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A link between the asymptotic expansions of the Bergman kernel and the Szegö kernel

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Introduction

Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$. Then the Bergman kernel K^{B} and the Szegö kernel K^{S} of Ω have singularities at the boundary diagonal. These singularities admit asymptotic expansions in powers and log of the defining function of Ω ([3], [2]) and, moreover, the coefficients of which can be expressed in terms of local invariants of the CR structure of the boundary $\partial\Omega$ as an application of the parabolic invariant theory developed in [4], [5], [1], [8], [6] and others. While these works provide a geometric algorithm of expressing the expansion of each kernel, it is not easy to read relations between them from this construction — for example, we can say very little about the relation between the log term coefficients of K^{B} and K^{S}, cf. §2.

In this note we present a method of relating these asymptotic expansions. Our strategy is to construct a meromorphic family of kernel functions K_s, $s \in \mathbb{C}$, such that K^{B} and K^{S} are realized as special values of K_s. In the case of the unit ball, $\{|z| < 1\}$, such a family is given by

$$K_s(z) = \pi^{-n}\Gamma(n-s)\,(1-|z|^2)^{s-n},$$

where $\Gamma(\alpha)$ is the gamma function, and K_{-1}, K_0 give K^{B}, K^{S}, respectively. Note that, for $s < 0$, K_s is characterized as the Bergman kernel for the weighted L^2 norm defined by the measure $(1-|z|^2)^{-s-1}/\Gamma(-s)dV$, see §1. For general strictly pseudoconvex domains, we begin by defining K_s for $s < 0$ as the weighted Bergman kernel, and then extend to $s \in \mathbb{C}$ by analytic continuation. Here we only consider the asymptotic expansion of K_s and define the analytic continuation as a meromorphic family of formal series, see §2. We then apply the invariant theory to express K_s in terms of geometric invariants of the boundary (Theorem 2). In these expansions, all K_s contain the same invariants up to universal

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constants depending polynomially on s. These formulae, in particular, give a relation between $K^{\mathrm{B}} = K_{-1}$ and $K^{\mathrm{S}} = K_0$.

Note that the kernel functions K_s for $s \in \mathbb{Z}$ have been introduced in Hirachi–Komatsu [7] and the present note is a continuation of that work. In [7], K_s are defined as the solutions of simple holonomic systems, which naturally arise from Kashiwara's microlocal analysis of the Bergman kernel [9]. While this point of view is not given explicitly in this note, this is also the main tool of the proofs of Theorems 1 and 2; the details will be given in my forthcoming paper.

§1. Weighted Bergman kernels

Let $\Omega \subset \mathbb{C}^n$ be a domain with C^∞ smooth boundary. Then there is a function $r \in C^\infty(\overline{\Omega})$, called a *defining function*, such that $\Omega = \{r > 0\}$ and $dr \neq 0$ on $\partial\Omega$. Fixing such an r, we define for $s < 0$ a weighted L^2 norm on Ω by

$$\|f\|_s^2 = \int_\Omega |f(z)|^2 \frac{r(z)^{-s-1}}{\Gamma(-s)} dV(z), \tag{1}$$

where $dV(z)$ is the standard Lebesgue measure on $\mathbb{C}^n$. Let

$$H_s(\Omega, r) := \{f \in \mathcal{O}(\Omega) : \|f\|_s < \infty\},$$

the Hilbert space of weighted L^2 holomorphic functions on Ω. If we take a complete orthonormal system $\{h_j\}_{j=0}^\infty$ of $H_s(\Omega, r)$, then the series

$$K_s[r](z, \overline{w}) := \sum_j h_j(z)\, \overline{h_j(w)}$$

converges for $(z, w) \in \Omega \times \Omega$ and define a function, which is shown to be independent of the choice of $\{h_j\}$. We call $K_s[r]$ the *weighted Bergman kernel*. Note that the Bergman kernel K^{B} is given by $K_{-1}[r]$, which is clearly independent of the choice of r.

In case $s = 0$, the right-hand side of (1) does not make sense because $\Gamma(-s)$ has simple poles at $s \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. However, we may define $\|\cdot\|_0$ by taking the limit

$$\lim_{s \to -0} \|f\|_s^2 = \int_{\partial\Omega} |f|^2 d\sigma(z), \quad f \in C^0(\overline{\Omega}),$$

where $d\sigma$ is the volume element on $\partial\Omega$ normalized by the condition

$$d\sigma \wedge dr = dV \quad \text{on } \partial\Omega.$$

Thus it is natural to define $H_0(\Omega, r) := \ker \overline{\partial}_b \subset L^2(\partial\Omega, d\sigma)$, where $\overline{\partial}_b$ is the tangential Cauchy–Riemann operator of $\partial\Omega$. Since each $f \in H_0(\Omega, r)$ admits an extension to $f \in \mathcal{O}(\Omega)$, we may also regard $H_0(\Omega, r) \subset \mathcal{O}(\Omega)$. The Szegö kernel is then defined by $K^{\mathrm{S}}[r](z, \overline{w}) := \sum_j h_j(z)\,\overline{h_j(w)}$, where $\{h_j\}_j$ is a complete orthonormal system of $H_0(\Omega, r)$.

Model case. In the case of the unit ball Ω_0, we may take $r(z) = 1 - |z|^2$. Then the monomials of z form a complete orthogonal system of $H_s(\Omega_0, r)$ (cf. [7]) and thus

$$K_s[r](z, \overline{w}) = \sum_\alpha \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|_s^2} = \frac{\Gamma(n-s)}{\pi^n}\,(1 - z \cdot \overline{w})^{s-n}.$$

The right-hand side is a meromorphic function of $s \in \mathbb{C}$ (with parameters $z, w \in \Omega$) and, thus $K_s[r]$ $(s < 0)$ can be analytically continued to a meromorphic function of $s \in \mathbb{C}$, which we also denote by $K_s[r]$. Then, in particular, $K_0[r]$ gives the Szegö kernel $K^{\mathrm{S}}[r]$.

§2. Asymptotic expansions of the weighted Bergman kernels

In what follows, we assume that Ω is strictly pseudoconvex, and mainly consider the restriction to the diagonal of the kernel functions $K_s[r](z) := K_s[r](z, \overline{z})$.

It is known from the work of Fefferman [3] that the boundary singularity of the Bergman kernel $K^{\mathrm{B}}(z)$ takes the form $\varphi r^{-n-1} + \psi \log r$, where $\varphi, \psi \in C^\infty(\overline{\Omega})$. Based on his analysis, G. Komatsu has shown that the weighted Bergman kernels $K_s[r]$ admit similar expansions.

Theorem ([10])**.** *For $s < 0$, the weighted Bergman kernel $K_s[r]$ admits the following asymptotic expansion at the boundary:*

$$K_s[r] = \begin{cases} \varphi^{(s)}[r]\, r^{s-n} + \psi^{(s)}[r] \log r & \text{if } s \in \mathbb{Z}, \\ \varphi^{(s)}[r]\, r^{s-n} & \text{if } s \notin \mathbb{Z}, \end{cases} \tag{2}$$

where $\varphi^{(s)}[r], \psi^{(s)}[r] \in C^\infty(\overline{\Omega})$.

If we introduce the functions

$$\Phi_s[r] = \begin{cases} \Gamma(-s)\, r^s & \text{if } s \in \mathbb{C} \setminus \mathbb{N}_0, \\ \frac{(-1)^{s+1}}{s!}\, r^s \log r & \text{if } s \in \mathbb{N}_0, \end{cases}$$

then we may rewrite the expansions (2) in a unified form:

$$K_s[r](z) = \sum_{j=0}^\infty \varphi_j^{(s)}[r](z) \Phi_{s-n+j}[r](z), \quad \varphi_j^{(s)}[r] \in C^\infty(\overline{\Omega}). \tag{3}$$

Here the coefficients $\varphi_j^{(s)}[r]$ are not uniquely determined because r and z are not independent.

Our basic result that enables us to define the meromorphic family $K_s[r]$, $s \in \mathbb{C}$, is the following

Theorem 1. *The coefficients* $\varphi_j^{(s)}[r]$ *of* (3) *can be chosen so that* $\varphi_j^{(s)}[r] = \sum_{k=0}^{2j} a_{j,k}[r]\, s^k$ *holds for functions* $a_{j,k}[r] \in C^\infty(\overline{\Omega})$ *that are independent of* s.

Taking $\varphi_j^{(s)}[r]$ as in the theorem above and then using the relation $s\, \Phi_{s+j}[r] = -r\, \Phi_{s+j-1}[r] - j\, \Phi_{s+j}[r]$, we may rewrite (3) in the form

$$K_s[r] = \sum_{j=-\infty}^{\infty} a_j[r] \Phi_{s-n+j}[r], \tag{4}$$

where $a_j[r] \in C^\infty(\overline{\Omega})$ are independent of s and satisfies $a_j[r] = O(r^{-2j})$ for $j < 0$ (hence the boundary singularity of $a_j[r]\Phi_{s-n+j}[r]$ gets weaker as $|j| \to \infty$). Note that $a_j[r]$ modulo $O(r^\infty)$ is now uniquely determined by r, and moreover it is shown that map $r \mapsto a_j[r]$ is given by a partial differential operator.

Now we *define* $K_s[r]$ for $s \in \mathbb{C} \setminus (-\infty, 0)$ by the formula (4), which is regarded as formal series. Then we can show, in particular, that $K_0[r]$ gives the asymptotic expansion of the Szegö kernel $K^{\mathrm{S}}[r]$.

§3. Transformation law and an invariant expansion of $K_s[r]$

We next examine the transformation law of $a_j[r]$ under biholomorphic maps $F \colon \widetilde{\Omega} \to \Omega$. Recall [3] that F can be extended to a diffeomorphism up to the boundary. So, for a defining function r of Ω, we may give a defining function of $\widetilde{\Omega}$ by

$$\widetilde{r} := |\det F'|^{-2/(n+1)} r \circ F, \tag{5}$$

where $\det F'$ is the holomorphic Jacobian of F. Now from the definition of the norm $\|\cdot\|_s$, we see that the weighted Bergman kernel transforms according to

$$K_s[\widetilde{r}] = |\det F'|^{2(n-s)/(n+1)} K_s[r] \circ F. \tag{6}$$

Thus, substituting these transformation laws into (4), we get

$$a_j[\widetilde{r}] = |\det F'|^{2j/(n+1)} a_j[r] \circ F \tag{7}$$

by the uniqueness of the expansion (4).

Our next task is to construct functionals of r that transform like this under biholomorphic maps — and hopefully express $a_j[r]$ in terms of these functionals. Here we utilize the ambient metric construction of [4]. Associated to each r, we first define a Lorentz-Kähler metric $g = g[r]$ on a neighborhood of $\mathbb{C}^* \times \partial\Omega \subset \mathbb{C}^* \times \mathbb{C}^n$ by $g[r] = \sum_{j,k=0}^n g_{j\overline{k}}\, dz_j d\overline{z}_k$, where $g_{j\overline{k}} = \partial^2 r_\# / \partial z_j \partial\overline{z}_k$. Let $R = R[r]$ be the curvature of g and $R^{(p,q)} = \overline{\nabla}^{q-2}\nabla^{p-2}R$ be its iterated covariant derivatives. Then consider complete contractions of the form

$$W_\# = \mathrm{contr}\Big(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_m,q_m)}\Big),$$

with $\sum p_l = \sum q_l = m + w$. Such a contraction $W_\#$ assigns to each r a smooth function $W[r] := W_\#[r]|_{z_0=0}$ on $\overline{\Omega}$ near $\partial\Omega$. We call the functional $r \mapsto W[r]$ a *Weyl functional of weight* w. If W has weight w, then under (5), we have the desired transformation law

$$W[\widetilde{r}] = |\det F'|^{2w/(n+1)} W[r] \circ F.$$

It is a natural hope that all a_j can be expressed in terms of these Weyl functionals. However, at this stage, it is hard to deal with the case of arbitrary r. So we here choose a good class of defining functions in such a way that we can apply the invariant theory of [4], [1], [6]. To specify a class of defining functions, following [6], we consider the following complex Monge-Ampère equation

$$(-1)^n \det\left(\partial^2 U / \partial z^j \partial\overline{z}^k\right)_{0 \le j,k \le n} = |z_0|^{2n}$$

for a function $U(z_0, z)$ on $\mathbb{C}^* \times \overline{\Omega}$. This equation admits asymptotic solutions along $\mathbb{C}^* \times \partial\Omega$ of the form

$$U = r_\# + r_\# \sum_{k=1}^{\infty} \eta_k \cdot (r^{n+1} \log r_\#)^k,$$

where r is a C^∞ defining function of Ω, $r_\#(z_0, z) = |z_0|^2 r(z)$ and $\eta_k \in C^\infty(\overline{\Omega})$. For such a solution U, the smooth part $r_\# = |z_0|^2 r$ is uniquely determined. So, for each Ω, we may define $\mathcal{F}_\Omega$ to be the totality of r that arises as the smooth part of an asymptotic solution U. This class $\mathcal{F}_\Omega$ is shown to be preserved under the pull-back (5).

Now we use Weyl functionals to express $K_s[r]$ for $r \in \mathcal{F}_\Omega$. The invariant theory of [6] implies that each $a_j[r]$ admits an asymptotic expansion

$$a_j[r] = \sum_{k=0}^{\infty} W_{j,k}[r]\, r^k, \qquad r \in \mathcal{F}_\Omega, \tag{8}$$

where $W_{j,k}$ is a linear combination of Weyl functionals of weight $j+k$. Hence, using $r\Phi_{s-m}[r] = (m-s)\Phi_{s-m+1}[r]$ to absorb all explicit r in (8) into other $\Phi_{s-l}[r]$, we get

Theorem 2. *If $r \in \mathcal{F}_\Omega$, then $K_s[r]$ admits an expansion*

$$K_s[r] = \sum_{j=0}^{\infty} W_j^{(s)}[r]\,\Phi_{s-n+j}[r], \tag{9}$$

where each $W_j^{(s)}$ is a linear combination of Weyl functionals of weight j whose coefficients are polynomials in s of degree $\le 2j$.

The first three terms of the expansion are given by

$$\pi^n K_s[r] = \Phi_{s-n}[r] + \frac{1}{24}\|R\|^2_{z_0=1}\Phi_{s-n+2}[r] + O(r^{s-n-3}).$$

Here the second term $W_{s-n+1}^{(s)}$ vanishes. Thus we see in particular that the Bergman and the Szegö kernels have the same expansion in $\Phi_s[r]$ up to this order.

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