CR INVARIANTS OF WEIGHT FIVE IN THE BERGMAN KERNEL

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Fefferman's program [11] of getting a biholomorphically invariant asymptotic expansion of the Bergman kernel for smoothly bounded strictly pseudoconvex domains is realized in dimension 2 with the identification of universal constants. According to the program, the expansion is in terms of an approximately invariant smooth defining function of the domain, which we refer to as Fefferman's defining function, and the coefficients are functions in the domain constructed by using derivatives of Fefferman's defining function. Consequently, the invariant expansion is necessarily a finite sum with a remainder term and the ambiguity estimate is crucial in the problem. We get an expansion such that the boundary values of the coefficients are CR invariants of weight ≤ 5. This refines earlier results of Graham [12] and the authors [15]. The refinement becomes possible by appropriate extensions inside the domain of the CR invariants of weight 4. Due to the ambiguity estimate of these extensions, our expansion is optimal as far as Fefferman's defining function is used. A similar result for the Szegö kernel is also obtained.

Introduction

The Bergman kernel of a domain Ω in $\mathbb{C}^n$ is by definition the reproducing kernel $K^{\mathrm{B}}(z, \overline{w})$ for $z, w \in \Omega$ associated with the space of square integrable holomorphic functions in Ω, so that any complete orthonormal system $\{h_j\}$ gives rise to the expression $K^{\mathrm{B}}(z, \overline{z}) = \sum |h_j(z)|^2$. This restriction to the diagonal is also referred to as the Bergman kernel and denoted by K^{B}, or K^{B}_{Ω} when the dependence on Ω is emphasized. Thus $K^{\mathrm{B}} = K^{\mathrm{B}}_{\Omega}$ is a domain functional, which is subject to a transformation law $K^{\mathrm{B}}_{\Omega_1} =$

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$K^{\mathrm{B}}_{\Omega_2} \circ \Phi \cdot |\det \Phi'|^2$ under a biholomorphic mapping $\Phi\colon \Omega_1 \to \Omega_2$, where $\det \Phi'$ stands for the holomorphic Jacobian of Φ. If we assume that the boundary is smooth, the Szegö kernel $K^{\mathrm{S}} = K^{\mathrm{S}}_{\Omega}$ is defined similarly, by using the space of holomorphic functions with L^2 boundary values. This time, the dependence on a surface element on $\partial\Omega$ must be taken into account, and we shall be concerned with a case where a canonical choice of surface element exists such that the biholomorphic transformation law $K^{\mathrm{S}}_{\Omega_1} = K^{\mathrm{S}}_{\Omega_2} \circ \Phi \cdot |\det \Phi'|^{2n/(n+1)}$ holds. Then K^{B} and K^{S} are biholomorphic invariants, a fact which leads to a problem of expressing these in terms of explicitly constructed invariants. The functions K^{B} and K^{S} are smooth (in fact, real analytic) in Ω, while the boundary behavior is complicated when $n > 1$; it depends on function-theoretic properties of Ω. If in particular Ω is strictly pseudoconvex, then $K^{\mathrm{B}}(z)$ and $K^{\mathrm{S}}(z)$ tend to $+\infty$ as z approaches to the boundary. Furthermore, the boundary behavior of K^{B} and K^{S} can be localized to a neighborhood of a boundary point of reference.

This paper concerns the local biholomorphic invariant theory, initiated by Fefferman [11], for the boundary singularities of the Bergman kernel K^{B} and the Szegö kernel K^{S} of a strictly pseudoconvex domain Ω in $\mathbb{C}^n$ with smooth boundary. Here, the surface element on $\partial\Omega$ which defines K^{S} is so chosen that K^{S} satisfies a biholomorphic transformation law analogous to that for K^{B}. Assuming $n = 2$, we shall explicitly identify the invariant asymptotic expansions of the singularities of K^{B} and K^{S} such that the boundary values of the coefficients are CR invariants of weight ≤ 5.

Our result is special to dimension two at some crucial points. To explain these, we begin by giving an overview of Fefferman's program of the invariant theory [11] in general dimension. Let r be a smooth defining function of $\partial\Omega$ such that $r > 0$ in Ω. Then, a theorem of Fefferman [9] (see also Boutet de Monvel-Sjöstrand [5]) states that

$$K^{\mathrm{B}} = \frac{n!}{\pi^n}\left(\frac{\varphi^{\mathrm{B}}}{r^{n+1}} + \psi^{\mathrm{B}} \log r\right), \qquad K^{\mathrm{S}} = \frac{(n-1)!}{\pi^n}\left(\frac{\varphi^{\mathrm{S}}}{r^{n}} + \psi^{\mathrm{S}} \log r\right),$$

where φ^{B}, ψ^{B}, φ^{S}, ψ^{S} are functions smooth up to the boundary such that the boundary values of φ^{B} and φ^{S} are given by those of $J[r]$ and $J[r]^{n/(n+1)}$, respectively. Here, $J[r]$ denotes the Levi determinant (also called the complex Monge-Ampère determinant) of r defined by

$$J[r] = (-1)^n \det \begin{pmatrix} r & \partial r/\partial \overline{z}_k \\ \partial r/\partial z_j & \partial^2 r/\partial z_j \partial \overline{z}_k \end{pmatrix},$$

where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. The expressions of K^{B} and K^{S} above are compared with the classical asymptotic expansion of the heat kernel on a compact d-dimensional Riemannian manifold M as the time $t \to +0$:

$$H_t(x,x) \sim t^{-d/2} \sum_{m=0}^{\infty} a_m(x)\, t^m \quad \text{for} \ \ x \in M.$$

Here, each coefficient $a_m(x)$ is a Riemannian invariant at x, i.e., an $\mathrm{O}(d)$-invariant polynomial of successive covariant derivatives of the curvature tensor R. According to the Weyl theory, $a_m(x)$ is a linear combination of complete contractions of the form

$$\operatorname{contr}\left(\nabla^{p_1-2}R\otimes\cdots\otimes\nabla^{p_s-2}R\right)\quad\text{with}\quad\sum_{j=1}^{s}p_j=2m.$$

The present counterpart of Riemannian geometry is local biholomorphic geometry, which is closely related to CR geometry on the boundary $\partial\Omega$. CR counterpart of $\mathrm{O}(d)$ is a parabolic subgroup of $\mathrm{SU}(n,1)$, and the Riemannian invariant has CR analogy, which we call the CR invariant, defined by using Moser's normal form $N(A)$, $A=(A^{\ell}_{\alpha\overline{\beta}})$, of $\partial\Omega$:

$$2\operatorname{Re}z_n=|z'|^2+\sum_{|\alpha|,|\beta|\geq 2}\sum_{\ell=0}^{\infty}A^{\ell}_{\alpha\overline{\beta}}\,z'_{\alpha}\,\overline{z'_{\beta}}\,(\operatorname{Im}z_n)^{\ell},\quad z=(z',z_n).$$

Here, the right side is a formal power series about the origin $z=0$ and α, β are ordered multi-indices. Let I_w^{CR} denote the totality of CR invariants of weight w. Then, an element of I_w^{CR} is a polynomial $P(A)$ in A which is subject to the transformation law

$$P(A)=P(\widetilde{A})\cdot|\det\Phi'(0)|^{2w/(n+1)}$$

under formal biholomorphic mappings $\Phi\colon N(A)\to N(\widetilde{A})$ such that $\Phi(0)=0$. Such a polynomial $P(A)$ can be identified with a smooth local boundary functional $K=K_{\partial\Omega}$ satisfying the transformation law

$$K_{\partial\Omega_1}=K_{\partial\Omega_2}\circ\Phi\cdot|\det\Phi'|^{2w/(n+1)}\quad\text{(on the boundary)}$$

under local (or formal) biholomorphic mappings $\Phi\colon\partial\Omega_1\to\partial\Omega_2$. Thus, examples of such $P(A)$ are realized by the boundary values of smooth local domain functionals $K=K_{\Omega}$ satisfying

$$K_{\Omega_1}=K_{\Omega_2}\circ\Phi\cdot|\det\Phi'|^{2w/(n+1)}\quad\text{(in the interior)}.$$

Rigorously, a technical condition concerning the polynomial dependence on A must be taken into account, though we have omitted it above for simplicity. Let us use tentative notation $\mathrm{w}^{\mathrm{TL}}(K)=w$ to mean that the above transformation law in the interior holds even when $K=K_{\Omega}$ is not smooth up to the boundary. Then $\mathrm{w}^{\mathrm{TL}}(K^{\mathrm{B}})=n+1$ and $\mathrm{w}^{\mathrm{TL}}(K^{\mathrm{S}})=n$, so that $\mathrm{w}^{\mathrm{TL}}(\varphi^{\mathrm{B}}/r^{n+1})=n+1$ and $\mathrm{w}^{\mathrm{TL}}(\varphi^{\mathrm{S}}/r^{n})=n$ modulo smooth functions up to the boundary, and $\mathrm{w}^{\mathrm{TL}}(\psi^{\mathrm{B}})=n+1$ and $\mathrm{w}^{\mathrm{TL}}(\psi^{\mathrm{S}})=n$ modulo flat functions along the boundary.

Fefferman's program is to invariantly express φ^{B} mod $O(r^{n+1})$, φ^{S} mod $O(r^n)$, ψ^{B} mod $O(r^{N-n})$, ψ^{S} mod $O(r^{N-n+1})$ for an integer $N \geq 0$ in terms of r and its derivatives, with an appropriate choice of a smooth local defining function r of $\partial\Omega$ as a local domain functional. Specifically, one seeks expansions of the form

$$\varphi^{\mathrm{B}} = \sum_{j=0}^{n} \varphi_j^{\mathrm{B}}\, r^j + O(r^{n+1}),\ \psi^{\mathrm{B}} = \sum_{j=n+1}^{N} \varphi_j^{\mathrm{B}}\, r^{j-n-1} + O(r^{N-n}),$$
$$\varphi^{\mathrm{S}} = \sum_{j=0}^{n-1} \varphi_j^{\mathrm{S}}\, r^j + O(r^n), \qquad \psi^{\mathrm{S}} = \sum_{j=n}^{N} \varphi_j^{\mathrm{S}}\, r^{j-n} + O(r^{N-n+1}),$$

where $\varphi_j^{\mathrm{B}} = \varphi_j^{\mathrm{B}}[r]$, $\varphi_j^{\mathrm{S}} = \varphi_j^{\mathrm{S}}[r]$ are locally constructed functions which are smooth up to the boundary. When $N \leq n$ or $N \leq n-1$, the above expansions are interpreted as

$$\varphi^{\mathrm{B}} = \sum_{j=0}^{N} \varphi_j^{\mathrm{B}}\, r^j + O(r^{N+1}), \qquad \varphi^{\mathrm{S}} = \sum_{j=0}^{N} \varphi_j^{\mathrm{S}}\, r^j + O(r^{N+1}).$$

It is natural to require $\mathrm{w}^{\mathrm{TL}}(r) = -1$, $J[r] = 1$ and expect $\mathrm{w}^{\mathrm{TL}}(\varphi_j^{\mathrm{B}}) = \mathrm{w}^{\mathrm{TL}}(\varphi_j^{\mathrm{S}}) = j$, $N = +\infty$. The hope is strengthened by a lemma by Fefferman [10] stating that

$$J[u_1] = J[u_2] \circ \Phi \quad \text{if} \quad u_1 = u_2 \circ \Phi \cdot |\det \Phi'|^{-2/(n+1)}.$$

Then we are naturally led to a local version of the boundary value problem for the complex Monge-Ampère equation

$$J[u] = 1 \ (u > 0) \quad \text{in} \ \ \Omega, \qquad u = 0 \quad \text{on} \ \ \partial\Omega. \tag{0.1}$$

However, an elementary construction in [10] of smooth local approximate solutions via successive approximation stops at a finite step, and this is compatible with later results of Cheng-Yau [6] and Lee-Melrose [17]. By [6] the boundary value problem (0.1) has a unique solution u^{MA} with finite differentiability up to the boundary, while by [17] the solution u^{MA} has an asymptotic expansion of the form

$$u^{\mathrm{MA}} \sim \rho \sum_{k=0}^{\infty} \eta_k \cdot (\rho^{n+1} \log \rho)^k \quad \text{with} \ \ \eta_k = \eta_k[\rho] \in C^\infty(\overline{\Omega})$$

for any smooth defining function ρ of Ω. We thus confine ourselves to the best possible smooth local approximate solution r of (0.1), which has ambiguity $O(r^{n+2})$ and satisfies

$$\mathrm{w}^{\mathrm{TL}}(r) = -1 \quad \text{mod}\ O(r^{n+2}), \qquad J[r] = 1 + O(r^{n+1}).$$

We write $r = r^{\mathrm{F}}$ for such a local defining function of the boundary. Let $r = r^{\mathrm{F}}$. Then $\mathrm{w}^{\mathrm{TL}}(\varphi^{\mathrm{B}}) = 0 \mod O(r^{n+1})$ and $\mathrm{w}^{\mathrm{TL}}(\varphi^{\mathrm{S}}) = 0 \mod O(r^n)$, so that we have hope of getting φ_j^{B} and φ_j^{S} for $j \le N$ such that

$$\mathrm{w}^{\mathrm{TL}}(\varphi_j^{\mathrm{B}}) = \mathrm{w}^{\mathrm{TL}}(\varphi_j^{\mathrm{S}}) = j \mod O(r^{N+1-j})$$

with some $N \le 2n+2$ for φ_j^{B} and $N \le 2n+1$ for φ_j^{S}. This is realized by giving for each $j \le N$ a vector space $I_{j,N}$ of locally constructed smooth functions with the following properties:

(1°) If $\varphi_j \in I_{j,N}$ then $\mathrm{w}^{\mathrm{TL}}(\varphi_j) = j \mod O(r^{N+1-j})$;
(2°) The restriction to the boundary gives a surjection $I_{j,N} \to I_j^{\mathrm{CR}}$.

Here, the polynomial dependence on Moser's normal form coefficients is again ignored. The space $I_{0,N}$ is trivially defined by the totality of absolute constants. Once such vector spaces $I_{j,N}$ are given, the expansion of $\varphi = \varphi^{\mathrm{B}}, \varphi^{\mathrm{S}}$ is obtained as follows. Define first φ_0 by the boundary value of φ. Then $\varphi_0 \in I_0^{\mathrm{CR}}$, so that $\varphi_0 \in I_{0,N}$. If we have an expansion of the form

$$\varphi = \varphi_0 + \varphi_1 r + \cdots + \varphi_{j-1} r^{j-1} + O(r^j)$$

with some $j < n+1$ or n ($j \le N$), then $\varphi_j \in I_{j,N}$ is defined by taking an extension of the boundary value of $(\varphi - \varphi_0 - \varphi_1 r - \cdots - \varphi_{j-1} r^{j-1})/r^j$. Similarly for the expansion of $\psi = \psi^{\mathrm{B}}, \psi^{\mathrm{S}}$ if $N \ge n+1$ or n.

To construct the spaces $I_{j,N}$ as above, Fefferman [11] developed a new theory, called the ambient metric construction, as follows. A Lorentz-Kähler metric $g = g[r]$ depending on $r = r^{\mathrm{F}}$ is defined on $\mathbb{C}^* \times \overline{\Omega}$ near $\mathbb{C}^* \times \partial\Omega$ by the potential $r^{\#}(z_0, z) = |z_0|^2 r(z)$, where $z_0 \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is an extra variable. The curvature tensor R of this g is used in constructing complete contractions of the form

$$W^{\#} = \operatorname{contr}\left(\overline{\nabla}^{q_1-2}\nabla^{p_1-2}R \otimes \cdots \otimes \overline{\nabla}^{q_s-2}\nabla^{p_s-2}R\right),$$
$$\sum_{j=1}^{s}(p_j - 1) = \sum_{j=1}^{s}(q_j - 1) = w.$$

Then, each $W^{\#} = W^{\#}[r]$ regarded as a functional of r takes the form

$$W^{\#}[r](z_0, z) = |z_0|^{-2w} W[r](z).$$

For w fixed, we tentatively refer to linear combinations of these $W = W[r]$ as Weyl functionals of weight w and use notation I_w^{W} for the totality of Weyl functionals of weight w. Counting the number of differentiation and developing the invariant theory, Fefferman [11] proved that $I_{j,N} = I_j^{\mathrm{W}}$ with $N = n$ satisfies the ambiguity estimate (1°) for $j \le n$ and the surjectivity

(2°) for $j \leq n-19$. The surjectivity (2°) was recently refined to $j \leq n$ by Bailey-Eastwood-Graham [1]. Consequently, the choice $I_{j,N} = I_j^{\mathrm{W}}$ with $N = n$ satisfies the conditions (1°) and (2°), so that the expansions of φ^{B}, φ^{S} and that of ψ^{S} mod $O(r)$ are obtained.

For $n \geq 3$, there are few explicit results. Christoffers [8] directly computed φ^{B} mod $O(r^3)$. The same result was obtained independently by Diederich (cf. [20]). Graham [12] later showed $\dim I_1^{\mathrm{CR}} = 0$ and that I_2^{CR} is generated by $\|A^0_{2\overline{2}}\|^2 = \sum_{|\alpha|=2} |A^0_{\alpha\overline{\alpha}}|^2$, so that if $r = r^{\mathrm{F}}$ then

$$\varphi^{\mathrm{B}} = 1 + c^{\mathrm{B}}_{2,n}\|A^0_{2\overline{2}}\|^2 r^2 + O(r^3), \quad \varphi^{\mathrm{S}} = 1 + c^{\mathrm{S}}_{2,n}\|A^0_{2\overline{2}}\|^2 r^2 + O(r^3),$$

where $c^{\mathrm{B}}_{2,n}$ and $c^{\mathrm{S}}_{2,n}$ are universal constants for $n \geq 3$. Christoffers' computation implies $c^{\mathrm{B}}_{2,n} \neq 0$. These constants $c^{\mathrm{B}}_{2,n}$, $c^{\mathrm{S}}_{2,n}$ were later identified in [15]; in particular, $c^{\mathrm{S}}_{2,n} \neq 0$. Consequently, a Weyl functional of weight 2 does appear in the above expansions of φ^{B} and φ^{S} for $n \geq 3$.

Let $n = 2$. The feature is different from the case $n \geq 3$. Graham [12] explicitly wrote down bases of I_j^{CR} for $j \leq 4$. In particular, $\dim I_1^{\mathrm{CR}} = \dim I_2^{\mathrm{CR}} = 0$, so that

$$\varphi^{\mathrm{B}} = 1 + O(r^3); \qquad \varphi^{\mathrm{S}} = 1 + O(r^2), \quad \psi^{\mathrm{S}} = O(r).$$

The first non-trivial space is I_3^{CR}. According to Graham's table [12], I_3^{CR} and I_4^{CR} are one dimensional. (We have $\dim I_5^{\mathrm{CR}} = 2$; see Proposition 1 in Section 1.) Once we are given spaces $I_{3,5}$ and $I_{4,5}$ satisfying the conditions (1°) and (2°), we get expansions of the form

$$\psi^{\mathrm{B}} = \sum_{j=3}^{5} \varphi^{\mathrm{B}}_j \, r^{j-3} + O(r^3), \qquad \psi^{\mathrm{S}} = \sum_{j=3}^{5} \varphi^{\mathrm{S}}_j \, r^{j-2} + O(r^4) \tag{0.2}$$

with $r = r^{\mathrm{F}}$, where $\varphi_j \in I_{j,5}$ for $j = 3,4,5$ for $\varphi_j = \varphi^{\mathrm{B}}_j$, φ^{S}_j. (We have taken $I_{1,5} = I_{2,5} = \{0\}$ and $I_{5,5} = I_5^{\mathrm{CR}}$.) Observe that the expansion (0.2) does not follow from the above-mentioned results of general dimension. The first difficulty is to discover a space $I_{3,5}$. (Though there exists a Weyl functional $W \in I_3^{\mathrm{W}}$ such that $0 \neq W|_{\partial\Omega} \in I_3^{\mathrm{CR}}$, the ambiguity estimate for W is bad. This is because W is linear in R, cf. Propositions 10 and 10′ in Section 6.) A breakthrough was made by Graham in [12] and [13]. Fixing $r = r^{\mathrm{F}}$, he considered in [13] the initial value problem corresponding to (0.1) and constructed a formal asymptotic solution of the form

$$u^{\mathrm{G}} = r \sum_{k=0}^{\infty} \eta^{\mathrm{G}}_k \cdot (r^{n+1}\log r)^k, \qquad \eta^{\mathrm{G}}_k \in C^\infty(\overline{\Omega}),$$

where the boundary value of $(\eta^{\mathrm{G}}_0 - 1)/r^{n+1}$ is prescribed arbitrarily as an extra initial data. Then $\mathrm{w}^{\mathrm{TL}}(\eta^{\mathrm{G}}_k) = k(n+1)$ mod $O(r^{n+1})$. It was proved in [12] that if $n = 2$ then the boundary value of η_1^{G} generates I_3^{CR}. Thus $I_{3,5}$ is defined by using η_1^{G} and we have expansions of the form

$$\psi^{\mathrm{B}} = c_3^{\mathrm{B}}\eta_1^{\mathrm{G}} + c_4^{\mathrm{B}}e_4^{\mathrm{CR}}r + O(r^2), \quad \psi^{\mathrm{S}} = c_3^{\mathrm{S}}\eta_1^{\mathrm{G}}r + c_4^{\mathrm{S}}e_4^{\mathrm{CR}}r^2 + O(r^3) \quad (0.3)$$

with universal constants c_j^{B} and c_j^{S} for $j = 3, 4$, where e_4^{CR} is a base of I_4^{CR}. Specifically, $\eta_1^{\mathrm{G}}|_{\partial\Omega} = 4A^0_{4\overline{4}}$ by [12], where the two dimensional notation for Moser's normal form coefficients $A^\ell_{p\overline{q}} = A^\ell_{\alpha\overline{\beta}}$ for $|\alpha| = p$, $|\beta| = q$ is used. Then $c_3^{\mathrm{B}} = -3$, $c_3^{\mathrm{S}} = -2$ (cf. [12], [15]). The constants c_4^{B} and c_4^{S} were identified in [15] with e_4^{CR} specified.

The purpose of this paper is to refine (0.3) a step further and get the expansions (0.2) explicitly. The point is the construction of the space $I_{4,5}$. For each $w \le 5$, we define a subspace I_w^{WF} of I_w^{W} by

$$I_w^{\mathrm{WF}} = \left\{ W \in I_w^{\mathrm{W}};\ \ \mathrm{w}^{\mathrm{TL}}(W) = w \mod O(r^{6-j}) \right\},$$

and refer to elements of I_w^{WF} as Weyl-Fefferman functionals of weight w. Thus the restriction $I_w^{\mathrm{WF}}|_{\partial\Omega}$ is a subspace of I_w^{CR}. Let us tentatively denote by d_w the dimension of $I_w^{\mathrm{WF}}|_{\partial\Omega}$. Our first main result (Theorem 1) states that

$$d_1 = d_2 = d_3 = 0, \quad d_4 = 1, \quad d_5 = 2.$$

and gives bases of I_4^{WF} and I_5^{WF}. Consequently, we may take $I_{j,5} = I_j^{\mathrm{WF}}$ for $j = 4, 5$ and get expansions (0.2). That is, once bases $e_4^{\mathrm{WF}} \in I_4^{\mathrm{WF}}$ and $e_{51}^{\mathrm{WF}}, e_{52}^{\mathrm{WF}} \in I_5^{\mathrm{WF}}$ are specified in such a way that $e_4^{\mathrm{WF}}|_{\partial\Omega} = e_4^{\mathrm{CR}}$, we get

$$\begin{aligned} \psi^{\mathrm{B}} &= c_3^{\mathrm{B}}\eta_1^{\mathrm{G}} + c_4^{\mathrm{B}}e_4^{\mathrm{WF}}r + \left(c_{51}^{\mathrm{B}}e_{51}^{\mathrm{WF}} + c_{52}^{\mathrm{B}}e_{52}^{\mathrm{WF}}\right)r^2 + O(r^3), \\ \psi^{\mathrm{S}} &= c_3^{\mathrm{S}}\eta_1^{\mathrm{G}}r + c_4^{\mathrm{S}}e_4^{\mathrm{WF}}r^2 + \left(c_{51}^{\mathrm{S}}e_{51}^{\mathrm{WF}} + c_{52}^{\mathrm{S}}e_{52}^{\mathrm{WF}}\right)r^3 + O(r^4), \end{aligned}$$

where c_{51}^{B}, c_{52}^{B}, c_{51}^{S}, c_{52}^{S} are universal constants (Theorem 2). It turns out that we have two natural expressions for the choice of e_4^{WF}, so that these constants differ accordingly. Our second main result (Theorem 3) gives the identification of these constants.

The Weyl-Fefferman functionals e_4^{WF}, e_{51}^{WF}, e_{52}^{WF} are nonlinear in R; these are squared norms of tensors of the form $\overline{\nabla}^{q-2}\nabla^{p-2}R$ with respect to the ambient Lorentz-Kähler metric g. The ambiguity estimates (1°) for these are proved by using that $A^\ell_{p\overline{q}} = 0$ for $p, q \le 3$, a fact which is specific to $n = 2$ and used throughout this paper. We deduce the surjectivity (2°) for e_4^{WF}, e_{51}^{WF}, e_{52}^{WF} from their explicit representations in Moser's normal form.

In determining the universal constants c_{51}^{B}, c_{52}^{B}, c_{51}^{S}, c_{52}^{S} above, we need explicit computations of Graham's asymptotic solution u^{G} as well as the singularities of the Bergman kernel K^{B} and the Szegö kernel K^{S}. For K^{B}

and K^{S}, we use algorithms based on the microlocal calculus of Kashiwara [16] and Boutet de Monvel [2–4]. These algorithms were used in [15] in the same context. For an algorithm of expanding u^{G} with respect to Moser's normal form coefficients, we consider a linearization of the Monge-Ampère operator $J[\,\cdot\,]$, which is different from that of Graham [13] in constructing u^{G}. All computations are simplified by using a special class of domains.

This paper is organized as follows. In Section 1, we first give lists of CR invariants and Weyl-Fefferman functionals of weight ≤ 5, and then state our main results on the Bergman kernel K^{B} and the Szegö kernel K^{S}. Our main results are reduced in Section 2 to several propositions which are proved in the subsequent sections. In Section 3, we introduce the notion of biweight which is a generalization of that defined by Boutet de Monvel [2–4], and review some known facts from a viewpoint of biweight. CR invariants of weight five are identified in Section 4. Section 5 is devoted to the study of η_1^{G}. Weyl-Fefferman functionals of weight ≤ 5 are identified in Section 6. In Section 7, we compute the singularities of K^{B} and K^{S} by using a method in [2–4].

Recently, the first author refines in [14] the result of this paper for K^{B} and that of Bailey-Eastwood-Graham [1] by giving a complete invariant expansion of the singularity of K^{B} for domains Ω in $\mathbb{C}^n$. The expansion is done with respect to a class of smooth defining functions of Ω, for which one can formulate a biholomorphic transformation law without ambiguity. This class consists of the smooth parts of certain asymptotic solutions of a Monge-Ampère equation lifted to $\mathbb{C}^* \times \Omega$, and is parametrized by a variable $C = (C^{\ell}_{\alpha\overline{\beta}})$ like Moser's normal form coefficients $A = (A^{\ell}_{\alpha\overline{\beta}})$. Then the boundary value of a Weyl-Fefferman functional is a polynomial in (A, C), which is referred to as a Weyl invariant depending on C. It turns out that any CR invariant is realized by a Weyl invariant independent of C, and vice versa.

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1. Statement of the results

1.1. CR invariants and Weyl-Fefferman functionals of weight ≤ 5. We begin by recalling the definition of CR invariants. Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^2$ with C^∞ boundary $\partial\Omega$. We shall be working near an arbitrarily fixed boundary point, say, the origin $0 \in \mathbb{C}^2$. Let us assume for a moment that the boundary is real analytic near 0. Then, after a holomorphic change of coordinates, Ω is locally given near $z = 0$ by $2u > |z_1|^2 + F(z_1, \overline{z}_1, v)$, with $z = (z_1, z_2) \in \mathbb{C}^2$ and $z_2 = u + iv$, where

F is a convergent power series of the form

$$F(z_1, \overline{z}_1, v) = \sum_{p,q \geq 2} A_{p\overline{q}}(v)\, z_1^p\, \overline{z}_1^q, \quad A_{p\overline{q}}(v) = \sum_{\ell=0}^{\infty} A_{p\overline{q}}^{\ell}\, v^{\ell} \tag{1.1}$$

satisfying $A_{q\overline{p}}(v) = \overline{A_{p\overline{q}}(v)}$ and $A_{2\overline{2}}(v) = A_{2\overline{3}}(v) = A_{3\overline{3}}(v) = 0$. In this case, we say that $\partial\Omega$ is in *Moser's normal form* at $0 \in \mathbb{C}^2$. Setting $A = (A_{p\overline{q}}^{\ell})$, we denote by $N(A)$ the (germ of) real hypersurface in Moser's normal form with coefficients $A_{p\overline{q}}^{\ell}$, and define $\mathcal{N}$ to be the totality of such A. We say that a real-valued polynomial $P = P(A)$ in $A \in \mathcal{N}$ is a *CR invariant of weight* w if

$$P(A) = P(\widetilde{A}) \cdot |\det \Phi'(0)|^{2w/3} \tag{1.2}$$

for any local biholomorphic mapping $\Phi \colon N(A) \to N(\widetilde{A})$ such that $\Phi(0) = 0$. By the construction of Moser's normal form in [7], we may regard each CR invariant as a real analytic function on $\partial\Omega$ near 0.

In case $\partial\Omega$ is not real analytic and merely C^∞, we consider formal power series. Then, after a *formally* holomorphic change of coordinates, $\partial\Omega$ is written *formally* in Moser's normal form at 0. If we also consider $\Phi \colon N(A) \to N(\widetilde{A})$ as above as a formal power series, then the CR invariants are defined as in the real analytic case. In this case, we may regard a CR invariant as a C^∞ function on $\partial\Omega$.

Let I_w^{CR} denote the complexification of the real vector space of all CR invariants of weight w, so that $I_0^{\mathrm{CR}} = \mathbb{C}$. (We also refer to elements of I_w^{CR} as CR invariants of weight w.) Then we have:

Proposition 1. *$I_1^{\mathrm{CR}} = I_2^{\mathrm{CR}} = \{0\}$, $\dim I_3^{\mathrm{CR}} = \dim I_4^{\mathrm{CR}} = 1$ and $\dim I_5^{\mathrm{CR}} = 2$. The spaces I_3^{CR} and I_4^{CR} are generated by $A_{4\overline{4}}^0$ and $|A_{4\overline{2}}^0|^2$, respectively. The space I_5^{CR} is spanned by $F_{0,1}^{\mathrm{CR}}$ and $F_{1,0}^{\mathrm{CR}}$, where*

$$F_{a,b}^{\mathrm{CR}}(A) = a\,|A_{5\overline{2}}^0|^2 + b\,|A_{4\overline{3}}^0|^2 + \mathrm{Re}\Big[\big(c\,A_{3\overline{5}}^0 - i\,d\,A_{2\overline{4}}^1\big)\,A_{4\overline{2}}^0\Big]$$

with $c = -2\,a + 10\,b/9$ and $d = -a + b/3$.

Remark 1.1. The results on I_w^{CR} for $w \leq 4$ above is due to Graham [12]. However, in the list of I_5^{CR} in [12] and also in [15], the term $A_{3\overline{5}}^0$ in $F_{a,b}^{\mathrm{CR}}(A)$ is missing, though I_5^{CR} was not used in [12] and [15] in the description of the Bergman kernel and the Szegö kernel.

We next define Weyl-Fefferman functionals. It is shown by Fefferman [10] that there exists a C^∞ defining function $r = r^{\mathrm{F}}$ of $\Omega = \{r > 0\}$ such that r^{F} is unique modulo $O^4(\partial\Omega)$ and satisfies $J[r^{\mathrm{F}}] = 1 + O^3(\partial\Omega)$, where $J[r]$ denotes the Levi determinant of r defined by

$$J[r] = \det \begin{pmatrix} r & \partial r/\partial \overline{z}_k \\ \partial r/\partial z_j & \partial^2 r/\partial z_j \partial \overline{z}_k \end{pmatrix}$$

and $O^m(\partial\Omega)$ for $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ stands for an error term which is smoothly divisible by the mth power of a C^∞ defining function of $\partial\Omega$. We refer to any one of r^{F} as a *Fefferman's defining function* of Ω.

For $r = r^{\mathrm{F}}$, we set

$$r^\#(z_0, z) = |z_0|^2\, r(z), \quad (z_0, z) \in \mathbb{C}^* \times \Omega \;\; (\mathbb{C}^* = \mathbb{C} \setminus \{0\}),$$

and define the *ambient metric* g by using Kähler potential $r^\#$, that is, $g_{j\overline{k}} = \partial^2 r^\# / \partial z_j \partial \overline{z}_k$ for $j, k = 0, 1, 2$. Then g is a Lorentz-Kähler metric near $\mathbb{C}^* \times \partial\Omega$ in $\mathbb{C}^* \times \overline{\Omega}$. Denoting by R the curvature tensor of g, we set $R^{(p,q)} = \overline{\nabla}^{q-2} \nabla^{p-2} R$ for $p, q \geq 2$, where ∇ and $\overline{\nabla}$ stand for the covariant differentiations. For each $w \in \mathbb{N} = \{1, 2, \dots\}$, we consider complete contractions of the form

$$W^\# = \operatorname{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\right) \tag{1.3}$$

such that $\sum_{j=1}^s p_j = \sum_{j=1}^s q_j = s + w$, and we set $\mathrm{w}(W^\#) = w$. There may be several ways of taking complete contraction, and we choose one of these. We consider a linear combination $W^\# = c_1\, W_1^\# + \cdots + c_N\, W_N^\#$ of complete contractions $W_j^\#$ as in (1.3) such that $\mathrm{w}(W_j^\#) = w$, and call $W^\#$ a *Weyl polynomial of weight* w. Here, we regard the variable of $W^\#$ to be the components of $R^{(p,q)}$ with $p, q \geq 2$.

We take a Weyl polynomial $W^\#$ of weight w. Then for each domain Ω and a Fefferman's defining function $r = r^{\mathrm{F}}$ of Ω, a function $W^\#[r]$ in a neighborhood of $\mathbb{C}^* \times \partial\Omega$ is defined by evaluating the curvature $R^{(p,q)}$ for the ambient metric $g[r]$. We see that $W^\#[r]$ takes the form

$$W^\#[r](z_0, z) = |z_0|^{-2w}\, W[r](z),$$

where $W[r]$ is a function in a neighborhood of $\partial\Omega$. Due to the ambiguity of the choice of $r = r^{\mathrm{F}}$, the function $W[r]$ is not uniquely determined. Taking account of the ambiguity, we define Weyl-Fefferman functionals as follows.

Definition. We say that W is a *Weyl-Fefferman functional of weight* w if W is well-defined modulo $O^{6-w}(\partial\Omega)$.

Let I_w^{WF} denote the totality of Weyl-Fefferman functionals of weight w. We shall see in Proposition 3 that if $W \in I_w^{\mathrm{WF}}$ then its boundary value is a CR invariant of weight w. We identify two Weyl-Fefferman functionals if these have the same boundary value. In other words, we consider the quotient space $\widetilde{I}_w^{\mathrm{WF}}$ of I_w^{WF} by the equivalence relation of having the same boundary value. Then one may regard $\widetilde{I}_w^{\mathrm{WF}}$ as a linear subspace of I_w^{CR}. Now we have:

Theorem 1. $\widetilde{I}_1^{\mathrm{WF}} = \widetilde{I}_2^{\mathrm{WF}} = \widetilde{I}_3^{\mathrm{WF}} = \{0\}$, $\dim \widetilde{I}_4^{\mathrm{WF}} = 1$ *and* $\dim \widetilde{I}_5^{\mathrm{WF}} = 2$. *The space* $\widetilde{I}_4^{\mathrm{WF}}$ *is generated by either one of* $\|R^{(4,2)}\|^2$ *and* $\|R^{(3,3)}\|^2$*, and the space* $\widetilde{I}_5^{\mathrm{WF}}$ *is spanned by* $\|R^{(5,2)}\|^2$ *and* $\|R^{(4,3)}\|^2$*, where* $\|R^{(a,b)}\|^2$ *denotes the squared norm of the tensor* $R^{(a,b)}$ *with respect to the Lorentz metric* g. (*The squared norm need not be non-negative.*)

Note by Proposition 1 and Theorem 1 that $0 = \dim \widetilde{I}_3^{\mathrm{WF}} < \dim I_3^{\mathrm{CR}} = 1$. Instead, $4\,A^0_{4\overline{4}} \in I_3^{\mathrm{CR}}$ is realized as the boundary value of η_1^{G}, where η_1^{G} is contained in Graham's asymptotic solution $u = u^{\mathrm{G}}$ of the complex Monge-Ampère boundary value problem

$$J[u] = 1 \quad (\text{and } u > 0) \quad \text{in } \ \Omega, \qquad u = 0 \quad \text{on } \ \partial\Omega. \tag{1.4}$$

More precisely, Graham [13] showed that if $r = r^{\mathrm{F}}$ is specified then for any $a \in C^\infty(\partial\Omega)$ there exists a unique formal series

$$u^{\mathrm{G}} = r \sum_{j=0}^{\infty} \eta_j^{\mathrm{G}} \cdot \left(r^3 \log r\right)^j, \qquad \eta_j^{\mathrm{G}} \in C^\infty(\overline{\Omega}), \tag{1.5}$$

such that $u = u^{\mathrm{G}}$ satisfies (1.4) formally and $\eta_0^{\mathrm{G}} = 1 + a\,r^3 + O^4(\partial\Omega)$; each η_j^{G} is independent of a modulo $O^3(\partial\Omega)$.

1.2. Invariant expansion of the Bergman kernel and the Szegö kernel. In addition to the assumption that Ω is a strictly pseudoconvex domain in $\mathbb{C}^2$ with C^∞ boundary, we assume that Ω is bounded. Recall that the Bergman kernel $K^{\mathrm{B}}(z,\overline{w})$ and the Szegö kernel $K^{\mathrm{S}}(z,\overline{w})$ ($z, w \in \Omega$) associated with Ω are defined by

$$K^{\mathrm{B}}(z,\overline{w}) = \sum_j h_j^{\mathrm{B}}(z)\,\overline{h_j^{\mathrm{B}}(w)}, \quad K^{\mathrm{S}}(z,\overline{w}) = \sum_j h_j^{\mathrm{S}}(z)\,\overline{h_j^{\mathrm{S}}(w)},$$

where $\{h_j^{\mathrm{B}}\}$ and $\{h_j^{\mathrm{S}}\}$ are complete orthonormal systems of the Hilbert spaces $H^{\mathrm{B}}(\Omega)$ and $H^{\mathrm{S}}(\Omega)$, respectively. Here, $H^{\mathrm{B}}(\Omega)$ is the totality of L^2 holomorphic functions, and $H^{\mathrm{S}}(\Omega)$ is the totality of holomorphic functions with L^2 boundary values, so that the space $H^{\mathrm{S}}(\Omega)$ depends on the choice of a surface element σ on $\partial\Omega$. We choose σ in such a way that the Szegö kernel is transformed by biholomorphic mappings invariantly in the sense of [11]. Specifically, we assume that σ satisfies

$$\sigma \wedge dr = J[r]^{1/3}\,dV(z) \quad \text{on } \ \partial\Omega, \quad \text{with } \ dV(z) = \bigwedge_{j=1}^{2} \frac{dz_j \wedge d\overline{z}_j}{-2\,i}\,,$$

where r is an arbitrary defining function of Ω. Then σ is determined independent of the choice of r. It is observed by Fefferman [9, 11] (see also [5]) that $K^{\mathrm{B}}(z) = K^{\mathrm{B}}(z,\overline{z})$ and $K^{\mathrm{S}}(z) = K^{\mathrm{S}}(z,\overline{z})$ satisfy

$$\frac{\pi^2}{2}\,K^{\mathrm{B}} = \frac{\varphi^{\mathrm{B}}}{r^3} + \psi^{\mathrm{B}}\,\log r, \quad \pi^2\,K^{\mathrm{S}} = \frac{\varphi^{\mathrm{S}}}{r^2} + \psi^{\mathrm{S}}\,\log r \tag{1.6}$$

near $\partial\Omega$, where φ^{B}, ψ^{B} and φ^{S}, ψ^{S} are functions C^∞ up to the boundary, and $\varphi^{\mathrm{B}} = \varphi^{\mathrm{S}} = J[r]$ on $\partial\Omega$. It is then shown by Graham [12] and in [15] that if $r = r^{\mathrm{F}}$ then

$$\begin{aligned} \varphi^{\mathrm{B}} &= 1 + O^3(\partial\Omega), \quad \psi^{\mathrm{B}} = -3\,\eta_1^{\mathrm{G}} + \frac{24}{5}\,|A^0_{4\overline{2}}|^2\, r + O^2(\partial\Omega), \\ \varphi^{\mathrm{S}} &= 1 + O^2(\partial\Omega), \quad \psi^{\mathrm{S}} = -2\,\eta_1^{\mathrm{G}}\, r + \frac{8}{15}\,|A^0_{4\overline{2}}|^2\, r^2 + O^3(\partial\Omega), \end{aligned} \tag{1.7}$$

with η_1^{G} in (1.5). We refine (1.7) a step further. That is, our main results are stated as follows.

Theorem 2. *Let $r = r^{\mathrm{F}}$ be a Fefferman's defining function of Ω. Then there exist real universal constants c_1^{B}, c_2^{B}, c_3^{B}, $\widetilde{c}_1^{\mathrm{B}}$, $\widetilde{c}_2^{\mathrm{B}}$, $\widetilde{c}_3^{\mathrm{B}}$ and c_1^{S}, c_2^{S}, c_3^{S}, $\widetilde{c}_1^{\mathrm{S}}$, $\widetilde{c}_2^{\mathrm{S}}$, $\widetilde{c}_3^{\mathrm{S}}$ independent of the choice of Ω such that ψ^{B} and ψ^{S} in* (1.6) *satisfy*

$$\begin{aligned} &\psi^{\mathrm{B}} + 3\,\eta_1^{\mathrm{G}} \\ &\quad = c_1^{\mathrm{B}}\|R^{(4,2)}\|^2\, r + \Big(c_2^{\mathrm{B}}\|R^{(5,2)}\|^2 + c_3^{\mathrm{B}}\|R^{(4,3)}\|^2\Big)\, r^2 + O^3(\partial\Omega) \\ &\quad = \widetilde{c}_1^{\mathrm{B}}\|R^{(3,3)}\|^2\, r + \Big(\widetilde{c}_2^{\mathrm{B}}\|R^{(5,2)}\|^2 + \widetilde{c}_3^{\mathrm{B}}\|R^{(4,3)}\|^2\Big)\, r^2 + O^3(\partial\Omega), \\ &\psi^{\mathrm{S}} + 2\,\eta_1^{\mathrm{G}}\, r \\ &\quad = c_1^{\mathrm{S}}\|R^{(4,2)}\|^2\, r^2 + \Big(c_2^{\mathrm{S}}\|R^{(5,2)}\|^2 + c_3^{\mathrm{S}}\|R^{(4,3)}\|^2\Big)\, r^3 + O^4(\partial\Omega) \\ &\quad = \widetilde{c}_1^{\mathrm{S}}\|R^{(3,3)}\|^2\, r^2 + \Big(\widetilde{c}_2^{\mathrm{S}}\|R^{(5,2)}\|^2 + \widetilde{c}_3^{\mathrm{S}}\|R^{(4,3)}\|^2\Big)\, r^3 + O^4(\partial\Omega). \end{aligned}$$

Here $\|R^{(a,b)}\|^2$ are regarded as functions on the base domain Ω by restricting to $z_0 = 1$.

Theorem 3. *The universal constants in Theorem 2 above are given by*

$$\begin{aligned} c_1^{\mathrm{B}} &= \frac{3}{1120}\,, & c_2^{\mathrm{B}} &= \frac{61}{141120}\,, & c_3^{\mathrm{B}} &= \frac{3}{7840}\,, \\ \widetilde{c}_1^{\mathrm{B}} &= \frac{1}{160}\,, & \widetilde{c}_2^{\mathrm{B}} &= \frac{1}{20160}\,, & \widetilde{c}_3^{\mathrm{B}} &= \frac{1}{560}\,, \\ c_1^{\mathrm{S}} &= \frac{1}{3360}\,, & c_2^{\mathrm{S}} &= \frac{1}{23520}\,, & c_3^{\mathrm{S}} &= \frac{1}{13230}\,, \\ \widetilde{c}_1^{\mathrm{S}} &= \frac{1}{1440}\,, & \widetilde{c}_2^{\mathrm{S}} &= 0, & \widetilde{c}_3^{\mathrm{S}} &= \frac{1}{4320}. \end{aligned}$$

2. Proofs of Theorems 2 and 3

2.1. Biholomorphic transformation laws. Given a domain functional $K = K_\Omega \in C^\infty(\overline{\Omega})$ which is well-defined modulo $O^k(\partial\Omega)$, we say that K

satisfies the *biholomorphic transformation law* of weight $w \in \mathbb{Z}$ modulo $O^k(\partial\Omega)$ if

$$K_{\Omega_1} = K_{\Omega_2} \circ \Phi \cdot |\det \Phi'|^{2w/3} \mod O^k(\partial\Omega)$$

for any biholomorphic mapping $\Phi\colon \Omega_1 \to \Omega_2$. This notion can be localized to a local domain functional K defined only near a boundary point, say $0 \in \mathbb{C}^2$, such that K satisfies the transformation law as above for any local biholomorphic mapping Φ which fixes the origin. For such K, we write $K \in I_w^{\mathrm{aux}}(k)$. If $K \in I_w^{\mathrm{aux}}(k)$ for all $k \in \mathbb{N}$, we write $K \in I_w^{\mathrm{aux}}(\infty)$.

Proposition 2. *Let ψ^{B} and ψ^{S} be as in* (1.6) *with $r = r^{\mathrm{F}}$. Then:*

(1°) *$\psi^{\mathrm{B}} \in I_3^{\mathrm{aux}}(\infty)$ and $\psi^{\mathrm{S}} \in I_2^{\mathrm{aux}}(\infty)$.*

(2°) *$r^{\mathrm{F}} \in I_{-1}^{\mathrm{aux}}(4)$ and $\eta_1^{\mathrm{G}} \in I_3^{\mathrm{aux}}(3)$.*

(3°) *If $W \in I_w^{\mathrm{WF}}$ with $w \le 5$ then $W \in I_w^{\mathrm{aux}}(6 - w)$.*

Proof. The statements (1°) and (2°) are not new, but we shall give the proofs in order to show the reasoning. To prove (1°), let us recall that K^{B} and K^{S} satisfy the biholomorphic transformation law of weight 3 and 2, respectively, without error. Noting that the singularities of K^{B} and K^{S} are localizable near a boundary point (cf. [5,9]), we have $\psi^{\mathrm{B}} \in I_3^{\mathrm{aux}}(\infty)$ and $\psi^{\mathrm{S}} \in I_2^{\mathrm{aux}}(\infty)$. To prove (2°), let us recall the transformation law for the Levi determinant:

$$J[u_1] = J[u] \circ \Phi \quad \text{if} \quad u_1 = u \circ \Phi \cdot |\det \Phi'|^{-2/3},$$

where $\Phi\colon \Omega_1 \to \Omega_2$ is biholomorphic and $u \in C^\infty(\Omega_2)$ is arbitrary. Since r^{F} is unique modulo $O^4(\partial\Omega)$ and satisfies $J[r^{\mathrm{F}}] = 1 + O^3(\partial\Omega)$, it follows that $r^{\mathrm{F}} \in I_{-1}^{\mathrm{aux}}(4)$. Similarly, we have $u^{\mathrm{G}} \in I_{-1}^{\mathrm{aux}}(\infty)$ formally, where the meaning will be apparent though $u^{\mathrm{G}} \notin C^\infty(\overline{\Omega})$ even formally. This yields $\eta_1^{\mathrm{G}} \in I_3^{\mathrm{aux}}(3)$.

It remains to prove (3°). Following Fefferman [10], we lift a biholomorphic mapping $\Phi\colon \Omega_1 \to \Omega_2$ to a biholomorphic mapping $\Phi^\#\colon \mathbb{C}^* \times \Omega_1 \to \mathbb{C}^* \times \Omega_2$ defined by

$$\Phi^\#(z_0, z) = \Big(z_0 \cdot [\det \Phi'(z)]^{-1/3}, \Phi(z)\Big).$$

If r_2 is a Fefferman's defining function on Ω_2, then $r_1 = r_2 \circ \Phi \cdot |\det \Phi'|^{-2/3}$ is a Fefferman's defining function on Ω_1 and $r_1^\# = r_2^\# \circ \Phi^\#$ holds. Thus $g_1 = (\Phi^\#)^* g_2$, where g_1 and g_2 are respectively ambient metrics with potentials $r_1^\#$ and $r_2^\#$. Consequently, if $W^\#[r_1]$ and $W^\#[r_2]$ are complete contractions of the form (1.3) constructed respectively from r_1 and r_2 in a same way, then

$$W^\#[r_1] = W^\#[r_2] \circ \Phi^\#, \text{ and thus } W[r_1] = W[r_2] \circ \Phi \cdot |\det \Phi'|^{2w/3}.$$

Thus, if $W \in I_w^{\mathrm{WF}}$ then $W \in I_w^{\mathrm{aux}}(6-w)$, because W is by definition well-defined modulo $O^{6-w}(\partial\Omega)$ independent of the choice of $r = r^{\mathrm{F}}$.

2.2. Polynomial dependence on Moser's normal form coefficients. Given an arbitrarily fixed boundary point, say, the origin $0 \in \mathbb{C}^2$, we assume for a moment that $\partial\Omega$ is real analytic near 0 and that $\partial\Omega$ is in Moser's normal form $N(A) = \{2u = |z_1|^2 + F(z_1, \overline{z}_1, v)\}$. Setting

$$U(z, \overline{z}) = U_0 - F(z_1, \overline{z}_1, v), \quad U_0 = 2\,u - |z_1|^2, \tag{2.1}$$

we make a real change of coordinates $(z_1, z_2) \to (z_1', z_2')$ defined by $z_1' = z_1$ and $z_2' = U + i\,v$, so that $u' = U$ and $v' = v$. Abusing notation, we write $(z_1, U + i\,v)$ in place of $(z_1', u' + i\,v')$. Given a local domain functional $K = K_\Omega \in I_w^{\mathrm{aux}}(k)$, we consider the Taylor expansion in the new coordinates

$$K \sim \sum_{m=0}^{\infty} c_m\,(z_1, \overline{z}_1, v)\,U^m,$$

where the coefficients c_m are formal power series in $(z_1, \overline{z}_1, v)$, so that these Taylor expansions make sense even when $\partial\Omega$ is not real analytic. We write $K \in I_w(k)$ if all the coefficients of the formal power series c_m for $m < k$ are polynomials in $A \in \mathcal{N}$. If $K \in I_w(k)$ for all $k \in \mathbb{N}$, we then write $K \in I_w(\infty)$.

Proposition 3.

(1°) *$\psi^{\mathrm{B}} \in I_3(\infty)$ and $\psi^{\mathrm{S}} \in I_2(\infty)$.*
(2°) *$r^{\mathrm{F}} \in I_{-1}(4)$ and $\eta_1^{\mathrm{G}} \in I_3(3)$.*
(3°) *If $W \in I_w^{\mathrm{WF}}$ with $w \leq 5$ then $W \in I_w(6-w)$.*

By virtue of Proposition 2, only the point of Proposition 3 is the polynomial dependence, which will be seen in Subsections 3.1, 4.1 and 5.1.

2.3. Proof of Theorem 2. Recall by Proposition 3 that $\psi^{\mathrm{B}} \in I_3(\infty)$ and $\eta_1^{\mathrm{G}} \in I_3(3)$, so that the boundary values of ψ^{B} and η_1^{G} are CR invariants of weight 3. It is shown by Graham [12] that $\eta_1^{\mathrm{G}} = 4\,A^0_{4\overline{4}}$ on $\partial\Omega$. It then follows from Proposition 1 that there exists a universal constant c_0^{B} such that $\psi^{\mathrm{B}} = c_0^{\mathrm{B}}\,\eta_1^{\mathrm{G}}$ on $\partial\Omega$. We can thus define $\psi_4^{\mathrm{B}} \in C^\infty(\overline{\Omega})$ by $\psi^{\mathrm{B}} - c_0^{\mathrm{B}}\,\eta_1^{\mathrm{G}} = \psi_4^{\mathrm{B}}\,r$. Recalling by Proposition 3 that $r \in I_{-1}(4)$, we see that $\psi_4^{\mathrm{B}} \in I_4(2)$. Setting $W_{41} = \|R^{(4,2)}\|^2$ and $W_{42} = \|R^{(3,3)}\|^2$, we have by Theorem 1 and Proposition 3 that $W_{41}, W_{42} \in I_4(2)$. Thus Proposition 1 and Theorem 1 imply the existence of universal constants c_1^{B} and $\widetilde{c}_1^{\mathrm{B}}$ such that $\psi_4^{\mathrm{B}} = c_1^{\mathrm{B}}\,W_{41} = \widetilde{c}_1^{\mathrm{B}}\,W_{42}$ on $\partial\Omega$. We can thus define $\psi_5^{\mathrm{B}}, \widetilde{\psi}_5^{\mathrm{B}} \in C^\infty(\overline{\Omega})$ by

$$\psi_4^{\mathrm{B}} = c_1^{\mathrm{B}}\,W_{41} + \psi_5^{\mathrm{B}}\,r = \widetilde{c}_1^{\mathrm{B}}\,W_{42} + \widetilde{\psi}_5^{\mathrm{B}}\,r.$$

Arguing as before, we have $\psi_5^{\mathrm{B}}, \widetilde{\psi}_5^{\mathrm{B}} \in I_5(1)$, so that

$$\psi_5^{\mathrm{B}} = c_2^{\mathrm{B}}\, W_{51} + c_3^{\mathrm{B}}\, W_{52} + O^1(\partial\Omega), \quad \widetilde{\psi}_5^{\mathrm{B}} = \widetilde{c}_2^{\mathrm{B}}\, W_{51} + \widetilde{c}_3^{\mathrm{B}}\, W_{52} + O^1(\partial\Omega)$$

with $W_{51} = \|R^{(5,2)}\|^2$ and $W_{52} = \|R^{(4,3)}\|^2$, where c_3^{B}, c_4^{B}, $\widetilde{c}_3^{\mathrm{B}}$, $\widetilde{c}_4^{\mathrm{B}}$ are universal constants. This completes the proof for ψ^{B}. The expansion for ψ^{S} is obtained in a same manner.

2.4. Explicit expressions in terms of Moser's normal coordinates. The proof of Theorem 3 requires explicit expressions in terms of Moser normal form coefficients.

Proposition 4. *Let $\gamma_t^{\#} = (1, 0, t/2) \in \mathbb{C}^* \times \mathbb{C}^2$. Then, as $t \to +0$,*

$$\begin{aligned}
\|R^{(4,2)}\|^2(\gamma_t^{\#}) &= 2^8 \cdot 7\, |A^0_{4\overline{2}}|^2 + 2^8\, F_{50,936,*,*}(A)\, t + O(t^2);\\
\|R^{(3,3)}\|^2(\gamma_t^{\#}) &= 2^8 \cdot 3\, |A^0_{4\overline{2}}|^2 + 2^5 \cdot 4!\, F_{25,243,*,*}(A)\, t + O(t^2);\\
\|R^{(5,2)}\|^2(\gamma_t^{\#}) &= -4 \cdot (5!)^2 F_{1,18,*,*}(A) + O(t);\\
\|R^{(4,3)}\|^2(\gamma_t^{\#}) &= -4 \cdot (5!)^2 F_{4/3,57/5,*,*}(A) + O(t),
\end{aligned}$$

where

$$F_{c_1,c_2,d_1,d_2}(A) = c_1\, |A^0_{5\overline{2}}|^2 + c_2\, |A^0_{4\overline{3}}|^2 + \mathrm{Re}\Big[(d_1\, A^0_{3\overline{5}} - i\, d_2\, A^1_{2\overline{4}})A^0_{4\overline{2}}\Big].$$

In particular, we have, for $r = r^{\mathrm{F}}$,

$$7\, \|R^{(3,3)}\|^2 - 3\, \|R^{(4,2)}\|^2 = \left(\frac{3}{7}\, \|R^{(5,2)}\|^2 - \frac{11}{7} \|R^{(4,3)}\|^2\right) r + O(r^2). \quad (2.2)$$

We now specialize the class of domains to

$$\Omega_{pq} = \big\{2u > |z_1|^2 + A^0_{p\overline{q}} z_1^p \overline{z}_1^q + A^0_{q\overline{p}} z_1^q \overline{z}_1^p\big\}$$

with $p + q \le 7$ and $p > q$.

Proposition 5. *Let $\gamma_t = (0, t/2) \in \mathbb{C}^2$. If $\Omega = \Omega_{pq}$ with $p + q \le 7$ and $p > q$, then*

$$\eta_1^{\mathrm{G}}(\gamma_t) = c_{pq}[\eta_1^{\mathrm{G}}]\, |A^0_{p\overline{q}}|^2\, t^{p+q-5} + O(t^3), \quad (2.3)$$

where $c_{42}[\eta_1^{\mathrm{G}}] = \dfrac{368}{5}$, $c_{52}[\eta_1^{\mathrm{G}}] = -\dfrac{680}{3}$, $c_{43}[\eta_1^{\mathrm{G}}] = -\dfrac{1956}{5}$.

Proposition 6. *If $\Omega = \Omega_{pq}$ with $p+q \le 7$ and $p > q$, then*

$$\begin{aligned} \psi^{\mathrm{B}}(\gamma_t) &= c_{pq}[\psi^{\mathrm{B}}] \left|A^0_{p\overline{q}}\right|^2 t^{p+q-5} + O^{p+q-4}(t), \\ \psi^{\mathrm{S}}(\gamma_t) &= c_{pq}[\psi^{\mathrm{S}}] \left|A^0_{p\overline{q}}\right|^2 t^{p+q-4} + O^{p+q-3}(t), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} &c_{42}[\psi^{\mathrm{B}}] = -216, \quad c_{52}[\psi^{\mathrm{B}}] = 660, \quad\ \ c_{43}[\psi^{\mathrm{B}}] = 1116, \\ &c_{42}[\psi^{\mathrm{S}}] = -\frac{440}{3}, \quad c_{52}[\psi^{\mathrm{S}}] = \frac{4040}{9}, \quad c_{43}[\psi^{\mathrm{S}}] = 760. \end{aligned}$$

2.5. Proof of Theorem 3. By virtue of Theorem 2, it remains only to determine universal constants c^{B}_1, c^{B}_2, c^{B}_3, $\widetilde{c}^{\mathrm{B}}_1$, $\widetilde{c}^{\mathrm{B}}_2$, $\widetilde{c}^{\mathrm{B}}_3$ and c^{S}_1, c^{S}_2, c^{S}_3, $\widetilde{c}^{\mathrm{S}}_1$, $\widetilde{c}^{\mathrm{S}}_2$, $\widetilde{c}^{\mathrm{S}}_3$. By (2.2), it suffices to identify c^{B}_1, c^{B}_2, c^{B}_3 and c^{S}_1, c^{S}_2, c^{S}_3. We thus specialize the class of domains to Ω_{pq} with $p+q \le 7$ and $p > q$. It then follows from Propositions 5 and 6 with Theorem 2 that

$$\begin{aligned} &\left(c_{pq}[\psi^{\mathrm{B}}] + 3\, c_{pq}[\eta^{\mathrm{G}}_1]\right) \left|A^0_{p\overline{q}}\right|^2 t^{p+q-5} \\ &\qquad = c^{\mathrm{B}}_1 \|R^{(4,2)}\|^2\, t + \Big(c^{\mathrm{B}}_2 \|R^{(5,2)}\|^2 + c^{\mathrm{B}}_3 \|R^{(4,3)}\|^2\Big)\, t^2 + O^3(t), \\ &\left(c_{pq}[\psi^{\mathrm{S}}] + 2\, c_{pq}[\eta^{\mathrm{G}}_1]\right) \left|A^0_{p\overline{q}}\right|^2 t^{p+q-4} \\ &\qquad = c^{\mathrm{S}}_1 \|R^{(4,2)}\|^2\, t^2 + \Big(c^{\mathrm{S}}_2 \|R^{(5,2)}\|^2 + c^{\mathrm{S}}_3 \|R^{(4,3)}\|^2\Big)\, t^3 + O^4(t), \end{aligned} \tag{2.5}$$

where the Weyl-Fefferman functionals in the right sides are restricted to $z = \gamma_t$, and the constants $c_{pq}[\eta^{\mathrm{G}}_1]$, $c_{pq}[\psi^{\mathrm{B}}]$, $c_{pq}[\psi^{\mathrm{S}}]$ in the left sides are given by Propositions 5 and 6. We now use Proposition 4. If $(p,q) = (4,2)$ then (2.5) with $\|R^{(4,2)}\|^2(0) = 2^8 \cdot 7 \left|A^0_{4\overline{2}}\right|^2$ yields

$$c^{\mathrm{B}}_1 = \frac{3}{1120}, \qquad c^{\mathrm{S}}_1 = \frac{1}{3360}.$$

If $(p,q) = (5,2)$, then

$$\|R^{(4,2)}\|^2(\gamma_t) = 2^8 \cdot 50\, |A^0_{5\overline{2}}|^2\, t + O(t^2)$$

and $\|R^{(5,2)}\|^2(0) = \frac{3}{4} \|R^{(4,3)}\|^2(0) = -4 \cdot (5!)^2\, |A^0_{5\overline{2}}|^2$, so that (2.5) yields

$$c^{\mathrm{B}}_2 + \frac{4}{3}\, c^{\mathrm{B}}_3 = \frac{19}{20160}\,, \quad c^{\mathrm{S}}_2 + \frac{4}{3}\, c^{\mathrm{S}}_3 = \frac{13}{90720}\,.$$

Similarly, if $(p,q) = (4,3)$, then (2.5) yields

$$c^{\mathrm{B}}_1 + \frac{19}{30}\, c^{\mathrm{B}}_2 = \frac{17}{25200}\,, \quad c^{\mathrm{S}}_1 + \frac{19}{30}\, c^{\mathrm{S}}_2 = \frac{41}{453600}\,.$$

Solving these, we get the desired result.

3. Weight and biweight with respect to dilations

Let us assume that $M \subset \partial\Omega$ is an open C^∞ portion and in Moser's normal form $M = N(A)$. For the coordinates $(z, \overline{z})$, we set $\mathcal{F}_A = \mathbb{C}[A][[z, \overline{z}]]$, the totality of formal power series in $(z, \overline{z})$ such that the coefficients are polynomials in A. In this section we introduce the notion of *biweight* on $\mathcal{F}_A$ as an obvious generalization of the weight for CR invariants and local domain functionals such as Weyl-Fefferman functionals, by weakening the transformation law under local biholomorphic mappings to that under dilations

$$\phi_\lambda(z_1, z_2) = (\lambda z_1, |\lambda|^2 z_2) \quad \text{for} \quad \lambda \in \mathbb{C}^*. \tag{3.1}$$

We also consider the biweight on classes containing $\mathcal{F}_A$.

3.1. Biweight for Moser's normal coefficients and coordinates. For $A = (A^\ell_{p\overline{q}})$, we define w^+-*biweight* of $A^\ell_{p\overline{q}}$ and that of constants by

$$\mathrm{w}_2^+(A^\ell_{p\overline{q}}) = (p + \ell - 1, q + \ell - 1) \tag{3.2}$$

and $\mathrm{w}_2^+(c) = (0, 0)$ for $c \in \mathbb{C}^*$. (We do not define $\mathrm{w}_2^+(0)$, but this will be naturally interpreted in each case, for instance, as $\mathrm{w}_2^+(0) = (-\infty, -\infty)$.) Then the notion of w^+-biweight extends to monomials in A in such a way that

$$\mathrm{w}_2^+(P_1(A)P_2(A)) = \mathrm{w}_2^+(P_1(A)) + \mathrm{w}_2^+(P_2(A)) \tag{3.3}$$

for monomials $P_1(A)$ and $P_2(A)$, where the sum of biweight is defined by $(w_1', w_1'') + (w_2', w_2'') = (w_1' + w_2', w_1'' + w_2'')$. For a polynomial $P(A)$, we write

$$P(A) = P_1(A) + \cdots + P_N(A) \quad \text{with} \quad \mathrm{w}_2^+(P_j(A)) = (w_j', w_j''), \tag{3.4}$$

where $P_j(A)$ are monomials constituting $P(A)$. We say that $P(A)$ is of w^+-*biweight* (w', w'') and write $\mathrm{w}_2^+(P(A)) = (w', w'')$ if $(w_j', w_j'') = (w', w'')$ for all j. When we do not specify (w', w''), we say that $P(A)$ is of *homogeneous* w^+-biweight. The equality (3.3) remains valid when $P_1(A)$ and $P_2(A)$ are polynomials of homogeneous w^+-biweight.

For Moser's normal coordinates $(z, \overline{z})$, we define w^--*biweight* for monomials in $(z, \overline{z})$ by setting

$$\mathrm{w}_2^-(z_1^p \overline{z}_1^q z_2^\ell \overline{z}_2^m) = (-p - \ell - m, -q - \ell - m) \tag{3.5}$$

and $\mathrm{w}_2^-(c) = (0, 0)$ for $c \in \mathbb{C}^*$. As in the case of w^+-biweight, the notion of w^--biweight extends to polynomials in $(z, \overline{z})$. In particular, $\mathrm{w}_2^-(U_0) = (-1, -1)$.

Let $\partial_z = (\partial_1, \partial_2)$ and $\partial_{\overline{z}} = (\partial_{\overline{1}}, \partial_{\overline{2}})$, where $\partial_j = \partial/\partial z_j$ and $\partial_{\overline{k}} = \partial/\partial \overline{z}_k$. We define w^--biweight for monomials in $(\partial_z, \partial_{\overline{z}})$ by setting

$$\mathrm{w}_2^-(\partial_1^p \partial_{\overline{1}}^q \partial_2^\ell \partial_{\overline{2}}^m) = (p + \ell + m, q + \ell + m), \tag{3.6}$$

and extend the notion of w^--biweight to polynomials in $(\partial_z, \partial_{\overline{z}})$. Since (3.6) is consistent with (3.5), the notion of w^--biweight extends to (linear partial) differential operators with polynomial coefficients. We note that w^--biweight defined by (3.5) and (3.6) differs by sign from biweight defined by Boutet de Monvel [2–4].

For monomials $P = P(A; z, \overline{z})$ in A and $(z, \overline{z})$, we define its *biweight* by setting $\mathrm{w}_2(P) = \mathrm{w}_2^+(P) + \mathrm{w}_2^-(P)$. Then the notion of biweight extends to polynomials in A and $(z, \overline{z})$, and hence, to the space $\mathcal{F}_A$ defined at the beginning of this section. In particular, $\mathrm{w}_2(U) = (-1, -1)$. If an element $P = P(A; z, \overline{z})$ of $\mathcal{F}_A$ is of homogeneous biweight, then $P_+(A) = P(A; 0, 0)$ and $P_-(z, \overline{z}) = P(0; z, \overline{z})$ are polynomials of homogeneous w^+-biweight and w^--biweight, respectively, and $\mathrm{w}_2^+(P_+) = \mathrm{w}_2(P) = \mathrm{w}_2^-(P_-)$. The notion of biweight extends to differential operators with coefficients in $\mathcal{F}_A$.

Let us give a remark on dilations in (3.1). Obviously, w^--biweight corresponds to the exponents of the homogeneity with respect to dilations:

$$P(\phi_\lambda(z), \overline{\phi_\lambda(z)}) = \lambda^{-w'} \overline{\lambda}^{-w''} P(z, \overline{z}) \quad \text{if} \ \ \mathrm{w}_2^-(P(z, \overline{z})) = (w', w''),$$

while w^+-biweight corresponds to the exponents of the transformation laws under dilations for Moser's normal form coefficients:

$$\widetilde{A}^\ell_{p\overline{q}} = \lambda^{1-p-\ell} \overline{\lambda}^{1-q-\ell} A^\ell_{p\overline{q}}, \quad \text{where} \ \ N(\widetilde{A}) = \phi_\lambda(N(A)).$$

Consequently, a series $P(A; z, \overline{z}) \in \mathcal{F}_A$ is of biweight (w', w'') if and only if

$$P(\widetilde{A}; \phi_\lambda(z), \overline{\phi_\lambda(z)}) = \lambda^{-w'} \overline{\lambda}^{-w''} P(A; z, \overline{z}) \quad \text{for every} \ \ \lambda \in \mathbb{C}^*.$$

3.2. Weight associated with biweight. We first define w^+-*weight* for polynomials $P(A)$ in A. We set $\mathrm{w}^+(P(A)) = (w' + w'')/2$ if $P(A)$ is of w^+-biweight (w', w''). In general, $P(A)$ admits a unique decomposition (3.4), where $P_j(A)$ are polynomials of homogeneous w^+-biweight. If $w'_j + w''_j = 2w$ for all j, we say that $P(A)$ is of w^+-weight w and write $\mathrm{w}^+(P(A)) = w$. As in the case of biweight, we say that $P(A)$ is of homogeneous w^+-weight when w is not specified. If $P(A) \in I_w^{\mathrm{CR}}$, then $\mathrm{w}_2^+(P(A)) = (w, w)$ and thus $\mathrm{w}^+(P(A)) = w$.

Similarly, w^--*weight* is defined for polynomials in $(z, \overline{z})$ and for differential operators with polynomial coefficients. (We note that w^--weight is $-1/2$ multiple of weight defined in [2–4].) Also, *weight* is defined on the space $\mathcal{F}_A$ and for differential operators with coefficients in $\mathcal{F}_A$. In particular, $\mathrm{w}(U) = \mathrm{w}^-(U_0) = -1$. We have seen that the notion of weight (with respect to dilations) is associated with that of biweight. Similarly for extensions of biweight which will be done in subsequent subsections. Therefore, once biweight is introduced, we *regard* that the corresponding weight is *defined*.

Let $w_j = (w'_j + w''_j)/2$ in the decomposition (3.4). If $w_j \geq w$ (resp. $\leq w$) for all j, we say that $P(A)$ is of w^+-weight $\geq w$ (resp. $\leq w$) and write $\mathrm{w}^+(P(A)) \geq w$ (resp. $\leq w$), though $\mathrm{w}^+(P(A))$ may not be determined as a number. This notation is justified by the fact that if $\mathrm{w}^+(P(A)) \geq w$ and $\mathrm{w}^+(P(A)) \leq w$ then $\mathrm{w}^+(P(A)) = w$. Note that if $\mathrm{w}^+(P_j(A)) \geq w_j$ (resp. $\leq w_j$) for $j = 1, 2$ then $\mathrm{w}^+(P_1(A)P_2(A)) \geq w_1 + w_2$ (resp. $\leq w_1 + w_2$). Similarly, we define the notion for polynomials $P(z, \overline{z})$ in $(z, \overline{z})$ to be of w^--weight $\leq w$ (resp. $\geq w$) and write $\mathrm{w}^-(P(z, \overline{z})) \leq w$ (resp. $\geq w$). These notions for $\mathrm{w}^\pm$-weight extend to $\mathcal{F}_A$ by regarding $(z, \overline{z})$ or A as parameters. Then, for an element $P = P(A; z, \overline{z})$ of $\mathcal{F}_A$ with $\mathrm{w}(P) = w$, P is of w^+-weight $\geq w_+$ (resp. $\leq w_+$) if and only if P is of w^--weight $\leq w - w_+$ (resp. $\geq w - w_+$).

3.3. Biweight for powers of U_0^{-1} and $\log U$. Setting $\mathrm{w}_2^-(U_0^m) = (-m, -m)$ for each negative integer m, we define biweight for formal power series of U_0^{-1} with coefficients in $\mathcal{F}_A$:

$$P(A; z, \overline{z}) = \sum_{m=-\infty}^{0} P_m(A; z, \overline{z})\, U_0^m, \qquad P_m \in \mathcal{F}_A,$$

by $\mathrm{w}_2(P) = (w', w'')$ if $\mathrm{w}_2(P_m) = (w' + m, w'' + m)$ for all integers $m \leq 0$. Expanding negative powers of $U = U_0(1 - F/U_0)$, we see that $\mathrm{w}_2(U^m) = (-m, -m)$ for $m \in \mathbb{Z}$. It should be noted that the expansion of P above is not unique, unless each P_m is normalized to be independent of U_0 in the coordinates $z_1, \overline{z}_1, v, U_0$. Nevertheless, the series for P with general P_m makes sense as an asymptotic series of increasing w^+-weight (or equivalently, decreasing w^--weight), as in the case of elements of $\mathcal{F}_A$. That is, for any integer $w > 0$, there exists an integer $m(w) < 0$ such that if $m < m(w)$ then $\mathrm{w}^+(P_m) \geq w$ and if $m(w) \leq m \leq 0$ then P_m modulo terms of w^+-weight $\geq w$ is uniquely determined by a polynomial in $(z, \overline{z})$ contained in $P_m = P_m(A; z, \overline{z})$.

We next set $\mathrm{w}_2((\log U)^k) = (0, 0)$ for $k \in \mathbb{N}$, and define biweight for formal power series of $\log U$ with coefficients in $\mathcal{F}_A$ by the additivity as before. Then the biweight of $\log U$ is consistent with that of U^{-1} via partial differentiation, and the restriction to $A = 0$ leads to the definition $\mathrm{w}_2^-(\log U_0) = (0, 0)$. This definition is also consistent with the biweight of $\log U - \log U_0 = \log(1 - F/U_0)$, where the right side is expanded as in the case of negative powers of $1 - F/U_0$.

Let us finally consider a formal defining function r of $M = N(A)$ which is regarded as an element of $\mathcal{F}_A$, that is, $r/U \in \mathcal{F}_A$ and $r/U > 0$ at $(z, \overline{z}) = (0, 0)$. If $\mathrm{w}_2(r) = (-1, -1)$, then $\log(r/U) \in \mathcal{F}_A$ is of biweight $(0, 0)$, and thus $\mathrm{w}_2((\log r)^k) = (0, 0)$ for $k \in \mathbb{N}$.

3.4. Ambiguity of Fefferman's defining function r^{F}. Let us recall that r^{F} is unique modulo $O(U^4)$. An elementary calculation shows that this ambiguity is indeed arbitrary. That is, if a function $\widetilde{r}$, C^∞ up to the boundary, satisfies $\widetilde{r} - r^{\mathrm{F}} = O(U^4)$ then $J[\widetilde{r}] = 1 + O(U^3)$. Thus, identifying r^{F} with its Taylor expansion with respect to the coordinates $z_1, \overline{z}_1, v, U$,

$$r^{\mathrm{F}} = \sum_{m=1}^{\infty} P_m(z_1, \overline{z}_1, v) U^m, \quad P_1 = 1,$$

we see that P_m for $m \geq 4$ are arbitrary; while P_m for $m \leq 3$ are shown to be elements of $\mathcal{F}'_A = \mathbb{C}[A][[z_1, \overline{z}_1, v]]$ by virtue of Fefferman's construction of r^{F} in [10]. We define $C = (C^{\ell m}_{p\overline{q}})$ by writing

$$P_m = \sum_{p,q,\ell \geq 0} C^{\ell m}_{p\overline{q}} z_1^p \overline{z}_1^q v^\ell \quad \text{for} \quad m \geq 4,$$

and set

$$r^{\mathrm{F}}_A = \sum_{m=1}^{3} P_m U^m, \qquad r^{\mathrm{F}} - r^{\mathrm{F}}_A = \sum_{m=4}^{\infty} P_m U^m. \tag{3.7}$$

Setting as in (3.2)

$$\mathrm{w}_2^+(C^{\ell m}_{p\overline{q}}) = (p + \ell + m - 1, q + \ell + m - 1), \tag{3.8}$$

we first extend the notion of w^+-biweight to polynomials in (A, C). Then the notion of biweight extends from $\mathcal{F}_A$ to $\mathcal{F}_{A,C}$ and $\mathcal{F}'_{A,C}$, where

$$\mathcal{F}_{A,C} = \mathbb{C}[A, C][[z, \overline{z}]] \quad \text{and} \quad \mathcal{F}'_{A,C} = \mathbb{C}[A, C][[z_1, \overline{z}_1, v]].$$

It follows that $P_m \in \mathcal{F}'_{A,C}$ and $\mathrm{w}_2(P_m) = (m - 1, m - 1)$ for all m, so that

$$\mathrm{w}_2(r^{\mathrm{F}}) = \mathrm{w}_2(r^{\mathrm{F}} - r^{\mathrm{F}}_A) = \mathrm{w}_2(r^{\mathrm{F}}_A) = (-1, -1). \tag{3.9}$$

3.5. Biweight for covariant derivatives of the curvature tensor. Setting $\mathrm{w}_2^-(z_0) = \mathrm{w}_2^-(\overline{z}_0) = (0, 0)$, we extend the notion of weight to the space $\mathbb{C}[A, C][[z_0, z_0^{-1}, z, \overline{z}_0, \overline{z}_0{}^{-1}, \overline{z}]]$, where the connection with dilation is ignored. Then

$$\mathrm{w}_2^-(\partial_0) = \mathrm{w}_2^-(\partial_{\overline{0}}) = (0, 0), \quad \text{where} \;\; \partial_0 = \partial/\partial z_0 \text{ and } \partial_{\overline{0}} = \partial/\partial \overline{z}_0.$$

It follows from (3.9) that $\mathrm{w}_2(r^\#) = (-1,-1)$ for $r = r^{\mathrm{F}}$. We set

$$\mathrm{w}_2^-(dz_j) = \mathrm{w}_2^-(z_j) \quad \text{and} \quad \mathrm{w}_2^-(d\overline{z}_k) = \mathrm{w}_2^-(\overline{z}_k) \quad \text{for} \quad j,k = 0,1,2.$$

It then follows from the definition of the ambient metric $g = (g_{j\overline{k}})$, which is a covariant tensor, that

$$\mathrm{w}_2(g) = \mathrm{w}_2\Big(\sum_{j,k=0}^{2} g_{j\overline{k}} dz_j \otimes d\overline{z}_k\Big) = (-1,-1).$$

(We also have $\mathrm{w}_2(\det(g_{j\overline{k}})) = (0,0)$.) This together with Cramer's formula implies

$$\mathrm{w}_2(\sum_{j,k=0}^{2} g^{j\overline{k}} \partial_j \otimes \partial_{\overline{k}}) = -\mathrm{w}_2(g) = (1,1),$$

where $(g^{j\overline{k}})$ is the inverse matrix of $g = (g_{j\overline{k}})$. These equalities are written componentwise as follows:

$$\mathrm{w}_2(g_{j\overline{k}}) = \mathrm{w}_2^-(j\overline{k}) - (1,1) = -\mathrm{w}_2(g^{j\overline{k}}) \quad \text{for} \quad j,k = 0,1,2, \tag{3.10}$$

where $\mathrm{w}_2^-(j\overline{k}) = \mathrm{w}_2^-(\partial_j \partial_{\overline{k}})$. More generally, we use the notation

$$\mathrm{w}_2^-(\alpha\overline{\beta}) = \mathrm{w}_2^-(\partial_\alpha \partial_{\overline{\beta}})$$

for (ordered) multi-indices $\alpha = \alpha_1 \cdots \alpha_a$ and $\beta = \beta_1 \cdots \beta_b$ with $\alpha_j, \beta_k \in \{0,1,2\}$, where $\partial_\alpha = \partial_{\alpha_1} \cdots \partial_{\alpha_a}$ and $\partial_{\overline{\beta}} = \partial_{\overline{\beta}_1} \cdots \partial_{\overline{\beta}_b}$. We also write

$$\mathrm{w}_2^-(\gamma) = \mathrm{w}_2^-(\alpha\overline{\beta}) = \mathrm{w}_2^-(\partial_\alpha \partial_{\overline{\beta}}),$$

where γ is a multi-index of *mixed type* obtained by a rearrangement of $\alpha\overline{\beta}$. For the later use, we set

$$|\alpha| = a, \quad |\overline{\beta}| = |\beta| = b, \quad |\gamma| = |\alpha| + |\overline{\beta}|.$$

For the (covariant) curvature tensor $R = (R_{j\overline{k}\ell\overline{m}})$ of the metric g, we have

$$\mathrm{w}_2(R) = (-1,-1), \text{ i.e., } \mathrm{w}_2(R_{j\overline{k}\ell\overline{m}}) = \mathrm{w}_2^-(j\overline{k}\ell\overline{m}) - (1,1), \tag{3.11}$$

a fact which is obtained by applying (3.10) to the expression

$$R_{j\overline{k}\ell\overline{m}} = \partial_j \partial_{\overline{k}} \partial_\ell \partial_{\overline{m}} r^\# - \sum_{p,q=0}^{2} g^{p\overline{q}} (\partial_{\overline{q}} \partial_j \partial_\ell r^\#)(\partial_p \partial_{\overline{k}} \partial_{\overline{m}} r^\#).$$

We also have, for the (iterated) covariant derivative $\nabla^c R = (\nabla_\gamma R_{j\bar{k}\ell\overline{m}})$ of order c (γ being multi-indices of mixed type with $|\gamma| = c$),

$$\mathrm{w}_2(\nabla^c R) = (-1,-1), \text{ i.e., } \mathrm{w}_2(\nabla_\gamma R_{j\bar{k}\ell\overline{m}}) = \mathrm{w}_2^-(\gamma j\bar{k}\ell\overline{m}) - (1,1). \quad (3.12)$$

More generally, if a covariant tensor S is of homogeneous biweight, so is its covariant derivative T of arbitrary order and $\mathrm{w}_2(T) = \mathrm{w}_2(S)$. We write this fact as follows:

$$\mathrm{w}_2(\nabla) = (0,0), \quad \text{or equaivalently, } \mathrm{w}_2^-(\nabla_\gamma) = \mathrm{w}_2^-(\gamma). \quad (3.13)$$

By virtue of (3.13), we see that (3.11) yields (3.12). This fact (3.13) is obtained by using $\mathrm{w}_2(\Gamma^\ell_{jk}) = \mathrm{w}_2^-(jk) - \mathrm{w}_2^-(\ell)$, where $\Gamma^\ell_{jk} = \sum_m g^{\ell\overline{m}}(\partial_j g_{k\overline{m}})$ are the Christoffel symbols; indeed, $\nabla_j \varphi_k = \partial_j \varphi_k - \sum_\ell \Gamma^\ell_{jk}\varphi_\ell$, $\nabla_j \varphi_{\overline{k}} = \partial_j \varphi_{\overline{k}}$, etc.

Suppose we are given a covariant tensor $S = (S_\gamma)$ of homogeneous biweight. Let us describe the effect of raising arbitrarily certain indices in γ via the (covariant) metric tensor $(g^{j\overline{k}})$. We first define the *dual* indices by $(0^*, 1^*, 2^*) = (2, 1, 0)$, and then extend the definition to multi-indices, so that we have α^* for α. Setting $\overline{\beta}^* = \overline{\beta^*}$, we get a mapping $\gamma \mapsto \gamma^*$ for multi-indices of mixed type. Then it follows from (3.10) that

$$\mathrm{w}_2(S^\gamma) = \mathrm{w}_2(S_{\overline{\gamma^*}}); \quad \text{in particular, } \mathrm{w}_2(S^{\alpha\overline{\beta}}) = \mathrm{w}_2(S_{\overline{\alpha^*}\beta^*}). \quad (3.14)$$

Similar equalities are valid when we raise indices partially. For instance,

$$\mathrm{w}_2(S_\gamma{}^\delta) = \mathrm{w}_2(S_{\gamma\overline{\delta^*}}), \quad \text{where } \gamma \text{ and } \delta \text{ are of mixed type.}$$

For a multi-index of mixed type $\gamma = \gamma_1 \cdots \gamma_c$, we define its *transpose* by ${}^t\gamma = \gamma_c \cdots \gamma_1$, and write $\nabla_\gamma R_{j\bar{k}\ell\overline{m}} = R_{j\bar{k}\ell\overline{m};{}^t\gamma}$. We also set

$$R_{\alpha\overline{\beta}} = R_{j\bar{k}\ell\overline{m};\alpha'\overline{\beta'}} \quad \text{with } \alpha = j\ell\alpha', \beta = km\beta'.$$

Then (3.12) yields $\mathrm{w}_2(R_{\alpha\overline{\beta}}) = \mathrm{w}_2^-(\alpha\overline{\beta}) - (1,1)$. Consequently, (3.14) and

$$\mathrm{w}_2^-(\gamma\overline{\gamma^*}) = \mathrm{w}_2^-(\gamma) + \mathrm{w}_2^-(\overline{\gamma^*}) = (|\gamma|, |\gamma|)$$

imply

$$\begin{gathered}\mathrm{w}_2(R_{\alpha\overline{\beta}}) + \mathrm{w}_2(R^{\overline{\beta}\alpha}) = (|\alpha| + |\beta| - 2, |\alpha| + |\beta| - 2), \\ \mathrm{w}_2(R_\alpha{}^\alpha) = (|\alpha| - 1, |\alpha| - 1).\end{gathered}$$

Let us finally give a remark on the change of coordinates. In Section 5 below, we shall compute covariant derivatives of the curvature tensor with respect to the projective coordinates $\zeta = (\zeta_0, \zeta_1, \zeta_2)$ defined by $\zeta_0 = z_0$, $\zeta_1 = z_0 z_1$ and $\zeta_2 = z_0 z_2$. Nevertheless, the facts stated in this subsection remain valid with respect to the coordinates ζ in place of (z_0, z), because

$$\mathrm{w}_2^-(\zeta_j) = \mathrm{w}_2^-(z_j) \quad \text{for} \quad j = 0, 1, 2.$$

3.6. Transformation laws for the singularities of K^{B} and K^{S}. So far in this section we have assumed $M = N(A) \in C^\infty$, because local biholomorphic transformation laws are apparent for locally constructed objects such as CR invariants, the Monge-Ampère asymptotic solutions and Weyl-Fefferman functionals, even when the mappings are given by formal power series. We have to be careful in treating the Bergman kernel K^{B} and the Szegö kernel K^{S}, since we consider the localizations of K^{B} and K^{S} which are defined globally and thus the transformation laws are not obvious.

Let us begin by recalling the asymptotic expansions of K^{B} and K^{S}. Assuming $\partial\Omega \in C^\infty$, we specify a defining function r of Ω arbitrarily. To unify the description, we set

$$K^{(3)} = (\pi^2/2) K^{\mathrm{B}}, \quad K^{(2)} = \pi^2 K^{\mathrm{S}},$$

and consider $K^{(w)}$ for $w = 2, 3$. According to Fefferman [9] and Boutet de Monvel–Sjöstrand [5], there exists a sequence $\{K_m^{(w)}\}_{m \geq 0}$ of functions of the form

$$K_m^{(w)} = \varphi r^{-w} + \psi_m \log r \quad \text{with} \quad \varphi, \psi_m \in C^\infty(\overline{\Omega}),$$

such that $\psi_m - \psi_{m-1} = O(r^m)$, and, as m becomes larger, $K^{(w)} - K_m^{(w)}$ is smoother and the vanishing order at $\partial\Omega$ is higher. Hence, if we let $m = \infty$ formally, then

$$K^{(w)} \sim \varphi r^{-w} + \psi \log r \quad \text{with} \quad \psi - \psi_m = O(r^{m+1}). \tag{3.15}$$

More precisely, ψ is realized as an element of $C^\infty(\overline{\Omega})$, and the difference between both sides of (3.15) belongs to $C^\infty(\overline{\Omega})$.

Fixing $p \in \partial\Omega$ arbitrarily, we take a local coordinate system z about the origin in such a way that $r = U_0 + O(|z|^3)$. Identifying φ modulo $O(r^w)$ and ψ with their Taylor expansions at the origin, we *regard* the right side of (3.15) as the (formal) singularity of $K^{(w)}$. We also identify r with its Taylor expansion. Then, in view of the constructions in [9] and [5], we see that the mappings of the Taylor coefficients of r to those of φ modulo $O(r^w)$ and ψ are well-defined and that these are polynomial mappings. In particular, the singularity of $K^{(w)}$ is localizable.

Let us next consider the local transformation law. Suppose we are given a biholomorphic mapping $\Phi\colon U \to U'$ such that

$$\Phi(M) = M', \quad \Phi(p) = p', \tag{3.16}$$

where $U \subset \mathbb{C}^2$ is a neighborhood of p and $M \subset U$ is a strictly pseudoconvex C^∞ real hypersurface containing p; similarly for $p' \in U'$. More rigorously, M and M' are germs of C^∞ hypersurfaces at p and p', respectively, and Φ is a germ of biholomorphic mapping satisfying (3.16). Shrinking M if necessary, we take arbitrarily a smoothly bounded strictly pseudoconvex domain $\Omega \subset\subset \mathbb{C}^2$ such that $M \subset \partial\Omega$. Denoting by $K_\Omega^{(w)}$ the kernel $K^{(w)}$ associated with Ω, we write the (formal) *singularity* of $K_\Omega^{(w)}$ as $K_M^{(w)}$. It then follows from the polynomial dependence above that $K_M^{(w)}$ is independent of the choice of Ω. Consequently, we may write, corresponding to (3.15),

$$K_M^{(w)} = \varphi r^{-w} + \psi \log r \quad \text{with} \quad \varphi, \psi \in \mathbb{C}[[z, \overline{z}]].$$

To get the transformation law for $K_M^{(w)}$, we first shrink M to M_0 so that $p \in M_0 \subset\subset M$, and take a smoothly bounded strictly pseudoconvex domain $\Omega_0 \subset \Omega \cap U$ in such a way that $M_0 \subset \partial\Omega_0$. Then $\Omega_0' = \Phi(\Omega_0)$ is a smoothly bounded strictly pseudoconvex domain, and $M_0' = \Phi(M_0)$ is a real hypersurface satisfying $p' \in M_0' \subset\subset M'$ and $M_0' \subset M'$. Hence, the global transformation law yields

$$K_{\Omega_0}^{(w)} = (K_{\Omega_0'}^{(w)} \circ \Phi)|\det \Phi'|^{2w/3}. \tag{3.17}$$

We consider (3.17) about $p \in M_0$. Let us write, in the sense of (3.17), as follows:

$$K_{\Omega_0}^{(w)} \sim \varphi r_0^{-w} + \psi \log r_0, \quad K_{\Omega_0'}^{(w)} \sim \varphi' {r_0'}^{-w} + \psi' \log r_0', \tag{3.18}$$

where r_0 and r_0' are defining functions of Ω_0 and Ω_0', respectively. (We may take $r_0 = r$ near M_0.) That is, if we regard M and M' as C^∞ germs, then the right sides of (3.18) are respectively $K_{\Omega_0}^{(w)}$ and $K_{\Omega_0'}^{(w)}$ in the C^∞ sense. Let us first assume that r_0' satisfies $r_0 = r_0' \circ \Phi |\det \Phi'|^{-2/3}$. Then (3.17) yields

$$\varphi = \varphi' \circ \Phi + O(r_0^w), \quad \psi = (\psi' \circ \Phi)|\det \Phi'|^{2w/3}, \tag{3.19}$$

which constitute the local biholomorphic transformation law for the singularity $K_M^{(w)}$. In case r_0' is a general defining function of Ω_0', the transformation law for ψ is unchanged while that for φ is subject to an obvious change. Even when Φ is given by a formal power series, (3.19) remains valid as a formal transformation law. Consequently, we may assume $M = N(A) \in C^\infty$

as in the previous subsections. Also, the polynomial dependence of ψ^{B} and ψ^{S} on A in the sense of Subsection 2.2 becomes apparent.

3.7. Biweight for simple holonomic singularities. Assuming $M = N(A) \in C^\infty$ as before, we use coordinates $(z_1, \overline{z}_1, U, v)$ and consider for $w \in \mathbb{Z}$ fixed a (formal) singularity of the form

$$K(z,\overline{z}) = \sum_{m=0}^{w-1} \varphi_m U^{m-w} + \sum_{m=w}^{\infty} \varphi_m U^{m-w} \log U, \tag{3.20}$$

where $\varphi_m \in \mathcal{F}'_A$ satisfy $\mathrm{w}_2(\varphi_m) = (m,m)$ for $m \geq 0$. We abbreviate these conditions by writing $\mathrm{w}_2(K) = (w,w)$. When $w \leq 0$, we agree to regard $\varphi_m = 0$ for $m < 0$. Observe that K does not involve (formal) smooth terms.

Recalling that U depends on $A \in \mathcal{N}$, let us expand $\log U$ and negative powers of U as in Subsection 3.3. Then (3.20) with $\mathrm{w}_2(K) = (w,w)$ yields

$$K(z,\overline{z}) = \sum_{m=-\infty}^{w-1} a_m U_0^{m-w} + \sum_{m=w}^{\infty} a_m U_0^{m-w} \log U_0 + \cdots, \tag{3.21}$$

where $a_m \in \mathcal{F}'_A$ satisfy $\mathrm{w}_2(a_m) = (m,m)$ for $m \in \mathbb{Z}$, and $\cdots$ stands for terms which belong to $\mathcal{F}_A$. We can recover $\{\varphi_m\}$ from $\{a_m\}$ via

$$\sum_{m=w}^{\infty} a_m U_0^{m-w} \log U_0 = \sum_{m=w}^{\infty} \varphi_m U^{m-w} \log U_0,$$

$$\sum_{m=-\infty}^{w-1} a_m U_0^{m-w} = \sum_{m=0}^{w-1} \varphi_m U^{m-w} + \sum_{m=w}^{\infty} \varphi_m U^{m-w} \log(1 - F/U_0) + \cdots.$$

The latter equality also yields $\mathrm{w}^-(a_m) \leq \mathrm{w}^-(F^m) \leq 3m \leq 2m$ for $m < 0$. Consequently,

$$\mathrm{w}^-(a_m U_0^m) \leq -|m| \quad \text{for} \quad m \in \mathbb{Z}. \tag{3.22}$$

Let us next recall the *complex normal form* of $M = N(A)$ introduced by Boutet de Monvel [2–4]. This is defined by solving the equation $U(z,\overline{z}) = 0$ for M with respect to the variables $\overline{z}_2$. Then M is given by

$$\overline{z}_2 = -z_2 + |z_1|^2 + H_B(z, \overline{z}_1), \quad \text{that is,} \quad U_B(z,\overline{z}) = 0$$

with $U_B(z,\overline{z}) = U_0(z,\overline{z}) - H_B(z,\overline{z}_1)$, where $H_B(z,\overline{z}_1)$ is a (formal) power series of the form

$$H_B(z,\overline{z}_1) = \sum_{p,q \geq 2} B_{p\overline{q}}(z_2) z_1^p \overline{z}_1^q \quad \text{with} \quad B_{p\overline{q}}(z_2) = \sum_{\ell=0}^{\infty} B_{p\overline{q}}^{\ell} z_2^{\ell}.$$

It follows that $B_{2\overline{2}}(z_2) = B_{2\overline{3}}(z_2) = B_{3\overline{2}}(z_2) = B_{3\overline{3}}(z_2) = 0$; thus the name complex normal form. We also have

$$\mathrm{w}_2^-(B^\ell_{p\overline{q}}) = \mathrm{w}_2^-(A^\ell_{p\overline{q}}), \quad \mathcal{F}_B = \mathcal{F}_A. \tag{3.23}$$

In fact, $A = (A^\ell_{p\overline{a}}) \mapsto B = (B^\ell_{p\overline{q}})$ is an injective polynomial mapping together with its inverse, and preserves w^--biweight; in particular, $B = 0$ if and only if $A = 0$.

We now use coordinates $(z, \overline{z}_1, U_B)$. Then (3.20) with $\mathrm{w}_2(K) = (w, w)$ and (3.23) yields

$$K(z, \overline{z}) = \sum_{m=0}^{w-1} \psi_m U_B^{m-w} + \sum_{m=w}^{\infty} \psi_m U_B^{m-w} \log U_B, \tag{3.24}$$

where $\psi_m \in \mathbb{C}[A][[z, \overline{z}_1]]$ satisfy $\mathrm{w}_2(\psi_m) = (m, m)$ for $m \geq 0$. The right side of (3.24) is the formal version of a simple holonomic singularity in the sense of Sato, Kawai and Kashiwara [19], cf. [16] and [2–4]. As in (3.21), we get by (3.24)

$$K(z, \overline{z}) = \sum_{m=-\infty}^{w-1} b_m U_0^{m-w} + \sum_{m=w}^{\infty} b_m U_0^{m-w} \log U_0 + \cdots, \tag{3.25}$$

where $b_m \in \mathbb{C}[A][[z, \overline{z}_1]]$ satisfy $\mathrm{w}_2(b_m) = (w, w)$ for $m \in \mathbb{Z}$, and $\cdots$ stands for an element of $\mathbb{C}[A][[z, \overline{z}_1, U]]$. As in (3.22), we have

$$\mathrm{w}^-(b_m U_0^m) \leq -|m| \quad \text{for} \quad m \in \mathbb{Z}. \tag{3.26}$$

Observe that $b_0(0, 0) = \psi_0(0, 0) = \varphi_0(0, 0, 0) = a_0(0, 0, 0)$, which is of w^--biweight $(0, 0)$ and thus a constant independent of A.

3.8. Biweight for microdifferential operators of infinite order. Let us define $[U_0]_m$ for $m \in \mathbb{Z}$ as the singularities by

$$[U_0]_m = \begin{cases} C_m U_0^m \log U_0, \quad C_m = 1/m! & \text{for} \quad m \geq 0, \\ C_m U_0^m, \quad C_m = (-1)^{m+1}(-m-1)! & \text{for} \quad m < 0; \end{cases}$$

thus $\partial_2^k [U_0]_m = [U_0]_{m-k}$ and $[U_0]_0 = \log U_0$. Then (3.25) with $\mathrm{w}_2(K) = (w, w)$ and (3.26) is written as follows:

$$K(z, \overline{z}) = \sum_{m=-\infty}^{\infty} c_m(z, \overline{z}_1)[U_0]_{m-w}, \tag{3.27}$$

$$\mathrm{w}_2(c_m) = (m, m), \quad \mathrm{w}^-(c_m[U_0]_m) \leq -|m| \quad \text{for} \quad m \in \mathbb{Z}, \tag{3.28}$$

where $c_m(z,\overline{z}_1) = (1/C_{m-w})b_m(z,\overline{z}_1)$. Observe that the operator ∂_2 acting on such singularities is invertible, and the inverse ∂_2^{-1} is determined by $\partial_2^{-1}[U_0]_m = [U_0]_{m+1}$ and

$$\partial_2^{-1}(f(z_2)[U_0]_m) = \sum_{j=0}^{\infty}(-1)^j f^{(j)}(z_2)[U_0]_{m+1+j} \quad \text{for} \quad f(z_2) \in \mathbb{C}[[z_2]].$$

In particular, $\partial_1\partial_2^{-1}[U_0]_m = -\overline{z}_1[U_0]_m$. We thus set

$$Q(z,\zeta) = \sum_{m=-\infty}^{\infty} c_m(z, -\zeta_1/\zeta_2)\zeta_2^{-m} \in \mathbb{C}[[z,\zeta,1/\zeta_2]],$$

where $\zeta = (\zeta_1, \zeta_2)$ stands for the dual variable of $z = (z_1, z_2)$. (Be careful that ζ here is different from that in the projective coordinates at the end of Subsection 3.5.) Then (3.27) is written as

$$K(z,\overline{z}) = Q(z,\partial_z)[U_0]_{-w}, \ Q(z,\partial_z) = \sum_{m=-\infty}^{\infty} c_m(z, -\partial_1\partial_2^{-1})\partial_2^{-m}, \quad (3.29)$$

and thus $Q(z,\zeta)$ is the total symbol of $Q(z,\partial_z)$.

We define the notion of $\mathrm{w}^{\pm}$-biweight and biweight for $Q(z,\partial_z)$ and $Q(z,\zeta)$ so as to be consistent with the definition for $K(z,\overline{z})$ via (3.28). This is done by setting $\mathrm{w}_2^-(\partial_2^{-1}) = (-1,-1)$ and

$$\mathrm{w}_2^-(\zeta_1) = (1,0), \quad \mathrm{w}_2^-(\zeta_2) = (1,1), \quad \mathrm{w}_2^-(1/\zeta_2) = (-1,-1).$$

Then the condition (3.28) is written as

$$\mathrm{w}_2(Q) = (0,0), \quad \mathrm{w}^-\left(c_m(z,-\zeta_1/\zeta_2)\zeta_2^{-m}\right) \le -|m| \quad \text{for} \quad m \in \mathbb{Z}. \quad (3.30)$$

In particular, if we write $Q = \sum Q_j$ with $\mathrm{w}^-(Q_j) = j \in \mathbb{Z}$ by arranging the terms of $Q(z,\zeta) \in \mathbb{C}[[z,\zeta,1/\zeta_2]]$ then $Q_j(z,\zeta) \in \mathbb{C}[z,\zeta,1/\zeta_2]$. We denote by $\mathcal{M}^\infty$ the totality of the formal operators $Q(z,\partial_z)$ as in (3.29) satisfying (3.30). Hence our singularities are parametrized by $\mathcal{M}^\infty$ via (3.29).

Given $Q \in \mathcal{M}^\infty$ and a sequence $\{Q_k\}$ in $\mathcal{M}^\infty$, we write $Q_k \to Q$ in $\mathcal{M}^\infty$ if $\mathrm{w}^-(Q - Q_k) \le w_k$ with a sequence $w_k \to -\infty$. Then the notion of limit with respect to w^--weight is defined on $\mathcal{M}^\infty$, and each element of $\mathcal{M}^\infty$ can be regarded as an asymptotic series via (3.30).

We now define a subclass $\mathcal{M}^{\mathrm{finite}} \subset \mathcal{M}^\infty$ as follows: $Q \in \mathcal{M}^{\mathrm{finite}}$ if in the series expression (3.29) there exists $m_0 = m_0(Q) \in \mathbb{Z}$ such that $c_m = 0$ for $m < m_0$. Hence each $Q \in \mathcal{M}^{\mathrm{finite}}$ is the formal version of a pseudodifferential operator of order $\le -m_0$, or rather, a holomorphic microdifferential operator in the sense of Sato, Kawai and Kashiwara [19], cf. [16] and [2–4]. Thus each $Q \in \mathcal{M}^\infty$ is called (the formal version of) holomorphic microdifferential operator *of infinite order*. In general, $Q \in \mathcal{M}^\infty$ is not pseudo-local and amounts to a Fourier integral operator. It follows from (3.30) that each $Q \in \mathcal{M}^\infty$ can be approximated with respect to w^--weight by a sequence in $\mathcal{M}^{\mathrm{finite}}$. Let us also note that $\mathcal{M}^{\mathrm{finite}}$ is closed under operations of taking the formal adjoint and composition, defined by the symbol relations

$$Q^*(z,\zeta) = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_z \cdot \partial_\zeta)^k Q(z,-\zeta), \quad \partial_z \cdot \partial_\zeta = \sum_{j=1}^{2} \frac{\partial^2}{\partial z_j \partial \zeta_j},$$

$$(Q_1 \circ Q_2)(z,\zeta) = \left(\sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{z'} \cdot \partial_{\zeta'})^k \{Q_1(z,\zeta') Q_2(z',\zeta)\} \right) \Bigg|_{(z',\zeta')=(z,\zeta)}.$$

It is seen that these operations extend consistently to $\mathcal{M}^\infty$.

Let us finally consider the inversion in $\mathcal{M}^\infty$. For $\lambda \in \mathbb{C}$, we denote by $\mathcal{M}^\infty(\lambda)$ the totality of $Q \in \mathcal{M}^\infty$ such that $c_0(0,0) = \lambda$ in the series expression (3.29). It is clear that if $Q \in \mathcal{M}^\infty(\lambda)$ then $Q^* \in \mathcal{M}^\infty(\lambda)$ and that if $Q \in \mathcal{M}^\infty(\lambda_j)$ for $j = 1, 2$ then $Q_1 Q_2 \in \mathcal{M}^\infty(\lambda_1 \lambda_2)$. It follows from (3.30) that if $Q \in \mathcal{M}^\infty(0)$ then $\mathrm{w}^-(Q) \le -1/2$ and thus $\mathrm{w}^-(Q^k) \le -k/2$ for $k \in \mathbb{N}$. Consequently, if $Q \in \mathcal{M}^\infty(1)$ then $1 - Q \in \mathcal{M}^\infty(0)$, so that

$$\sum_{k=0}^{\infty} (1-Q)^k = \lim_{N \to \infty} \sum_{k=0}^{N} (1-Q)^k \in \mathcal{M}^\infty(1)$$

is well-defined and gives the inverse Q^{-1} of Q. Since $\mathcal{M}^\infty(\lambda) = \lambda \mathcal{M}^\infty(1)$ for $\lambda \in \mathbb{C}^*$, it follows that the union of $\mathcal{M}^\infty(\lambda)$ over $\lambda \in \mathbb{C}^*$ constitutes invertible elements in $\mathcal{M}^\infty$, and that $Q^{-1} \in \mathcal{M}^\infty(1/\lambda)$ when $Q \in \mathcal{M}^\infty(\lambda)$.

4. CR invariants of weight ≤ 5

4.1. Polynomials of homogeneous biweight. Recall that a CR invariant of weight w is a polynomial in Moser's normal form coefficients $A = (A^\ell_{p\overline{q}})$ of (homogeneous) biweight (w, w) in the sense of Subsection 3.1. To prove Proposition 3, we thus begin by determining all such polynomials for $w \le 5$.

Lemma 4.1. *Let $w \le 5$, and let $P_w(A)$ be a real polynomial in $A = (A^\ell_{p\,\overline{q}})$ of biweight (w, w). Then*

$$P_0(A) = a, \quad P_1(A) = P_2(A) = 0, \quad P_3(A) = a\, A^0_{4\overline{4}},$$
$$P_4(A) = F_{abc}(A), \quad P_5(A) = F_{abcde\alpha\beta}(A),$$

with $F_{abc}(A) = a\,|A^0_{4\bar 2}|^2 + b\,A^0_{5\bar 5} + c\,A^1_{4\bar 4}$ *and*

$$F_{abcde\alpha\beta}(A) = a\,|A^0_{5\bar 2}|^2 + b\,|A^0_{4\bar 3}|^2 + \mathrm{Re}[(\alpha\,A^0_{3\bar 5} + \beta\,A^1_{2\bar 4})A^0_{4\bar 2}] \\ + c\,A^0_{6\bar 6} + d\,A^1_{5\bar 5} + e\,A^2_{4\bar 4},$$

where $a, b, c, d, e \in \mathbb{R}$ *and* $\alpha, \beta \in \mathbb{C}$ *are arbitrary constants.*

Proof. Let V_k denote the vector space of real homogeneous polynomials in A of degree $k \geq 0$. Obviously, $0 \neq P_w(A) \in V_0$ if and only if $w = 0$, and this fact yields in particular $P_0(A) = a$. Recall next that $A^\ell_{2\bar 2} = A^\ell_{3\bar 3} = 0$ and that if $\mathrm{w}_2(A^\ell_{p\bar q}) = (w, w)$ then $q = p$ and $\ell = w - p + 1$. We then see that $P_w(A) \in V_1$ is possible for $w = 1, \ldots, 5$ only when $P_1(A) = 0$, $P_2(A) = 0$ and

$$P_3(A) = a\,A^0_{4\bar 4}, \quad P_4(A) = F_{0bc}(A), \quad P_5(A) = F_{00cde00}(A),$$

respectively. Let us now assume that $0 \neq P_w(A) \in V_k$ for $k \geq 2$. Recall that if $A^\ell_{p\bar q} \neq 0$ then $\mathrm{w}(A^\ell_{p\bar q}) \geq 2$, and that $\mathrm{w}(A^\ell_{p\bar q}) = 2$ if and only if $A^\ell_{p\bar q} = A^0_{2\bar 4}$ or $A^0_{4\bar 2}$. Consequently, $w = \mathrm{w}(P_w(A)) \geq 2k$, and in particular $w \geq 4$. (This yields in general $P_1(A) = P_2(A) = 0$ and $P_3(A) = a\,A^0_{4\bar 4}$.) If $w = 4$ then $k = 2$, while $P_4(A) \in V_2$ is a real quadratic polynomial in $A^0_{2\bar 4}$ and $A^0_{4\bar 2}$, so that $P_4(A) = a\,|A^0_{4\bar 2}|^2$. (Thus $P_4(A) = F_{abc}(A)$ in general.) It only remains to consider the case $w = 5$, in which case we have again $k = 2$. What we need to show is that the assumption $0 \neq P_5(A) \in V_2$ implies $P_5(A) = F_{ab000\alpha\beta}(A)$. Observe that $P_5(A)$ is a linear combination of monomials of the form $Q_1(A)Q_2(A)$, where the following two cases are possible for $Q_j = Q_j(A)$:

$$\mathrm{w}(Q_1) = \mathrm{w}(Q_2) = 5/2; \quad \text{or} \quad \mathrm{w}(Q_1) = 3 \text{ and } \mathrm{w}(Q_2) = 2.$$

In the former case, Q_1Q_2 is a constant multiple of $|A^0_{p\bar q}|^2$ with $(p, q) = (5, 2)$ or $(4, 3)$. In the latter case, the possible choices are $Q_1Q_2 = \alpha\,A^0_{3\bar 5}A^0_{4\bar 2}$, $\beta\,A^1_{2\bar 4}A^0_{4\bar 2}$ and their complex conjugates. Therefore, $P_5(A) = F_{ab000\alpha\beta}(A)$ as desired. □

4.2. A group action on polynomials in A. To describe the nonuniqueness of Moser's normal form, we recall Moser's construction of normal forms. For $A, \widetilde{A} \in \mathcal{N}$, let $\mathcal{B}(A, \widetilde{A})$ denote the set of all formal biholomorphic mapping Φ near the origin such that $\Phi(0) = 0$ and $\Phi(N(A)) = N(\widetilde{A})$. Let H denote the isotropy group of the Siegel domain $2u > |z_1|^2$. Thus, H consists of automorphisms of the Siegel domain which fix the origin. In [7] (see also [18]), a group action

$$H \times \mathcal{N} \in (h, A) \mapsto h.A \in \mathcal{N}$$

is constructed in such a way that for each $(h, A) \in H \times \mathcal{N}$ there exists $\Phi = \Phi_{(h,A)} \in \mathcal{B}(A, h.A)$ with $\Phi'(0) = h'(0)$ having the following properties:

(i) If $h(N(A)) = N(\widetilde{A})$ for some $\widetilde{A} \in \mathcal{N}$, then $\Phi_{(h,A)} = h$;

(ii) For each $\widetilde{\Phi} \in \mathcal{B}(A, \widetilde{A})$, there exists a unique $h \in H$ such that $\widetilde{A} = h.A$ and $\widetilde{\Phi} = \Phi_{(h,A)}$.

Therefore, formally holomorphic equivalence classes of $\mathcal{N}$ are realized as H-orbits of $\mathcal{N}$. Consequently, the transformation law (1.2) is equivalent to

$$P(h.A) = P(A)|\det h'(0)|^{-2w/3} \quad \text{for any} \quad h \in H. \tag{4.1}$$

It is convenient to rewrite the transformation law (4.1) by the Lie group H in terms of the Lie algebra of H. In fact, we have:

Lemma 4.2. *A real polynomial $P(A)$ of biweight (w, w) is a CR invariant of weight w if and only if*

$$\frac{d}{dt} P(\psi_{t\xi}.A)\bigg|_{t=0} = 0 \quad \textit{for any} \quad \xi \in \mathbb{C}, \tag{4.2}$$

where $\psi_\xi(z_1, z_2) = (z_1 - \xi\, z_2, z_2)/(1 - \bar{\xi}\, z_1 + |\xi|^2 z_2/2)$.

Proof. Setting $H_0 = \{\psi \in H; \det \psi'(0) = 1\}$, we observe that every $h \in H$ admits a unique decomposition $h = \phi_\lambda \circ \psi$ with $\lambda \in \mathbb{C}^*$ and $\psi \in H_0$. Let us recall that H_0 is isomorphic to the Heisenberg group and that each element of $\psi \in H_0$ takes the form

$$\psi(z_1, z_2) = \frac{(z_1 - \xi\, z_2, z_2)}{1 - \bar{\xi}\, z_1 + \eta\, z_2}, \quad \text{where} \quad \eta = \frac{1}{2}|\xi|^2 + i\, r, \ (\xi, r) \in \mathbb{C} \times \mathbb{R}.$$

Recalling that the Heisenberg group is generated by elements of the form $(\xi, r) = (\xi, 0)$, we see that H is generated by

$$\phi_\lambda, \ \psi_\xi \quad \text{with} \quad (\lambda, \xi) \in \mathbb{C}^* \times \mathbb{C}.$$

Hence, the transformation law (4.1) can be written as follows:

$$P(\phi_\lambda.A) = |\lambda|^{-2w} P(A) \quad \text{for} \quad \lambda \in \mathbb{C}^*, \tag{4.1a}$$

$$P(\psi_\xi.A) = P(A) \qquad \text{for} \quad \xi \in \mathbb{C}. \tag{4.1b}$$

The first condition (4.1a) says by definition that $P(A)$ is of biweight (w, w). The second condition (4.1b) is equivalent to

$$\frac{d}{dt} P(\psi_{t\xi}.A) = 0 \quad \text{for} \quad (t, \xi) \in \mathbb{R} \times \mathbb{C}, \tag{4.1b$'$}$$

because $\psi_0(z_1, z_2) = (z_1, z_2)$. Noting that $\psi_{t\xi} \circ \psi_{t'\xi} = \psi_{(t+t')\xi}$ for any $t, t' \in \mathbb{R}$, we see that (4.1b)$'$ is equivalent to (4.2). $\square$

4.3. End of the proof of Proposition 1. Let us compute the left side of (4.2) for polynomials $P(A) = P_w(A)$ in Lemma 4.1.

Lemma 4.3. *With $\xi \in \mathbb{C}$ arbitrarily fixed, let $A(t) = \psi_{t\,\xi}.A$. Then,*

$$\frac{d}{dt} A^0_{4\bar4}(t)\Big|_{t=0} = 0, \tag{1°}$$

$$\frac{d}{dt} F_{abc}(A(t))\Big|_{t=0} = \mathrm{Re}\Big[\overline{\xi}(c_1\, A^0_{4\bar5} + c_2\, A^1_{3\bar4})\Big], \tag{2°}$$

$$\begin{aligned}\frac{d}{dt} F_{abcde\alpha\beta}(A(t))\Big|_{t=0} &= \mathrm{Re}\Big[(c_3\,\overline{\xi}\, A^0_{2\bar5} + c_4\,\xi\, A^0_{3\bar4}) A^0_{4\bar2}\Big] \\ &\quad + \mathrm{Re}\Big[\overline{\xi}\,(c_5\, A^0_{5\bar6} + c_6\, A^1_{4\bar5} + c_7\, A^2_{3\bar4})\Big],\end{aligned} \tag{3°}$$

where c_2 (resp. c_7) is a linear combination of b, c (resp. c, d and e), and

$$c_1 = -b - 10i\, c, \quad c_3 = -2a + \frac{3}{2}\,\alpha - 5i\,\beta,$$
$$c_4 = \frac{2}{3} b - \frac{3}{2}\,\alpha + 3i\,\beta, \quad c_5 = -2c - 12i\, d, \quad c_6 = \frac{i}{2}\, c - 2d - 10i\, e.$$

Proof. It is proved by Graham in [12], Lemma 2.8, that

$$\begin{aligned}
&A^0_{2\bar4}(t) = A^0_{2\bar4}, \qquad && A^0_{2\bar5}(t) = A^0_{2\bar5} - \xi\, t\, A^0_{2\bar4},\\
&A^0_{4\bar4}(t) = A^0_{4\bar4}, && A^0_{3\bar4}(t) = A^0_{3\bar4} + \frac{1}{3}\,\overline{\xi}\, t\, A^0_{2\bar4},\\
&A^0_{3\bar5}(t) = A^0_{3\bar5} + \frac{3}{2}\,\overline{\xi}\, t\, A^0_{2\bar5} - \frac{3}{2}\,\xi\, t\, A^0_{3\bar4} - |\xi\, t|^2 A^0_{2\bar4}, &&\\
&A^1_{2\bar4}(t) = A^1_{2\bar4} - 5i\,\overline{\xi}\, t\, A^0_{2\bar5} + 3i\,\xi\, t\, A^0_{3\bar4} + 3i\,|\xi\, t|^2 A^0_{2\bar4}. &&
\end{aligned}$$

Then $(1)^\circ$ is obvious and $|A^0_{4\bar2}(t)|^2$ does not contribute to (2°). The derivative at $t = 0$ of the nonlinear part of $F_{abcde\alpha\beta}(A(t))$ in $A(t)$ is given by $\mathrm{Re}\Big[(c_3\,\overline{\xi} A^0_{2\bar5} + c_4\,\xi A^0_{3\bar4}) A^0_{4\bar2}\Big]$. It remains to consider the linear parts of $F_{abc}(A)$ and $F_{abcde\alpha\beta}(A)$, which we denote by $Q_4(A)$ and $Q_5(A)$, respectively. That is,

$$Q_4(A) = b\, A^0_{5\bar5} + c\, A^1_{4\bar4}, \quad Q_5(A) = c\, A^0_{6\bar6} + d\, A^1_{5\bar5} + e\, A^2_{4\bar4}.$$

We are concerned with $Q_4(\dot A)$ and $Q_5(\dot A)$, where $\dot A = (\dot A^\ell_{p\bar q})$ is defined by $\dot A^\ell_{p\bar q} = d\, A^\ell_{p\bar q}(t)/dt\big|_{t=0}$. It is elementary to verify $\dot A^\ell_{q\bar p} = \overline{\dot A^\ell_{p\bar q}}$ and that each $\dot A^\ell_{p\bar q}$ is a polynomial in $A, \xi, \overline{\xi}$ of homogeneous biweight $(p+\ell-1, q+\ell-1)$ if biweight for ξ, $\overline{\xi}$ is defined by

$$\mathrm{w}_2(\xi) = (0, 1), \quad \mathrm{w}_2(\overline{\xi}) = (1, 0).$$

It then follows from the trace conditions on Moser's normal form that

$$\begin{aligned}
\dot A^0_{5\overline{5}} &= \mathrm{Re}\Big[\overline{\xi}\,(c_{11}A^0_{4\overline{5}} + c_{21}A^1_{3\overline{4}})\Big],\\
\dot A^1_{4\overline{4}} &= \mathrm{Re}\Big[\overline{\xi}\,(c_{12}A^0_{4\overline{5}} + c_{22}A^1_{3\overline{4}})\Big],\\
\dot A^0_{6\overline{6}} &= \mathrm{Re}\Big[\overline{\xi}\,(c_{51}A^0_{5\overline{6}} + c_{61}A^1_{4\overline{5}} + c_{71}A^2_{3\overline{4}})\Big],\\
\dot A^1_{5\overline{5}} &= \mathrm{Re}\Big[\overline{\xi}\,(c_{52}A^0_{5\overline{6}} + c_{62}A^1_{4\overline{5}} + c_{72}A^2_{3\overline{4}})\Big],\\
\dot A^2_{4\overline{4}} &= \mathrm{Re}\Big[\overline{\xi}\,(c_{53}A^0_{5\overline{6}} + c_{63}A^1_{4\overline{5}} + c_{73}A^2_{3\overline{4}})\Big]
\end{aligned}$$

with absolute constants $c_{11}, \dots, c_{73}$. We need to determine c_{11}, c_{12} and c_{5j}, c_{6j} for $j = 1, 2, 3$. Let us recall another result of Graham in [12], Lemma 2.4, a); it is shown that if a surface in Moser's normal form is defined by

$$N(A) = \Big\{2u = |z_1|^2 + 2\,\mathrm{Re}\,(z_1^k \overline{z}_1^{k+1} v^\ell)\Big\} \quad \text{with} \quad k \geq 4,\ \ell \geq 0,$$

then $N(\psi_{t\xi}.A)$ is given by

$$\begin{aligned}
2u = |z_1|^2 + 2\,\mathrm{Re}\Big[&z_1^k \overline{z}_1^{k+1} v^\ell + \frac{i}{4}\ell\,(t\overline{\xi}\,|z_1|^{2k+4} - t\xi z_1^{k+1}\overline{z}_1^{k+3}) v^{\ell-1}\\
&- \frac{1}{2}(k+\ell-3)t\overline{\xi}\,|z_1|^{2k+2} v^\ell - (k+\ell)t\xi z_1^k \overline{z}_1^{k+2} v^\ell\\
&- i(k+1)t\overline{\xi}\,|z_1|^{2k} v^{\ell+1} + i\,k\,t\xi\, z_1^{k-1}\overline{z}_1^{k+1} v^{\ell+1}\Big] + \cdots,
\end{aligned}$$

where $\cdots$ stands for terms of weight $< -k-\ell-1$. In other words, if we start from $A \in \mathcal{N}$ such that all $A^\ell_{p\overline{q}} = 0$ except for $A^\ell_{k\,\overline{k+1}} = 1$ with $k \geq 4, \ell \geq 0$, then

$$\begin{aligned}
&\dot A^{\ell-1}_{k+2\,\overline{k+2}} = \mathrm{Re}\Big[\frac{i}{2}\,\ell\,\overline{\xi}\Big], \quad \dot A^{\ell}_{k+1\,\overline{k+1}} = \mathrm{Re}\Big[-(k+\ell-3)\overline{\xi}\Big],\\
&\dot A^{\ell+1}_{k\,\overline{k}} = \mathrm{Re}\Big[-2i\,(k+1)\,\overline{\xi}\Big], \quad \dots.
\end{aligned}$$

Using this result for $(k,\ell) = (4,0)$, $(5,0)$, $(4,1)$ we get

$$\begin{aligned}
&c_{11} = -1, \quad c_{12} = -10\,i,\\
&c_{51} = -2, \quad c_{52} = -12\,i, \quad c_{53} = 0,\\
&c_{61} = \frac{i}{2}, \quad c_{62} = -2, \quad c_{63} = -10\,i.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Q_4(\dot A) &= \mathrm{Re}\Big[\overline{\xi}(c_1\,A^0_{4\overline{5}} + c_2\,A^1_{3\overline{4}})\Big],\\
Q_5(\dot A) &= \mathrm{Re}\Big[\overline{\xi}\,(c_5\,A^0_{5\overline{6}} + c_6\,A^1_{4\overline{5}} + c_7\,A^2_{3\overline{4}})\Big],
\end{aligned}$$

as desired. $\square$

Proof of Proposition 1. For $w \leq 5$, a CR invariant of weight w is exactly the polynomial $P_w(A)$ in Lemma 4.1 which satisfies the condition (4.2) in Lemma 4.2. Note that the case $w \leq 2$ is trivial. Using Lemma 4.3, $(1^\circ), (2^\circ)$ and (3°), we consider the case $3 \leq w \leq 5$. If $w = 3$, then by (1°) the condition (4.2) for $P_w(A)$ is automatically satisfied. If $w = 4$, then (4.2) holds if and only if $c_1 = c_2 = 0$ in (2°), a condition which is equivalent to $b = c = 0$. Let $w = 5$, and thus (4.2) holds if and only if

$$c_j = 0 \quad \text{for} \;\; 3 \leq j \leq 7 \quad \text{in } (3^\circ). \tag{4.3}$$

Note that the condition $c_5 = c_6 = 0$ is equivalent to $c = d = e = 0$, in which case we have $c_7 = 0$. Consequently, (4.3) holds if and only if $c_3 = c_4 = 0$ and $c = d = e = 0$. Solving the equations $c_3 = c_4 = 0$, we get, as desired,

$$\alpha = -2a + \frac{10}{9}\, b, \qquad \beta = i\, a - \frac{i}{3}\, b.$$

5. Fefferman's defining functions and Graham's asymptotic solutions

In this section, we prove Proposition 3, (2°) and Proposition 5.

Let us recall that the point of Proposition 3, (2°) is the polynomial dependence of r^{F} and η_1^{G} on Moser's normal form coefficients. We thus reformulate in Subsection 5.1 Graham's construction of asymptotic solutions in (1.6) of the complex Monge-Ampère boundary value problem (1.5), in such a way that the polynomial dependence is obvious.

Once Proposition 3, (2°) is established, the proof of Proposition 5 is reduced to identifying the universal constants $c_{pq}[\eta_1^{\mathrm{G}}]$ for $(p, q) = (4, 2), (5, 2)$ and $(4, 3)$, by virtue of the following lemma.

Lemma 5.1. *Let $\Omega = \Omega_{pq}$ with $p + q \leq 7$ and $p > q$. If $K \in I_w(6 - w)$ with $w \leq 5$, then there exists a constant c_{pq} such that*

$$K_\Omega(\gamma_t) = c_{pq} |A^0_{p\overline{q}}|^2\, t^{p+q-5} + O(t^{6-w}).$$

Proof. Let us write $K_{\partial\Omega}$ for K_Ω. By definition, we have the expansion

$$K_{N(A)}(\gamma_t) = \sum_{m=0}^{5-w} P_m(A)\, t^m + O(t^{6-w}),$$

where each coefficient $P_m(A)$ is of biweight $(w{+}m, w{+}m)$. For the surface $\partial\Omega_{pq} = N(A)$, Lemma 4.1 yields $P_{p+q-2}(A) = c_{pq} |A^0_{p\overline{q}}|^2$ and $P_m(A) = 0$ for $m \neq p + q - 2$. $\square$

Remark 5.1. For Fefferman's defining function r^{F}, we have

$$r^{\mathrm{F}}(\gamma_t) = t + O(t^4).$$

In fact, if we consider the expansion $r^{\mathrm{F}}(\gamma_t) = \sum_{m=1}^{3} P_m(A)\, t^m + O(t^4)$, then each coefficient $P_m(A)$ is of biweight $(m-1, m-1)$, so that Lemma 3.1 implies $P_2 = P_3 = 0$. Note also that P_1 is a universal constant independent of A, and the value $P_1(A) = 1$ is determined by using the Siegel domain $2u > |z_1|^2$.

To determine $c_{pq}[\eta_1^{\mathrm{G}}]$, we consider in Subsection 5.3 an asymptotic expansion with respect to Moser's normal form coefficients A. Namely, in addition to the filtration with respect to the vanishing order on the boundary, we also consider a filtration relative to the degree of polynomials in A. This enables us to do explicit computations in Subsection 5.4.

5.1. Polynomial dependence. As in Section 3, we set

$$\mathcal{F}_A = \mathbb{C}[A][[z,\overline{z}]] = \mathbb{C}[A][[z_1,\overline{z}_1,U,v]] \quad \text{and} \quad \mathcal{F}'_A = \mathbb{C}[A][[z_1,\overline{z}_1,v]].$$

Let $\mathcal{B}$ denote the ring of formal series of the form

$$f = \sum_{k=0}^{\infty} \eta_k \cdot (U^3 \log U)^k \quad \text{with} \quad \eta_k = \eta_k[f] \in \mathcal{F}_A. \tag{5.1}$$

We now construct formal solutions of (1.4) in the form $u = UV_0(1+f)$ with $f \in \mathcal{B}$, where $V_0 = J[U]^{-1/3}$. Recall by Fefferman's construction of r^{F} in [9] that $J[UV_0] = 1 + O(U)$, so that

$$J[UV_0(1+\eta_0)] = (1+\eta_0)^3 + O(U).$$

We thus require $\eta_0 = O(U)$, that is, $f|_{U=0} = 0$. Then the condition $u > 0$ in Ω is formally satisfied, and (1.4) is formally written as follows:

$$\mathcal{M}[f] = 1, \quad \text{where} \quad \mathcal{M}[f] = J[UV_0(1+f)], \quad V_0 = J[U]^{-1/3}. \tag{5.2}$$

We now have, as a refinement of [12], Theorem 2.11, the following result.

Proposition 7. *For every $a \in \mathbb{C}[[z_1,\overline{z}_1,v]]$, there exists a unique solution $f = f[a] \in \mathcal{B}$ of (5.2) such that $\eta_0 = \eta_0[f] \in \mathcal{F}_A$ satisfies*

$$\partial_U^3 \eta_0|_{U=0} = 3!\, a \quad \textit{and} \quad \eta_0 = O(U). \tag{5.3}$$

Furthermore, $f = f[a]$ depends polynomially on $A \in \mathcal{N}$ and the coefficients of the series a, that is,

$$\eta_k \in \mathbb{C}[A, C^0][[z_1,\overline{z}_1,U,v]] \quad \textit{for} \quad a = \sum C^{\ell}_{p\overline{q}} z_1^p \overline{z}_1^q v^\ell \ \textit{with}\ C^0 = (C^{\ell}_{p\overline{q}}).$$

Before proving Proposition 7 above, let us observe that this implies the polynomial dependence of r^{F} and η_1^{G} on $A \in \mathcal{N}$.

Proof of Proposition 3, (2°). We first note that u^{G} is uniquely determined by specifying $a \in \mathbb{C}[[z_1, \overline{z}_1, v]]$. Then (5.2) yields $u^{\mathrm{G}} = UV_0(1+f)$ with $f = f[a]$ in Proposition 7. Comparing the smooth part of both sides, we get

$$r^{\mathrm{F}} = UV_0(1 + \eta_0[f]) + O(U^4),$$

which, together with Proposition 2, implies $r^{\mathrm{F}} \in I_{-1}(4)$. Comparing next the coefficients of $\log r^{\mathrm{F}}$ in the expansion with respect to r^{F}, we obtain

$$\eta_1^{\mathrm{G}} = \eta_1[f] V_0 (U/r^{\mathrm{F}})^4 = \eta_1[f] V_0^{-3} (1 + \eta_0[f])^{-4} + O(U^3).$$

Since $\eta_1[f] V_0^{-3}(1+\eta_0[f])^{-4} \in \mathcal{F}_A$, Proposition 2 implies $\eta_1^{\mathrm{G}} \in I_3(3)$. $\square$

In order to prove Proposition 7 above, we begin by determining the linear part of the operator $\mathcal{M}$ in (5.2). We set

$$\mathcal{D}_A = \mathcal{F}_A[\partial_{z_1}, \partial_{\overline{z}_1}, U\partial_U, \partial_v],$$

the ring of linear differential operators generated by ∂_{z_1}, $\partial_{\overline{z}_1}$, $U\partial_U$, ∂_v with coefficients in $\mathcal{F}_A$; thus $\mathcal{D}_A$ acts on $\mathcal{B}$. We then have:

Proposition 8. *Let $V \in \mathcal{F}_A$ satisfy $V|_{A=0} = 1$. Then there exists an operator $\mathcal{L} \in \mathcal{D}_A$ acting on $\mathcal{B}$ such that*

$$J[UV(1+\varphi)] = J[UV](1 - \mathcal{L}\varphi) + \Psi(P_1\varphi, \ldots, P_\ell\varphi), \tag{5.4}$$

where $P_1, \ldots, P_\ell \in \mathcal{D}_A$ and $\Psi(x) \in \mathbb{C}[x]$ with $x = (x_1, \ldots, x_\ell)$ satisfies $\Psi(x) = O(|x|^2)$. The operator $\mathcal{L}$ takes the form

$$\mathcal{L} = I(U\partial_U) + UP_0 \quad \text{with} \quad I(\tau) = (\tau+1)(\tau-3),$$

where $P_0 \in \mathcal{D}_A$ satisfies $P_0|_{A=0} = Q_0$ with

$$Q_0 = \partial_{z_1}\partial_{\overline{z}_1} - \frac{i}{2} z_1 \partial_{z_1}\partial_v + \frac{i}{2}\overline{z}_1 \partial_{\overline{z}_1}\partial_v + \frac{1}{4}(U + |z_1|^2)\partial_v^2.$$

Remark 5.2. A similar result holds in the n dimensional case. In fact, the formula (5.4) is valid with $I(\tau) = (\tau+1)(\tau-n-1)$ and

$$Q_0 = \sum_{j=1}^{n-1} \left(\partial_{z_j}\partial_{\overline{z}_j} - \frac{i}{2} z_j \partial_{z_j}\partial_v + \frac{i}{2}\overline{z}_j \partial_{\overline{z}_j}\partial_v + \frac{1}{4}|z_j|^2 \partial_v^2 \right) + \frac{1}{4} U \partial_v^2,$$

where the partial derivatives are taken with respect to the real coordinates $z_1, \ldots, z_{n-1}, \overline{z}_1, \ldots, \overline{z}_{n-1}, U, v$.

Postponing the proof of Proposition 8 until the next subsection, let us continue the argument of proving Proposition 7. We use Proposition 8 with $V = V_0 = J[U]^{-1/3}$. Then

$$\mathcal{M}[f] = J[UV_0](1 - \mathcal{L}f) + \Psi(P_1 f, \ldots, P_\ell f). \tag{5.5}$$

Note that $\mathcal{M}[f]$ is a polynomial in Pf, $P \in \mathcal{D}_A$. Thus, $\mathcal{M}$ consists of (nonlinear) totally characteristic operators in the sense of [17].

It is convenient to introduce a filtration $\mathcal{B} = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \cdots$ by setting

$$\mathcal{B}_k = \sum_{j=k}^{\infty} \mathbf{B}_j, \quad \text{where } \mathbf{B}_j = \Big\{ U^j \sum_{k=0}^{[j/3]} \eta_{j,k} \cdot (\log U)^k ; \eta_{j,k} \in \mathcal{F}'_A \Big\}.$$

Then $f \in \mathcal{B}_1$ if and only if $\eta_0[f] = O(U)$. Consequently, the condition (5.3) is equivalent to $f \in \mathcal{B}(a) \cap \mathcal{B}_1$, where

$$\mathcal{B}(a) = \left\{ f \in \mathcal{B}; \partial_U^3 \eta_0[f]|_{U=0} = 3!\, a \right\}.$$

We shall construct a solution $f \in \mathcal{B}(a) \cap \mathcal{B}_1$ of (5.2) in the form

$$f = \lim_{k\to\infty} f_k \quad \text{with } f_k = \sum_{j=1}^{k} \lambda_j \quad \text{and } \lambda_j \in \mathbf{B}_j, \tag{5.6}$$

where we require $\lambda_3 \in \mathbf{B}_3(a) := \mathbf{B}_3 \cap \mathcal{B}(a)$. Observe that this limit makes sense as a formal series $f = \sum_{j\geq 1} \lambda_j$. To define f_j for $j \geq 1$ successively, we first linearize the operator $\mathcal{M}$ at f_{j-1}.

Lemma 5.2. *If $f \in \mathcal{B}_1$ is given as in* (5.6)*, then*

$$\mathcal{M}[f_m] = \mathcal{M}[f_{j-1}] - I(U\partial_U)\lambda_j \quad \text{mod } \mathcal{B}_{j+1} \tag{5.7}$$

for $1 \leq j \leq m \leq \infty$, where $f_0 = 0$ and $f_\infty = f$.

Proof. Noting that $P\lambda_j \in \mathcal{B}_j$ for $P \in \mathcal{D}_A$, we have

$$\mathcal{L}f_m = \mathcal{L}f_{j-1} + I(U\partial_U)\lambda_j \quad \text{mod } \mathcal{B}_{j+1}.$$

On the other hand, since $\mathcal{B}_j \cdot \mathcal{B}_k \subset \mathcal{B}_{j+k}$, it follows from the assumption $f \in \mathcal{B}_1$ that

$$\Psi(P_1 f_m, \ldots, P_\ell f_m) = \Psi(P_1 f_{j-1}, \ldots, P_\ell f_{j-1}) \quad \text{mod } \mathcal{B}_{j+1}.$$

Hence (5.5) implies

$$\mathcal{M}[f_m] - \mathcal{M}[f_{j-1}] = -J[UV_0]\, I(U\partial_U)\lambda_j \quad \text{mod } \mathcal{B}_{j+1}.$$

Recalling that $J[UV_0] = 1$ mod $\mathcal{B}_1$, we obtain (5.7). ◻

We next solve the linear equations for $\lambda_j \in \mathbf{B}_j$ ($j \geq 1$):

$$I(U\partial_U)\lambda_j = \mu_j \in \mathbf{B}_j. \tag{5.8$_j$}$$

Lemma 5.3. *If $j \neq 3$, then $(5.8)_j$ always has a unique solution. For $j = 3$, the equation $(5.8)_3$ has a solution $\lambda_3 \in \mathbf{B}_3$ if and only if μ_3 does not contain $\log U$, that is, $\mu_3 \in \mathcal{F}_A$. Furthermore, the solution is unique under the restriction $\lambda_3 \in \mathbf{B}_3(a)$, for each $a \in \mathbb{C}[[z_1, \overline{z}_1, v]]$ prescribed.*

Proof. For $\lambda_j, \mu_j \in \mathbf{B}_j$, we write

$$\lambda_j = \sum_{k=0}^{[j/3]} \lambda_{j,k} U^j (\log U)^k, \quad \mu_j = \sum_{k=0}^{[j/3]} \mu_{j,k} U^j (\log U)^k$$

with $\lambda_{j,k}, \mu_{j,k} \in \mathcal{F}'_A$. Then, $(5.8)_j$ holds if and only if

$$I(j)\lambda_{j,k} + (k+1)I'(j)\lambda_{j,k+1} + (k+2)(k+1)\lambda_{j,k+2} = \mu_{j,k}. \tag{5.9$_j$}$$

Notice that $\lambda_{j,k} = 0$ for $k > [j/3]$. If $j \neq 3$, then $I(j) \neq 0$, and thus $(5.9)_j$ uniquely determines $\lambda_{j,k}$ for all $k \geq 0$. Consequently, $(5.8)_j$ for $j \neq 3$ always has a unique solution. For $j = 3$, we note that $I(3) = 0$ and $\lambda_{3,k} = 0$ for $k > 1$. Thus $(5.9)_3$ is equivalent to

$$4\lambda_{3,1} = \mu_{3,0}, \quad \mu_{3,k} = 0 \quad \text{for} \quad k \geq 1.$$

Therefore, the equation $(5.8)_3$ has a solution if and only if $\mu_{3,k} = 0$ for $k \geq 1$, and the solution is in general of the form $\lambda_3 = U^3(\lambda_{3,0} + \lambda_{3,1} \log U)$ with $\lambda_{3,1} = \mu_{3,0}/4$. Hence, the solution becomes unique by specifying $\lambda_{3,0} = a$. $\square$

Using Lemmas 5.2 and 5.3, we can prove Proposition 7 as follows.

Proof of Proposition 7. Suppose at first we are given $f \in \mathcal{B}(a) \cap \mathcal{B}_1$ arbitrarily, and let f_j, λ_k for $j \geq 0$, $k \geq 1$ be defined by (5.6), where $f_0 = 0$. It then follows from Lemma 5.2 that (5.2) is equivalent to

$$\mathcal{M}[f_{j-1}] = 1 \mod \mathcal{B}_j \quad \text{for all} \;\; j \geq 1, \tag{5.10}$$

which holds if and only if $(5.8)_j$ is valid for each $j \geq 1$, where μ_j denotes the $\mathbf{B}_j$ component of $\mathcal{M}[f_{j-1}]$. In this case, we must have by Lemma 5.3 that $\mu_3 \in \mathcal{F}_A \cap \mathbf{B}_3$, which together with the condition $f \in \mathcal{B}(a) \cap \mathcal{B}_1$ implies $\lambda_3 \in \mathbf{B}_3(a)$ and then the uniqueness of the solution f of (5.2) in $\mathcal{B}(a) \cap \mathcal{B}_1$. Furthermore, Lemma 5.3 with Lemma 5.2 permits us to construct f_j for $j \geq 1$ as in (5.6) successively so that (5.10) holds, where $f_0 = 0$. In fact, the condition $\mu_3 \in \mathcal{F}_A \cap \mathbf{B}_3$ is satisfied because

$$f_2 = \lambda_1 + \lambda_2 \in \mathbf{B}_1 + \mathbf{B}_2 \subset \mathcal{F}_A$$

and thus $\mathcal{M}[f_2] \in \mathcal{F}_A$. Therefore, (5.2) has a unique solution $f \in \mathcal{B}(a) \cap \mathcal{B}_1$, while, as we have remarked before, the condition $f \in \mathcal{B}(a) \cap \mathcal{B}_1$ is equivalent to (5.3). $\square$

5.2. Proof of Proposition 8. We consider the n dimensional case as in Remark 5.2. For $\mathcal{F}_A$ and $\mathcal{D}_A$ as in the previous subsection, we set

$$\mathcal{F}_A^1 = \{\varphi \in \mathcal{F}_A;\ \varphi|_{A=0} = 0\}, \quad \mathcal{D}_A^1 = \{P \in \mathcal{D}_A;\ P|_{A=0} = 0\}.$$

Let us recall that our coordinates z_α, $\overline{z}_\beta$ $(\alpha, \beta = 1, \dots, n-1)$, U, v are obtained by a change of variables from the standard ones z_j, $\overline{z}_k$ $(j, k = 1, \dots, n)$ in $\mathbb{C}^n$. Denoting by D_j, $D_{\overline{k}}$ the partial derivatives with respect to the original coordinates z_j, $\overline{z}_k$, respectively, we also recall that

$$J[U\Phi] = (-1)^n \det(H_0\Phi), \qquad H_0\Phi = \begin{pmatrix} U & D_{\overline{k}}U \\ D_jU & D_jD_{\overline{k}}U \end{pmatrix} \Phi,$$

where U acts as a multiplication operator. We introduce artificial notation

$$\partial_n = -\frac{i}{2}\frac{\partial}{\partial v}, \quad \partial_{\overline{n}} = \frac{i}{2}\frac{\partial}{\partial v} \quad \text{(and thus } \partial_n + \partial_{\overline{n}} = 0\text{)},$$

together with usual abbreviation $\partial_\alpha = \partial/\partial z_\alpha$, $\partial_{\overline{\beta}} = \partial/\partial \overline{z}_\beta$, $\partial_U = \partial/\partial U$. Writing $F_j = \partial_j F$, $F_{\overline{k}} = \partial_{\overline{k}} F$ and $F_{j\overline{k}} = \partial_j\partial_{\overline{k}}F$, we also set

$$a_\alpha = -\overline{z}_\alpha - F_\alpha, \quad a_{\overline{\beta}} = -z_\beta - F_{\overline{\beta}}, \quad a_n = 1 - F_n, \quad a_{\overline{n}} = 1 - F_{\overline{n}}.$$

Then,

$$\begin{gathered} D_j = \partial_j + a_j\partial_U, \quad D_{\overline{k}} = \partial_{\overline{k}} + a_{\overline{k}}\partial_U, \\ D_jD_{\overline{k}} = \partial_j\partial_{\overline{k}} + X_{j\overline{k}}\,\partial_U + a_ja_{\overline{k}}\,\partial_U^2, \end{gathered}$$

where $X_{j\overline{k}} = a_j\partial_{\overline{k}} + a_{\overline{k}}\partial_j - \delta'_{j\overline{k}} - F_{j\overline{k}}$ with

$$\delta'_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}, \quad \delta'_{j\overline{n}} = \delta'_{n\overline{k}} = 0.$$

Setting $b_\alpha = a_\alpha/a_n$ and $b_{\overline{\beta}} = a_{\overline{\beta}}/a_{\overline{n}}$, we apply elementary operations on $H_0\Phi$. First, we subtract the last row multiplied by b_α from the $\alpha+1$st one, and then the last column multiplied by $b_{\overline{\beta}}$ from the $\beta+1$st one. Next, we divide the first row by U and multiply the last column by U. Let $H_1\Phi$ denote the resulting matrix. Then,

$$H_1\Phi = \begin{pmatrix} 1 & U^{-1}(D_{\overline{\beta}} - b_{\overline{\beta}}D_{\overline{n}})U & D_{\overline{n}}U \\ (D_\alpha - b_\alpha D_n)U & h_{\alpha\overline{\beta}} & U(D_\alpha - b_\alpha D_n)D_{\overline{n}}U \\ D_nU & (D_{\overline{\beta}} - b_{\overline{\beta}}D_{\overline{n}})D_nU & UD_nD_{\overline{n}}U \end{pmatrix} \Phi,$$

with

$$h_{\alpha\overline{\beta}} = \Big(D_\alpha D_{\overline{\beta}} - b_\alpha D_nD_{\overline{\beta}} - b_{\overline{\beta}}D_\alpha D_{\overline{n}} + b_\alpha b_{\overline{\beta}}D_nD_{\overline{n}}\Big)\,U.$$

We set $d_{\alpha\overline{\beta}} = -(X_{\alpha\overline{\beta}} - b_\alpha X_{n\overline{\beta}} - b_{\overline{\beta}} X_{\alpha\overline{n}} + b_\alpha b_{\overline{\beta}} X_{n\overline{n}})$ and

$$
\begin{aligned}
Y_\alpha = \partial_\alpha - b_\alpha \partial_n, \quad Y_{\overline{\beta}} = \partial_{\overline{\beta}} - b_{\overline{\beta}} \partial_{\overline{n}}, \\
Y_{\alpha\overline{\beta}} = \partial_\alpha \partial_{\overline{\beta}} - b_\alpha \partial_n \partial_{\overline{\beta}} - b_{\overline{\beta}} \partial_\alpha \partial_{\overline{n}} + b_\alpha b_{\overline{\beta}} \partial_n \partial_{\overline{n}}.
\end{aligned}
$$

Then $Y_\alpha = D_\alpha - b_\alpha D_n$, $Y_{\overline{\beta}} = D_{\overline{\beta}} - b_{\overline{\beta}} D_{\overline{n}}$, and each $d_{\alpha\overline{\beta}}$ is a multiplication operator given by the function

$$
d_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} + Y_{\alpha\overline{\beta}} F.
$$

Consequently,

$$
H_1 \Phi = \begin{pmatrix} 1 & Y_{\overline{\beta}} & D_{\overline{n}} U \\ U Y_\alpha & U Y_{\alpha\overline{\beta}} - d_{\alpha\overline{\beta}} \partial_U U & U Y_\alpha D_{\overline{n}} U \\ D_n U & Y_{\overline{\beta}} D_n U & U D_n D_{\overline{n}} U \end{pmatrix} \Phi.
$$

Setting $\Phi = V(1+\varphi)$, we write $H_1 \Phi = H_2 + H_3 \varphi$, where $H_2 = H_1 V$ and $H_3 \varphi = H_1(V\varphi)$. Since $V - 1 \in \mathcal{F}_A^1$, it follows that H_2 takes the form

$$
H_2 = \begin{pmatrix} V & Y_{\overline{\beta}} V & a_{\overline{n}} V \\ 0 & -d_{\alpha\overline{\beta}} V & 0 \\ a_n V & Y_{\overline{\beta}} V & 0 \end{pmatrix} + U\mathcal{F}_A^1,
$$

where $U\mathcal{F}_A^1$ stands for a matrix of which each entry belongs to $U\mathcal{F}_A^1$. Noting that $a_n - 1$, $a_{\overline{n}} - 1$ and $d_{\alpha\overline{\beta}} - \delta_{\alpha\overline{\beta}}$ all belong to $\mathcal{F}_A^1$, we have

$$
H_2^{-1} = \frac{1}{V} \begin{pmatrix} 0 & \mathcal{F}_A^1 & 1/a_n \\ 0 & -d^{-1}_{\alpha\overline{\beta}} & 0 \\ 1/a_{\overline{n}} & \mathcal{F}_A^1 & -|a_n|^{-2} \end{pmatrix} + U\mathcal{F}_A^1,
$$

where $\left(d^{-1}_{\alpha\overline{\beta}}\right)$ is the inverse matrix of $\left(d_{\alpha\overline{\beta}}\right)$. Similarly,

$$
H_3 = V \begin{pmatrix} 1 & \mathcal{D}_A & D_{\overline{n}} U \\ U\mathcal{D}_A & U Y_{\alpha\overline{\beta}} - d_{\alpha\overline{\beta}} \partial_U U & U\mathcal{D}_A \\ D_n U & \mathcal{D}_A & U D_n D_{\overline{n}} U \end{pmatrix} + U\mathcal{D}_A^1,
$$

where $U\mathcal{D}_A^1$ stands for a matrix valued operator of which each entry belongs to $U\mathcal{D}_A^1$. Recalling $J[U\Phi] = (-1)^n \det(H_1 \Phi)$ and $J[UV] = (-1)^n \det H_2$, we have

$$
\begin{aligned}
J[UV(1+\varphi)] &= J[UV] \cdot \det(1 + H_2^{-1} H_3 \varphi) \\
&= J[UV] \cdot \bigl(1 + \operatorname{tr}(H_2^{-1} H_3)\varphi\bigr) + \Psi(P_1\varphi, \dots, P_\ell \varphi),
\end{aligned}
$$

where $\mathrm{tr}(H_2^{-1}H_3)$ denotes the trace of the matrix valued operator $H_2^{-1}H_3$. Thus we get (5.4) with

$$\mathcal{L} = -\mathrm{tr}\left(H_2^{-1}H_3\right).$$

We have

$$\begin{aligned}\mathcal{L} = -a_n^{-1}D_nU - a_{\overline{n}}^{-1}D_{\overline{n}}U + |a_n|^{-2}UD_nD_{\overline{n}}U \\ + U\sum_{\alpha,\beta=1}^{n-1} d_{\beta\overline{\alpha}}^{-1}Y_{\alpha\overline{\beta}} - (n-1)\partial_U U + U\mathcal{D}_A^1.\end{aligned}$$

Let us note that $I(U\partial_U) = \partial_U U(\partial_U U - n - 2)$. Using $\partial_n + \partial_{\overline{n}} = 0$ and $d_{\alpha\overline{\beta}}^{-1} - \delta_{\alpha\overline{\beta}} \in \mathcal{F}_A^1$, we get

$$\mathcal{L} - I(U\partial_U) = U^2\partial_n\partial_{\overline{n}} + U\sum_{\alpha=1}^{n-1} Y_{\alpha\overline{\alpha}} + U\mathcal{D}_A^1.$$

Writing the right side as UP_0, we obtain the desired result.

Remark 5.3. Let us consider the special case in which the function F is independent of the variable v. Then we have a subclass $\mathcal{N}'$ of $\mathcal{N}$ consisting of $A = A'$ of the form $A' = (A^0_{p\overline{q}})$, and we may write

$$\mathcal{F}_{A'} = \mathbb{C}[A'][[z_1, \overline{z}_1, U]], \quad \mathcal{F}'_{A'} = \mathbb{C}[A'][[z_1, \overline{z}_1]].$$

In this case, if $a \in \mathbb{C}[[z_1, \overline{z}_1]]$ in Proposition 7, then the solution $f = f[a]$ of (5.2) is independent of v in the sense that $\eta_k[f] \in \mathcal{F}_{A'}$ for $k \geq 0$. This fact is seen by inspecting the proof of Proposition 7 as follows. We write $\mathcal{B}$, $\mathbf{B}_j$, $\mathcal{B}_j$, $\ldots$ as $\mathcal{B}'$, $\mathbf{B}'_j$, $\mathcal{B}'_j$, $\ldots$ when $A \in \mathcal{N}$ is replaced by $A' \in \mathcal{N}'$. We set $\mathcal{D}_{A'} = \mathcal{F}_{A'}[\partial_{z_1}, \partial_{\overline{z}_1}, U\partial_U]$, which acts on $\mathcal{B}'$. Then Proposition 8 remains valid if we replace $\mathcal{F}_A$, $\mathcal{D}_A$, $\mathcal{B}$ and Q_0 by $\mathcal{F}_{A'}$, $\mathcal{D}_{A'}$, $\mathcal{B}'$ and

$$\Delta_1 = -\partial_{z_1}\partial_{\overline{z}_1}, \tag{5.11}$$

respectively. Similarly, Lemmas 5.1 and 5.2 can be modified in an obvious manner, where $\mathbf{B}_j, \mathcal{B}_j, \ldots$ are replaced by $\mathbf{B}'_j, \mathcal{B}'_j, \ldots$. Thus we have

$$f[a] = \sum_{j=0}^{\infty} \lambda_j, \quad \lambda_j = \lambda_j[a] \in \mathbf{B}'_j.$$

We shall use this fact, without comment, in Subsections 5.4 and 5.5.

5.3. Expansion with respect to the normal form coefficients. For each $m \in \mathbb{N}_0$, let $\mathbf{A}_m$ denote the totality of $f \in \mathcal{B}$ as in (5.1) such that if we write

$$\eta_k[f] = \sum_{j=0}^{\infty} \eta_{j,k}[f]\, U^j \quad \text{with} \quad \eta_{j,k}[f] \in \mathcal{F}'_A,$$

then $\eta_{j,k}[f]$ are polynomials of homogeneous degree m in A. Thus the dependence of U on A is not taken into account. We then get a filtration

$$\mathcal{B} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots, \quad \text{where} \quad \mathcal{A}_n = \sum_{m=n}^{\infty} \mathbf{A}_m. \tag{5.12}$$

For $f = f[a] \in \mathcal{B}(a) \cap \mathcal{B}_1$ in Proposition 7, we set $g = V_0(1+f[a])-1 \in \mathcal{B}$. Then

$$\mathcal{M}_1[g] = 1, \quad \text{where} \quad \mathcal{M}_1[g] = J[U(1+g)].$$

Note that $1 + \eta_0[g] = V_0(1 + \eta_0[f])$ and $\eta_k[g] = V_0\eta_k[f]$; in particular, $\eta_0[g] = O(U)$. We thus write

$$g = g[b], \quad \text{where} \quad 3!\, b = \partial_U^3 \eta_0[g]|_{U=0},$$

so that $g \in \mathcal{B}(b)$. We also have $g \in \mathcal{B}(b) \cap \mathcal{A}_1$, because $J[U] = 1 + \mathcal{A}_1$ and thus $f[a] \in \mathcal{A}_1$.

As in (5.6), let us consider the following expansion of $g = g[b]$:

$$g = \lim_{n\to\infty} g_n \quad \text{with} \quad g_n = \sum_{m=1}^{n} \theta_m \quad \text{and} \quad \theta_m \in \mathbf{A}_m, \tag{5.13}$$

and thus $g = \sum_{m\geq 1} \theta_m$. Then we have the following analogue of Lemma 5.2 for the filtration (5.12).

Lemma 5.4. *If $g \in \mathcal{B}(b) \cap \mathcal{A}_1$ is given as in* (5.13)*, then*

$$\mathcal{M}_1[g_j] = \mathcal{M}_1[g_{m-1}] - (I(U\partial_U) + UQ_0)\theta_m \quad \mathit{mod} \ \ \mathcal{A}_{m+1} \tag{5.14}$$

for $1 \leq m \leq j \leq \infty$, where $g_0 = 0$ and $g_\infty = g$. The operator $I(U\partial_U)$ and Q_0 are given in Proposition 8.

Proof. Setting $V = 1$ in (5.4), we have

$$\mathcal{M}_1[g_j] = J[U](1 - \mathcal{L}g_j) + \Phi(P_1 g_j, \ldots, P_\ell g_j).$$

Let us observe by definition that

$$\mathcal{A}_m \cdot \mathcal{A}_n \subset \mathcal{A}_{m+n}, \quad \mathcal{D}_A \mathcal{A}_m \subset \mathcal{A}_m, \quad \mathcal{D}_A^1 \mathcal{A}_m \subset \mathcal{A}_{m+1}.$$

It follows from $g_j \in \mathcal{A}_1$ that

$$\Phi(P_1 g_j, \dots, P_\ell g_j) = \Phi(P_1 g_{m-1}, \dots, P_\ell g_{m-1}) \mod \mathcal{A}_{m+1}.$$

Recalling the definition of $\mathcal{L}$, we next get

$$\mathcal{L}(g_j - g_{m-1}) = (I(U\partial_U) + UQ_0)\theta_m \mod \mathcal{A}_{m+1}.$$

Using $J[U] = 1 + \mathcal{A}_1$, we obtain the desired result. □

For $g = g[b]$, let $g_n = g_n[b]$ and $\theta_m = \theta_m[b]$ be defined by (5.13). Then

$$(I(U\partial_U) + UQ_0)\theta_m = \gamma_m \in \mathbf{A}_m \quad (\theta_m \in \mathbf{A}_m) \tag{5.15$_m$}$$

for $m \geq 1$, where $\gamma_m = \gamma_m[b]$ denotes the $\mathbf{A}_m$ component of $\mathcal{M}_1[g_{m-1}]$. We now need an expression of θ_m in terms of γ_m, which gives an analogue of Lemma 5.3 for the filtration (5.12). Let us first recall by Lemma 5.3 that the operator $I(U\partial_U): \mathcal{B} \to \mathcal{B}^+$ admits a right inverse $I_b^-: \mathcal{B}^+ \to \mathcal{B}(b)$, where $\mathcal{B}^+$ denotes the space of elements $\varphi \in \mathcal{B}$ such that the $\mathbf{B}_3$ component of φ does not contain $\log U$. Using the operator I_b^-, we have:

Lemma 5.5. *For every $m \geq 1$, the equation $(5.15)_m$ has a solution if and only if $\gamma_m \in \mathcal{B}^+$. The solution is unique under the restriction $\theta_m \in \mathcal{B}(b)$, where $b \in \mathbb{C}[[z', \overline{z'}, v]]$ is arbitrarily prescribed. The solution operator, denoted by $K_b: \mathcal{B}^+ \to \mathcal{B}(b)$, is given by*

$$K_b\gamma = \sum_{j=0}^{\infty} (-I_b^- UQ_0)^j I_b^- \gamma = \sum_{j=0}^{\infty} I_b^- (-UQ_0 I_b^-)^j \gamma.$$

Proof. Since $U\mathcal{B} \subset \mathcal{B}^+$ and $I(U\partial_U)\mathcal{B} \subset \mathcal{B}^+$, the validity of $(5.15)_m$ implies $\gamma_m \in \mathcal{B}^+$. Conversely, suppose we are given $\gamma_m \in \mathcal{B}^+$. Observe that the series defining K_b are well-defined. In fact, if $\gamma \in \mathcal{B}_k \cap \mathcal{B}^+$ then $I_b^-\gamma \in \mathcal{B}_k$ and thus $UQ_0 I_b^- \gamma \in \mathcal{B}_{k+1} \cap \mathcal{B}^+$. Consequently, for $\gamma \in \mathcal{B}^+$ and $k \geq 0$, the $\mathbf{B}_k$ components of $K_b\gamma$ are determined successively. Setting

$$\theta_m[b] = K_b\gamma_m = \sum_{j=0}^{\infty} (-I_b^- UQ_0)^j I_b^- \gamma_m \quad \text{for} \quad m \geq 1, \tag{5.16}$$

we shall show that $\theta_m = \theta_m[b]$ is a unique solution of $(5.15)_m$ such that $\theta_m \in \mathcal{B}(b)$. Since $I(U\partial_U)I_b^-$ is the identity operator on $\mathcal{B}^+$, it follows that

$$(I(U\partial_U) + UQ_0) \sum_{j=0}^{n} I_b^- (-UQ_0 I_b^-)^j \gamma_m = \gamma_m - (-UQ_o I_b^-)^{n+1} \gamma_m.$$

Thus $\theta_m = \theta_m[b] \in \mathcal{B}(b)$ in (5.16) satisfies $(5.15)_m$. Since $I_b^- I(U\partial_U)$ is the identity operator on $\mathcal{B}(b)$, it follows that, for any $\theta_m \in \mathcal{B}(b)$,

$$\sum_{j=0}^{n}(-I_b^- UQ_0)^j I_b^- (I(U\partial_U) + UQ_0)\theta_m = \theta_m - (-I_b^- UQ_0)^{n+1}\theta_m.$$

This implies the uniqueness of the solution of $(5.15)_m$ in $\mathcal{B}(b)$. $\square$

Using Lemmas 5.4 and 5.5, we argue as in the proof of Proposition 7. Then we see that each term θ_m in the expansion (5.13) of $g = g[b]$ is given by (5.16), where γ_m denotes the $\mathbf{A}_m$ component of $\mathcal{M}_1[g_{m-1}]$ with $g_0 = 0$. Note that g_{m-1} and γ_m depend on b.

Remark 5.4. We shall apply the argument of this subsection to a class of surfaces in normal form characterized by the condition

$$A^{\ell}_{p\overline{q}} = 0 \quad \text{whenever} \quad (p, q, \ell) \notin \Lambda,$$

where Λ is an index set. In this case, it is convenient to set

$$A' = (A^{\ell}_{p\overline{q}})_{(p,q,\ell)\in\Lambda}$$

and consider A' instead of A. That is, we define $\mathbf{A}'_m$ by $\mathbf{A}_m$ with A' in place of A, so that we get a filtration

$$\mathcal{A}'_0 \supset \mathcal{A}'_1 \supset \cdots, \quad \text{where} \ \ \mathcal{A}'_n = \sum_{m=n}^{\infty} \mathbf{A}'_m.$$

Then, Lemmas 5.3 and 5.4 remain valid if the spaces $\mathcal{A}_m$ are replaced by $\mathcal{A}'_m$. Therefore, we have (5.13) with $\theta_m \in \mathbf{A}'_m$, and each $\theta_m = \theta_m[b]$ is given by (5.16), where the $\mathbf{A}'_m$ component γ_m of $\mathcal{M}_1[g_{m-1}]$ belongs to $\mathbf{A}'_m$.

5.4. Explicit computation for special domains. We now restrict ourselves to the class of domains Ω_{pq} as in Proposition 5. Let us compute $g_2 = \theta_1 + \theta_2$ explicitly to the extent we need in the proof of Proposition 5.

For each $(p, q) \in \{(2, 4), (2, 5), (3, 4)\}$, we set $A' = (A^0_{p\overline{q}}, A^0_{q\overline{p}})$ as in Remark 5.4 so that $\Omega_{pq} = N(A')$ and write $F_{pq} = F_{A'}$ for the function $F = F_A$ in (1.1). For simplicity of the notation, we shall drop primes in $\mathbf{B}'_j, \mathcal{B}'_j, \ldots$ of Remark 5.3 and write $\mathbf{B}_j, \mathcal{B}_j, \ldots$ instead. We also write $\mathbf{A}_j, \mathcal{A}_j, \ldots$ in place of $\mathbf{A}'_j, \mathcal{A}'_j, \ldots$. Assume

$$b = 0$$

and thus $g = g[0]$. Then, (5.16) gives

$$\theta_m = K_0\gamma_m = \sum_{j=0}^{\infty}(I_0^- U\Delta_1)^j I_0^- \gamma_m \quad (m \geq 1) \tag{5.17}$$

for Δ_1 in (5.11), where γ_m is the $\mathbf{A}_m$ component of $\mathcal{M}_1[g_{m-1}]$. Using (5.17), we first have:

Lemma 5.6. *Let $\theta_1 = \theta_1^{pq}$ for $\Omega = \Omega_{pq}$. Then*

$$\begin{aligned}
\theta_1^{42} &= -4\,\mathrm{Re}\Big[A^0_{4\overline{2}}(U\,z_1^2 + \frac{4}{3}\,z_1^3\,\overline{z}_1)\Big],\\
\theta_1^{52} &= -\frac{20}{3}\,\mathrm{Re}\Big[A^0_{5\overline{2}}(U\,z_1^3 + z_1^4\,\overline{z}_1)\Big],\\
\theta_1^{43} &= -4\,\mathrm{Re}\Big[A^0_{4\overline{3}}(2\,U^2\,z_1 + 3\,U\,z_1^2\,\overline{z}_1 + 2\,z_1^3\,\overline{z}_1^2)\Big].
\end{aligned}$$

Proof. Since $F = F_{pq}$ is independent of the variable v, it follows that $J[U] = 1 - \Delta_1 F$. Recalling that $g_0 = 0$ and thus $\mathcal{M}_1[g_0] = J[U]$, we get $\gamma_1 = -\Delta_1 F$. Since $[\Delta_1, I_0^- U] = 0$, $[\Delta_1, I_0^-] = 0$ and $\Delta_1^{q+1} F = 0$, it follows from (5.17) that

$$\theta_1 = -\sum_{j=0}^{q-1} (I_0^- U)^j\, I_0^-\, \Delta_1^{j+1} F.$$

Noting $q \le 3$ and $I_0^- U^j = U^j / I(j)$ for $j \in \mathbb{N}_0 \setminus \{3\}$, we get

$$\theta_1 = -\sum_{j=0}^{q-1} \frac{U^j \Delta_1^{j+1} F}{I(0) I(1) \cdots I(j)}.$$

Evaluating the right side explicitly, we obtain the desired result. $\square$

Let us next consider the $\mathbf{A}_2$ component $\gamma_2 = \gamma_2^{pq}$ of $\mathcal{M}_1[g_1]$. We may write

$$\gamma_2^{pq} = |A^0_{p\overline{q}}|^2 \varphi^{pq} + 2\,\mathrm{Re}\Big[(A^0_{p\overline{q}})^2 \psi^{pq}\Big], \tag{5.18}$$

where $\varphi^{pq}, \psi^{pq} \in \mathcal{B}$ are independent of A'. Using Lemma 5.5, we can identify φ^{pq} as follows.

Lemma 5.7. *Let $\varphi^{pq}, \psi^{pq} \in \mathcal{B}$ satisfy (5.18). Then,*

$$\begin{aligned}
\varphi^{42} &= -\frac{256}{3}\,|z_1|^8 - \frac{512}{9}\,|z_1|^6\,U + 48\,|z_1|^4\,U^2,\\
\varphi^{52} &= -\frac{400}{3}\,|z_1|^{10} - \frac{700}{9}\,|z_1|^8\,U + \frac{2000}{9}\,|z_1|^6\,U^2,\\
\varphi^{43} &= -192\,|z_1|^{10} - 368\,|z_1|^8\,U - 48\,|z_1|^6\,U^2 + 288\,|z_1|^4\,U^3 - 48\,U^5.
\end{aligned}$$

Proof. We set $\tau = A^0_{p\overline{q}}$, $\overline{\tau} = A^0_{q\overline{p}}$ and $\Theta = 1 + \theta_1$ with $\theta_1 = \theta_1^{pq}$. Recall that $\mathcal{M}_1[\theta_1] = J[U\Theta]$ is a polynomial in τ and $\overline{\tau}$ such that the coefficient of $|\tau|^2$ is φ^{pq}. We follow the procedure in the proof of Proposition 8 for $n = 2$ with Θ in place of Φ. Since F and Θ are independent of the variable v, we have

$$F_2 = F_{\overline{2}} = 0, \quad \Theta_2 = \Theta_{\overline{2}} = 0,$$

where we used the notation $F_j = \partial_j F$, etc. Then $J[U\Theta] = \det H^1$, where

$$H^1 = \begin{pmatrix} \Theta & \Theta_{\overline{1}} & \partial_U U\Theta \\ U\Theta_1 & U\Theta_{1\overline{1}} - (1 + F_{1\overline{1}})\partial_U U\Theta & U\partial_U U\Theta_1 \\ \partial_U U\Theta & \partial_U U\Theta_{\overline{1}} & U\partial_U^2 U\Theta \end{pmatrix}.$$

Since the entries of H^1 are at most quadratic in τ and $\overline{\tau}$, we may write

$$H^1 = H^2 + \tau H^3 + \overline{\tau} H^4 + |\tau|^2 H^5 + \mathcal{E}(\tau^2, \overline{\tau}^2),$$

where $\mathcal{E}(\tau^2, \overline{\tau}^2)$ stands for an error term of the form $O(\tau^2) + O(\overline{\tau}^2)$. Then

$$H^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (H^2)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

We set $\widetilde{H}^m = (H^2)^{-1} H^m$ for $m = 3, 4, 5$. Noting $\det H^2 = 1$, we have

$$\begin{aligned} J[U\Theta] &= \det\left(1 + \tau\widetilde{H}^3 + \overline{\tau}\widetilde{H}^4 + |\tau|^2\widetilde{H}^5\right) + \mathcal{E}(\tau^2, \overline{\tau}^2) \\ &= 1 + \tau\,\mathrm{tr}(\widetilde{H}^3) + \overline{\tau}\,\mathrm{tr}(\widetilde{H}^4) + |\tau|^2\varphi^{pq} + \mathcal{E}(\tau^2, \overline{\tau}^2), \end{aligned}$$

where

$$\varphi^{pq} = \mathrm{tr}(\widetilde{H}^5) + \mathrm{tr}(\widetilde{H}^3)\mathrm{tr}(\widetilde{H}^4) - \mathrm{tr}(\widetilde{H}^3\widetilde{H}^4). \tag{5.19}$$

We know that $\mathrm{tr}(\widetilde{H}^3) = \mathrm{tr}(\widetilde{H}^4) = 0$, a fact which is also seen directly from Lemma 5.6 and the expressions

$$\begin{aligned} \widetilde{H}^3 &= \begin{pmatrix} \partial_U U\theta' & \partial_U U\theta'_{\overline{1}} & U\partial_U^2 U\theta' \\ -U\theta'_1 & -F'_{1\overline{1}} + U\theta'_{1\overline{1}} - \partial_U U\theta' & -U\partial_U U\theta'_1 \\ -U\partial_U\theta' & -U\partial_U\theta'_{\overline{1}} & (1 - (U\partial_U)^2)\theta' \end{pmatrix}, \\ \widetilde{H}^4 &= \begin{pmatrix} \partial_U U\theta'' & \partial_U U\theta''_{\overline{1}} & U\partial_U^2 U\theta'' \\ -U\theta''_1 & -F''_{1\overline{1}} + U\theta''_{1\overline{1}} - \partial_U U\theta'' & -U\partial_U U\theta''_1 \\ -U\partial_U\theta'' & -U\partial_U\theta''_{\overline{1}} & (1 - (U\partial_U)^2)\theta'' \end{pmatrix}, \end{aligned}$$

where we wrote $F = \tau F' + \overline{\tau} F''$ and $\theta_1 = \tau\theta' + \overline{\tau}\theta''$. Thus

$$\varphi^{pq} = (\mathrm{I})^{pq} - (\mathrm{II})^{pq},$$

where $(\mathrm{I})^{pq} = \mathrm{tr}(\widetilde{H}^5)$ and $(\mathrm{II})^{pq} = \mathrm{tr}(\widetilde{H}^3\widetilde{H}^4)$. We have

$$\widetilde{H}^5 = -H^5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & F'_{1\overline{1}}\partial_U U\theta'' + F''_{1\overline{1}}\partial_U U\theta' & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We also have $\mathrm{tr}(\widetilde{H}^3\widetilde{H}^4) = T_1 + T_2 + T_3$, where

$$\begin{aligned}
T_1 &:= \bigl|\partial_U U\theta'\bigr|^2 + \bigl|(U\partial_U - 1)\partial_U U\theta'\bigr|^2 - 2\,\mathrm{Re}\,\bigl[(U\partial_U^2 U\theta')(U\partial_U\theta'')\bigr],\\
T_2 &:= U\bigl(|U\partial_U\theta_1'|^2 - |\theta_1'|^2\bigr) + U\bigl(|U\partial_U\theta_{\bar 1}'|^2 - |\theta_{\bar 1}'|^2\bigr),\\
T_3 &:= \bigl|F_{1\bar 1}' - U\theta_{1\bar 1}' + \partial_U U\theta'\bigr|^2.
\end{aligned}$$

Using these expressions, we get, by direct computation,

$$\begin{aligned}
(\mathrm{I})^{42} &= -64\,U|z_1|^6 - \frac{128}{3}\,|z_1|^8,\\
(\mathrm{I})^{52} &= -\frac{400}{3}\,U|z_1|^8 - \frac{200}{3}\,|z_1|^{10},\\
(\mathrm{I})^{43} &= -288\,U^2|z_1|^6 - 288\,U|z_1|^8 - 96\,|z_1|^{10},\\
(\mathrm{II})^{42} &= -48\,U^2|z_1|^4 - \frac{64}{9}\,U|z_1|^6 + \frac{128}{3}\,|z_1|^8,\\
(\mathrm{II})^{52} &= -\frac{2000}{9}\,U^2|z_1|^6 - \frac{500}{9}\,U|z_1|^8 + \frac{200}{3}\,|z_1|^{10},\\
(\mathrm{II})^{43} &= 48\,U^5 - 288\,U^3|z_1|^4 - 240\,U^2|z_1|^6 + 80\,U|z_1|^8 + 96\,|z_1|^{10}.
\end{aligned}$$

These together with (5.19) yield the desired result. $\square$

Let us finally consider $\theta_2 = \theta_2^{pq}$. By (5.17) and (5.18), we have

$$\theta_2^{pq} = |A_{p\bar q}^0|^2\,K_0\varphi^{pq} + 2\,\mathrm{Re}\Bigl[(A_{p\bar q}^0)^2 K_0\psi^{pq}\Bigr]. \tag{5.20}$$

We are concerned with $K_0\varphi^{pq}$ restricted to $z_1 = 0$.

Lemma 5.8. *For each (p,q), there exists a constant c_{pq} such that, for $K_0\varphi^{pq}$ in* (5.20),

$$K_0\varphi^{pq} = U^{p+q-2}(c_{pq} + d_{pq}\log U) \quad \text{at} \;\; z_1 = 0,$$

where

$$d_{24} = \frac{368}{5}, \quad d_{25} = -\frac{680}{3}, \quad d_{34} = -\frac{1956}{5}.$$

Proof. Observe by Lemma 5.7 that each φ^{pq} is a linear combination of $|z_1|^{2j}U^k$ with $3 \le j+k \le 5$. For such terms, we have

$$\begin{aligned}
K_0(|z_1|^{2j}\,U^k) &= (\Delta_1^j|z_1|^{2j})(I_0^- U)^j I_0^- U^k + \cdots\\
&= (-1)^j (j!)^2\,(I_0^- U)^j I_0^- U^k + \cdots,
\end{aligned}$$

where $\cdots$ stands for terms which vanish at $z_1 = 0$. Note that

$$(I_0^- U)^j I_0^- U^k = U^{j+k}(c_{jk}' + d_{jk}'\log U), \quad j+k \ge 3,$$

where c'_{jk} are constants and d'_{jk} are given by

$$d'_{jk} = \frac{1}{4} \prod_{\substack{k \le m \le j+k \\ m \ne 3}} \frac{1}{I(m)} \quad \text{for} \ \ k \le 3, \quad d'_{jk} = 0 \quad \text{for} \ \ k > 3.$$

We thus get $K_0(|z_1|^{2j} U^k) = (-1)^j (j!)^2 U^{j+k} (c'_{jk} + d'_{jk} \log U)$ at $z_1 = 0$. Using this formula and Lemma 5.7, we obtain the conclusion. □

5.5. Proof of Proposition 5. The existence of the constants $c_{pq}[\eta_1^{\mathrm{G}}]$ in (2.3) follows from Proposition 3, (2°) and Lemma 5.1. Let us identify these constants by using Lemma 5.8.

We begin by recalling that $u^{\mathrm{G}} = U(1 + g[b])$, where b corresponds to the ambiguity of u^{G}. Since this ambiguity does not affect the values of $c_{pq}[\eta_1^{\mathrm{G}}]$, we take $b = 0$. Then,

$$u^{\mathrm{G}} = U(1 + \theta_1^{pq} + \theta_2^{pq}) \mod \mathcal{A}_3.$$

Let us restrict ourselves to $z_1 = 0$. Then, Lemma 5.8 with (5.20) implies

$$\theta_2^{pq} = d_{pq} \, |A^0_{p\overline{q}}|^2 U^{p+q-2} \log U + \cdots ,$$

where the dots $\cdots$ stands for terms irrelevant to our purpose. Using Lemma 5.6 and noting $\theta_1|_{z_1=0} = 0$, $U(\gamma_t) = t$, we get

$$u^{\mathrm{G}}(\gamma_t) = t\Big(1 + d_{pq} \, |A^0_{p\overline{q}}|^2 t^{p+q-5} (t^3 \log t)\Big) + \cdots .$$

Recalling that $r^{\mathrm{F}}(\gamma_t) = 4 + O(t^4)$, we obtain

$$\eta_1^{\mathrm{G}}(\gamma_t) = d_{pq} \, |A^0_{p\overline{q}}|^2 t^{p+q-5} + \cdots .$$

This implies, as desired, $c_{pq}[\eta_1^{\mathrm{G}}] = d_{pq}$.

6. Weyl-Fefferman functionals of weight ≤ 5

This section is devoted to the proof of Theorem 1, Proposition 3, (3°) and Proposition 4. We first consider the polynomial dependence on $A \in \mathcal{N}$, together with the ambiguity caused by that of $r = r^{\mathrm{F}}$. We have:

Proposition 9. *Let $W^\#$ be a complete contraction of weight $w \le 5$ that is not linear in R. If $w \le 5$, then $W \in I_w^{\mathrm{WF}}$; moreover $W \in I_w(6 - w)$. If $w \le 3$, then $W = O^{4-w}(\partial\Omega)$.*

Proposition 10. *Let $W \in I_w^{\mathrm{WF}}$ with $w \leq 5$. If $W^\#$ is linear in R, then $W = O^{6-w}(\partial\Omega)$.*

We prove Propositions 9 and 10 above in Subsections 6.1 and 6.2, respectively. Observe that these propositions imply Proposition 3, (3°). Furthermore, Theorem 1 follows from these propositions if we assume the validity of Proposition 4, which is proved in Subsections 6.3, 6.4 and 6.5.

In what follows, we shall be concerned with surfaces in Moser's normal form $N(A)$, real analytic or C^∞, where $A \in \mathcal{N}$ varies.

6.1. Proof of Proposition 9. As in Subsection 3.4, we decompose Fefferman's defining function $r = r^{\mathrm{F}}$ of $N(A)$ as $r^{\mathrm{F}} = r^{\mathrm{F}}_A + (r^{\mathrm{F}} - r^{\mathrm{F}}_A)$, where $r^{\mathrm{F}} - r^{\mathrm{F}}_A = O(U^4)$ describes the ambiguity of r^{F}. Let $g = (g_{j\overline{k}})$ be the ambient metric with potential $r^\#$ and define $R^{(a,b)}$ for g. We sometimes write $R^{(a,b)} = R^{(a,b)}[r]$ in order to emphasize the dependence on r. Making a change of coordinates

$$(z_0, z_1, z_2) \to \zeta = (\zeta_0, \zeta_1, \zeta_2) \quad \text{in} \quad \mathbb{C}^* \times \mathbb{C}^2$$

defined by $z_0 = \zeta_0, z_1 = \zeta_1/\zeta_0$ and $z_2 = \zeta_2/\zeta_0$, we consider the components of $R^{(a,b)}[r]$ with respect to $d\zeta_j, d\overline{\zeta}_j$ $(j = 0, 1, 2)$, and regard these as formal power series in $\zeta, \overline{\zeta}$ about the point $e_0 = (1, 0, 0)$. Then we have an expansion in $\zeta, \overline{\zeta}$ about e_0:

$$g = g_0 + \sum_{|\alpha|+|\beta|\geq 1} c_{\alpha\overline{\beta}}[g](\zeta - e_0)^\alpha(\overline{\zeta} - e_0)^\beta,$$

where

$$g_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = g_0^{-1}.$$

Each coefficient $c_{\alpha\overline{\beta}}[g]$ is a matrix such that the entries are polynomials in A and C.

Lemma 6.1. *For any component of $R^{(a,b)}[r]$, the coefficients of the expansion in $\zeta, \overline{\zeta}$ about e_0 are polynomials in $A = (A^\ell_{p\overline{q}})$ and $C = (C^{\ell m}_{p\overline{q}})$.*

Proof. For any component Q of $R^{(a,b)}[r]$, we consider the expansion

$$Q = \sum_{\alpha,\beta} c_{\alpha\overline{\beta}}[Q](\zeta - e_0)^\alpha(\overline{\zeta} - e_0)^\beta.$$

Then each coefficient $c_{\alpha\overline{\beta}}[Q]$ is a polynomial in $(c_{\alpha\overline{\beta}}[g])$ and $(g^{j\overline{k}}(e_0))$. Thus, $c_{\alpha\overline{\beta}}[Q]$ is a polynomial in A and C. □

Let us next consider the dependence on C.

Lemma 6.2. *Let $W^{\#}$ be a complete contraction of weight w. Then the coefficient P_m of the expansion*

$$W(\gamma_t) \sim \sum_{m=0}^{\infty} P_m(A,C)\, t^m, \quad \text{where } \ \gamma_t = (0, t/2), \tag{6.1}$$

is a polynomial in A and C of biweight $(w+m, w+m)$.

Proof. It follows from the transformation law under dilations that $W \in \mathcal{F}_{A,C}$ is of biweight (w,w). Since $t = (z_2 + \overline{z}_2)/2$ has biweight $(-1,-1)$, we see that $P_m(A,C)$ has biweight $(w+m, w+m)$. □

Proposition 9 now follows from:

Lemma 6.3. *Let P be a monomial of degree ≥ 2 in A and C. If P is of biweight (w,w) with $w \leq 5$, then P is independent of C. If $w \leq 3$, then $P = 0$.*

Proof. Let us first recall that $\mathrm{w}(A^{\ell}_{p\overline{q}}) \geq 2$ and $\mathrm{w}(C^{\ell m}_{p\overline{q}}) \geq 3$. Hence, if P depends on C, then $w = \mathrm{w}(P) \geq 5$ and thus $w = 5$. Consequently, $P = A^0_{2\overline{4}} C^{04}_{0\overline{0}}$ or $A^0_{4\overline{2}} C^{04}_{0\overline{0}}$ up to scalar multiples. This is absurd, because

$$\mathrm{w}_2(A^0_{2\overline{4}} C^{04}_{0\overline{0}}) = (4,6), \quad \mathrm{w}_2(A^0_{4\overline{2}} C^{04}_{0\overline{0}}) = (6,4).$$

Thus P is independent of C. The second statement follows from Lemma 4.1, or, the proof is already obvious by the argument above. □

6.2. Proof of Proposition 10. Starting from $r = r^{\mathrm{F}}$, we form a linear complete contraction

$$W^{\#} = W^{\#}[r] = \mathrm{contr} R^{(p,p)} \quad \text{for } \ p = w + 1 \geq 2. \tag{6.2}$$

There are several ways to make a complete contraction, and we fix *any one of these*; for instance,

$$\mathrm{contr} R^{(p,p)} = \sum_{|\alpha| = p} R_{\alpha}{}^{\alpha} \qquad (p = w+1).$$

Our results below are independent of the definition of $\mathrm{contr} R^{(p,p)}$.

Proposition 10′. *The following statements hold for $W^{\#}$ in* (6.2).

(1°) *If $w \leq 2$ then $W^{\#}(e_0) = 0$.*
(2°) *If $w = 3$ then W modulo $O(U^2)$ depends on C.*
(3°) *If $w \geq 4$ then $W^{\#}(e_0)$ depends on C.*
(4°) *If $w = 3$ then $W^{\#}(e_0) = -(4!)^2 A^0_{4\overline{4}}$.*

Observe that Proposition 10 follows from Proposition 10′, where (4°) is not used.

Let us prove Proposition $10'$. It follows from Lemma 6.2 that $W^{\#}(e_0)$ is a polynomial $P(A, C)$ of biweight (w, w). If $w \leq 2$ then $P(A, C)$ is independent of C because $\mathrm{w}(C^{\ell m}_{p\overline{q}}) \geq 3$. Consequently, Lemma 4.1 implies $(1°)$.

It remains to prove $(2°)$, $(3°)$ and $(4°)$. Denoting by $\varphi = \varphi[r]$ the linear (homogeneous) part of r with respect to A and C, we set $\varphi^{\#} = |\zeta_0|^2\varphi$, which is regarded as a formal power series in $\zeta, \overline{\zeta}$ about e_0. Then

$$g_{j\overline{k}} = (g_0)_{j\overline{k}} + \frac{\partial^2\varphi^{\#}}{\partial\zeta_j\partial\overline{\zeta}_k} + O^2(A, C),$$

where $O^s(A, C)$ stands for a formal power series in $\zeta, \overline{\zeta}$ such that the coefficients are polynomial in A, C which do not contain terms of degree $< s$. Thus noting that g_0 is a constant matrix, we get

Lemma 6.4. $R_{\alpha\overline{\beta}} = \partial^{\alpha}_{\zeta}\partial^{\beta}_{\overline{\zeta}}\varphi^{\#} + O^2(A, C)$.

By virtue of Lemma 6.4 above, the following proof of Proposition $10'$ is valid independently of the definition of $\mathrm{contr}R^{(p,p)}$.

Since $g = g_0 + O^1(A, C)$, it follows from Lemma 6.4 that

$$W^{\#} = (\Delta^{\#}_0)^p\varphi^{\#} + O^2(A, C), \tag{6.3}$$

where $\Delta^{\#}_0$ denotes the Laplacian with respect to g_0. Specifically,

$$\Delta^{\#}_0 = \frac{\partial^2}{\partial\zeta_0\partial\overline{\zeta}_2} + \frac{\partial^2}{\partial\zeta_2\partial\overline{\zeta}_0} - \frac{\partial^2}{\partial\zeta_1\partial\overline{\zeta}_1}.$$

The following lemma is useful.

Lemma 6.5. *Let* $\varphi_{m,\ell} = |\zeta_0|^{2\ell}(U^{\#}_0)^m$, *where* $U^{\#}_0 = |\zeta_0|^2 U_0$. *If* $m, n \in \mathbb{N}$ *satisfy* $n \leq m$, *then*

$$(\Delta^{\#}_0)^n\varphi_{m,\ell} = C_{n,m,\ell}\,\varphi_{m-n,\ell},$$

where $C_{n,m,\ell} = \prod_{j=0}^{n-1}(m-j)(m-j+2\ell+2)$.

Proof. Setting $Z = \sum_{j=0}^{2}\zeta_j\partial/\partial\zeta_j$, we have

$$[\Delta^{\#}_0, (U^{\#}_0)^q] = q\,(U^{\#}_0)^{q-1}(Z + \overline{Z} + q + 2).$$

Since $Z|\zeta_0|^{2\ell} = \overline{Z}|\zeta_0|^{2\ell} = \ell\,|\zeta_0|^{2\ell}$ and $\Delta^{\#}_0|\zeta_0|^{2\ell} = 0$, it follows that

$$\Delta^{\#}_0\varphi_{q,\ell} = [\Delta^{\#}_0, (U^{\#}_0)^q]|\zeta_0|^{2\ell} + (U^{\#}_0)^q\Delta^{\#}_0|\zeta_0|^{2\ell} = q(q+2\ell+2)\varphi_{q-1,\ell}.$$

Using this for $q = m, m-1, \dots, m-n+1$, we obtain the result. $\square$

With the aid of Lemmas 6.4 and 6.5, let us prove (2°). We consider

$$W_3^{\#} = \mathrm{contr} R^{(4,4)}[r_4], \quad \text{where} \ \ r_4 = U + C^{05}_{0\bar{0}}\, U^5.$$

It follows from (6.3) that $W_3^{\#} = C^{05}_{0\bar{0}} (\Delta_0^{\#})^4 \varphi_{5,-4} + O^2(C) + O^1(A)$. On the other hand, Lemma 6.5 yields

$$(\Delta_0^{\#})^4 \varphi_{5,-4} = C_{4,5,-4} \varphi_{1,-4} = 5!\, 4!\, \varphi_{1,-4}.$$

Therefore, W_3 modulo $O(U^2)$ depends on $C^{05}_{0\bar{0}}$ as desired.

The proof of (3°) is similar as follows. We set, for $w = p - 1 \geq 4$,

$$W_w^{\#} = \mathrm{contr} R^{(p,p)}[r_w], \quad \text{where} \ \ r_w = U + C^{0p}_{0\bar{0}}\, U^p.$$

Then (6.3) implies $W_w^{\#} = C^{0p}_{0\bar{0}}\, (\Delta_0^{\#})^p \varphi_{p,1-p} + O^2(C) + O^1(A)$, while Lemma 6.5 yields

$$(\Delta_0^{\#})^p \varphi_{p,1-p} = C_{p,p,1-p} |\zeta_0|^{2-2p}, \quad C_{p,p,1-p} = (-1)^p \frac{(2p-5)!\, p!}{(p-3)!} \neq 0.$$

Therefore, $W_w^{\#}(e_0)|_{A=0}$ modulo $O^2(C)$ is a non-zero multiple of $C^{0p}_{0\bar{0}}$, and thus $W_w^{\#}(e_0)$ depends on $C^{0p}_{0\bar{0}}$.

It remains to prove (4°). We recall by Lemma 6.2 that $W^{\#}(e_0)$ is of biweight $(3,3)$. Then, by Lemma 6.3, $W^{\#}(e_0)$ must be linear in A and C, so that

$$W^{\#}(e_0) = c_1\, A^0_{4\bar{4}} + c_2\, C^{04}_{0\bar{0}}, \tag{6.4}$$

where c_1 and c_2 are constants. Hence (4°) is equivalent to

$$c_1 = -(4!)^2, \quad c_2 = 0.$$

Let us first prove $c_2 = 0$. We restrict ourselves to the case $A = 0$. Setting $r = U_0 + C^{04}_{0\bar{0}}\, U_0^4$, we have, as in the proof of (3°),

$$W^{\#}[r] = C^{04}_{0\bar{0}}\, (\Delta_0^{\#})^4 \varphi_{4,-3} = C^{04}_{0\bar{0}}\, C_{4,4,-3}\, |\zeta_0|^{-6}$$

while $C_{4,4,-3} = 0$. Therefore, $c_2 = 0$.

Let us next identify the constant c_1. We restrict ourselves to the case $\Omega = \Omega_{44}$, so that $U = U_0 - A^0_{4\bar{4}} |z_1|^8$. Recall that the expansion of $r = r^{\mathrm{F}}$ in the sense of Subsection 5.3 is given by the formally smooth part of

$$U(1+g) \quad \text{with} \ \ g = g[b] = \theta_1 + \theta_2 + \cdots .$$

We take $b = 0$ and write $\theta_1 = \theta_1^{44}$. Arguing as in the proof of the Lemma 5.6 and using $I_0^- U^3 = (U^3/4)\log U$, we get

$$\theta_1^{44} = 4\,A^0_{4\overline{4}}\left(-\frac{4}{3}\,|z_1|^6 - 3\,|z_1|^4\,U - 4\,|z_1|^2\,U^2 + U^3\log U\right).$$

Collecting linear terms in $A^0_{4\overline{4}}$ for the smooth part of $U(1+\theta_1^{44})$, we obtain

$$\varphi = -A^0_{4\overline{4}}\left(|z_1|^8 + \frac{16}{3}\,|z_1|^6\,U_0 + 12\,|z_1|^4\,U_0^2 + 16\,|z_1|^2\,U_0^3\right).$$

Direct computation yields $\Delta_0^\# \varphi^\# = 16\,A^0_{4\overline{4}}\,\varphi_{3,-3}$. Thus Lemma 6.5 implies

$$(\Delta_0^\#)^4\varphi^\# = 16\,A^0_{4\overline{4}}(\Delta_0^\#)^3\varphi_{3,-3} = -(4!)^2\,A^0_{4\overline{4}}|\zeta_0|^{-6}.$$

From this, we get $W^\#(e_0) = -(4!)^2\,A^0_{4\overline{4}}$, and thus $c_1 = -(4!)^2$. Hence, the proof of $(4°)$ is finished. $\square$

6.3. Proof of Proposition 4. We reduce the proof of Proposition 4 to that of the following:

Proposition 4′. *For constants c_1, c_2, d_1 and d_2, let*

$$\begin{aligned}\widetilde{F}[c_1,c_2,d_1,d_2] = {}& c_1\,|R_{11\overline{1}\overline{2}\overline{2}}|^2 + c_2\,|R_{12\overline{2}\overline{2}}|^2\\ &+ \mathrm{Re}\Big[R_{22\overline{1}\overline{1}}(d_1\,R_{112\overline{2}\overline{2}} + d_2\,R_{11\overline{2}\overline{2}\overline{2}})\Big],\end{aligned}$$

where the right side is evaluated at e_0. If $(a,b) = (4,2)$ or $(3,3)$ then

$$\|R^{(a,b)}\|^2(\gamma_t^\#) = c_0^{ab}\,|R_{11\overline{2}\overline{2}}(e_0)|^2 + \widetilde{F}[c_1^{ab},c_2^{ab},d_1^{ab},d_2^{ab}]\,t + O(t^2),$$

and if $(a,b) = (5,2)$ or $(4,3)$ then

$$\|R^{(a,b)}\|^2(e_0) = \widetilde{F}[c_1^{ab},c_2^{ab},d_1^{ab},d_2^{ab}],$$

where d_1^{ab}, d_2^{ab} are complex constants independent of A, C and

$$\begin{aligned}c_0^{42} &= 28, & c_1^{42} &= 8, & c_2^{42} &= 416,\\ c_0^{33} &= 12, & c_1^{33} &= 12, & c_2^{33} &= 324,\\ && c_1^{52} &= -36, & c_2^{52} &= -1800,\\ && c_1^{43} &= -48, & c_2^{43} &= -1140.\end{aligned}$$

In this subsection, we assume the validity of Proposition 4′ and prove Proposition 4. To express $R_{\alpha\overline{\beta}}(e_0)$ as above in terms of A, let us begin with a general observation.

Lemma 6.6. *Let* $\mathrm{w}_2(R_{\alpha\overline{\beta}}) = (w', w'')$ *with* $2w = w' + w'' \leq 6$ *and* $(w', w'') \neq (3,3)$. *Then* $R_{\alpha\overline{\beta}}(e_0)$ *is a linear combination of* $A^{\ell}_{p\overline{q}}$ *with* $\mathrm{w}_2(A^{\ell}_{p\overline{q}}) = (w', w'')$. *Furthermore,* $R_{\alpha\overline{\beta}}(e_0)$ *is symmetric in the entries of* α *(resp.* β*) and satisfies* $\overline{R_{\alpha\overline{\beta}}(e_0)} = R_{\beta\overline{\alpha}}(e_0)$.

Proof. It follows from Lemma 6.2 that $R_{\alpha\overline{\beta}}(e_0)$ is a polynomial in A and C of biweight (w', w''). Since $\mathrm{w}(A^{\ell}_{p\overline{q}}) \geq 2$ and $\mathrm{w}_2(C^{04}_{0\overline{0}}) = (3,3)$, it follows that $R_{\alpha\overline{\beta}}(e_0)$ must be linear in A and cannot contain C. Then, Lemma 6.4 implies the symmetry in the entries of α (resp. β). It remains to prove the Hermitian symmetry of $R_{\alpha\overline{\beta}}(e_0)$. Recalling that the metric g is Kählerian, we see that

$$\overline{R_{\alpha\overline{\beta}}} = R_{\beta_1\overline{\alpha_1}\beta_2\overline{\alpha_2};\overline{\alpha_3}\cdots\overline{\alpha_p}\beta_3\cdots\beta_q} \quad \text{for} \quad R_{\alpha\overline{\beta}} = R_{\alpha_1\overline{\beta_1}\alpha_2\overline{\beta_2};\alpha_3\cdots\alpha_p\overline{\beta_3}\cdots\overline{\beta_q}}.$$

Therefore, the desired result follows, as before, from Lemma 6.4. $\square$

The following lemma is crucial.

Lemma 6.7. $R_{11\overline{2}\overline{2}}, R_{111\overline{2}\overline{2}}$ *and* $R_{12\overline{2}\overline{2}}$ *evaluated at* e_0 *are respectively given by* $-8\,A^0_{4\overline{2}}$, $-40\,A^0_{5\overline{2}}$ *and* $-24\,A^0_{4\overline{3}}$.

Proof. Let $\alpha\overline{\beta} = 11\overline{2}\overline{2}, 111\overline{2}\overline{2}$ or $12\overline{2}\overline{2}$. Then $\mathrm{w}_2(R_{\alpha\overline{\beta}}) = (3,1), (4,1)$ or $(3,2)$, respectively, so that Lemma 6.6 applies. Recalling the conditions on $A \in \mathcal{N}$, we see that $R_{\alpha\overline{\beta}}(e_0)$ are constant multiples of $A^0_{4\overline{2}}$, $A^0_{5\overline{2}}$ or $A^0_{4\overline{3}}$, respectively, say, $R_{\alpha\overline{\beta}}(e_0) = c_1\,A^0_{4\overline{2}}, c_2\,A^0_{5\overline{2}}$ or $c_3\,A^0_{4\overline{3}}$, respectively. In order to identify these constants, let us restrict ourselves to $\Omega = \Omega_{pq}$ with $(p,q) = (4,2), (5,2)$ or $(4,3)$. Note by Lemma 6.4 that

$$R_{\alpha\overline{\beta}}(e_0) = \partial^{\alpha}_{\zeta}\partial^{\beta}_{\overline{\zeta}}\varphi^{\#}(e_0) = \partial^{\alpha}_{z}\partial^{\beta}_{\overline{z}}\varphi(0),$$

because α and β do not contain 0 in their entries. We may take $\varphi = \varphi^{pq}$ to be the linear part of $U(1+\theta^{pq}_1)$ in $(A^0_{p\overline{q}}, A^0_{q\overline{p}})$. Then we can easily identify c_1, c_2 and c_3 by using Lemma 5.6. $\square$

We now prove that Proposition 4 follows from Proposition 4′. In view of Lemmas 6.6 and 6.7, we only have to show that both $R_{112\overline{2}\overline{2}}(e_0)$ and $R_{11\overline{2}\overline{2}\overline{2}}(e_0)$ are linear combinations of $A^0_{5\overline{3}}$ and $A^1_{4\overline{2}}$. This fact is obtained just as at the beginning of the proof of Lemma 6.7.

6.4. Preliminaries for the proof of Proposition 4′. In order to prove Proposition 4′, we need to compute $\|R^{(a,b)}\|^2$ at e_0 for $a+b = 6, 7$ and $(d/dt)\|R^{(a,b)}\|^2(\gamma^{\#}_t)$ at $t = 0$ for $a+b = 6$. Recall by definition that

$$\|R^{(a,b)}\|^2 = \sum_{|\alpha|=a, |\beta|=b} R_{\alpha\overline{\beta}} R^{\alpha\overline{\beta}} \tag{6.5}$$

with

$$R^{\alpha\overline{\beta}} = \sum_{|\alpha'|=a, |\beta'|=b} g^{\alpha\overline{\alpha'}} g^{\beta'\overline{\beta}}\, \overline{R_{\alpha'\overline{\beta'}}},$$

where $g^{\alpha\overline{\alpha'}} = g^{\alpha_1\overline{\alpha'_1}} \cdots g^{\alpha_a\overline{\alpha'_a}}$ for $\alpha = \alpha_1 \cdots \alpha_a$ and $\alpha' = \alpha'_1 \cdots \alpha'_a$ with $g^{-1} = (g^{j\overline{k}})$, and similarly for $g^{\beta'\overline{\beta}}$.

We first evaluate both sides of (6.5) at e_0 for $a+b = 6, 7$. Then the sum in the right side is considerably simplified by using the following lemma.

Lemma 6.8. *Let α, β be multi-indices with $|\alpha|, |\beta| \geq 2$.*

(1°) *If either* $\mathrm{w}(R_{\alpha\overline{\beta}}) < 2$ *or* $\mathrm{w}_2(R_{\alpha\overline{\beta}}) = (2,2)$*, then $R_{\alpha\overline{\beta}} = 0$ at e_0.*

(2°) *$R_{j0\overline{\beta}} = 0$ and $R_{\alpha\overline{j0}} = 0$ at e_0 for $j = 0,1,2$. $R_{\alpha 0\overline{\beta}} = (1-|\alpha|)R_{\alpha\overline{\beta}}$ and $R_{\alpha\overline{\beta 0}} = (1-|\beta|)R_{\alpha\overline{\beta}}$ at e_0.*

Proof. (1°) follows from Lemma 6.6, because $\mathrm{w}(A^{\ell}_{p\overline{q}}) \geq 2$ and $\mathrm{w}_2(A^{\ell}_{p\overline{q}}) \neq (2,2)$.

To prove (2°), we follow Fefferman in [11], pp. 175-179, and introduce a new coordinate system by

$$z'_0 = \log z_0, \quad z'_1 = z_1, \quad z'_2 = z_2,$$

and set $Z_j = \partial/\partial z'_j$ for $j = 0,1,2$, so that $Z_j = \partial/\partial\zeta_j$ at e_0. Then $Z_0 r^{\#} = r^{\#}$, and thus $L_{Z_0} g = g$, where L_{Z_0} denotes the Lie differentiation along Z_0. Hence,

$$\nabla_{Z_0} Z_j = Z_j, \quad \nabla_{Z_0} \overline{Z}_j = 0, \quad \text{for } \ j = 0,1,2, \tag{6.6}$$

$$L_{Z_0} R^{(p,q)} = R^{(p,q)} \quad \text{for } \ R^{(p,q)} = (R_{\alpha\overline{\beta}})_{|\alpha|=p, |\beta|=q}. \tag{6.7}$$

Using (6.6), we get $R_{0j\overline{k\ell}} = 0$ as follows:

$$\sum_{\ell=0}^{2} R_{0j\overline{k}}{}^{\ell} Z_\ell = \Big[\nabla_{Z_0}, \nabla_{\overline{Z}_k}\Big] Z_j = 0.$$

Applying $\overline{\nabla}^{\beta'}$ with $\beta = k\ell\beta'$, we obtain $R_{0j\overline{\beta}} = 0$. Arguing with $\overline{Z}_0$ in place of Z_0, we get $R_{\alpha\overline{0j}} = 0$.

To prove the latter statement of (2°), we set

$$Z_\alpha = (Z_{\alpha_1}, \ldots, Z_{\alpha_p}), \quad \overline{Z}_\beta = (\overline{Z}_{\beta_1}, \ldots, \overline{Z}_{\beta_q})$$

for $\alpha = \alpha_1 \cdots \alpha_p$ and $\beta = \beta_1 \cdots \beta_q$. Using (6.6),

$$\begin{aligned}
&(\nabla_{\overline{Z}_0} R^{(p,q)})(Z_\alpha, \overline{Z}_\beta) - (L_{\overline{Z}_0} R^{(p,q)})(Z_\alpha, \overline{Z}_\beta) \\
&\qquad = -\sum_{s=1}^{q} R^{(p,q)}(Z_\alpha, \overline{Z}_{\beta_1}, \ldots, \nabla_{\overline{Z}_0} \overline{Z}_{\beta_s}, \ldots, \overline{Z}_{\beta_q}) \\
&\qquad = -q\, R^{(p,q)}(Z_\alpha, \overline{Z}_\beta).
\end{aligned}$$

This together with (6.7) yields $\nabla_{\overline{Z}_0} R^{(p,q)} = (1-q)R^{(p,q)}$, that is, $R_{\alpha\overline{\beta 0}} = (1-|\beta|)R_{\alpha\overline{\beta}}$. Arguing with $\overline{Z}_0$ in place of Z_0, we get $\nabla_{Z_0} R^{(p,q)} = (1-p)R^{(p,q)}$. Thus, using (6.6), we obtain

$$\begin{aligned} R_{\alpha 0\overline{\beta}} &= (R_{\alpha_1\alpha_2\overline{\beta}_1\overline{\beta}_2})_{\alpha_3\cdots\alpha_p 0\overline{\beta}_3\cdots\overline{\beta}_q} \\ &= (R_{\alpha_1\alpha_2\overline{\beta}_1\overline{\beta}_2})_{\alpha_3\cdots\alpha_p\overline{\beta}_3\cdots\overline{\beta}_q 0} = (1-p)R_{\alpha\overline{\beta}}. \qquad \square \end{aligned}$$

Lemma 6.9. *Let* $6 \le a+b \le 7$ *in* (6.5)*. Then* $R_{\alpha\overline{\beta}}R^{\alpha\overline{\beta}} = 0$ *at* e_0 *unless both of the following two conditions are satisfied.*

(i) $2 \le \mathrm{w}(R_{\alpha\overline{\beta}}) \le a+b-4$ *and* $w' \ne w''$*, where* $(w', w'') = \mathrm{w}_2(R_{\alpha\overline{\beta}})$.

(ii) $n_0(\gamma)+n_1(\gamma) \ge 2$ *and* $n_1(\gamma)+n_2(\gamma) \ge 2$ *for* $\gamma = \alpha, \beta$*, where* $n_j(\gamma)$ *denotes the number of* j*'s contained in a multi-index* γ*.*

Proof. Assuming $R_{a\overline{\beta}}R^{\alpha\overline{\beta}} \ne 0$ at e_0, let us prove (i) and (ii). Recall that $\|R^{(a,b)}\|^2$ is of biweight (w_0, w_0) with $w_0 = a+b-2$ and that $\mathrm{w}_2(R_{\alpha\overline{\beta}}) + \mathrm{w}_2(R^{\alpha\overline{\beta}}) = (w_0, w_0)$. It then follows from Lemma 6.8, (1°) that

$$4 \le w' + w'' \le 2\,w_0 - 4 \quad \text{and} \quad (w', w'') \ne (2,2), (w_0-2, w_0-2),$$

a condition which is equivalent to (i). To prove (ii), we use Lemma 6.8, (2°). The condition $n_1(\gamma)+n_2(\gamma) \ge 2$ for $\gamma = \alpha, \beta$ follows from $R_{\alpha\overline{\beta}} \ne 0$ at e_0, while the assumption $R^{\alpha\overline{\beta}} \ne 0$ at e_0 implies $n_0(\gamma)+n_1(\gamma) \ge 2$. $\square$

Observe that the condition (i) is symmetric in the entries of α (resp. β). The same applies to $R_{\alpha\overline{\beta}}$ and $R^{\alpha\overline{\beta}}$ by virtue of Lemma 6.6, because the condition (i) implies $2\,\mathrm{w}(R_{\alpha\overline{\beta}}) \le 6$ and $2\,\mathrm{w}(R^{\alpha\overline{\beta}}) \le 6$. Consequently, denoting by $\sigma(\alpha)$ the number of permutations of a multi-index α, we have

$$\|R^{(a,b)}\|^2 = \sideset{}{'}\sum_{|\alpha|=a, |\beta|=b} \sigma(\alpha)\sigma(\beta) R_{\alpha\overline{\beta}}R^{\alpha\overline{\beta}} \quad \text{at} \;\; e_0, \tag{6.8}$$

where the notation $\sum'$ means that the summation only extends over non-decreasing multi-indices. Hence, Lemma 6.9 is restated as follows.

Lemma 6.9′. *If* $6 \le a+b \le 7$*, then* (6.8) *holds, where the summation only extends over* α *and* β *satisfying the conditions* (i) *and* (ii) *in Lemma* 6.9*. Furthermore, for* $R^{\alpha\overline{\beta}}$ *in the right side of* (6.8)*,*

$$R^{\alpha\overline{\beta}} = (-1)^{n_1(\alpha)+n_1(\beta)} R_{\beta^*\overline{\alpha^*}} \quad \textit{at} \;\; e_0, \tag{6.9}$$

where α^* *and* β^* *are the dual indices of* α *and* β *defined in Subsection 3.5.*

The latter part of Lemma 6.9′ above follows from the Hermitian symmetry in Lemma 6.6 and the formula $g_0^{j\overline{k}} = (-1)^{n_1(j)}\delta^{j\,k^*}$, where $\delta^{i\,j}$ is

Kronecker's delta. Again, $R_{\beta^*\overline{\alpha^*}}$ in (6.9) is symmetric in the entries α^* (resp. β^*).

We next consider, for $a+b=6$,

$$\frac{d}{dt}\|R^{(a,b)}\|^2(\gamma_t^{\#})\Big|_{t=0} = \mathrm{Re}\left(\frac{\partial}{\partial\zeta_2}\|R^{(a,b)}\|^2\right)(e_0),$$

and compute the right side. Note that $\partial/\partial\zeta_2$ above can be replaced by ∇_2, the covariant differentiation along $\partial/\partial\zeta_2$. We have

$$\nabla_2(R_{\alpha\overline{\beta}}R^{\alpha\overline{\beta}}) = (\nabla_2 R_{\alpha\overline{\beta}})R^{\alpha\overline{\beta}} + R_{\alpha\overline{\beta}}(\nabla_2 R^{\alpha\overline{\beta}}),$$

while $\nabla_2 R_{\alpha\overline{\beta}} = R_{\alpha 2\overline{\beta}}$ at e_0 and $R_{\alpha\overline{\beta}}(\nabla_2 R^{\alpha\overline{\beta}}) = R^{\beta^*\overline{\alpha^*}}R_{\beta^* 2\overline{\alpha^*}}$ at e_0. Consequently,

$$\begin{aligned}\frac{d}{dt}\|R^{(a,b)}\|^2(\gamma_t^{\#})\Big|_{t=0} &= \mathrm{Re}\Bigg(\sum_{|\alpha|=a,|\beta|=b} R_{\alpha 2\overline{\beta}}(e_0)R^{\overline{\beta}\alpha}(e_0)\Bigg)\\ &\quad + \mathrm{Re}\Bigg(\sum_{|\beta|=a,|\alpha|=b} R_{\alpha 2\overline{\beta}}(e_0)R^{\overline{\beta}\alpha}(e_0)\Bigg).\end{aligned} \tag{6.10}$$

Now the proof of Lemma 6.9 yields the following lemma.

Lemma 6.10. *Let $a+b=6$. Then* (6.10) *holds, where $R_{\alpha 2\overline{\beta}}R^{\alpha\overline{\beta}} = 0$ at e_0 unless both of the following two conditions are satisfied.*

(i) $2 \le \mathrm{w}(R_{\alpha 2\overline{\beta}}) \le 3$ *and $w' \ne w''$, where $(w', w'') = \mathrm{w}_2(R_{\alpha 2\overline{\beta}})$.*

(ii) $n_0(\gamma)+n_1(\gamma) \ge 2$ *and $n_1(\gamma)+n_2(\gamma) \ge 2$ for $\gamma = \alpha 2, \beta$.*

In view of the symmetry again in the entries of α (resp. β), we get:

Lemma 6.10′. *If $a+b=6$, then*

$$\begin{aligned}&\frac{d}{dt}\|R^{(a,b)}\|^2(\gamma_t^{\#})\Big|_{t=0}\\ &\quad = \mathrm{Re}\Bigg({\sum_{|\alpha|=a,|\beta|=b}}' + {\sum_{|\beta|=a,|\alpha|=b}}'\Bigg)\sigma(\alpha)\sigma(\beta)R_{\alpha 2\overline{\beta}}R^{\alpha\overline{\beta}} \quad \text{at} \quad e_0,\end{aligned} \tag{6.11}$$

where both summations extend over α and β satisfying the conditions (i) *and* (ii) *in Lemma* 6.10*.*

We have thus obtained the expressions (6.8) and (6.11), where the summations are subject to the restrictions given by Lemmas 6.9′ and 6.10′, respectively. Our next task is to express the right sides of (6.8) and (6.11) in terms of $R_{11\overline{22}}$, $R_{111\overline{22}}$, $R_{12\overline{22}}$, $R_{112\overline{22}}$, $R_{11\overline{222}}$ and their complex conjugates evaluated at e_0. Let us first consider the terms in (6.8), and suppose that $R_{\alpha\overline{\beta}}R^{\alpha\overline{\beta}} \neq 0$ at e_0. Then α and β satisfy the conditions (i) and (ii) in Lemma 6.9. Recalling that $6 \leq |\alpha| + |\beta| \leq 7$, we find by inspection that both $\mathrm{w}_2(R_{\alpha\overline{\beta}})$ and $\mathrm{w}_2(R^{\alpha\overline{\beta}})$ must be one of the followings:

$$\begin{aligned} &(3,1),\ (4,1),\ (3,2),\ (4,2),\\ &(1,3),\ (1,4),\ (2,3),\ (2,4). \end{aligned} \tag{6.12}$$

Similarly for the terms in (6.11). That is, if $R_{\alpha 2\overline{\beta}}R^{\alpha\overline{\beta}} \neq 0$ at e_0 with $|\alpha| + |\beta| = 6$, then the possible values of $\mathrm{w}_2(R_{\alpha 2\overline{\beta}})$ and $\mathrm{w}_2(R^{\alpha\overline{\beta}})$ are given by (6.12).

Observe that the latter four cases in (6.12) are reduced to the former four cases by virtue of the Hermitian symmetry.

Lemma 6.11. *If* $\mathrm{w}(R_{1\alpha\overline{1\beta}}) \leq 3$ *and* $\mathrm{w}_2(R_{1\alpha\overline{1\beta}}) \neq (3,3)$, *then*

$$R_{1\alpha\overline{1\beta}} = R_{2\alpha\overline{0\beta}} + R_{0\alpha\overline{2\beta}} \quad \text{at} \ \ e_0. \tag{6.13}$$

Proof. By the definition of the Ricci tensor,

$$(\mathrm{Ric}_{\alpha_1\overline{\beta_1}})_{\alpha'\overline{\beta'}} = \left(\sum_{j,k=0}^{2} g^{j\overline{k}} R_{j\alpha_1\overline{k\beta_1}}\right)_{\alpha'\overline{\beta'}} = \sum_{j,k=0}^{2} g^{j\overline{k}} R_{j\alpha\overline{k\beta}},$$

where $\alpha = \alpha_1\alpha'$ and $\beta = \beta_1\beta'$. Thus, (6.13) is equivalent to

$$(\mathrm{Ric}_{\alpha_1\overline{\beta_1}})_{\alpha'\overline{\beta'}} = 0 \quad \text{at} \ \ e_0. \tag{6.13$'$}$$

Since the metric $g = g[r]$ is Kählerian, it follows from the relation $\det g = |z_0|^4 J[r]$ that

$$\mathrm{Ric}_{i\overline{j}} = \partial_{\zeta_i}\partial_{\overline{\zeta}_j} \log(\det g) = \partial_{\zeta_i}\partial_{\overline{\zeta}_j} \log J[r].$$

Recalling that $r = r^{\mathrm{F}}$ satisfies $J[r] = 1 + O(U^3)$, we see that $\log J[r]$ is of the form $f\,U^3$ with f smooth. Hence, (6.13)$'$ is equivalent to

$$\Big(\partial_{\alpha_1}\partial_{\overline{\beta}_1}(f\,U^3)\Big)_{\alpha'\overline{\beta'}} = 0 \quad \text{at} \ \ e_0. \tag{6.14}$$

By the assumption $\mathrm{w}(R_{1\alpha\overline{1\beta}}) \leq 3$, we have $n_2 \leq 3$ and $n_1 \leq 6 - 2n_2$, where $n_j = n_j(1\alpha\overline{1\beta})$. If $n_1(\alpha) = n_1(\beta) = 3 - n_2$, then $\mathrm{w}_2(R_{1\alpha\overline{1\beta}}) = (3,3)$, a contradiction. In other cases, (6.14) holds, and the proof is complete. $\square$

By using Lemmas 6.8 and 6.11, we have:

Lemma 6.12. *If* $\mathrm{w}_2(R_{\alpha\overline{\beta}}) = (3,1), (4,1), (3,2), (4,2)$ *then*

$$R_{\alpha\overline{\beta}} = c_{\alpha\overline{\beta}} R_{11\overline{22}}, \quad c_{\alpha\overline{\beta}} R_{111\overline{22}}, \quad c_{\alpha\overline{\beta}} R_{12\overline{22}}, \quad c_{\alpha\overline{\beta}} R_{112\overline{22}} + d_{\alpha\overline{\beta}} R_{11\overline{222}}$$

at e_0, respectively, where $c_{\alpha\overline{\beta}}$ and $d_{\alpha\overline{\beta}}$ are constants. Specifically,

$$\begin{gathered} c_{111\overline{12}} = -1, \quad c_{1111\overline{11}} = 2, \quad c_{1111\overline{12}} = -2, \quad c_{11111\overline{11}} = 6, \\ c_{11\overline{122}} = c_{112\overline{12}} = -1, \quad c_{111\overline{112}} = c_{1112\overline{11}} = 2, \quad c_{1111\overline{111}} = -6. \end{gathered}$$

Proof. Setting $(w', w'') = \mathrm{w}_2(R_{\alpha\overline{\beta}})$ and $w = \mathrm{w}(R_{\alpha\overline{\beta}})$, we recall that

$$n_1(\alpha) - n_1(\beta) = w' - w'', \quad n_2(\alpha) + n_2(\beta) = w + 1 - \frac{1}{2}\left(n_1(\alpha) + n_1(\beta)\right).$$

We eliminate $0, \overline{0}$ and $1\overline{1}$ in $\alpha\overline{\beta}$ repeatedly by using Lemmas 6.8 and 6.11. By the procedure of eliminating $1\overline{1}$, $n_1(\alpha) - n_1(\beta)$ is unchanged and $n_2(\alpha) + n_2(\beta)$ increases by 1. Both are invariant when 0 or $\overline{0}$ is eliminated. Hence, $R_{\alpha\overline{\beta}}$ is a linear combination of $R_{\alpha'\overline{\beta'}}$ with $\mathrm{w}_2(R_{\alpha'\overline{\beta'}}) = (w', w'')$ such that

$$n_1(\beta') = 0; \quad 0 \notin \alpha', \beta'; \quad |\alpha'|, |\beta'| \geq 2.$$

Enumerating possible $\alpha'\overline{\beta'}$ for each (w', w''), we obtain the former conclusion, the existence of $c_{\alpha\overline{\beta}}$ and $d_{\alpha\overline{\beta}}$. The latter half is elementary. ☐

6.5. Proof of Proposition 4′. We first prove the existence of the constants $c_0^{ab}, c_j^{ab}, d_j^{ab}$ for $j = 1, 2$, and then identify these except d_j^{ab}. In what follows, all the quantities $R_{\alpha\overline{\beta}}$, $R^{\alpha\overline{\beta}}$ and $\|R^{(a,b)}\|^2$ are evaluated at e_0.

Step 1 (*existence of the constants*). Let us first prove the existence of c_0^{ab} for $(a,b) = (4,2), (3,3)$. We use Lemma 6.9′, and find that if $R_{\alpha\overline{\beta}} R^{\alpha\overline{\beta}} \neq 0$ in (6.8) then $(w', w'') = \mathrm{w}_2(R_{\alpha\overline{\beta}})$ is either $(3,1)$ or $(1,3)$. Noting that $\mathrm{w}_2(R_{\alpha\overline{\beta}} R^{\alpha\overline{\beta}}) = (4,4)$, we see by Lemma 6.12 the existence of c_0.

Let us next prove the existence of $c_1^{ab}, c_2^{ab}, d_1^{ab}, d_2^{ab}$ for $(a,b) = (5,2)$, $(4,3)$, $(4,2)$, $(3,3)$. We begin with the case $(a,b) = (5,2)$ or $(4,3)$, and thus $\mathrm{w}_2(R_{\alpha\overline{\beta}} R^{\alpha\overline{\beta}}) = (5,5)$ in (6.8). Note by Lemma 6.9′ that $4 \leq w' + w'' = 2\,\mathrm{w}(R_{\alpha\overline{\beta}}) \leq 6$. If $w' + w'' = 4$ or 6 then $(w', w'') = (3,1), (1,3)$ or $(4,2), (2,4)$, respectively, and thus Lemma 6.12 implies

$$R_{\alpha\overline{\beta}} R^{\alpha\overline{\beta}} = \widetilde{F}[0, 0, d_1^{\alpha\overline{\beta}}, d_2^{\alpha\overline{\beta}}] \tag{6.15}$$

with some constants $d_1^{\alpha\overline{\beta}}$ and $d_2^{\alpha\overline{\beta}}$. If $w' + w'' = 5$ then $(w', w'') = (4,1)$, $(3,2)$, $(2,3)$, $(1,4)$, so that again Lemma 6.12 applies. We thus get the existence of c_j^{ab} and d_j^{ab} for $j = 1, 2$ in the case $(a,b) = (5,2)$ and $(4,3)$.

It remains to consider the case $(a, b) = (4, 2)$ or $(3, 3)$. We use Lemma 6.10$'$ in place of Lemma 6.9$'$. Note that $\mathrm{w}_2(R_{\alpha 2\overline{\beta}} R^{\alpha\overline{\beta}}) = (5, 5)$ and that $4 \le 2\,\mathrm{w}(R_{\alpha 2\overline{\beta}}) \le 6$. As before, if $2\,\mathrm{w}(R_{\alpha 2\overline{\beta}}) = 4$ or 6 then

$$R_{\alpha 2\overline{\beta}} R^{\alpha\overline{\beta}} = \widetilde{F}[0, 0, d_3^{\alpha\overline{\beta}}, d_4^{\alpha\overline{\beta}}] \tag{6.16}$$

with some constants $d_3^{\alpha\overline{\beta}}$ and $d_4^{\alpha\overline{\beta}}$. If $2\,\mathrm{w}(R_{\alpha 2\overline{\beta}}) = 5$ then Lemma 6.12 again applies, and the case $(a, b) = (4, 2)$ or $(3, 3)$ is also done. This completes the proof of the existence of the constants in Proposition 4$'$.

Step 2 (*listing possible* $\alpha\overline{\beta}$ *and* $\alpha 2\overline{\beta}$). For the terms in (6.8), we find by inspection that $R_{\alpha\overline{\beta}} R^{\alpha\overline{\beta}} = 0$ except for the following cases.

$$\begin{array}{lll}
(a,b) & = (4,2): & \alpha\overline{\beta} = 0022\overline{11},\ 1111\overline{11}; \\
(a,b) & = (3,3): & \alpha\overline{\beta} = 012\overline{111},\ 111\overline{012}; \\
(a,b) & = (5,2): & 2\,\mathrm{w}(R_{\alpha\overline{\beta}}) = 4, 6 \text{ or} \\
 & & \alpha\overline{\beta} = 11111\overline{11},\ 01112\overline{11},\ 00122\overline{11}; \\
(a,b) & = (4,3): & 2\,\mathrm{w}(R_{\alpha\overline{\beta}}) = 4, 6 \text{ or} \\
 & & \alpha\overline{\beta} = 1112\overline{011},\ 0112\overline{012},\ 1111\overline{111},\ 0111\overline{112},\ 1111\overline{012}, \\
 & & \qquad 0122\overline{011},\ 0022\overline{012},\ 0112\overline{111},\ 0012\overline{112},\ 0022\overline{111}.
\end{array}$$

If $a + b = 7$ and $2\,\mathrm{w}(R_{\alpha\overline{\beta}}) = 4, 6$ then $R_{\alpha\overline{\beta}} R^{\alpha\overline{\beta}}$ takes the form (6.15).

Similarly, we find for the terms in (6.11) that $R_{\alpha 2\overline{\beta}} R^{\alpha\overline{\beta}} = 0$ except for the following cases.

$$\begin{array}{lll}
(a,b) & = (4,2): & 2\,\mathrm{w}(R_{\alpha 2\overline{\beta}}) = 4, 6 \text{ or} \\
 & & \alpha 2\overline{\beta} = 01112\overline{11},\ 00122\overline{11},\ 1120\overline{012},\ 0120\overline{022}, \\
 & & \qquad 1120\overline{111},\ 0120\overline{112},\ 0121\overline{111}; \\
(a,b) & = (3,3): & 2\,\mathrm{w}(R_{\alpha 2\overline{\beta}}) = 4, 6 \text{ or} \\
 & & \alpha 2\overline{\beta} = 0022\overline{111},\ 0112\overline{012},\ 1112\overline{011},\ 0122\overline{011}, \\
 & & \qquad 0022\overline{012},\ 0112\overline{111},\ 0012\overline{112}.
\end{array}$$

If $2\,\mathrm{w}(R_{\alpha 2\overline{\beta}}) = 4, 6$ then $R_{\alpha 2\overline{\beta}} R^{\alpha\overline{\beta}}$ takes the form (6.16).

Step 3 (*identifying the constants*). Noting the Hermitian symmetry stated in Lemma 6.6, we have by Lemma 6.9$'$ and Step 2 above that

$$\begin{aligned}
\|R^{(4,2)}\|^2 &= 6\,|R_{0022\overline{11}}|^2 + |R_{1111\overline{11}}|^2 = 28\,|R_{11\overline{22}}|^2, \\
\|R^{(3,3)}\|^2 &= 6\,|R_{012\overline{111}}|^2 + 6\,|R_{111\overline{012}}|^2 = 12\,|R_{11\overline{22}}|^2,
\end{aligned}$$

where Lemma 6.8, (2°) and Lemma 6.12 are used in getting the second equalities. We thus get $c_0^{42} = 28$ and $c_0^{33} = 12$. Similarly, if $(a, b) = (5, 2)$ or (4,3) then

$$\|R^{(a,b)}\|^2 = T_1^{ab} + T_2^{ab} + \widetilde{F}[0, 0, d_1^{ab}, d_2^{ab}],$$

where $T_1^{52} = -|R_{11111\overline{11}}|^2$, $T_2^{52} = -30\,|R_{00122\overline{11}}|^2 - 20\,|R_{01112\overline{11}}|^2$ and

$$\begin{aligned} T_1^{43} = &- 6\,|R_{0022\overline{111}}|^2 - 6\,|R_{1111\overline{012}}|^2, \\ T_2^{43} = &- 36\,|R_{0022\overline{012}}|^2 - 72\,|R_{0112\overline{012}}|^2 \\ &- |R_{1111\overline{111}}|^2 - 12\,R_{1112\overline{011}}R_{112\overline{0111}} \\ &- 36\,R_{0012\overline{112}}R_{011\overline{0122}} - 12\,R_{0111\overline{112}}R_{011\overline{1112}} \\ &- 12\,|R_{0112\overline{111}}|^2 - 36\,R_{0122\overline{011}}R_{112\overline{0012}}. \end{aligned}$$

We then get $T_1^{ab} = c_1^{ab}|R_{111\overline{22}}|^2$ and $T_2^{ab} = c_2^{ab}|R_{12\overline{22}}|^2$ with $c_1^{52} = -36$, $c_2^{52} = -1800$, $c_1^{43} = -48$ and $c_2^{43} = -1140$.

It remains to identify c_1^{ab} and c_2^{ab} for $(a, b) = (4, 2)$ or $(3, 3)$. We now use Lemma 6.11 in place of Lemma 6.10. Then

$$\frac{d}{dt}\|R^{(a,b)}\|^2(\gamma_t^{\#})\Big|_{t=0} = \mathrm{Re}\Big(T_1^{ab} + T_2^{ab} + F[0, 0, d_1^{ab}, d_2^{ab}]\Big),$$

where $T_1^{42} = -2\,R_{012\overline{1111}}R_{1111\overline{12}}$, $T_1^{33} = -6\,R_{0022\overline{111}}R_{111\overline{022}}$ and

$$\begin{aligned} T_2^{42} = &- 4\,R_{01112\overline{11}}R_{11\overline{1112}} - 4\,R_{112\overline{0111}}R_{1112\overline{11}} \\ &- 12\,R_{012\overline{0022}}R_{0022\overline{12}} - 12\,R_{00122\overline{11}}R_{11\overline{0122}} \\ &- 12\,R_{112\overline{0012}}R_{0122\overline{11}} - 24\,R_{012\overline{0112}}R_{0112\overline{12}}, \\ T_2^{33} = &- 6\,R_{1112\overline{011}}R_{112\overline{111}} - 36\,R_{0112\overline{012}}R_{012\overline{112}} \\ &- 6\,R_{0112\overline{111}}R_{111\overline{112}} - 18\,R_{0012\overline{112}}R_{011\overline{122}} \\ &- 36\,R_{0022\overline{012}}R_{012\overline{022}} - 36\,R_{0122\overline{011}}R_{112\overline{012}}. \end{aligned}$$

As before, we then get $T_1^{ab} = c_1^{ab}|R_{111\overline{22}}|^2$ and $T_2^{ab} = c_2^{ab}|R_{12\overline{22}}|^2$ with $c_1^{42} = 8, c_2^{42} = 416, c_1^{33} = 12$ and $c_2^{33} = 324$. Therefore, all necessary constants are identified, and the proof of Proposition 4′ is complete.

7. Microlocal calculus of the Bergman and the Szegö kernels

7.1. Method of computation. Let us use the notations in Subsections 3.6–3.8, so that $K_M^{(3)}$ and $K_M^{(2)}$ are the singularities of $(\pi^2/2)K^{\mathrm{B}}$ and $\pi^2 K^{\mathrm{S}}$, respectively, for $M = N(A)$. Recalling the polynomial dependence of $K_M^{(w)}$ for $w = 2, 3$ on A, we assume that M is real analytic. To prove Proposition 6, we use the following formulas in [2–4] and [15]:

$$K_M^{(3)}(z, \overline{z}) = (A^{\mathrm{B}})^{*\,-1}U_0^{-3}, \quad K_M^{(2)}(z, \overline{z}) = (A^{\mathrm{S}})^{*\,-1}U_0^{-2}, \tag{7.1}$$

where $A^{\rm B} = A^{\rm B}(z, \partial_z)$ and $A^{\rm S} = A^{\rm S}(z, \partial_z)$ are elements of $\mathcal{M}_A^\infty$ defined by

$$\begin{aligned} A^{\rm B}(z,\zeta) &= \exp[-H_B(z, -\zeta_1/\zeta_2)\zeta_2], \\ A^{\rm S}(z,\zeta) &= V^{\rm S}(z, -\zeta_1/\zeta_2) A^{\rm B}(z,\zeta) \end{aligned} \tag{7.2}$$

with $V^{\rm S}(z, \overline{z}_1) = J[U_B]^{1/3}$ which is independent of the variable $\overline{z}_2$; in fact,

$$J[U_B] = \det \begin{pmatrix} \partial U_B/\partial z_1 & \partial^2 U_B/\partial z_1 \partial \overline{z}_1 \\ \partial U_B/\partial z_2 & \partial^2 U_B/\partial z_2 \partial \overline{z}_1 \end{pmatrix}.$$

Recall that $Q \in \mathcal{M}_A^\infty(1)$ for $Q = A^{\rm B}, A^{\rm S}$ implies the existence of $Q^{*\,-1} \in \mathcal{M}_A^\infty(1)$ which is given by the Neumann series $\sum (1-Q^*)^k$. This expression of $Q^{*\,-1}$ enables us to compute explicitly the asymptotic expansion of $K_M^{(w)}$ for $w = 3, 2$. (Thus the real analyticity assumption on M can be removed.)

Let us add a remark on the formulas in (7.1). It is easy to see that the operators $A^{\rm B}$, $A^{\rm S}$ defined by (7.2) satisfy

$$A^{\rm B}(z, \partial_z) \log U_0 = \log U_B, \quad A^{\rm S}(z, \partial_z) U_0^{-1} = V^{\rm S}(z, \overline{z}_1) U_B^{-1}, \tag{7.3}$$

while the point proved in [2–4] and [15] is that (7.3) implies (7.1). This implication for $K_M^{(3)}$ is a consequence of Kashiwara's characterization in [16] of constant multiples of $K^{(3)}$ by a simple holonomic system of holomorphic microdifferential equations, and the same idea applies also to $K_M^{(2)}$. In (7.3), $\log U_B$ represents microlocally a constant multiple of the Heaviside function of a domain Ω with $M \subset \partial\Omega$, and similarly for $V^{\rm S}(z, \overline{z}_1) U_B^{-1}$ which corresponds to the delta measure supported on M with respect to the invariant surface element defining the Szegö kernel $K^{\rm S}$, cf. [2–4] and [15].

7.2. Proof of Proposition 6. By virtue of Lemma 5.1, we see that (2.4) holds with some constants $c_{pq}[\psi^{\rm B}]$ and $c_{pq}[\psi^{\rm S}]$. It remains only to determine these constants. We shall show that

$$\begin{aligned} c_{pq}[\psi^{\rm B}] = &\ \frac{(-1)^{p+q+1}}{2\cdot(p+q-5)!} \left\{ (p+q)! - \frac{(p!)^2}{(p-q)!} \right\}, \\ c_{pq}[\psi^{\rm S}] = &\ \frac{(-1)^{p+q}}{(p+q-4)!} \left\{ (p+q)! - \frac{2\,p\,q}{3}\,(p+q-1)! \right. \\ &\left. \qquad - \frac{2\,(p\,q)^2}{9}\,(p+q-2)! - \left(1 - \frac{q}{3}\right)^2 \frac{(p!)^2}{(p-q)!} \right\}. \end{aligned}$$

Let us begin by noting that $H(z, \overline{z}_1) = F_{pq}(z_1, \overline{z}_1)$ (cf. Subsection 5.4 for the notation F_{pq}), so that the symbols $A^{\rm B}(z,\zeta)$ and $A^{\rm S}(z,\zeta)$ are independent of the variable z_2, because

$$J[U] = \Delta_1 U = 1 - \Delta_1 F_{pq}.$$

We thus write $V^{\mathrm{S}}(z_1,\overline{z}_1)$, $A^{\mathrm{B}}(z_1,\zeta)$ and $A^{\mathrm{S}}(z_1,\zeta)$ in place of $V^{\mathrm{S}}(z,\overline{z}_1)$, $A^{\mathrm{B}}(z,\zeta)$ and $A^{\mathrm{S}}(z,\zeta)$, respectively. Setting

$$x_1 = A^0_{p\overline{q}}\, z_1^p(-\zeta_1/\zeta_2)^q\zeta_2, \qquad x_1' = p\,q\,A^0_{p\overline{q}}\, z_1^{p-1}(-\zeta_1/\zeta_2)^{q-1},$$
$$x_2 = A^0_{q\overline{p}}\, z_1^q(-\zeta_1/\zeta_2)^p\zeta_2, \qquad x_2' = p\,q\,A^0_{q\overline{p}}\, z_1^{q-1}(-\zeta_1/\zeta_2)^{p-1},$$

we have $A^{\mathrm{B}}(z_1,\zeta) = \exp[-(x_1+x_2)] = 1-(x_1+x_2)+x_1\,x_2+\cdots$ and

$$V^{\mathrm{S}}(z_1,-\zeta_1/\zeta_2) = (1+x_1'+x_2')^{1/3} = 1+\frac{1}{3}(x_1'+x_2')-\frac{2}{9}x_1'\,x_2'+\cdots,$$

where $\cdots$ stands for terms containing $(A^0_{p\overline{q}})^2$ or $(A^0_{q\overline{p}})^2$. Consequently,

$$A^{\mathrm{S}}(z_1,\zeta) = V^{\mathrm{S}}(z_1,-\zeta_1/\zeta_2)A^{\mathrm{B}}(z_1,\zeta) = 1-y_1-y_2-y_3+\cdots,$$

where $y_1 = x_1 - x_1'/3$, $y_2 = x_2 - x_2'/3$ and

$$y_3 = -x_2\,x_2+\frac{2}{9}\,x_1'\,x_2'+\frac{1}{3}(x_1'\,x_2+x_2'\,x_1).$$

Recalling that each y_j is a function of (z_1,ζ), we denote by Y_j' the formal adjoint of $y_j(z_1,\partial_z)$. Then $(A^{\mathrm{S}})^*(z_1,\partial_z) = 1-Y_1'-Y_2'-Y_3'+\cdots$, so that

$$(A^{\mathrm{S}})^{*\,-1}(z_1,\partial_z) = 1+Y_1'+Y_2'+(Y_3'+Y_1'\,Y_2'+Y_2'\,Y_1')+\cdots.$$

So far, we have only neglected higher order terms in $A^0_{p\overline{q}}$ and $A^0_{q\overline{p}}$. If we denote by $\cdots$ also terms which do not contain $|A^0_{p\overline{q}}|^2$, then

$$(A^{\mathrm{S}})^{*\,-1}(z_1,\partial_z) = Y_3'+Y_1'\,Y_2'+Y_2'\,Y_1'+\cdots. \tag{7.4}$$

A similar expression for $(A^{\mathrm{B}})^{*\,-1}(z_1,\partial_z)$ is obtained by formally setting $x_1' = x_2' = 0$ in (7.4).

Let us compute the right side of (7.4) at $z_1 = 0$. In order to treat the Bergman kernel case simultaneously with the Szegö kernel case, it is convenient to set

$$B^0_{p\overline{q}} = \frac{p\,q}{3}\,A^0_{p\overline{q}} = \overline{B^0_{q\overline{p}}}, \tag{7.5}$$

so that $x_1'/3 = B^0_{p\overline{q}}\,z_1^{p-1}(-\zeta_1/\zeta_2)^{q-1}$, $x_2'/3 = B^0_{q\overline{p}}\,z_1^{q-1}(-\zeta_1/\zeta_2)^{p-1}$ and thus

$$\begin{aligned}
y_1 =& A^0_{p\overline{q}}\,z_1^p(-\zeta_1/\zeta_2)^q\zeta_2 - B^0_{p\overline{q}}\,z_1^{p-1}(-\zeta_1/\zeta_2)^{q-1},\\
y_2 =& A^0_{q\overline{p}}\,z_1^q(-\zeta_1/\zeta_2)^p\zeta_2 - B^0_{q\overline{p}}z_1^{q-1}(-\zeta_1/\zeta_2)^{p-1},\\
y_3 =& -|A^0_{p\overline{q}}|^2 z_1^{p+q}(-\zeta_1/\zeta_2)^{p+q}\zeta_2^2 + 2|B^0_{p\overline{q}}|^2 z_1^{p+q-2}(-\zeta_1/\zeta_2)^{p+q-2}\\
&+(B^0_{p\overline{q}}\,A^0_{q\overline{p}} + B^0_{q\overline{p}}\,A^0_{p\overline{q}})z_1^{p+q-1}(-\zeta_1/\zeta_2)^{p+q-1}\zeta_2.
\end{aligned}$$

(The notation $B^0_{p\overline{q}}$ here is a tentative one and not for the complex normal form coefficient.) Then,

$$
\begin{aligned}
Y_1' =& (A^0_{p\overline{q}}\partial_1^q z_1^p - B^0_{p\overline{q}}\,\partial_1^{q-1} z_1^{p-1})(-\partial_2)^{1-q},\\
Y_2' =& (A^0_{q\overline{p}}\partial_1^p\, z_1^q - B^0_{q\overline{p}}\partial_1^{p-1}\, z_1^{q-1})(-\partial_2)^{1-p},\\
Y_3' =& \Big\{-|A^0_{p\overline{q}}|^2\partial_1^{p+q} z_1^{p+q} + 2|B^0_{p\overline{q}}|^2\partial_1^{p+q-2} z_1^{p+q-2}\\
&+ (B^0_{p\overline{q}}A^0_{q\overline{p}} + B^0_{q\overline{p}}A^0_{p\overline{q}})\partial_1^{p+q-1} z_1^{p+q-1}\Big\}(-\partial_2)^{2-p-q},
\end{aligned}
$$

where the powers of z_1 act as multiplication operators. We can now evaluate Y_3', $Y_1'\,Y_2'$ and $Y_2'\,Y_1'$ at $z_1 = 0$. Recalling $p > q$, we see that $Y_1' = 0$ and thus $Y_1'Y_2' = 0$ both at $z_1 = 0$. We also have, at $z_1 = 0$,

$$
\begin{aligned}
Y_3' =& \Big\{ -(p+q)!\,|A^0_{p\overline{q}}|^2 + 2\cdot(p+q-2)!\,|B^0_{p\overline{q}}|^2\\
&+ (p+q-1)!\,(B^0_{p\overline{q}}A^0_{q\overline{P}} + B^0_{q\overline{p}}\,A^0_{p\ q})\Big\}(-\partial_2)^{2-p-q},\\
Y_2' =& \left\{\frac{p!}{(p-q)!}\,A^0_{q\overline{p}} - \frac{(p-1)!}{(p-q)!}\,B^0_{q\overline{p}}\right\}\partial_1^{p-q}(-\partial_2)^{1-p},\\
Y_2'\,Y_1' =& \frac{1}{(p-q)!}|p\,A^0_{p\overline{q}} - (p-1)!\,B^0_{p\overline{q}}|^2(-\partial_2)^{2-p-q}.
\end{aligned}
$$

It then follows from (7.4) that, at $z_1 = 0$,

$$
(A^{\mathrm{S}})^{*\,-1}(0,\partial_z) = (Y_3' + Y_2'\,Y_1')\Big|_{z_1=0} = \widetilde{c}_{pq}[\psi^{\mathrm{S}}](-\partial_2)^{2-p-q}, \tag{7.6}
$$

where, with $B^0_{p\overline{q}}$ and $B^0_{q\overline{p}}$ as in (7.5),

$$
\begin{aligned}
\widetilde{c}_{pq}[\psi^{\mathrm{S}}] =& -(p+q)!\,|A^0_{p\overline{q}}|^2 + 2\cdot(p+q-2)!\,\left|B^0_{p\overline{q}}\right|^2\\
&+ (p+q-1)!\left(B^0_{p\overline{q}}\,A^0_{q\overline{p}} + B^0_{q\overline{p}}\,A^0_{p\overline{q}}\right)\\
&+ \frac{1}{(p-q)!}\left|p!\,A^0_{p\overline{q}} - (p-1)!\,B^0_{p\overline{q}}\right|^2.
\end{aligned}
$$

If we formally set $B^0_{p\overline{q}} = B^0_{q\overline{p}} = 0$ in (7.6) above, we get

$$
\left(A^{\mathrm{B}}\right)^{*\,-1}(0,\partial_z) = \widetilde{c}_{pq}[\psi^{\mathrm{B}}]\,(-\partial_2)^{2-p-q}, \tag{7.7}
$$

where

$$
\widetilde{c}_{pq}[\psi^{\mathrm{B}}] = \left\{-(p+q)! + \frac{(p!)^2}{(p-q)!}\right\}\left|A^0_{p\overline{q}}\right|^2.
$$

The conclusion follows from (7.5)–(7.7) via the well-known formula

$$
\partial_2^{-\ell}U_0^{-m} = \frac{(-1)^{m+1}}{(m-1)!\,(\ell-m)!}\,U_0^{\ell-m}\log U_0 \quad \text{for} \quad \ell > m > 0.
$$

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