Omega)$.

The simplest case is $s = 0$. If $(\,\cdot\,,\,\cdot\,)_0$ is the standard L^2 inner product

$$(h_1,h_2)_0 = \int_\Omega h_1(z)\,\overline{h_2(z)}\,dV(z), \quad dV = \bigwedge_{j=1}^n \frac{dz_j \wedge d\overline{z}_j}{-2i}\,,$$

then $K^0 = K^{\mathrm{B}}$, where K^{B} denotes the Bergman kernel. When we wish to emphasize the dependence on the domain Ω, we write $K^{\mathrm{B}} = K^{\mathrm{B}}_\Omega$. In fact, the Bergman kernel is a domain functional, and it is elementary that if $\Phi\colon \Omega_1 \to \Omega_2$ is biholomorphic then

$$K^{\mathrm{B}}_{\Omega_1}(z) = K^{\mathrm{B}}_{\Omega_2}(\Phi(z))\,|\det \Phi'(z)|^2,$$

where $\Phi' = \partial\Phi/\partial z$ and thus $\det \Phi'$ is the holomorphic Jacobian of Φ. More generally, we follow Fefferman and make the following:

Definition 1.2. If a domain functional $K = K_\Omega$ satisfies

$$K_{\Omega_1}(z) = K_{\Omega_2}(\Phi(z))\,|\det \Phi'(z)|^{2w/(n+1)} \tag{1.1}$$

whenever $\Phi\colon \Omega_1 \to \Omega_2$ is biholomorphic, then we say that K satisfies the *transformation law of weight* w, and write $\mathrm{w}^{\mathrm{TL}}(K) = w$.

Another well-known example is the Szegö kernel K^1. Here, we may choose an inner product on $H^{1/2}(\Omega)$ to be given by

$$(h_1, h_2)_{1/2} = \int_{\partial\Omega} h_1(z)\,\overline{h_2(z)}\,\sigma(z),$$

where σ is a surface element on $\partial\Omega$. Thus $H^{1/2}(\Omega)$ depends on σ. It is possible to choose σ in such a way that K^1 satisfies the transformation law, as follows.

Let us take a smooth positive defining function $\rho \in C^\infty(\overline{\Omega})$, and thus

$$\Omega = \{z \in \mathbb{C}^n;\ \rho(z) > 0\}, \qquad d\rho(z) \neq 0 \quad \text{for} \quad z \in \partial\Omega.$$

Let $J[\,\cdot\,]$ denote the *Levi determinant* or the (complex) *Monge-Ampère operator* defined by

$$J[\rho] = (-1)^n \det \begin{pmatrix} \rho & \partial\rho/\partial\overline{z}_k \\ \partial\rho/\partial z_j & \partial^2\rho/\partial z_j\partial\overline{z}_k \end{pmatrix} \quad (j, k = 1, \ldots, n). \tag{1.2}$$

We then have $\mathrm{w}^{\mathrm{TL}}(K^1) = n$, provided the surface element σ is subject to the normalization

$$\sigma \wedge d\rho = J[\rho]^{1/(n+1)}\,dV \quad \text{on} \quad \partial\Omega.$$

In this case, we write K^1 as K^{S} and call it the *invariant Szegö kernel* or just the *Szegö kernel.* Thus

$$\mathrm{w}^{\mathrm{TL}}(K^{\mathrm{B}}) = n + 1, \quad \mathrm{w}^{\mathrm{TL}}(K^{\mathrm{S}}) = n.$$

These numbers coincide with the magnitude of the singularities. In fact, according to a celebrated theorem of Fefferman [F1] (see also Boutet de Monvel and Sjöstrand [BS]), there exist functions $\varphi^{\mathrm{B}} = \varphi^{\mathrm{B}}[\rho]$ and $\psi^{\mathrm{B}} = \psi^{\mathrm{B}}[\rho]$ in $C^\infty(\overline{\Omega})$ such that

$$\frac{\pi^n}{n!}\,K^{\mathrm{B}} = \frac{\varphi^{\mathrm{B}}}{\rho^{n+1}} + \psi^{\mathrm{B}}\log\rho, \quad \Big(\varphi^{\mathrm{B}} - J[\rho]\Big)\Big|_{\partial\Omega} = 0. \tag{1.3}$$

Similarly, there exist $\varphi^{\mathrm{S}} = \varphi^{\mathrm{S}}[\rho]$ and $\psi^{\mathrm{S}} = \psi^{\mathrm{S}}[\rho]$ in $C^\infty(\overline{\Omega})$ such that

$$\frac{\pi^n}{(n-1)!}\,K^{\mathrm{S}} = \frac{\varphi^{\mathrm{S}}}{\rho^n} + \psi^{\mathrm{S}}\log\rho, \quad \Big(\varphi^{\mathrm{S}} - J[\rho]^{n/(n+1)}\Big)\Big|_{\partial\Omega} = 0. \tag{1.4}$$

(Note that $J[\rho] > 0$ on $\partial\Omega$ by the strict pseudoconvexity.)

We are interested in the Sobolev-Bergman kernel K^s satisfying

$$\mathrm{w}^{\mathrm{TL}}(K^s) = w(s), \quad \text{where} \quad w(s) = n + 1 - s \in \mathbb{Z}. \tag{1.5}$$

It will be also natural to require the existence of $\varphi^s, \psi^s \in C^\infty(\overline{\Omega})$ such that

$$c^{\mathrm{SB}}_{s,n}\, K^s = \begin{cases} \varphi^s\, \rho^{-w(s)} + \psi^s \log\rho & \text{for } \ w(s) > 0, \\ \varphi^s\, \rho^{-w(s)} \log\rho & \text{for } \ w(s) \le 0, \end{cases} \tag{1.6}$$

where $c^{\mathrm{SB}}_{s,n} \neq 0$ are normalization constants so chosen that

$$\Big(\varphi^s - J[\rho]^{w(s)/(n+1)}\Big)\Big|_{\partial\Omega} = 0.$$

Definition 1.3. A Sobolev-Bergman kernel K^s is said to be *weakly invariant* if the condition (1.5) holds. If in addition the condition (1.6) holds, then K^s is said to be *invariant*.

If the conditions (1.5) and (1.6) are not taken into account, it is easy to give examples of Sobolev-Bergman kernels K^s for any $s \in \mathbb{R}$, by specifying an inner product $(\,\cdot\,,\,\cdot\,)_{s/2}$. For instance, if $s/2 > 0$ is an integer, then we may take, with the usual (commutative) multi-index notation,

$$(h_1, h_2)_{s/2} = \int_\Omega \sum_{|\alpha|+|\beta|\le s/2} \Big(\partial_z^\alpha \partial_{\overline{z}}^\beta\, h_1\Big)\Big(\partial_{\overline{z}}^\alpha \partial_z^\beta\, \overline{h}_2\Big)\, dV, \tag{1.7}$$

though the condition (1.5) breaks down.

Remark 1. In case Ω is a ball in $\mathbb{C}^n$ and $H^{s/2}(\Omega)$ is specified by the inner product (1.7) with $s/2 \in \mathbb{N}_0$, Boas [Bo] showed that the reproducing kernel K^s takes the form (1.6), where the logarithmic terms appear even for $0 < s < n$. It is easy to define an inner product of $H^s(\Omega)$, for each domain Ω which is biholomorphic to a ball, in such a way that the transformation law (1.1) with $w = w(s)$ holds for the reproducing kernels $K = K^s$ of such domains. However, such reproducing kernels are not defined for domains which are not biholomorphic to a ball. In other words, a domain functional $K^s = K^s_\Omega$ is not determined as a weakly invariant Sobolev-Bergman kernel.

In case $s < 0$ is a real number, we can define a weakly invariant Sobolev-Bergman kernel as follows. An inner product on $H^{s/2}_{\mathrm{top}}(\Omega)$ is given by

$$(h_1, h_2)_{s/2} = \int_\Omega h_1(z)\, \overline{h_2(z)}\, \rho(z)^{-s}\, dV(z)$$

for any smooth defining function $\rho > 0$ of Ω. Moreover, we may replace ρ by any continuous function $u > 0$ of the same magnitude as ρ to have

$$(h_1, h_2)_{s/2} = \int_\Omega h_1(z)\, \overline{h_2(z)}\, u(z)^{-s}\, dV(z). \tag{1.8}$$

We have:

Proposition 1.1. *Let* $0 > s \in \mathbb{R}$. *If* $u = u_\Omega$ *satisfies*

$$\mathrm{w}^{\mathrm{TL}}(u) = -1, \quad 0 < \inf u/\rho \le \sup u/\rho < +\infty, \tag{1.9}$$

then the Sobolev-Bergman kernel K^s *defined by* (1.8) *is weakly invariant.*

Proof. If $\Phi\colon \Omega_1 \to \Omega_2$ is biholomorphic, then

$$u_1 = (u_2 \circ \Phi)\, |\det \Phi'|^{-2/(n+1)}, \quad u_\ell = u_{\Omega_\ell} \ \ (\ell = 1, 2).$$

It then follows that an isometry $\Phi^*\colon H^{s/2}(\Omega_2) \to H^{s/2}(\Omega_1)$ is given by

$$\Phi^* \widetilde{h} = (\widetilde{h} \circ \Phi)(\det \Phi')^{w(s)/(n+1)}.$$

If $\{\widetilde{h}_j\}$ is a complete orthonormal system of $H^{s/2}(\Omega_2)$, then a complete orthonormal system $\{h_j\}$ of $H^{s/2}(\Omega_1)$ is defined by $h_j = \Phi^* \widetilde{h}_j$. Thus, the transformation law (1.1) for $K^s = K^s_\Omega$ follows from

$$\sum_j |h_j|^2 = \sum_j |\widetilde{h}_j \circ \Phi|^2\, |\det \Phi'|^{2w(s)/(n+1)}.$$

□

Examples of $u = u_\Omega$ satisfying (1.9) are given by

$$u^{\mathrm{B}} = \left(c^{\mathrm{SB}}_{n,n}\, K^{\mathrm{B}}\right)^{-1/(n+1)} \quad \text{or} \quad u^{\mathrm{S}} = \left(c^{\mathrm{SB}}_{n-1,n}\, K^{\mathrm{S}}\right)^{-1/n}.$$

Another important example is given by the solution $u = u^{\mathrm{MA}}$ of the boundary value problem

$$J[u] = 1 \ \text{ and } \ u > 0 \ \text{ in } \ \Omega; \qquad u = 0 \quad \text{on } \ \partial\Omega, \tag{1.10}$$

The unique existence of a solution of (1.10) in $C^\infty(\Omega) \cap C^{n+3/2-\varepsilon}(\overline{\Omega})$ was proved by Cheng and Yau [CY]. Thus the first relation in (1.9) follows from the fact (see [F2]) that if $\Phi\colon \Omega_1 \to \Omega_2$ is biholomorphic then

$$J[u_1] = J[u_2] \circ \Phi, \quad \text{where} \quad u_1 = (u_2 \circ \Phi) \cdot |\det \Phi'|^{-2/(n+1)}.$$

The second relation in (1.9) follows from the asymptotic expansion due to Lee and Melrose [LM]:

$$u^{\mathrm{MA}} \sim \rho \sum_{k=0}^{\infty} \eta_k \cdot (\rho^{n+1} \log \rho)^k, \quad \eta_k \in C^\infty(\overline{\Omega})$$

where $\eta_0|_{\partial\Omega} > 0$. Thus $u = u^{\mathrm{MA}}$ satisfies (1.9).

If Ω is the unit ball Ω_{ball}, then

$$u^{\mathrm{B}}_{\Omega_{\mathrm{ball}}}(z) = K^{\mathrm{S}}_{\Omega_{\mathrm{ball}}}(z) = u^{\mathrm{MA}}_{\Omega_{\mathrm{ball}}} = 1 - |z|^2.$$

In contrast to Boas' result in Remark 1, we have:

Proposition 1.2. *Let $K^s = K^s_\Omega$ be a weakly invariant Sobolev-Bergman kernel of order $s/2$, $0 > s \in \mathbb{R}$, defined by the inner product* (1.8) *with either one of $u = u^{\mathrm{B}}$, u^{S} or u^{MA}. Then*

$$K^s_{\Omega_{\mathrm{ball}}}(z) = \frac{\Gamma(w(s))}{\pi^n\,\Gamma(1-s)}\,\frac{1}{(1-|z|^2)^{w(s)}}\,.$$

Proof. For $n = 1$, the result follows by using the fact that monomials form a complete orthogonal system of $H^{s/2}(\Omega_{\mathrm{ball}})$. For $n \geq 2$, we consider

$$K^s_{\mathrm{aux}}(z) = \sum_{\alpha\in\mathbb{N}_0^n} |h_\alpha(z)|^2, \qquad h_\alpha(z) := \frac{z^\alpha}{\|z^\alpha\|_{s/2}}\,,$$

where $\|\cdot\|_{s/2}$ is the norm corresponding to the inner product $(\,\cdot\,,\,\cdot\,)_{s/2}$. It suffices to show that

$$K^s_{\mathrm{aux}}(z) = \frac{\Gamma(w(s))}{\pi^n\,\Gamma(1-s)}\,\frac{1}{(1-|z|^2)^{w(s)}}\,, \qquad K^s_{\mathrm{aux}}(z) = K^s_{\Omega_{\mathrm{ball}}}(z).$$

The first equality is obtained by direct computation using the result for $n = 1$. The second one is equivalent to the completeness of the orthonormal system $\{h_\alpha\}$, and the proof of this fact is done by noting that

$$K^s_{\Omega_{\mathrm{ball}}}(z) = \sup\left\{|h(z)|^2\big/\|h\|^2_{s/2};\quad 0 \neq h \in H^{s/2}(\Omega_{\mathrm{ball}})\right\},$$

just as in the proof for $s = 0$ given by Hörmander [Hö]. □

2. Definition of local Sobolev-Bergman kernels

In this section, we consider the local Sobolev-Bergman kernel of order $s/2$ for $s \in \mathbb{Z}$ and the invariance in the sense of (1.5). We begin with the motivation because the definition is somewhat technical.

An important fact is that the singularities of the Bergman kernel $K^{\mathrm{B}}(z)$ and the (invariant) Szegö kernel $K^{\mathrm{S}}(z)$ as in (1.3) and (1.4) can be localized to any boundary point, say $p \in \partial\Omega$. That is, if $\Omega_1 \cap U = \Omega_2 \cap U$ for a neighborhood $U \subset \mathbb{C}^n$ of p, then $K_{\Omega_1} - K_{\Omega_2}$ for $K = K^{\mathrm{B}}$ or K^{S} is *smooth* near $p \in \partial\Omega$, where smooth means C^∞ or C^ω (real analytic) in accordance with the regularity of $\partial\Omega$ near p. Furthermore, one can define local kernels $K_{\mathrm{loc}} = K^{\mathrm{B}}_{\mathrm{loc}}$ and $K^{\mathrm{S}}_{\mathrm{loc}}$ by requiring the following three conditions:

(i) $K_{\mathrm{loc}}(z, \overline{w})$ is holomorphic in z and anti-holomorphic in w for $z, w \in \Omega \cap U$. Two local kernels K_{loc} and $\widetilde{K}_{\mathrm{loc}}$ are identified when the difference is smooth in $\mathbb{C}^n$ near p. Thus U can be shrunk arbitrarily.

(ii) $K_{\mathrm{loc}} = K^{\mathrm{B}}_{\mathrm{loc}}$ and $K^{\mathrm{S}}_{\mathrm{loc}}$ have singularities of the form (1.3) and (1.4), respectively, where $\varphi = \varphi^{\mathrm{B}}$, φ^{S} and $\psi = \psi^{\mathrm{B}}$, ψ^{S} are smooth in $\Omega \cap U$.

(iii) Reproducing properties modulo smooth errors hold, that is,

$$\int_{\Omega\cap U} K^{\mathrm{B}}_{\mathrm{loc}}(z, \overline{w})\, f_1(w)\, dV(w) - f_1(z) \sim 0,$$

$$\int_{\partial\Omega\cap U} K^{\mathrm{S}}_{\mathrm{loc}}(z, \overline{w})\, f_2(w)\, \sigma(w) - f_2(z) \sim 0,$$

for holomorphic functions f_1 and f_2 in U_0, where $U_0 \subset \mathbb{C}^n$ is an open set satisfying $p \in U \Subset U_0$, and each f_2 is regarded as the boundary value. In case $\partial\Omega$ is C^∞ near p, $f_1(z)$ and $f_2(z)$ are required to be of polynomial growth in $1/\mathrm{dist}\,(z, \partial\Omega)$. (If $\partial\Omega$ is C^ω near p, then no restriction on f_1 and f_2 is necessary, provided the pairings are interpreted in the sense of hyperfunctions, cf. Kaneko [Kan].)

The local kernels $K^{\mathrm{B}}_{\mathrm{loc}}$ and $K^{\mathrm{S}}_{\mathrm{loc}}$ are uniquely determined by the requirements (i)–(iii). We wish to define the local Sobolev-Bergman kernel K^s_{loc} for $s \in \mathbb{N}$ in a similar way. Our main concern is the invariance in the sense of (1.5) under local biholomorphic mappings. However, the condition (iii) uses the inner products, and we don't know how to define $(\,\cdot\,,\,\cdot\,)_s$ for $s \in \mathbb{N}$ such that the Sobolev-Bergman kernel K^s is invariant. We thus abandon (iii) and instead adopt Kashiwara's characterization of the local Bergman kernel $K^{\mathrm{B}}_{\mathrm{loc}}$, a method which applies equally to the local Szegö kernel $K^{\mathrm{S}}_{\mathrm{loc}}$.

In this section, we assume that $\partial\Omega$ is C^ω near p. We are only concerned with local kernels $K_{\mathrm{loc}}(z, \overline{w})$ defined near $(z, w) = (p, p)$, and thus the subscript loc will be omitted.

2.1. Kashiwara's transformation. We fix a local defining function r of Ω near $p \in \partial\Omega$ and assume for a moment that r is real analytic. Then r has a holomorphic extension to a neighborhood of $M \times \overline{M}$ in $\mathbb{C}^n \times \mathbb{C}^n$, where $M \subset \partial\Omega$ is a neighborhood of p (or more precisely, a germ of $\partial\Omega$ at p). Denoting it again by r, we set, for $m \in \mathbb{Z}$,

$$\widehat{K}_m[r] = \begin{cases} \dfrac{1}{m!}\, r^m \log r & \text{for } \ m \geq 0, \\ (-1)^{m+1}(-m-1)!\, \dfrac{1}{r^{-m}} & \text{for } \ m < 0, \end{cases}$$

and consider singularities of the form

$$K = \begin{cases} \varphi \widehat{K}_{-w}[r] + \psi \widehat{K}_0[r] & \text{if} \quad w > 0, \\ \varphi \widehat{K}_{-w}[r] & \text{if} \quad w \le 0, \end{cases}$$

where φ an ψ are holomorphic in $(z, \overline{z})$ near $M \times \overline{M}$ for some M. We denote by $\mathcal{C}_p^\times$ the totality of K such that $\varphi \neq 0$ near $M \times \overline{M}$. By [SKK], if $\widehat{K} \in \mathcal{C}_p^\times$ then, for any holomorphic microdifferential operator $P = P(z, \partial_z)$, there exists an antiholomorphic microdifferential operator $Q = Q(\overline{z}, \partial_{\overline{z}})$ such that $P\widehat{K} = Q\widehat{K}$. Furthermore, if $P_j = P_j(z, \partial_z)$ for $j = 1, \ldots, 2n$ are chosen independently then $\widehat{K}$ is determined up to a multiplicative constant by

$$P_j(z, \partial_z)\widehat{K}(z, \overline{z}) = Q_j(\overline{z}, \partial_{\overline{z}})\widehat{K}(z, \overline{z}) \quad \text{for} \quad j = 1, \ldots, 2n. \tag{2.1}$$

(A more rigorous description of [SKK] will be given in Appendix A.) Let us consider another system of microdifferential equations for $K \in \mathcal{C}_p^\times$

$$P_j^*(z, \partial_z)K(z, \overline{z}) = Q_j^*(\overline{z}, \partial_{\overline{z}})K(z, \overline{z}) \quad \text{for} \quad j = 1, \ldots, 2n, \tag{2.2}$$

where P_j^*, Q_j^* are formal adjoints of P_j, Q_j, respectively. The independence of P_j implies that of P_j^*, so that the solution of (2.2), if it exists, is unique up to a multiplicative constant. Now

Kashiwara's theorem ([Kas])**.** *If* $\widehat{K} = \widehat{K}_0[r] = \log r$ *in* (2.1) *then* (2.2) *is satisfied by the local Bergman kernel* $K = K^{\mathrm{B}}$.

By [SKK], if $\widehat{K} \in \mathcal{C}_p^\times$ then there exists a unique invertible holomorphic microdifferential operator $\mathbf{A}[\widehat{K}]$ such that

$$\widehat{K}(z, \overline{z}) = \mathbf{A}(z, \partial_z)\widehat{K}_0(z, \overline{z}) \quad \text{with} \quad \mathbf{A} = \mathbf{A}[\widehat{K}], \quad \widehat{K}_0 = \widehat{K}_0[r]. \tag{2.3}$$

Thus, Kashiwara's theorem yields

Lemma 2.1. *If* (2.1) *and* (2.3) *hold for* $\widehat{K} \in \mathcal{C}_p^\times$, *then* (2.2) *is satisfied by*

$$K(z, \overline{z}) = \mathbf{A}^*(z, \partial_z)^{-1} K^{\mathrm{B}}(z, \overline{z}).$$

Proof. Since $\mathbf{A}$ is a holomorphic operator and thus $Q_j\mathbf{A} = \mathbf{A}Q_j$, it follows that $P_j\mathbf{A}\widehat{K}_0 = Q_j\mathbf{A}\widehat{K}_0 = \mathbf{A}Q_j\widehat{K}_0$, that is, $\mathbf{A}^{-1}P_j\mathbf{A}\widehat{K}_0 = Q_j\widehat{K}_0$, so that Kashiwara's theorem yields $Q_j^*K^{\mathrm{B}} = \mathbf{A}^*P_j^*\mathbf{A}^{*-1}K^{\mathrm{B}} = \mathbf{A}^*P_j^*K$. Using $\mathbf{A}^{*-1}Q_j^* = Q_j^*\mathbf{A}^{*-1}$, we get $P_j^*K = \mathbf{A}^{*-1}Q_j^*K^{\mathrm{B}} = Q_j^*K$. □

Since $\mathbf{A}[K^{\mathrm{B}}]^* = \mathbf{A}[K^{\mathrm{B}}]$, it follows that

$$\mathcal{C}_p^\times \ni \widehat{K} \mapsto K \in \mathcal{C}_p^\times \tag{2.4}$$

given by Lemma 2.1 is an involution. We refer to it as *Kashiwara's transformation.*

Definition 2.1. Let r be a real analytic local defining function of Ω near $p \in \partial\Omega$. For $s \in \mathbb{Z}$, we define $K^s[r] = K$ by $\widehat{K} = \widehat{K}_{-s}[r]$ in (2.4) and call $K^s[r]$ the *local Sobolev-Bergman kernel of order* $s/2$ *with respect to* r.

By the definition via Kashiwara's theorem, we have $K^0[r] = (\mathrm{const.})\, K^{\mathrm{B}}$ independently of the choice of r. We also have $K^1[r] = (\mathrm{const.})\, K^{\mathrm{S}}$ if $J[r] = 1$ on $\partial\Omega$.

2.2. Biholomorphic transformation law. We wish to define a local Sobolev-Bergman kernel of Sobolev order $s/2$ for $s \in \mathbb{Z}$ as a local domain functional $K^s = (K^s_\Omega)_\Omega$ near the reference points $p_\Omega \in \partial\Omega$, say $p_\Omega = 0 \in \mathbb{C}^n$, where we continue to assume that $\partial\Omega$ is real analytic near 0. In the definition, we require three conditions of which the first two are:

Condition SB1. Each K^s_Ω is of the form $K^s_\Omega = K^s[r_\Omega]$, where r_Ω is a local defining function of Ω near $0 \in \mathbb{C}^n$. That is, K^s_Ω is the local Sobolev-Bergman kernel with respect to r_Ω.

Condition SB2. The family $r = r_\Omega$ is so chosen that $K^s = K^s_\Omega$ satisfies the transformation law of weight $w(s)$

$$K^s_\Omega = (K^s_{\widetilde{\Omega}} \circ \Phi)|\det \Phi'|^{2w(s)/(n+1)} \quad \text{with} \quad w(s) = n + 1 - s. \tag{2.5}$$

under local biholomorphic mappings $\Phi\colon \Omega \to \widetilde{\Omega}$ defined near the origin such that $\Phi(0) = 0$.

The third condition is somewhat complicated, and the precise statement is postponed to the next subsection. That condition is motivated by the result of this subsection.

Assume Condition SB1. Then the validity of Condition SB2 depends on the approximate transformation law of weight -1 for the family $r = (r_\Omega)_\Omega$:

$$r_\Omega = (r_{\widetilde{\Omega}} \circ \Phi)|\det \Phi'|^{-2/(n+1)} \mod O(r^N_\Omega), \tag{2.6}$$

where $O(r^N_\Omega)$ stands for terms which are smoothly divisible by r^N_Ω. In fact, we have:

Proposition 2.1. *Assume there exists* $N_0 \in \mathbb{N}$ *such that* $r = r_\Omega$ *satisfies the transformation law* (2.6) *for* $N = N_0$ *but not for* $N = N_0 + 1$. *Then the transformation law* (2.5) *is valid if and only if* $0 \le s \le N_0 - 1$.

This is consistent with the independence of K^{B} and the dependence of K^{S} on $r = (r_\Omega)$.

In the proof of Proposition 2.1, we need the following property of Kashiwara's transformation.

Lemma 2.2. *Assume Condition SB1. Then* $K_\Omega = K^s_\Omega$ *satisfies* (2.5) *if and only if* $\widehat{K}_\Omega$ *satisfies*

$$\widehat{K}_\Omega = (\widehat{K}_{\widetilde{\Omega}} \circ \Phi)|\det \Phi'|^{2s/(n+1)}. \tag{2.7}$$

Proof. We shall show that (2.7) implies (2.5). The proof of the converse is similar. What we have to show is that

$$\widehat{K}_\Omega = |f|^{2s}\Phi^*\widehat{K}_{\widetilde{\Omega}} \quad \text{implies} \quad K_\Omega = |f|^{2w(s)}\Phi^* K_{\widetilde{\Omega}},$$

where $f = (\det \Phi')^{1/(n+1)}$ and Φ^* stands for the pull-back by Φ. Let us abbreviate by writing $\mathbf{A}_\Omega = \mathbf{A}[\widehat{K}_\Omega]$, and similarly for $\widetilde{\Omega}$ in place of Ω. Then the assumption (2.7) is further written as

$$\mathbf{A}_\Omega \log r_\Omega = |f|^{2s}\Phi^*\mathbf{A}_{\widetilde{\Omega}}(\Phi^{-1})^*\Phi^* \log r_{\widetilde{\Omega}}.$$

The right side is simplified by setting $\widetilde{\mathbf{A}} = \Phi^*\mathbf{A}_{\widetilde{\Omega}}(\Phi^{-1})^*$, using $\Phi^* \log r_{\widetilde{\Omega}} = \log r_\Omega$, and choosing a holomorphic microdifferential operator $P = P(z, \partial_z)$ such that $P \log r_\Omega = \overline{f}^s \log r_\Omega$. Since $\widetilde{\mathbf{A}}$ is a holomorphic operator, it follows that

$$\mathbf{A}_\Omega \log r_\Omega = f^s\widetilde{\mathbf{A}}P \log r_\Omega, \quad \text{so that} \quad \mathbf{A}_\Omega = f^s\widetilde{A}P$$

(see Appendix A). Using $\Phi^*\mathbf{A}^*_{\widetilde{\Omega}}(\Phi^{-1})^* = f^{-n-1}\widetilde{\mathbf{A}}^* f^{n+1}$, we get

$$(\mathbf{A}^*_\Omega)^{-1} = f^{w(s)}\Phi^*(\mathbf{A}^*_{\widetilde{\Omega}})^{-1}(\Phi^{-1})^* f^{-n-1}(P^*)^{-1},$$

where f^{-n-1} acts as a multiplication operator. We apply both sides to K^{B}_Ω. Noting that $P^*K^{\mathrm{B}}_\Omega = \overline{f}^s K^{\mathrm{B}}_\Omega$, we have

$$(P^*)^{-1}K^{\mathrm{B}}_\Omega = \overline{f}^{-s}K^{\mathrm{B}}_\Omega = f^{n+1}\overline{f}^{w(s)}\Phi^*K^{\mathrm{B}}_{\widetilde{\Omega}}.$$

Since $\overline{f}^{w(s)}$ commutes with a holomorphic operator, it follows that

$$K_\Omega = |f|^{2w(s)}\Phi^*(\mathbf{A}^*_{\widetilde{\Omega}})^{-1}K^{\mathrm{B}}_{\widetilde{\Omega}} = |f|^{2w(s)}\Phi^* K_{\widetilde{\Omega}},$$

which is the desired conclusion (2.5). □

Proof of Proposition 2.1. Let us abbreviate by writing $\widetilde{r} = r_{\widetilde{\Omega}} \circ \Phi$ and $f = (\det \Phi')^{1/(n+1)}$. In case $s \leq 0$, (2.7) is written as

$$r_\Omega^{-s} \log r_\Omega = \widetilde{r}^{-s} |f|^{2s/(n+1)} \log \widetilde{r}.$$

For $s = 0$, this is always the case. For $s < 0$, this is valid if and only if (2.6) holds for any positive integer m. In case $s > 0$, (2.7) is written as

$$r_\Omega^{-s} = \widetilde{r}^{-s} |f|^{2s/(n+1)}.$$

This is valid if and only if (2.6) holds for $N \geq s + 1$. Thus the desired result follows from Lemma 2.2. □

2.3. Definition of local Sobolev-Bergman kernel. We are in a position to state a condition on the family $r = (r_\Omega)_\Omega$ to be called Condition SB3. This consists of the approximate transformation law (2.6) for $m = s+1$ and the polynomial dependence on Moser's normal form coefficients.

Recall that Moser's normal form is a real hypersurface of the form

$$N(A): \quad \rho_A = 2u - |z'|^2 - \sum_{\ell=0}^{\infty} \sum_{|\alpha|,|\beta|\geq 2} A^\ell_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, v^\ell = 0,$$

with normal coordinates $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $u = \operatorname{Re} z_n$, $v = \operatorname{Im} z_n$, such that $A = (A^\ell_{\alpha\overline{\beta}})$ is subject to the following conditions:

(N1) Each $A^\ell_{p\overline{q}} = (A^\ell_{\alpha\overline{\beta}})_{|\alpha|=p,|\beta|=q}$ is a bisymmetric tensor of type (p, q) on $\mathbb{C}^{n-1}$. That is, α, β are ordered multi-indices such as $\alpha = \alpha_1 \ldots \alpha_p$, $1 \leq \alpha_j \leq n-1$, and $A^\ell_{\alpha\overline{\beta}}$ is unchanged under permutation of α and that of β.

(N2) $A^\ell_{\alpha\overline{\beta}}$ is Hermitian symmetric, that is $\overline{A^\ell_{\alpha\overline{\beta}}} = A^\ell_{\beta\overline{\alpha}}$.

(N3) $\operatorname{tr} A^\ell_{2\overline{2}} = 0$, $(\operatorname{tr})^2 A^\ell_{2\overline{3}} = 0$, $(\operatorname{tr})^3 A^\ell_{3\overline{3}} = 0$, where tr stands for the usual tensorial trace taken with respect to $\delta^{j\overline{k}}$.

Some notation is in order. By $\mathcal{N}$, we denote the totality of $A = (A^\ell_{\alpha\overline{\beta}})$ satisfying the conditions (N1)–(N3). We define $\mathcal{N}^\omega$ to be the set of $A \in \mathcal{N}$ such that $N(A)$ is real analytic. (In general, $N(A)$ is a formal surface.) The strictly pseudoconvex side $\rho_A > 0$ of $N(A)$ is denoted by $\Omega(A)$, which makes sense near the origin. We use the coordinates $(z', \overline{z'}, \rho_A, v)$ for functions on $\Omega \cup N(A)$.

We have assumed that each $\partial\Omega$ is real analytic near the origin, so that we can place it locally in Moser's normal form $N(A)$ with $A \in \mathcal{N}^\omega$.

More precisely, there exists a local biholomorphic mapping Φ_A such that $\Phi_A(\Omega) = \Omega(A)$ and $\Phi_A(\partial\Omega) = N(A)$ locally. For r_Ω, we set

$$r_A = (r_\Omega \circ \Phi_A^{-1}) |\det \Phi_A'|^{2/(n+1)},$$

and consider the Taylor expansion about the origin

$$r_A = \sum_{k=1}^{N-1} c_k\Big(z', \overline{z'}, v\Big)\, \rho_A^k + O(\rho_A^N). \tag{2.8}$$

More precisely, we require that the family $(r_A)_{r\in\mathcal{N}^\omega}$ is well-defined in the sense of (2.8). Now we pose:

Condition SB3. In case $s > 0$, the family $r = (r_\Omega)_\Omega$ satisfies (2.6) for $N = s + 1$. Furthermore, in (2.8) for $N = s + 1$, any coefficient of the Taylor expansion of $c_k(z', \overline{z'}, v)$ about the origin is a universal polynomial in $A \in \mathcal{N}^\omega$. In case $s < 0$, the requirements above hold for any $N \in \mathbb{N}$. In case $s = 0$, no requirement is imposed.

Definition 2.2. By a local Sobolev-Bergman kernel of order $s/2$, $s \in \mathbb{Z}$, we mean a local domain functional $K^s = (K^s_\Omega)$ satisfying Conditions SB1–3.

By virtue of Proposition 2.1, the existence of a local Sobolev-Bergman kernel is reduced to that of a family of defining functions $r = (r_\Omega)_\Omega$ satisfying Condition SB3. Our main result of this paper is:

Theorem 1. *A local Sobolev-Bergman kernel of order* $s/2$ ($s \in \mathbb{Z}$) *exists if and only if* $0 \le s \le n + 1$.

The non-existence part of Theorem 1 is a consequence of:

Theorem 2. *There does not exist a family of* C^∞ *local defining functions* $r = (r_\Omega)$ *satisfying the requirements in Condition SB3 with* $N = n + 3$.

The proof of Theorem 2 is given in Section 5. Let us observe that Theorem 1 follows from Theorem 2. It suffices to show the existence of $r = (r_\Omega)$ satisfying Condition SB3 with $N = n + 2$ in place of $N = s + 1$. But this has been done by Fefferman [F2]. He constructed r_Ω satisfying $J[r_\Omega] = 1 + O(r_\Omega^{n+1})$ and (2.6) for $N = n + 2$. Specifically, one starts from an arbitrary smooth local defining function ρ of Ω, and defines ρ_s for $s = 1, \dots, n + 1$ successively by

$$\rho_1 = J[\rho]^{-1/(n+1)}\rho, \qquad \frac{\rho_s}{\rho_{s-1}} = 1 + \frac{1 - J[\rho_{s-1}]}{c_s}, \qquad c_s = s(n + 2 - s). \tag{2.9}$$

Then $J[\rho_s] = 1 + O(\rho^s)$, and ρ_s satisfies the approximate transformation law (2.6) for $N = s + 1$. Thus, we may set $r_\Omega = \rho_{n+1}$. It is clear that r_Ω

is real analytic dependence on $A \in \mathcal{N}$ as in Condition SB3 is examined if we locally place $\partial\Omega$ in normal form $N(A)$ and start from $\rho = \rho_A$. In fact, the universality of the polynomials in Condition SB3 follows from the transformation law (2.6) for $N = n + 2$.

3. Local Sobolev-Bergman kernels (the C^∞ case)

3.1. Polynomial dependence in the real analytic case. In order to define local Sobolev-Bergman kernels in the C^∞ category, we rewrite Condition SB3 under Conditions SB1 and SB2. That is, we need to state the polynomial dependence on Moser's normal form coefficients $A = (A^\ell_{\alpha\overline{\beta}})$ more explicitly.

Let us first recall the notion of biweight on $A^\ell_{\alpha\overline{\beta}}$ for $A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N}$ defined by

$$\mathrm{w}_2(A^\ell_{\alpha\overline{\beta}}) = (|\alpha| + \ell - 1, |\beta| + \ell - 1).$$

This comes from the transformation law under dilations

$$\phi_\lambda(z', z_n) = (\lambda z', |\lambda|^2 z_n) \qquad \text{for} \quad \lambda \in \mathbb{C}^*.$$

The notion for polynomials in A to be of (homogeneous) biweight is defined by

$$\mathrm{w}_2(P_1(A)P_2(A)) = \mathrm{w}_2(P_1(A)) + \mathrm{w}_2(P_2(A))$$

for monomials $P_1(A)$ and $P_2(A)$. If $P(A)$ is a polynomial of biweight (w', w''), we write

$$\mathrm{w}_2(P(A)) = (w', w''), \quad \mathrm{w}^{\mathrm{dil}}(P(A)) = \frac{1}{2}(w' + w''),$$

and call $\mathrm{w}^{\mathrm{dil}}(P(A))$ the weight of $P(A)$ with respect to dilations. Then, a polynomial in A is of weight w with respect to dilations if and only if it is a linear combination of polynomials of biweight (w', w'') such that $w' + w'' = 2w$. We have no essential change if we replace $\mathcal{N}$ by $\mathcal{N}^\omega$.

Let $K^s = (K^s_\Omega)$ be the local Sobolev-Bergman kernel of order $s/2$ in Definition 2.2, so that each $\partial\Omega$ is real analytic near the reference point assumed to be the origin $0 \in \mathbb{C}^n$. As in the previous section, we locally place $\partial\Omega$ in normal form $N(A)$, and write $K^s = (K^s_A)_{A \in \mathcal{N}^\omega}$, where each K^s_A corresponds to $\Omega(A)$. In fact, (K^s_A) is a subfamily of (K^s_Ω), but there is no loss of information via the transformation law

$$K^s_A = (K^s_\Omega \circ \Phi_A^{-1}) |\det \Phi'_A|^{-2w(s)/(n+1)} \tag{3.1}$$

for Φ_A in Subsection 2.3. Note that (3.1) is consistent with (2.5). As in (2.8), we have

$$K^s_A = \sum_{m=0}^{\infty} \sum_{\alpha,\beta,\ell} P^{\ell m}_{\alpha\overline{\beta}}(A)\, z'_\alpha\, \overline{z'_\beta}\, v^\ell\, \widehat{K}_{m-w(s)}[\rho_A], \tag{3.2}$$

where $P^{\ell m}_{\alpha\overline{\beta}}(A)$ are universal polynomials in $A \in \mathcal{N}$ determined by $K^s = (K^s_A)$. Furthermore,

$$\mathrm{w}^{\mathrm{dil}}(P^{\ell m}_{\alpha\overline{\beta}}(A)) = \frac{1}{2}(|\alpha| + |\beta|) + \ell + m. \tag{3.3}$$

As before, we refer to the universality of the polynomials $P^{\ell m}_{\alpha\overline{\beta}}(A)$ in (3.2) as the polynomial dependence of $K^s = (K^s_\Omega)$ on A. This follows from Condition SB3 and the construction in Subsection 2.1. Here, a crucial fact is the polynomial dependence of the local Bergman kernel $K^0 = K^{\mathrm{B}}$ on A, a fact which has been examined in [HKN1].

Let us restrict ourselves to the half line $z = \gamma_t$ for $t > 0$ small defined by $\gamma_t = (0, t/2) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Then (3.2) implies

$$K^s_A(\gamma_t) = \sum_{m=0}^{\infty} P_m(A)\, \widehat{K}_{m-w(s)}[t], \tag{3.4}$$

where $P_m(A) = P^{0m}_{0\overline{0}}(A)$. Thus (3.3) yields

$$\mathrm{w}^{\mathrm{dil}}(P_m(A)) = m. \tag{3.5}$$

Since $w^{\mathrm{dil}}(A^\ell_{\alpha\overline{\beta}}) > 0$, it follows from (3.5) that:

Lemma 3.1. *Each polynomial $P_m(A)$ in* (3.4) *depends only on $A^\ell_{\alpha\overline{\beta}}$ such that $\mathrm{w}^{dil}(A^\ell_{\alpha\overline{\beta}}) \le m$.*

A crucial fact is the following.

Proposition 3.1. *The expansion* (3.4) *determines $K^s = (K^s_A)$.*

Proof. We first take a small neighborhood $M \subset \partial\Omega$ of the origin. For any $q \in M$ fixed, we then place M about q in normal form $N(A)$ with some $A \in \mathcal{N}^\omega$. By [CM], we may take the local biholomorphic mappings $\Phi_{q,A} \colon M \to N(A)$ with $\Phi_{q,A}(q) = 0$ to depend on $q \in M$ real analytically. Setting

$$K^s_{q,A} = (K^s_\Omega \circ \Phi^{-1}_{q,A})|\det \Phi'_{q,A}|^{-2w(s)/(n+1)},$$

we have, as in (3.4),

$$K^s_{q,A}(\gamma_t) = \sum_{m=0}^{\infty} P_m(A)\, \widehat{K}_{m-w(s)}[t]. \tag{3.6}$$

The point is that $P_m(A)$ in (3.6) are independent of $q \in M$. This fact follows from the universality of $P^{\ell m}_{\alpha\overline{\beta}}(A)$ in (3.2). The expansion (3.2) about the origin is recovered from (3.6) by varying $q \in M$. Thus (3.4) determines (3.2). □

3.2. Definition of local Sobolev-Bergman kernels in the C^∞ category. Let us define local Sobolev-Bergman kernels $K^s = (K^s_\Omega)$ near $0 \in \partial\Omega$ in case each $\partial\Omega$ is merely C^∞. We regard each K^s_Ω as a formal singularity. In other words, we ignore the difference by flat functions. As before, it suffices to specify $K^s = (K^s_A)_{A \in \mathcal{N}}$ given by the transformation law (3.1). This is done by real analytic approximation. More precisely, we first truncate $A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N}$ by neglecting $A^\ell_{\alpha\overline{\beta}}$ such that $\mathrm{w}^{\mathrm{dil}}(A^\ell_{\alpha\overline{\beta}}) > N$ for $N \in \mathbb{N}$ large, and denote the results by A_N. Then $N(A_N)$ are algebraic real hypersurfaces, for which we can consider an expansion of the form (3.2). By Proposition 3.1, this expansion is determined by an expansion of the form (3.4). In this new expansion, the coefficients $P_m(A)$ for $m \le N$ are determined by A_N, a fact which follows from Lemma 3.1. In other words, these $P_m(A)$ are unchanged if A_N are replaced by A_{N+1}. Consequently, we have the expansions (3.4) and (3.2) for any $A \in \mathcal{N}$ even when $A \notin \mathcal{N}^\omega$. Therefore, $K^s = (K^s_\Omega)$ for $\partial\Omega \in C^\infty$ near $0 \in \mathbb{C}^n$ is well-defined.

Remark 2. Let $s \in \mathbb{Z}$ and $s \notin [0, n+1]$. Then by Theorem 2, there does not exist a local Sobolev-Bergman kernel of order $s/2$. Nevertheless, we can define a similar local domain functional $K^s = (K^s_\Omega)$ with ambiguity. We require $K^s = (K^s_\Omega)$ to satisfy Conditions SB1–3, but the exact transformation law (2.5) in Condition SB2 is replaced by an approximate one. The existence of $K^s = (K^s_\Omega)$ in the real analytic category is proved as in the exact kernels case $s \in [0, n+1]$, though we have to be more careful in inspecting the construction in Subsection 2.1. The ambiguity of K comes from that of $\widehat{K}$ via that of $\mathbf{A}(z, \partial_z)$. The definition of $K^s = (K^s_\Omega)$ in the C^∞ category is also similar to that in the exact kernels case $s \in [0, n+1]$ in the previous subsection. We have (3.1) if each K^s_Ω is regarded as an equivalence class with respect to the ambiguity. For the approximate kernels as above, one can develop Fefferman's invariant theory as in the next section.

4. Invariant expansions of local Sobolev-Bergman kernels

4.1. Ambient metric construction. Let $K^s = (K^s_\Omega)$ be the local Sobolev-Bergman kernel of order $s/2$ in the C^∞ category, so that $s \in \mathbb{Z}$ satisfies $0 \le s \le n+1$. As before, we set $w(s) = n+1-s$. Let $r = (r_\Omega)$ be a family of C^∞ local defining functions satisfying Condition SB3 with $N = n+2$ in place of $N = s+1$. It has been known for $s = 0, 1$ (that is, for the Bergman kernel and the Szegö kernel) that K^s admits an expansion of the form

$$K_\Omega^s = \sum_{m=0}^{n} W_m^s[r_\Omega]\,\widehat{K}_{m-w(s)}[r_\Omega] \qquad \mathrm{mod}\ O(\widehat{K}_s[r_\Omega]), \tag{4.1}$$

where $W_m^s = W_m^s[r_\Omega]$ are Weyl functionals of weight m given by the ambient metric construction (cf. [F3], [BEG], [HKN1], [Hi]). Terminology will be reviewed below in this subsection (Definitions 4.1 and 4.2). If $n = 2$ and $s = 0, 1$ then (4.1) is refined as follows (cf. [G2], [HKN1], [HKN2]):

$$K_\Omega^s = \sum_{m=0}^{5} W_m^s[r_\Omega]\,\widehat{K}_{m-w(s)}[r_\Omega] \qquad \mathrm{mod}\ O(\widehat{K}_{s+3}[r_\Omega]), \tag{4.2}$$

where $W_m^s = W_m^s[r_\Omega]$ $(m \neq 3)$ are Weyl-Fefferman functionals of weight m. Here, the case $m = 3$ is exceptional and we explain it at the end of this subsection. The proof of these facts yields the following:

Proposition 4.1. *An expansion of the form* (4.1) *holds in general for* $0 \le s \le n+1$.

Proposition 4.2. *An expansion of the form* (4.2) *for* $n = 2$ *holds in general for* $0 \le s \le 3$.

In fact, we have defined the local Sobolev-Bergman kernel in such a way that Propositions 4.1 and 4.2 are obvious. In order to explain it, we begin by recalling the ambient metric construction. For simplicity of notation, we drop the subscript Ω in r_Ω and write $K^s[r]$ for K_Ω^s. Though our description below looks global near $\partial\Omega$, it is obvious as before how to localize or formalize to a neighborhood of a boundary point of reference.

The ambient metric $g = g[r]$ is defined by the potential $r_\#(z_0, z) = |z_0|^2 r(z)$ on $\mathbb{C}^* \times \overline{\Omega}$, where $z_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is an extra variable. That is, g is a Lorentz-Kähler metric in a neighborhood of $\mathbb{C}^* \times \partial\Omega$, inside Ω. Specifically,

$$g = \sum_{j,k=0}^{n} g_{j\overline{k}}\, dz_j d\overline{z}_k = \sum_{j,k=0}^{n} \frac{\partial^2 r_\#}{\partial z_j \partial \overline{z}_k}\, dz_j d\overline{z}_k.$$

Denoting by $R = R[r]$ the curvature tensor of g, we consider successive covariant derivatives $R^{(p,q)} = \overline{\nabla}^{q-2}\nabla^{p-2}R$. Regarding components of $R^{(p,q)}$ as independent variables, we manufacture complete contractions, with respect to g, of the form

$$W_\# = \mathrm{contr}\Big(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_m,q_m)}\Big), \tag{4.3}$$

where $\sum p_\ell = \sum q_\ell = 2(m+w)$, the definition of w called the weight of $W_\#$. By a *Weyl polynomial* $W_\#$ of weight w, we mean a linear combination of complete contractions of the form (4.3) of weight w. Here, $W_\#$ is regarded as a polynomial in components of $R^{(p,q)}$ for all $p, q \geq 2$.

Given a Weyl polynomial $W_\#$ of weight w, we now regard it as a functional of r and write $W_\# = W_\#[r]$. Setting $W[r] = W_\#[r]|_{z_0=1}$, we have

$$W_\#[r](z_0, z) = |z_0|^{2w} W[r](z).$$

Using the terminology in [HKN2], we pose:

Definition 4.1. $W = W[r]$ is called a *Weyl functional* of weight w.

If $W = W[r]$ is a Weyl functional of weight w, then the following transformation law holds under biholomorphic mappings $\Phi : \Omega_1 \to \Omega_2$

$$W[r_1] = (W[r_2] \circ \Phi) |\det \Phi'|^{2w/(n+1)}, \tag{4.4}$$

provided r_j are defining functions of Ω_j, subject to the restriction at the beginning of this section, such that $r_1 = (r_2 \circ \Phi)|\det \Phi'|^{-2/(n+1)}$. Furthermore, (4.4) holds modulo $O(r^{n+1-w})$, without assuming the relation between r_1 and r_2. Consequently, it follows from the construction that if $w \leq n$ then the boundary value of $W[r]$ is a CR invariant of weight w. This is a consequence of the polynomial dependence of $W[r]$ on $A \in \mathcal{N}$ in the sense as before.

Definition 4.2 (cf. [HKN2]). Let $n = 2$. We say that a Weyl functional $W = W[r]$ of weight w is a *Weyl-Fefferman functional* if $W[r]$ modulo $O(r^{6-w})$ is independent of the choice of r.

If $n = 2$ and $W = W[r]$ is a Weyl-Fefferman functional of weight w, then (4.4) holds modulo $O(r^{6-w})$. Hence, if $w \leq 5$ then the boundary value of $W[r]$ is a CR invariant of weight w.

By a CR invariant of weight w, we mean a polynomial $P(A)$ in $A \in \mathcal{N}$ satisfying the transformation law

$$P(A) = P(\widetilde{A}) |\det \Phi'(0)|^{2w/(n+1)}$$

under any local biholomorphic mapping $\Phi \colon N(A) \to N(\widetilde{A})$ such that $\Phi(0) = 0$. We denote the totality of these $P(A)$ by I_w^{CR}. Any CR invariant can be regarded as a smooth function on $\partial\Omega$. Propositions 4.1 and 4.2 are consequences of the following fact, except for $W_3^s[r_\Omega]$ in (4.2).

Proposition 4.3. *If* $n \geq 3$ *and* $w \leq n$, *then any* CR *invariant of weight* w *is realized by the boundary value of a Weyl functional of weight* w. *If*

$n = 2$, $w \le 5$ and $w \neq 3$, then any CR *invariant of weight w is realized by the boundary value of a Weyl-Fefferman functional of weight w.*

For the proof, see [BEG] and [HKN2].

Remark 3. Let us say that a Weyl functional is linear (resp. nonlinear) if the corresponding Weyl polynomial is linear (resp. nonlinear).

(1°) Let $n = 2$ and $w \le 5$. Then, any nonlinear Weyl functional of weight w is a Weyl-Fefferman functional and any linear Weyl-Fefferman functional of weight w is trivial. Now let $W \neq 0$ be a linear Weyl functional of weight w. If $w \le 2$ then the boundary value of W is zero, whereas if $w = 3$ then the boundary value of W is nonzero and gives rise to a CR invariant. The vector space of CR invariants of weight 3 is one dimensional, and thus a base is realized by the boundary value of a linear Weyl functional, though the ambiguity estimate is too rough. (Cf. [HKN2] for the detail.)

(2°) Let $n \ge 3$ and $w \le n+1$. It is plausible that any Weyl functional of weight w has the ambiguity modulo $O(r^{n+2-w})$ and that any CR invariant of weight w is realized by the boundary value of a Weyl functional of weight w. If $w \le n$ then any linear Weyl functional of weight w is trivial (cf. [F3]). It is desirable to define the notion of Weyl-Fefferman functionals as in the case of $n = 2$ by the optimal ambiguity estimate for nonlinear Weyl functionals.

(3°) According to the theory developed in [Hi] and roughly explained in the next section, the Weyl functionals $W_w = W_w[r]$ of arbitrary weight w make sense as functionals of a special family of defining functions r, where the ambiguity of r is measured by a parameter and its effect on $W_w = W_w[r]$ is taken into account. In this sense, Propositions 4.1 and 4.2 can be refined in such a way that (4.1) and (4.2) are infinite asymptotic series. Here, we don't need a refinement of Proposition 4.3, which is stated in Subsection 5.3.

We conclude this subsection by explaining what is $W_3^s[r]$ in (4.3), where the subscript Ω in r_Ω is dropped. For each s, this is a constant multiple of $\eta_1^{\mathrm{G}} = \eta_1^{\mathrm{G}}[r]$ which appears in Graham's asymptotic solution of $J[u] = 1$:

$$u^{\mathrm{G}} = r \sum_{k=0}^{\infty} \eta_k^{\mathrm{G}} \cdot (r^{n+1} \log r)^k, \quad \eta_k^{\mathrm{G}} \in C^\infty(\overline{\Omega})$$

in the general case of dimension $n \ge 2$. This is a formal series, and the difference of flat functions along $\partial\Omega$ is ignored in determining η_k^{G}. We have

$$\eta_0^{\mathrm{G}} = 1 + ar^{n+1} + O(r^{n+2}) \qquad \text{with} \quad a \in C^\infty(\partial\Omega),$$

and u^{G} is uniquely constructed by specifying a. We have approximate transformation laws

$$\eta^{\mathrm{G}}_{k,\Omega_1} = (\eta^{\mathrm{G}}_{k,\Omega_2} \circ \Phi)|\det \Phi'|^{2k} \mod O(r^{n+1})$$

under (local) biholomorphic mappings $\Phi\colon \Omega_1 \to \Omega_2$. In particular, each η^{G}_k modulo $O(r^{n+1})$ is independent of a and r, as far as r is subject to the condition at the beginning of this subsection. By construction, the polynomial dependence on $A \in \mathcal{N}$ is valid as before. Thus, η^{G}_1 for $n = 2$ behaves like a Weyl-Fefferman functional of weight 3.

4.2. Explicit result in dimension ≥ 3. Let $n \geq 3$. It is proved in [G2] that $I^{\mathrm{CR}}_0 = \mathbb{C}$, $I^{\mathrm{CR}}_1 = \{0\}$ and that I^{CR}_2 is generated by

$$\|A^0_{2\overline{2}}\|^2 = \sum_{|\alpha|=|\beta|=2} |A^0_{\alpha\overline{\beta}}|^2.$$

Consequently, we have for $W^s_m = W^s_m[r_\Omega]$ in the expansion (4.1),

$$W^s_0 = 1, \quad W^s_1 = 0, \quad W^s_2[r_\Omega]\Big|_{\partial\Omega} = c^s(n)\|A^0_{2\overline{2}}\|^2, \tag{4.5}$$

where $c^s(n)$ are universal constants. By [HKN1],

$$c^0(n) = \frac{2}{3(n-1)n}, \quad c^1(n) = \frac{2}{3(n-2)(n-1)}.$$

By a similar proof, we have:

Proposition 4.4. *The constants $c^s(n)$ in (4.5) are given by*

$$c^s(n) = \frac{2}{3(n-s-1)(n-s)} \quad \mathit{for} \quad s \neq n-1, n,$$

and $c^{n-1}(n) = -2/3$, $c^n(n) = 2/3$.

4.3. Explicit results in dimension two. Let $n = 2$. We first note by [G2] that $I^{\mathrm{CR}}_0 = \mathbb{C}$ and that I^{CR}_1 and I^{CR}_2 are trivial. Consequently, we have for $W^s_m = W^s_m[r_\Omega]$ in the expansion (4.2),

$$W^s_0 = 1, \quad W^s_1 = 0, \quad W^s_2 = 0.$$

It remains to determine $\psi^s = W^s_3 + W^s_4 r + W^s_5 r^2$, where we abbreviated by writing r and W^s_j in place of r_Ω and $W^s_j[r_\Omega]$, respectively. By [G2] and [HKN2], we have

$$\dim I^{\mathrm{CR}}_3 = \dim I^{\mathrm{CR}}_4 = 1, \quad \dim I^{\mathrm{CR}}_5 = 2.$$

More precisely, $I_3^{\rm CR}$ and $I_4^{\rm CR}$ are generated by $A^0_{4\bar 4}$ and $|A^2_{2\bar 4}|^2$, respectively; $I_5^{\rm CR}$ is spanned by $F_5^{\rm CR}(1,0)$ and $F_5^{\rm CR}(0,1)$, where

$$F_5^{\rm CR}(a,b) = F(a,b,-2a+(10/9)b,-a+b/3)$$

with $F(a,b,c,d) = a|A^0_{5\bar 2}|^2 + b|A^0_{4\bar 3}|^2 + {\rm Re}\{(cA^0_{3\bar 5} - idA^1_{2\bar 4})A^0_{4\bar 2}\}$. By Graham [G2], the boundary value of $\eta_1^{\rm G}$ is $4A^0_{4\bar 4}$. It is proved in [HKN2] that if $p+q-2 = 4,5$ then $\|R^{(p,q)}\|^2$ is a Weyl-Fefferman functional of weight $w = p+q-2$, where $\|R^{(p,q)}\|^2$ stands for the squared norm of the tensor $R^{(p,q)}$ with respect to the ambient metric g restricted to $z_0 = 1$. (The squared norm need not be non-negative because g is a Lorentz metric.) Furthermore, the boundary values of $\|R^{(5,2)}\|^2$ and $\|R^{(4,3)}\|^2$ are linearly independent as CR invariants. Consequently, we may set

$$\psi^s = c_0^s\eta_1^{\rm G} + c_1^s\|R^{(4,2)}\|^2 r + \Big(c_2^s\|R^{(5,2)}\|^2 + c_3^s\|R^{(4,3)}\|^2\Big)r^2 + O(r^3), \quad (4.6)$$

where c_j^s for $j = 0,\dots,3$ are universal constants.

Proposition 4.5. *The constants* c_j^s *in* (4.6) *are given by*

$$\begin{array}{llll}
c_0^0 = -3, & c_1^0 = 3/1120, & c_2^0 = 61/141120, & c_3^0 = 3/7840,\\
c_0^1 = -2, & c_1^1 = 1/3360, & c_2^1 = 1/23520, & c_3^1 = 1/13230,\\
c_0^2 = -1, & c_1^2 = -1/10080, & c_2^2 = -1/70560, & c_3^2 = -1/169344,\\
c_0^3 = 1, & c_1^3 = 1/4480, & c_2^3 = 1/33075, & c_3^3 = 1/31360.
\end{array}$$

The proof of Proposition 4.5 is done by locally placing $\partial\Omega$ in normal form $N(A)$ and restricting both sides of (4.6) to the half line $\gamma_t = (0, t/2)$, $t > 0$. By [HKN2], we have

$$\begin{aligned}
&\|R^{(4,2)}\|^2(\gamma_t) = 2^8\, q_1(7,0) + 2^8\, q_2(117,435,936,0,50,0)\, t + O(t^2),\\
&\|R^{(5,2)}\|^2(\gamma_t) = 4\cdot(5!)^2 q_2(5/2,9,18,0,1,0) + O(t), \qquad (4.7)\\
&\|R^{(4,3)}\|^2(\gamma_t) = 4\cdot(5!)^2 q_2(37/30,5,57/5,0,4/3,0) + O(t),
\end{aligned}$$

where

$$\begin{aligned}
&q_1(d_1,d_2) = d_1|A^0_{2\bar 4}|^2 + d_2A^0_{5\bar 5},\\
&q_2(d_1,d_2,d_3,d_4,d_5,d_6) = {\rm Re}\Big(2d_1iA^1_{2\bar 4}A^0_{4\bar 2} + 2d_2A^0_{3\bar 5}A^0_{4\bar 2} + d_3|A^0_{3\bar 4}|^2\\
&\qquad\qquad + d_4A^2_{4\bar 4} + d_5|A^0_{2\bar 5}|^2 + d_6A^0_{6\bar 6}\Big).
\end{aligned}$$

Though [HKN2] does not give the expansion of $\eta_1^{\rm G}(\gamma_t)$ for general $N(A)$, an algorithm of computation is provided. Computer-aided calculation yields

Lemma 4.1.

$$\begin{aligned}\eta_1^{\mathrm{G}}(\gamma_t) = {} & 4\,A^0_{4\bar 4} + q_1(368/5, -20)\,t \\ & + q_2(226/15, -312, -1956/5, 2, -680/3, 60)\,t^2 + O(t^3).\end{aligned}$$

A method of computation of $\psi^s(\gamma_t)$ is given in Appendix B. Again, computer-aided calculation yields

Lemma 4.2. *With q_1 and q_2 as in Lemma 4.1,*

$$\begin{aligned}\psi^0(\gamma_t) &= -12A^0_{4\bar 4} + q_1(-216, 60)\,t \\ &\quad + q_2(-36, 900, 1116, -6, 660, -180)\,t^2 + O(t^3), \\ \psi^1(\gamma_t) &= -8A^0_{4\bar 4} + q_1(-440/3, 40)\,t \\ &\quad + q_2(-248/9, 1840/3, 760, -4, 4040/9, -120)\,t^2 + O(t^3), \\ \psi^2(\gamma_t) &= -4A^0_{4\bar 4} + q_1(-664/9, 20)\,t \\ &\quad + q_2(-131/9, 310, 386, -2, 680/3, -60)\,t^2 + O(t^3) \\ \psi^3(\gamma_t) &= 4A^0_{4\bar 4} + q_1(74, -20)\,t \\ &\quad + q_2(15, -312, -390, 2, -228, 60)\,t^2 + O(t^3).\end{aligned}$$

Proposition 4.5 is proved by using Lemmas 4.1 and 4.2, together with (4.7) and the result for $s = 0$ or $s = 1$ given in [HKN2].

4.4. A construction of CR invariants of weight five in dimension two. As an implication of Lemmas 4.1 and 4.2, we now give a linear relation satisfied by η_1^{G} and local Sobolev-Bergman kernels of order $s/2$ for $s = 0, 1, 2, 3$. Let us first normalize by setting

$$\eta_{\mathrm{I}} = \frac{\eta_1^{\mathrm{G}}}{4}, \quad \psi^0_{\mathrm{I}} = \frac{\psi^0}{4}, \quad \psi^1_{\mathrm{I}} = -\frac{\psi^1}{4}, \quad \psi^2_{\mathrm{I}} = -\frac{\psi^2}{8}, \quad \psi^3_{\mathrm{I}} = -\frac{\psi^3}{12}.$$

so that the evaluation at $z = 0$ gives rise to $\eta_{\mathrm{I}} = \psi^s_{\mathrm{I}} = A^0_{4\bar 4}$ for $s = 0, 1, 2, 3$. To get a CR invariant of weight four, we next set

$$\eta_{\mathrm{II}} = \frac{5}{2}\,\frac{\eta_{\mathrm{I}} - \psi^3_{\mathrm{I}}}{r}, \quad \psi^0_{\mathrm{II}} = 2\,\frac{\psi^0_{\mathrm{I}} - \psi^3_{\mathrm{I}}}{r}, \quad \psi^1_{\mathrm{II}} = \frac{9}{4}\,\frac{\psi^1_{\mathrm{I}} - \psi^3_{\mathrm{I}}}{r}, \quad \psi^2_{\mathrm{II}} = 3\,\frac{\psi^2_{\mathrm{I}} - \psi^3_{\mathrm{I}}}{r}.$$

Then $\eta_{\mathrm{II}} = \psi^s_{\mathrm{II}} = |A^0_{2\bar 4}|^2$ at $z = 0$ for $s = 0, 1, 2$. We thus set

$$\eta_{\mathrm{III}} = \frac{\eta_{\mathrm{II}} - \psi^2_{\mathrm{II}}}{r}, \quad \psi^0_{\mathrm{III}} = 6\,\frac{\psi^0_{\mathrm{II}} - \psi^2_{\mathrm{II}}}{r}, \quad \psi^1_{\mathrm{III}} = \frac{48}{5}\,\frac{\psi^1_{\mathrm{II}} - \psi^2_{\mathrm{II}}}{r}.$$

Then

$$\eta_{\rm III}|_{z=0} = q_2(7/12, -5/2, -6, 0, -5/6, 0),$$
$$\psi^0_{\rm III}|_{z=0} = \psi^1_{\rm III}|_{z=0} = q_2(1, -6, -18, 0, -4, 0).$$

The right sides are CR invariants of weight five which are linearly independent. In particular, we see that $\dim I_5^{\rm CR} \geq 2$. This observation was indeed used as a motivation of getting results in [HKN2] about $I_5^{\rm CR}$.

5. Proof of Theorem 2.

5.1. Non-existence of exactly invariant defining functions. We prove Theorem 2 stated in Subsection 2.3. This is done by using the non-existence of a local defining function $r = r_\Omega$ satisfying exact transformation law of weight -1. To state it more precisely, we introduce spaces $\mathcal{F}^m_{\rm def}$ of local defining functions for $m \geq 3$ ($m \in \mathbb{Z}$) as follows. Recall first that $C^\infty_{\rm def}(\overline{\Omega})$ is the totality of functions $r \in C^\infty(\overline{\Omega})$ such that $r > 0$ in Ω and $dr \neq 0$ on $\partial\Omega$. Localizing it, we have a sheaf of (smooth) local defining functions $C^\infty_{{\rm def},\partial\Omega}(\overline{\Omega}) = (C^\infty_{{\rm def},p}(\overline{\Omega}))_{p\in\partial\Omega}$. If $\partial\Omega = N(A)$ with $A \in \mathcal{N}$, we write $C^\infty_{{\rm def},A} = C^\infty_{{\rm def},0}(\overline{\Omega})$, where we disregard the difference by flat functions at the origin. Then, $C^\infty_{\rm def} = (C^\infty_{{\rm def},A})_{A\in\mathcal{N}}$ is a space of local domain functionals which represent local defining functions. We denote by

$$\mathcal{F}^m_{\rm def} = (\mathcal{F}^m_{{\rm def},A})_{A\in\mathcal{N}} \quad \text{for} \quad m \geq 3 \quad (m \in \mathbb{Z}),$$

the totality of $r = (r_A)_{A\in\mathcal{N}} \in C^\infty_{\rm def}$ such that r satisfies the transformation law of weight -1 modulo $O(r^m)$ and that if

$$r_A(\gamma_t) = \sum_{j=1}^{m-1} P_j(A)t^j + O(t^m)$$

in Moser's normal coordinates then $P_j(A) \in I_j^{\rm CR}$. Then,

Proposition 5.1. $\mathcal{F}^{n+3}_{\rm def} = \emptyset$.

Postponing the proof for a moment, we first observe that Theorem 2 follows from this.

Proof of Theorem 2. We may assume $w \leq -1$ by considering Kashiwara's transformation. Assume there exists a local Sobolev-Bergman kernel of weight w, $K = \varphi r^{-w} \log r$, where $r \in C^\infty_{\rm def}$ and $\varphi \in C^\infty$ with $\varphi(0) \neq 0$. Setting $\rho = \varphi^{-1/w} r$, we have $K = \rho^{-w} \log \rho$ and $\rho \in C^\infty_{\rm def}$. Furthermore, $\rho \in \cap \mathcal{F}^m_{\rm def}$, but this contradicts with Proposition 5.1. □

The proof of Proposition 5.1 requires some results in [Hi]. In [Hi], a subclass $\mathcal{F}$ of $\mathcal{F}^{n+2}_{\rm def}$ is defined so that

$$\mathcal{F} = (\mathcal{F}_A)_{A\in\mathcal{N}} \not\subset \mathcal{F}^{n+3}_{\rm def}, \tag{5.1}$$

and that the ambient metric construction gives rise to Weyl functionals $W = W[r]$ of arbitrary weight $w \in \mathbb{N}_0$ on the class $\mathcal{F}$. We have the following two lemmas.

Lemma 5.1. *If $\rho \in \mathcal{F}^m_{\mathrm{def}}$ with $m \geq 3$, then*

$$\rho = cr + \sum_{j=1}^{m-2} W_j[r]r^{j+1} + O(r^m) \quad \text{for} \quad r \in \mathcal{F},$$

where $c > 0$ is a universal constant and $W_j = W_j[r]$ are Weyl functionals of weight j on $\mathcal{F}$.

Lemma 5.2. *If $W = W[r]$ is a Weyl functional of weight $w \in \mathbb{N}_0$ on $\mathcal{F}$, then $r^w W[r]$ modulo $O(r^{n+3})$ is independent of $r \in \mathcal{F}$.*

In the proof of Proposition 5.1, only these lemmas and (5.1) are used. Even the definition of $\mathcal{F}$ is not necessary.

Proof of Proposition 5.1. Assuming $\mathcal{F}^{n+3}_{\mathrm{def}} \neq \emptyset$, we pick $\rho \in \mathcal{F}^{n+3}_{\mathrm{def}}$. It then follows from Lemma 5.1 that

$$\rho = cr + \sum_{j=1}^{n+1} W_j[r]r^{j+1} + O(r^{n+3}) \quad \text{for} \quad r \in \mathcal{F}. \tag{5.2}$$

We set $\phi[r] = \sum W_j[r]r^j$. It then follows from Lemma 5.2 that $\phi[r]$ modulo $O(r^{n+3})$ is independent of $r \in \mathcal{F}$. This also holds for $r\phi[r]$, because $\mathcal{F} \subset \mathcal{F}^{n+2}_{\mathrm{def}}$ and $\phi[r] = O(r)$. Thus, (5.2) with $\rho \in \mathcal{F}^{n+3}_{\mathrm{def}}$ implies $r \in \mathcal{F}^{n+3}_{\mathrm{def}}$, but this contradicts with (5.1). □

5.2. Definition of the class $\mathcal{F}$ and a review of [Hi]. Before proving Lemmas 5.1 and 5.2 with (5.1), let us give the definition of $\mathcal{F}$. It suffices to fix Ω and define a subclass $\mathcal{F}_{\partial\Omega}$ of $C^\infty_{\mathrm{def}}(\overline{\Omega})$ so that the localization of $\mathcal{F}_{\partial\Omega}$ gives rise to $\mathcal{F}$. We begin by considering the boundary value problem

$$J_\#[U] = |z^0|^{2n} \ \text{ and } \ U > 0 \ \text{ in } \ \mathbb{C}^* \times \Omega, \quad U = 0 \ \text{ on } \ \mathbb{C}^* \times \partial\Omega \tag{5.3}$$

for functions $U = U(z^0, z)$, where

$$J_\#[U] = (-1)^n \det(U_{j\bar{k}})_{0 \leq j,k \leq n}, \quad U_{j\bar{k}} = \partial^2 U/\partial z^j \partial \overline{z}^k.$$

This is a lift of the Monge-Ampère operator in the sense that if $U(z^0, z) = |z^0|^2 u(z)$ then $J_\#[U] = |z^0|^{2n} J[u]$. But we are concerned with asymptotic solutions of (5.3) of the form

$$U = r_\# + r_\# \sum_{k=1}^{\infty} \eta_k \cdot (r^{n+1} \log r_\#)^{n+1} \quad \text{with} \quad \eta_k \in C^\infty(\overline{\Omega}), \tag{5.4}$$

where $r_{\#}(z^0, z) = |z^0|^{2n} r(z)$ with $r \in C^\infty_{\mathrm{def}}(\overline{\Omega})$. Note that r is not prescribed but determined together with U. We call r the smooth part of U and denote the totality of these r by $\mathcal{F}_{\partial\Omega}$. The fact $\mathcal{F}_{\partial\Omega} \neq \emptyset$ is proved by solving a formal initial value problem for (5.3) near $\partial\Omega$ with an extra initial condition

$$X^{n+2} r|_{\partial\Omega} = a \in C^\infty(\partial\Omega),$$

where X is a real vector field which is transversal to $\partial\Omega$. The unique existence of the asymptotic solution U for each data $a \in C^\infty(\partial\Omega)$ is valid and the operation of taking the smooth part $U \mapsto r$ is injective, provided we ignore the difference by flat functions along $\partial\Omega$. Thus $a \mapsto r$ is essentially a bijection $C^\infty(\partial\Omega) \to \mathcal{F}_{\partial\Omega}$. The construction is local near a boundary point, or even formal, as we explain at the end of this subsection.

An important fact is that one can formulate an exact transformation law

$$r = (\widetilde{r} \circ \Phi)|\det \Phi'|^{-2/(n+1)} \tag{5.5}$$

under biholomorphic mappings $\Phi\colon \Omega \to \widetilde{\Omega}$. Specifically, if $\widetilde{r} \in \mathcal{F}_{\partial\widetilde{\Omega}}$ and if r is defined by (5.5) then $r \in \mathcal{F}_{\partial\Omega}$. In this sense, Weyl functionals, $W = W[r]$ for $r \in \mathcal{F}_{\partial\Omega}$, of weight w satisfies the exact transformation law

$$W[r] = (W[\widetilde{r}] \circ \Phi)|\det \Phi'|^{2w/(n+1)}. \tag{5.6}$$

A main result of [Hi] states that if ψ^{B} is regarded as a functional of $r \in \mathcal{F}_{\partial\Omega}$ then

$$\psi^{\mathrm{B}}[r] = \sum_{k=0}^{m-1} W_{k+n+1}[r] r^k + O(r^m) \quad \text{for any} \quad m \in \mathbb{N},$$

where $W_j = W_j[r]$ are Weyl functionals of weight j. The proof of this fact applies without change to Lemma 5.1. We thus regard Lemma 5.1 as proved, where the localization is taken into account as follows.

In the definition of the local space $\mathcal{F} = (\mathcal{F}_A)_{A\in\mathcal{N}}$, we may set $X = \partial/\partial\rho$ for Moser's normal coordinates. Then each $\mathcal{F}_A$ is parametrized by a space of formal power series as follows:

$$(\partial^{n+2} r/\partial\rho^{n+2})\Big|_{\rho=0} = \sum_{\alpha,\beta,\ell} C^{\ell}_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, v^\ell \quad \text{for} \quad r \in \mathcal{F}_A.$$

We thus have a bijection $\mathcal{C} \ni C \mapsto r = r_{A,C} \in \mathcal{F}_A$ for each $A \in \mathcal{N}$, where $\mathcal{C}$ denotes the totality of $C = (C^{\ell}_{\alpha\overline{\beta}})$. This bijection is the localization of the composition operator $C^\infty(\partial\Omega) \to \mathcal{F}_{\partial\Omega}$ given by $a \mapsto U$ and $U \mapsto r$. Consequently, we have a bijection

$$\mathcal{N} \times \mathcal{C} \ni (A, C) \mapsto r_{A,C} \in \mathcal{F}_A, \tag{5.7}$$

where C parametrizes the ambiguity of $r_{A,C}$. Setting $r_C = (r_{A,C})_{A\in\mathcal{N}}$, we denote by $\mathcal{F}$ the totality of r_C for $C \in \mathcal{C}$. Then $\mathcal{F} \subset C^\infty_{\mathrm{def}}$. It is easy to see that $\mathcal{F} \subset \mathcal{F}^{n+2}_{\mathrm{def}}$ (see [Hi]), and (5.1) is clear from the definition. Abusing notation, we write r in place of r_C, so that selecting $r \in \mathcal{F}$ is equivalent to specifying $C \in \mathcal{C}$. The point of introducing the class $\mathcal{F}$ is the exact transformation laws (5.5) and (5.6), where $C \in \mathcal{C}$ must vary. It is therefore necessary to regard the space $\mathcal{F}$ itself as a family of local domain functionals parametrized by $C \in \mathcal{C}$.

5.3. Reduction to the boundary. We have justified (5.1) and Lemma 5.1. To prove Lemma 5.2, we need to consider the boundary value of each Weyl functional on $\mathcal{F}$, say $W = W[r]$, where $r = (r_{A,C})_{A\in\mathcal{N}}$ with $r_{A,C}$ in (5.7). More precisely, we take the restriction of $W[r]$ to the origin $0 \in N(A)$. Denoting it by $P_W = P_W(A, C)$, we see by inspecting the construction that P_W is a polynomial in $(A, C) \in \mathcal{N} \times \mathcal{C}$. Let $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C})$ denote the totality of such polynomials which come from Weyl functionals of weight w on $\mathcal{F}$. We define a subspace $I^{\mathrm{W}}_w(\mathcal{N})$ of $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C})$ to be the totality of $P_W(A, C)$ which are independent of $C \in \mathcal{C}$. Then, another main result of [Hi] states that

$$I^{\mathrm{W}}_w(\mathcal{N}) = I^{\mathrm{CR}}_w \quad \text{for} \quad w \in \mathbb{N}_0 \tag{5.8}$$

and that if $n \geq 3$ (resp. $n = 2$) then

$$I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N}) \quad \text{for} \quad w \leq n + 2 \quad (\text{resp. } w \leq 5), \tag{5.9}$$

where the weight restriction in (5.9) is optimal. In the following, Lemma 5.2 is proved by using (5.8), while (5.9) shows that the error estimate in Lemma 5.2 is optimal.

Proof of Lemma 5.2. This is a refinement of Fefferman's Ambiguity Lemma in [F3]. As in [F3], the problem is reduced to the case $\partial\Omega = N(A)$ with $A \in \mathcal{N}$, via the transformation law for r and $W = W[r]$. In Moser's normal coordinates, we investigate the behavior of $r^w W[r]$ along the half line $\gamma_t = (0, t) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $t > 0$. We have

$$(r^w W[r])(\gamma_t) = \sum_{j=m}^{n+2} P_j(A, C) t^j + O(t^{n+3}),$$

where $P_j(A, C)$ are polynomials in $(A, C) \in \mathcal{N} \times \mathcal{C}$. Furthermore, $P_j(A, C)$ is of weight j. It suffices to show that $P_j(A, C)$ are independent of $C \in \mathcal{C}$. Assume that $P_j(A, C)$ depends on C. Since

$$\mathrm{w}(A^{\ell}_{\alpha\overline{\beta}}) \geq 2, \quad \mathrm{w}(C^{\ell}_{\alpha\overline{\beta}}) \geq n + 1$$

for $A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N}$ and $C = (C^\ell_{\alpha\overline{\beta}}) \in \mathcal{C}$, it follows that $P_j(A, C)$ is linear homogeneous. Consequently, the Weyl polynomial $W_\#$ must be linear, so that we may assume $W_\# = \mathrm{tr}(\nabla^{(p,p)}R)$. By the linearity of $P_j(A, C)$, the assumption implies that $P_j(0, C) \neq 0$, so that we are reduces to the case $A = 0 \in \mathcal{N}$. In this case, $N(A)$ is the boundary of a Siegel domain, and any asymptotic solution of (5.3) of the form (5.4) is (formally) smooth. Consequently, any ambient metric is Ricci-flat, so that $W_\#$ must vanish. We thus have $P_j(0, C) = 0$, a contradiction. □

Appendix

Appendix A. Holomorphic microfunctions. Proofs of the facts stated below are found for instance in a textbook by Schapira [S].

Let X be a complex manifold and Y a complex hypersurface. Then Y is locally given by the zeros of a holomorphic function $f(z)$ such that $df \neq 0$. A germ of holomorphic microfunction at $p \in Y$ is, by definition, an equivalence class modulo $\mathcal{O}_{X,p}$ of a germ of (multi-valued) holomorphic function in $X \setminus Y$ of the form

$$\varphi f^{-m} + \psi \log f \quad \text{with} \quad m \in \mathbb{Z}, \quad \varphi, \psi \in \mathcal{O}_{X,p}.$$

Let $\mathcal{C}_{Y|X,p}$ denote the vector space of those equivalence classes. Then a sheaf of holomorphic microfunctions is defined by $\mathcal{C}_{X|Y} = (\mathcal{C}_{Y|X,p})_{p\in Y}$. For $L \in \mathcal{C}_{Y|X,p}$, the singular support of L is contained in

$$N = T^*_Y X \setminus 0 = \{(p, \xi) \in T^*X;\ p \in Y,\ \xi = c\, df|_{z=p},\ c \in \mathbb{C}^*\},$$

the conormal bundle of $Y \subset X$. (In [SKK], $\mathcal{C}_{Y|X}$ is defined to be a sheaf on the projective conormal bundle $N/\mathbb{C}^*$, which can be identified with Y.) The sheaf $\mathcal{E}_X$ of microdifferential operators is defined in such a way that a germ $P(z, \partial_z) \in \mathcal{E}_{X,\hat{p}}$ acts on $\mathcal{C}_{Y|X,p}$, where $\hat{p} = (p, \xi) \in N$. Specifically, $\mathcal{E}_{X,\hat{p}}$ is a ring generated by

$$z_1, \dots, z_n,\ \partial_{z_1}, \dots, \partial_{z_n} \quad \text{and} \quad \partial_{z_n}^{-1},$$

where $z = (z_1, \dots, z_n)$ is a local coordinate system of X such that $z_n = f$. The action of $\partial_{z_n}^{-1}$ on $L \in \mathcal{C}_{Y|X,p}$ is given by a curvilinear integral

$$\partial_{z_n}^{-1} L(z) = \int_{p'}^{z} L(z)\, dz_n,$$

where $p' \in X \setminus Y$ is chosen so close to p that the right side (modulo $\mathcal{O}_{X,p}$) is independent of the choice of p'.

We say that $L \in \mathcal{C}_{Y|X,p}$ is *nondegenerate* if L is represented by a function of the form

$$\varphi\, f^{-m} + \psi \log f \quad \text{for} \quad m > 0, \quad \text{or} \quad \varphi\, r^{-m} \log f \quad \text{for} \quad m \leq 0, \qquad (A.1)$$

where φ is non-vanishing. If $L \in \mathcal{C}_{Y|X,p}$ is of the form $L = P \log f$ with $P = P(z, \partial_z) \in \mathcal{E}_{X,\hat{p}}$, then L is nondegenerate if and only if P is elliptic (i.e. invertible).

In what follows, we consider the case $X = \mathbb{C}^n \times \overline{\mathbb{C}^n}$, the complexification of the diagonal $\{(z, w) \in X;\ w = \overline{z}\} = \mathbb{C}^n \cong \mathbb{R}^{2n}$. Let Ω be a domain in $\mathbb{C}^n$ such that the boundary is locally given by a real-analytic defining function $\rho(z, \overline{z})$ near a boundary point of reference. Then the complexification of the boundary $\partial\Omega$ is locally given by $Y = \{(z, w) \in X;\ \rho(z, w) = 0\}$.

Lemma A.1. *If Ω is strictly pseudoconvex locally, then every holomorphic microfunction $L \in \mathcal{C}_{Y|X,(z_0,w_0)}$ is written as*

$$L(z, w) = P(z, \partial_z) \log \rho(z, w) = Q(w, \partial_w) \log \rho(z, w), \qquad (A.2)$$

where $P \in \mathcal{E}_{\mathbb{C}^n,(z_0,d_z\rho)}$ and $Q \in \mathcal{E}_{\overline{\mathbb{C}^n},(w_0,d_w\rho)}$ are microdifferential operators determined uniquely by L.

In this lemma, we may replace $\log \rho$ by any nondegenerate holomorphic microfunction K with support Y. It then follows that for any $P(z, \partial_z) \in \mathcal{E}_{\mathbb{C}^n,(z_0,d_z\rho)}$ there exits a unique $Q(w, \partial_w) \in \mathcal{E}_{\overline{\mathbb{C}^n},(w_0,d_w\rho)}$ such that

$$P(z, \partial_z)K = Q(w, \partial_w)K.$$

Let Q^* denote the formal adjoint of Q. Then the correspondence $P \mapsto Q^*$ gives rise to an isomorphism of rings $\mathcal{E}_{\mathbb{C}^n,(z_0,d_z\rho)} \to \mathcal{E}_{\overline{\mathbb{C}^n},(w_0,-d_w\rho)}$, which is called the *quantized contact transformation with kernel K*. It is clear from Lemma A.1 that

Lemma A.2. *If two kernels $K, \widetilde{K} \in \mathcal{C}_{Y|X,(z_0,w_0)}$ give the same quantized contact transformation, then $K = c\,\widetilde{K}$ with some constant $c \in \mathbb{C}^*$.*

If $K \mapsto \widehat{K}$ is Kashiwara's transformation, then

$$P(z, \partial_z)K = Q(w, \partial_w)K \ \text{ if and only if } \ P^*(z, \partial_z)\widehat{K} = Q^*(w, \partial_w)\widehat{K}.$$

In particular, the quantized contact transformation $P(z, \partial_z) \mapsto Q^*(w, \partial_w)$ with kernel K is given by the inverse of the quantized contact transformation $Q(w, \partial_w) \mapsto P^*(z, \partial_z)$ with kernel $\widehat{K}$.

The proof of Lemma A.1 (e.g., in Shapira [S]) simply yields the following lemma, which was used in the proof of Lemma 3.2.

Lemma A.3. *If L in* (A.2) *is of the form* (A.1) *with ρ in place of f and with φ non-vanishing, then P and Q are operators of order $\leq m$.*

Appendix B. Method of computing the asymptotic expansion. We here explain the method of computing the expansion of K^s.

Let us first recall the procedure for computing the Bergman kernel K^0 due to Boutet de Monvel. We take a $\mathbb{C}$-valued defining function of the complexification of $\partial\Omega$ of the form $U(z,\overline{z}) = z_n + \overline{z}_n - z' \cdot \overline{z'} - H(z,\overline{z'})$, where

$$H(z,\overline{z'}) = \sum_{|\alpha|,|\beta|\geq 2,\ell\geq 0} B^\ell_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, z^\ell_n.$$

Then each $B^\ell_{\alpha\overline{\beta}}$ is a polynomial in $A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N}$. Let $\mathbf{A}_0 = \mathbf{A}_0(z,\partial_z)$ be a microdifferential operator of infinite order given by the total symbol

$$\mathbf{A}_0(z,\zeta) = \exp\left(-H(z,-\zeta'/\zeta_n)\zeta_n\right).$$

We define weight by

$$\mathrm{w}(z_j) = -\mathrm{w}(\partial_{z_j}) = -1/2 \quad (j < n), \quad \mathrm{w}(z_n) = -\mathrm{w}(\partial_{z_n}) = -1.$$

(For more about the notion of weight, cf. Section 3 of [HKN2].) Then $\mathbf{A}_0$ can be regarded as an asymptotic series as weight tends to $-\infty$. We can verify $\log U = \mathbf{A}_0(z,\partial_z)\log\rho_0$ by using $\partial_{z_j}\partial^{-1}_{z_n}\log\rho_0 = -\overline{z_j}\log\rho_0$. Therefore the Bergman kernel $K^0[r]$ for Ω (up to a constant multiple $(-\pi)^n$) is given by

$$K^0[r] = \mathbf{A}_0^{*-1}(z,\partial_z)\,\widehat{K}_{-n-1}[\rho_0]. \qquad (B.1)$$

Here the inverse of $\mathbf{A}_0^*$ is defined by $\mathbf{A}_0^{*-1} = \sum_{k=0}^\infty (1-\mathbf{A}_0^*)^k$, which is an asymptotic series as weight tends to $-\infty$ because each term of $1-\mathbf{A}_0^*$ has negative weight.

We generalize (B.1) to K^s for $s > 0$. First, write

$$\widehat{K}_s[r] = \sum_{\ell=1}^s a_\ell(z,\overline{z'})\,\widehat{K}_\ell[U]$$

and define a microdifferential operator of infinite order by the total symbol

$$\mathbf{A}_s(z,\zeta) = \mathbf{A}_0(z,\zeta)\sum_{\ell=1}^s a_\ell(z,-\zeta'/\zeta_n)\zeta_n^\ell.$$

Then we get $\widehat{K}_s[r] = \mathbf{A}_s(z,\partial_z)\log\rho_0$ by using $\mathbf{A}_0(z,\partial_z)\partial^\ell_{z_n}\log\rho_0 = \widehat{K}_\ell[U]$. Thus we have

$$K^s[r] = \mathbf{A}_s^{*-1}(z,\partial_z)\,K^0[\rho_0].$$

Here $\mathbf{A}_s^{*-1}$ is defined by the series

$$\mathbf{A}_s^{*-1} = \partial_{z_n}^{-s}\sum_{k=0}^\infty(1-\mathbf{A}_s^*\partial_{z_n}^{-s})^k,$$

in which each term in $1 - \mathbf{A}_s^* \partial_{z_n}^{-s}$ has negative weight.

Method of proving Lemma 4.2. We only need to know the first five terms in

$$\mathbf{A}_s^{*-1}(z,\zeta)\Big|_{z_1=\zeta_1=0} = \sum_{k=-s}^{\infty} c_k\, \zeta_2^{-k},$$

that is, the terms of weight $\geq -s-5$ in the right-hand side. Such terms can be computed from the the terms of $\mathbf{A}_s$ that have weight $\geq s-5$. Details of this computation are discussed in [HKN2]. □

Proof of Proposition 4.4. We only need to compute $K^s(\gamma_t)$ for a surface in normal form for which $\|R^{(2,2)}\|^2(0) = \|A^0_{2\overline{2}}\|^2 \neq 0$. We here take the surface $\rho = \rho_0 - F = 0$, where $F = z_1^2\overline{z}_2^2 + z_2^2\overline{z}_1^2$, for which $\|A^0_{2\overline{2}}\|^2 = 2$. Starting from this ρ, we set ρ_1, ρ_2 and ρ_3 as in Subsection 2.3. Then we have $r = \rho_3 + O(\rho^3)$. Since each term in ρ^3 has weight less than -3, we see that $r = r_3 +$ (terms of weight < -3). Thus we have

$$r = \rho + \left(\frac{16|z_1 z_2|^2\rho_0}{n+1} - \frac{8(|z_1|^2+|z_2|^2)\rho_0^2}{(n+1)n} + \frac{16\rho_0^3}{3(n+1)n(n-1)}\right) + (\text{terms of weight} < -3). \quad (B.2)$$

In particular, we get

$$r(\gamma_t) = t + 2c'\,t^3 + O(t^4) \quad \text{with} \quad c' = \frac{8}{3(n+1)n(n-1)}. \quad (B.3)$$

Next we write $\widehat{K}_s[r] = \mathbf{A}_s(z,\partial_z)\log\rho_0$. Then from (B.2) we get

$$\mathbf{A}_s(z,\zeta) = \zeta_n^s - \widetilde{F}\zeta_n^{s-1} - \Big(-\frac{\widetilde{F}^2}{2} + \frac{16s}{n+1}\, z_1 z_2 \zeta_1\zeta_2 + \frac{8s(s-1)}{(n+1)n}(z_1\zeta_1 + z_2\zeta_2) + \frac{16s(s-1)(s-2)}{3(n+1)n(n-1)}\Big)\zeta_n^{s-2} + (\text{terms of weight} < s-3),$$

where $\widetilde{F} = z_1^2\zeta_2^2 + z_2^2\zeta_1^2$. Thus we have

$$\mathbf{A}_s^{*-1}(z,\zeta)\Big|_{z'=\zeta'=0} = \zeta_n^{-s} + \widetilde{c}^s\zeta_n^{-s-2} + (\text{terms of weight} < -s-3),$$

where

$$\widetilde{c}^s = -4 + \frac{16s}{n+1} + \frac{16s(s-1)}{(n+1)n} + \frac{16s(s-1)(s-2)}{3(n+1)n(n-1)}.$$

Therefore we get, for $s = 0, 1, \ldots, n-2$,

$$K^s(\gamma_t) = t^{s-n-1}\left(1 + \frac{\widetilde{c}^s t^2}{(n-s)(n-s-1)} + O(t^3)\right), \quad (B.4)$$

and

$$\begin{aligned} K^{n-1}(\gamma_t) &= t^{-2} + \Big(-\widetilde{c}^{n-1} + O(t)\Big)\log t, \\ K^{n}(\gamma_t) &= t^{-1} + \Big(\widetilde{c}^{n} t + O(t^2)\Big)\log t, \\ K^{n+1}(\gamma_t) &= \Big(1 + \widetilde{c}^{n+1} t^2/2 + O(t^3)\Big)\log t. \end{aligned} \tag{B.5}$$

Using (B.3) and (B.4), we have $\varphi_s(\gamma_t) = 1 + 2((n-s+1)c' + \widetilde{c}^s)t^2 + O(t^3)$ for $m = 0, 1, \dots, n-2$. Thus we get

$$c_s = (n-s+1)c' + \widetilde{c}^s = \frac{2}{3(s-n+1)(s-n)}.$$

The constants c_s for $s \geq n-1$ are determined by using (B.5) in the same manner. □

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