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Invariant theory of the Bergman kernel

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Dedicated to Professor M. Kuranishi on his 70th birthday

Introduction

This article is a brief report of recent developments in Fefferman's program, proposed and initiated in [F3], concerning invariant expression of the singularity of the Bergman kernel K^{B} on the diagonal of a strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$ with smooth boundary. It was proved by Fefferman in [F1] that

$$K^{\mathrm{B}} = \frac{\varphi^{\mathrm{B}}}{r^{n+1}} + \psi^{\mathrm{B}} \log r \quad \text{with } \varphi^{\mathrm{B}}, \psi^{\mathrm{B}} \in C^\infty(\overline{\Omega}), \tag{0.1}$$

where $r \in C^\infty$ is a defining function of the boundary $\partial\Omega$ such that $r > 0$ in Ω and $dr \neq 0$ on $\partial\Omega$. The problem is to choose r appropriately and express φ^{B} modulo $O^{n+1}(r)$ and ψ^{B} modulo $O^\infty(r)$ invariantly in the sense of local biholomorphic geometry. This can be compared with the asymptotic expansion of the heat kernel associated with the diagonal of a compact Riemannian manifold, where the time variable corresponds to the function r in (0.1). The boundary $\partial\Omega$ is approximated at every point by a sphere (hyperquadric), and carries a differential geometric structure, called the CR (or pseudoconformal) structure.

Let us employ an extrinsic approach due to Chern and Moser in [CM], [M], and put the boundary $\partial\Omega$ (formally) in Moser's normal form $N(A)$ with $A = (A^\ell_{\alpha\overline{\beta}})$ given by

$$2\,\mathrm{Re}\, z_n = |z'|^2 + \sum_{|\alpha|,|\beta|\geq 2} \sum_{\ell=0}^{\infty} A^\ell_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, (\mathrm{Im}\, z_n)^\ell,$$

where $z = (z', z_n) = (z_1, \ldots, z_{n-1}, z_n) \in \mathbf{C}^n$. (For the notations z'_α and $|\alpha|$ with ordered multi-indices α, see Subsection 1.1, (B) below.)

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Then CR invariants of weight $w \in \mathbf{N}_0 = \{0, 1, 2, \dots\}$ are defined as polynomials $P = P(A)$ satisfying the transformation law

$$P(A) = P(\widetilde{A})\,|\det \Phi'(0)|^{2w/(n+1)} \tag{0.2}$$

for local (or formal) biholomorphic mappings Φ such that $\Phi(N(A)) = N(\widetilde{A})$ and $\Phi(0) = 0$. We wish to express the asymptotic expansions

$$\begin{aligned} \varphi^{\mathrm{B}} &= \sum_{k=0}^{n} \varphi_k\, r^k \mod O^{n+1}(r), \qquad && \varphi_k \in C^\infty(\overline{\Omega}), \\ \psi^{\mathrm{B}} &= \sum_{k=0}^{\infty} \psi_k\, r^k \mod O^{\infty}(r), \qquad && \psi_k \in C^\infty(\overline{\Omega}), \end{aligned} \tag{0.3}$$

of φ^{B} and ψ^{B} in (0.1) in terms of CR invariants. We thus consider local (or localizable) domain functionals $K = K_\Omega$ near a reference point at the boundary $\partial\Omega$ satisfying a transformation law of weight $w \in \mathbf{Z}$:

$$K_{\Omega_1} = K_{\Omega_2} \circ \Phi\,|\det \Phi'|^{2w/(n+1)} \tag{0.4}$$

for local biholomorphic mappings $\Phi : \Omega_1 \to \Omega_2$ preserving the reference points, cf. (0.2).

The Bergman kernel K^{B} satisfies (0.4) with $w = n+1$. If one could find a defining function r satisfying (0.4) with $w = -1$, then there would be a hope to have expansions as in (0.3) such that φ_k for $k \le n$ and ψ_{k-n-1} for $k \ge n+1$ satisfy (0.4) with $w = k$. According to Hörmander [Hö], the boundary value of φ^{B} agrees with that of the Levi determinant

$$J[r] = (-1)^n \det \begin{pmatrix} r & \partial r/\partial \overline{z}_k \\ \partial r/\partial z_j & \partial^2 r/\partial z_j\, \partial \overline{z}_k \end{pmatrix}$$

multiplied by $n!/\pi^n$. Thus we are led to the zero Dirichlet boundary value problem for the complex Monge-Ampère equation

$$J[u] = 1 \quad \text{and} \quad u > 0 \quad \text{in} \quad \Omega; \qquad u = 0 \quad \text{on} \quad \partial\Omega. \tag{0.5}$$

According to Fefferman [F2], any solution of $J[u] = 1$ satisfies (0.4) with $w = -1$. However, the solution of (0.5), of which the unique existence is guaranteed by Cheng and Yau in [CY], has a finite differentiability up to the boundary. This fact is seen from the asymptotic expansion below due to Lee and Melrose in [LM] (cf. also Graham [G2]):

$$u = r \sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1} \log r)^k, \qquad \eta_k \in C^\infty(\overline{\Omega}), \tag{0.6}$$

with a C^∞ defining function r as before.

There are C^∞ approximate solutions $r = r^{\mathrm{F}}$ of (0.5) satisfying

$$J[r^{\mathrm{F}}] = 1 + O^{n+1}(r) \quad \text{near} \ \ \partial\Omega \qquad (r = r^{\mathrm{F}} > 0 \quad \text{in} \ \ \Omega).$$

In [F2], Fefferman gave an explicit algorithm of constructing such a function r^{F}. Let us refer to these r^{F} as Fefferman's defining functions. After reviewing quickly in Section 1 the background of the problem which contains expositions of CR invariants, the Bergman kernel and the complex Monge-Ampère boundary value problem, we state in Section 2 Fefferman's main results in [F3], which were supplemented recently by Bailey, Eastwood and Graham in [BEG], on the expansion of φ^{B} in (0.3) by using Fefferman's defining function $r = r^{\mathrm{F}}$. Local domain functionals, called Weyl invariants, of weight $\le n$ are defined by using the curvature of the Lorentz-Kähler metric with potential function $|z_0|^2\, r^{\mathrm{F}}(z)$ on a bundle $\mathbf{C}^* \times \overline{\Omega}$ (or a neighborhood of $\mathbf{C}^* \times \partial\Omega$) with an extra variable $z_0 \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$. It is proved in [F3] and [BEG] that any CR invariant of weight $\le n$ is realized as the boundary value of a Weyl invariant and that the expansion of φ^{B} in (0.3) with $r = r^{\mathrm{F}}$ is valid, where each φ^{B}_k is a Weyl invariant of weight k. Proofs of these results are outlined in Section 5.

The two dimensional case is exceptional and it is possible to obtain a very precise result by using Fefferman's defining function $r = r^{\mathrm{F}}$. We overview in Section 3 the work of Graham in [G1] and [G2] supplemented by the authors' joint work with Nakazawa in [HKN1] and [HKN2]. There are no nonzero CR invariants of weight 1, 2, and the expansion of φ^{B} in (0.3) with $r = r^{\mathrm{F}}$ is trivial, that is, $\varphi^{\mathrm{B}} = 2/\pi^2 + O^3(r)$. For ψ^{B} in (0.1), it is shown in [G1] and [HKN2] that

$$\psi^{\mathrm{B}} = \frac{2}{\pi^2}\left(-3\,\eta_1 + W_4\, r + W_5\, r^2\right) + O^3(r) \quad \text{with} \ \ r = r^{\mathrm{F}},$$

where W_k for $k = 4, 5$ are Weyl invariants of weight k and η_1 is that in (0.6) with $r = r^{\mathrm{F}}$. This result is best possible as far as Fefferman's defining function is used. Explicit determination of W_4 and W_5 is also done in [HKN2] (partial results are found in [G1] and [HKN1]). In order to identify universal constants appearing in W_4 and W_5, it is necessary to express the singularity of the Bergman kernel in terms of Moser's normal form coefficients. This is done in [HKN1] and [HKN2] by using microlocal calculus due to Kashiwara in [Kas] and Boutet de Monvel in [B1]–[B3]. We explain this method in Section 4.

In order to get a complete expansion of ψ^{B} as in (0.3), it is necessary to take account of the ambiguity of $r = r^{\mathrm{F}}$. In [Hi], a special family of Fefferman's defining functions parametrized by $C^\infty(\partial\Omega)$ (or rather the space of formal power series) is so defined as to satisfy (0.4) with $w=-1$. This family leads to the definition of Weyl invariants with ambiguity measured by $C^\infty(\partial\Omega)$. It is proved in [Hi] that the space CR invariants of arbitrary weight exactly corresponds to that of Weyl invariants without ambiguity and that the expansion of ψ^{B} in (0.3) with a Fefferman's defining function r parametrized by $C^\infty(\partial\Omega)$ is valid, where each ψ^{B}_k is a Weyl invariant, with ambiguity, of weight $k+n+1$. This expansion of ψ^{B} is invariant in the sense that each Weyl invariant with ambiguity measured by $C^\infty(\partial\Omega)$ is a universal polynomial of $A=(A^\ell_{\alpha\overline{\beta}})$ and $C=(C^\ell_{\alpha\overline{\beta}})$, where $A^\ell_{\alpha\overline{\beta}}$ are Moser's normal form coefficients and $C^\ell_{\alpha\overline{\beta}}$ appear as the coefficients of the power series expansion of an element $f\in C^\infty(\partial\Omega)$, that is,

$$f(z',\overline{z'},\operatorname{Im} z_n)=\sum_{|\alpha|,|\beta|\geq 0}\sum_{\ell=0}^{\infty} C^\ell_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\,(\operatorname{Im} z_n)^\ell.$$

In Section 6, we state these results more precisely and outline the proofs.

In this article, we restrict ourselves to the local analysis of the Bergman kernel associated with a general strictly pseudoconvex domain, and do not refer to related topics. Here we only mention two of these. The first one is an analogue of Fefferman's program above for the Szegö kernel associated with an invariant surface element on the boundary of a strictly pseudoconvex domain. This problem was also posed in [F3], and the analysis of the Bergman kernel presented in this article applies to the Szegö kernel as well, after a slight modification (cf. [HKN1], [HKN2]). Another topic is a conformal analogue of the construction of CR invariants in terms of Weyl invariants. This problem was posed by Fefferman and Graham in [FG]. For recent progress of this topic, the reader should see the papers by Bailey-Eastwood-Graham [BEG] and by Eastwood-Graham [EG]; there are also comprehensive survey articles by Graham [G3] and by Bailey [Ba].

§1 Backgrounds

1.1 CR invariants

(A) Local boundary equivalence problem. A remarkable phenomenon in Several Complex Variables is the existence of a domain Ω (in fact, many domains) such that all holomorphic functions in Ω extend holomorphically across a part of the boundary $\partial\Omega$ to a larger domain simultaneously. If such a phenomenon does never occur for Ω, then Ω is called a *domain of holomorphy*. Assume for simplicity that Ω is a domain in $\mathbf{C}^n$ with C^∞ boundary. That is, $\Omega = \{r > 0\}$, where $r \in C^\infty(\mathbf{C}^n, \mathbf{R})$ is a *defining function* of the boundary $\partial\Omega$ and thus $|dr| > 0$ on $\partial\Omega$. A well-known theorem of Oka states that Ω is a domain of holomorphy if and only if it is *pseudoconvex* at every boundary point. The pseudoconvexity at $z \in \partial\Omega$ is by definition the non-negativity of the eigenvalues of the *Levi form* of r at $z = (z_1, \dots, z_n)$ given by

$$L_{r,z}(\xi, \overline{\xi}) = -\sum_{j,k=1}^{n} \frac{\partial^2 r(z)}{\partial z_j \partial \overline{z}_k}\, \xi_j\, \overline{\xi}_k \quad \text{for } \xi = (\xi_1, \dots, \xi_n) \in T_z^{1,0}(\partial\Omega),$$

where $T_z^{1,0}(\partial\Omega) = \left\{\xi \in \mathbf{C}^n;\ \sum_{j=1}^n \xi_j\, \partial r(z)/\partial z_j = 0\right\}$, and thus each element $\xi \in T_z^{1,0}(\partial\Omega)$ is identified with a $(1,0)$-vector $\sum \xi_j\, \partial/\partial z_j$ which is tangential to $\partial\Omega$ at z. If the Levi form is positive-definite on $\partial\Omega$, then Ω is said to be *strictly pseudoconvex*. The notion of (strict) pseudoconvexity is defined independently of the choice of r.

Let Ω_1 and Ω_2 be strictly pseudoconvex domains in $\mathbf{C}^n$ with C^∞ boundaries. If there exists a biholomorphic mapping $\Phi : \Omega_1 \to \Omega_2$, then Ω_1 and Ω_2 are said to be *holomorphically equivalent*. When are Ω_1 and Ω_2 holomorphically equivalent? A necessary condition is formulated via a theorem of Fefferman [F1] which states that if Φ as above exists then Φ extends to a C^∞ diffeomorphism from $\overline{\Omega}_1$ to $\overline{\Omega}_2$. (If the boundaries are real analytic, then Φ extends biholomorphically across the boundaries, cf. Lewy [L2].) Thus one can compare the boundaries. The boundary value of Φ is a diffeomorphism $\Phi_0 : \partial\Omega_1 \to \partial\Omega_2$ such that the components are CR functions, those functions which are annihilated by differentiation with respect to sections of the bundle $T^{0,1}(\partial\Omega) = \overline{T^{1,0}(\partial\Omega)}$. Suppose now we are given Φ_0, a CR diffeomorphism. If the boundaries are real analytic, then Φ_0 has an analytic extension to a full neighborhood of $\partial\Omega_1$. In general, Φ_0 extends holomorphically to Ω_1 according to a theorem of Lewy [L1]. These are in fact local results, and one is led to a *local boundary equivalence problem* of comparing open portions M_j of $\partial\Omega_j$ $(j = 1, 2)$, which are strictly pseudoconvex real hypersurfaces. That is, one asks when there exists a CR diffeomorphism $\Phi_0 : M_1 \to M_2$ such that $\Phi_0(p_1) = p_2$, where the pairs (M_j, p_j) with $p_j \in M_j$ are prescribed. In what follows, we mainly consider the real analytic case, and identify Φ_0 with its holomorphic extension Φ. More precisely, we regard Φ as a germ of mapping between germs of surface (M_j, p_j). In the C^∞ case, we regard Φ as a formal mapping given by formal power series between C^∞ surfaces (M_j, p_j), and thus we are only concerned with the Taylor expansions of defining functions of M_j about the reference points

$p_j \in M_j$.

(B) Moser's normal form. Let $M \subset \mathbf{C}^n$ be a strictly pseudoconvex real hypersurface containing the origin $0 \in \mathbf{C}^n$ as a reference point, and assume first that M is real analytic. To study the biholomorphic equivalence problem of M in the previous subsection, Moser [M], [CM] introduced the notion of normal form of M defined as follows.

For the standard coordinate system $z = (z_1, \ldots, z_n)$ in $\mathbf{C}^n$, we write $z = (z', z_n)$ and set $z'_\alpha = z_{\alpha_1} \cdots z_{\alpha_a}$, where $\alpha = (\alpha_1, \ldots, \alpha_a)$ is an ordered multi-index of length $|\alpha| = a$, that is, $\alpha_j \in \{1, \ldots, n-1\}$ for $j = 1, \ldots, a$. After a holomorphic change of coordinates, M is locally written near the origin as

$$2u = |z'|^2 + F_A(z', \overline{z'}, v), \qquad z_n = u + i\,v, \tag{1.1}$$

where F_A is a real analytic function having the Taylor expansion

$$F_A(z', \overline{z'}, v) = \sum_{|\alpha|+|\beta|+2\ell \geq 3} A^\ell_{\alpha\overline{\beta}}\, z'_\alpha\, \overline{z'_\beta}\, v^\ell = \sum_{\alpha,\,\beta} A_{\alpha\overline{\beta}}(v)\, z'_\alpha\, \overline{z'_\beta}\,.$$

(The meaning of the subscript A in F_A will be made clear just after the definition of Moser's normal form.) We say that M given by (1.1) is in *pre-normal form* if $\overline{A_{\alpha\overline{\beta}}(v)} = A_{\beta\overline{\alpha}}(v)$ hold for all α, β and each $A_{\alpha\overline{\beta}}(v)$ is unchanged under permutation of α and that of β. These normalizations are always possible.

By another change of coordinates, M in pre-normal form is made to satisfy $A_{\alpha\overline{\beta}}(v) = 0$ when $|\alpha| < 2$ or $|\beta| < 2$, and thus

$$F_A(z', \overline{z'}, v) = \sum_{|\alpha|,|\beta| \geq 2} A_{\alpha\overline{\beta}}(v)\, z'_\alpha\, \overline{z'_\beta}, \quad A_{\alpha\overline{\beta}}(v) = \sum_{\ell=0}^{\infty} A^\ell_{\alpha\overline{\beta}}\, v^\ell. \tag{1.2}$$

Definition. A surface M in pre-normal form given by (1.1) is said to be in *Moser's normal form* if (1.2) holds and the following trace conditions are fulfilled:

$$\operatorname{tr} A_{2\overline{2}}(v) = 0, \quad (\operatorname{tr})^2 A_{2\overline{3}}(v) = 0, \quad (\operatorname{tr})^3 A_{3\overline{3}}(v) = 0. \tag{1.3}$$

Here, $A_{a\bar{b}}(v) = (A_{\alpha\overline{\beta}}(v))_{|\alpha|=a, |\beta|=b}$, and $(\operatorname{tr})^m A_{a\bar{b}}(v)$ for $m = 1, 2, 3$ means that the contractions with respect to Kronecker's delta $\delta^{j\bar{k}}$ are taken m times for the indices α, β in $A_{\alpha\overline{\beta}}(v)$ with $|\alpha| = a$, $|\beta| = b$.

If M is a surface in Moser's normal form, we write $M = N(A)$ and $A \in \mathcal{N}$, where $A = (A^\ell_{\alpha\overline{\beta}})$ is a collection of the coefficients in (1.2). Thus

$\mathcal{N}$ is the vector space of all collections A giving Moser's normal forms. We may identify a surface $N(A)$ with $A \in \mathcal{N}$.

The existence of Moser's normal form is guaranteed as follows.

Theorem 1.1 ([CM], [M]). *For any M in pre-normal form, there exists a holomorphic change of coordinates $w = \Phi(z)$ such that $\Phi(M)$ is in Moser's normal form. The mapping Φ is unique under the conditions*

$$\Phi(0) = 0, \quad \Phi'(0) = \text{identity}, \quad \operatorname{Im}\left(\partial^2 w_n(0)/\partial z_n^2\right) = 0,$$

where Φ' denotes the holomorphic differential of Φ.

According to Theorem 1.1, there exists a holomorphic coordinate system $z = (z', z_n)$ such that M is in Moser's normal form $N(A)$. We refer to z', z_n as *Moser's normal coordinates.* These give "real" coordinates z', $\overline{z'}$, u, v with $z_n = u + i\,v$. We rather use coordinates z', $\overline{z'}$, ρ_A, v, where

$$\rho_A = 2u - |z'|^2 - F_A(z', \overline{z'}, v),$$

so that $N(A)$ is given by the equation $\rho_A = 0$.

In general, Moser's normal form of a surface M is not unique; M has a unique normal form if and only if M is locally equivalent to a sphere, in which case the normal form is given by

$$M_0 = \partial\Omega_0 = \{2u = |z'|^2\}, \quad \text{where } \Omega_0 = \{2u > |z'|^2\}.$$

The model domain Ω_0 is a Siegel domain which is biholomorphic to a ball. Elements of $\operatorname{Aut}(\Omega_0)$, the group of holomorphic automorphisms of Ω_0, are linear fractional transformations. The non-uniqueness of the normal form is measured by using the *isotropy group* H of $\operatorname{Aut}(\Omega_0)$ at the origin 0 defined by $H = \{h \in \operatorname{Aut}(\Omega_0);\ h(0) = 0\}$; elements of H are biholomorphic at 0. In fact, there is a group action

$$H \times \mathcal{N} \ni (h, A) \mapsto h.A \in \mathcal{N} \tag{1.4}$$

such that equivalence classes of $\mathcal{N}$ are realized by H-orbits of $\mathcal{N}$. The action (1.4) is defined by $N(h.A) = M$ with $M = h(N(A))$ when M is in Moser's normal form. In general, M is merely in pre-normal form, but Theorem 1.1 guarantees the unique existence of a local biholomorphic mapping Φ such that $\Phi(M)$ is close to M and in Moser's normal form. Then the action (1.4) is defined by $N(h.A) = \Phi(M)$. That is,

$$N(h.A) = E_{h,A}(N(A)), \quad \text{where} \quad E_{h,A} = \Phi \circ h. \tag{1.4$'$}$$

Observe that $E'_{h,A}(0) = h'(0)$.

Let us finally give remarks on the case where the original real hypersurface $M \subset \mathbf{C}^n$, being strictly pseudoconvex, is not real analytic but merely C^∞. In the category of formal power series, the notions of pre-normal form and Moser's normal form make sense. After a formal change of variables, M can be always put in pre-normal form, and Theorem 1.1 has an obvious analogue. We continue to use the notations $N(A)$ and $A \in \mathcal{N}$. (We have a larger class $\mathcal{N}^\infty \supsetneqq \mathcal{N}$ but abuse notation by writing both $\mathcal{N}$ and $\mathcal{N}^\infty$ as $\mathcal{N}$.) Then the action (1.4) remains well-defined.

Remark 1.1. Let a surface M with a reference point $p \in M$ be real analytic or C^∞. Then by Theorem 1.1, there exists a (formal) biholomorphic mapping Φ_p such that $\Phi_p(p) = 0$ and $\Phi_p(M) = N(A)$ for some $A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N}$. We now regard each $A^\ell_{\alpha\overline{\beta}}$ as a function of $p \in M$. Then a family $\{\Phi_p\}_{p\in M}$ can be chosen in such a way that $A^\ell_{\alpha\overline{\beta}}$ is real analytic or C^∞. This fact is contained in the proof of Theorem 1.1.

(C) Local scalar invariants. Given a surface M with a reference point $p \in M$, local scalar invariants of M at p are defined as follows. For $A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N}$, we regard components $A^\ell_{\alpha\overline{\beta}}$ as variables and consider functions of A.

Definition. A polynomial $P(A)$ in $A \in \mathcal{N}$ is called a *CR invariant of weight* $w \in \mathbf{N}_0$ if

$$P(A) = |\det h'(0)|^{2w/(n+1)}\, P(h.A) \qquad \text{for any} \quad h \in H. \tag{1.5}$$

We denote by I_w^{CR} the totality of CR invariants of weight w, and thus I_w^{CR} is the complexification of a real vector space.

Each $P(A) \in I_w^{\mathrm{CR}}$ determines a functional $M \mapsto P_M$ defined by

$$P_M(p) = |\det \Phi_p'(p)|^{2w/(n+1)}\, P(A) \quad \text{with} \;\; \Phi_p(M) = N(A),$$

where Φ_p is a mapping in Remark 1.1. The function P_M is real analytic or C^∞ according to the regularity assumption on M, and the value $P_M(p)$ is independent of the choice of Φ_p. We have a transformation law under biholomorphic mappings Φ:

$$P_M(p) = |\det \Phi'(p)|^{2w/(n+1)}\, P_{\Phi(M)}(\Phi(p)) \qquad (p \in M).$$

Conversely, given a functional $P_M(p)$ of a pair (M, p) satisfying the law above, if $P_{N(A)}(0)$ is a polynomial in $A \in \mathcal{N}$ then $P_{N(A)}(0) \in I_w^{\mathrm{CR}}$.

Every $P(A) \in I_w^{\mathrm{CR}}$ is a polynomial in $A \in \mathcal{N}$ of homogeneous weight w, if we define the *weight* of $A^{\ell}_{\alpha\overline{\beta}}$ by

$$\mathrm{w}(A^{\ell}_{\alpha\overline{\beta}}) = \mathrm{w}(\alpha\overline{\beta}\ell) = (|\alpha| + |\beta|)/2 + \ell - 1.$$

This fact is seen by using dilations $\phi_r \in H$ defined by $\phi_r(z', z_n) = (rz', r^2 z_n)$ for $r > 0$. We have $P(A) = r^{2w} P(\phi_r.A)$, while the action $\phi_r.A = \widetilde{A}$ is given by $\widetilde{A}^{\ell}_{\alpha\overline{\beta}} = r^{-|\alpha|-|\beta|-2\ell+2} A^{\ell}_{\alpha\overline{\beta}}$.

1.2 The Bergman kernel

For a general domain $\Omega \subset \mathbf{C}^n$, we denote by $H^{\mathrm{B}}(\Omega)$ the Hilbert space of L^2 holomorphic functions in Ω with the norm $\|\cdot\|_{\mathrm{B}}$. Then the *Bergman kernel* associated with Ω is defined by

$$K^{\mathrm{B}}(z) = K^{\mathrm{B}}(z, \overline{z}) = \sum_j |h_j(z)|^2 \quad \text{for} \quad z \in \Omega,$$

where $\{h_j\}_j$ is an arbitrary complete orthonormal system of $H^{\mathrm{B}}(\Omega)$. The series $\sum |h_j(z)|^2$ converges uniformly on every compact subset ω of Ω, by virtue of the following inequality with a constant $C_\omega > 0$:

$$|h(z)| \le C_\omega \, \|h\|_{\mathrm{B}} \quad \text{for} \quad z \in \omega, \quad h \in H^{\mathrm{B}}(\Omega).$$

(In fact, $\sum |h_j(z)|^2$ is the square of the norm of the evaluation functional $h \mapsto h(z)$ on $H^{\mathrm{B}}(\Omega)$.) Thus, a complex extension of $K^{\mathrm{B}}(z) = K^{\mathrm{B}}(z, \overline{z})$ is given by

$$K^{\mathrm{B}}(z, \overline{w}) = \sum_j h_j(z) \, \overline{h_j(w)} \quad \text{for} \quad z, w \in \Omega,$$

which is holomorphic in $(z, \overline{w})$. This function $K^{\mathrm{B}}(z, \overline{w})$, which is also referred to as the *Bergman kernel*, is the *reproducing kernel* associated with the Hilbert space $H^{\mathrm{B}}(\Omega)$ in the sense that

$$K^{\mathrm{B}}(\,\cdot\,, \overline{w}) \in H^{\mathrm{B}}(\Omega) \quad \text{for} \quad w \in \Omega \ \text{ fixed},$$

$$\overline{K^{\mathrm{B}}(z, \overline{w})} = K^{\mathrm{B}}(w, \overline{z}) \quad \text{for} \quad z, w \in \Omega,$$

$$h(z) = \int_\Omega K^{\mathrm{B}}(z, \overline{w}) \, h(w) \, dV(w) \quad \text{for} \quad h \in H^{\mathrm{B}}(\Omega), \ \ z \in \Omega,$$

where $dV(w)$ denotes the standard volume element of $\mathbf{C}^n$ at w.

When we wish to emphasize the dependence on Ω, we write $K^{\mathrm{B}}(z,\overline{w})$ as $K^{\mathrm{B}}_{\Omega}(z,\overline{w})$. Recall that each element $h \in H^{\mathrm{B}}(\Omega)$ is identified with a holomorphic n-form $\omega_h(z) = h(z)\,dz_1 \wedge \cdots \wedge dz_n$, and

$$\frac{i^{n^2}}{2^n}\int_{\Omega} \omega_h \wedge \overline{\omega_h} = \|h\|^2_{\mathrm{B}} < +\infty.$$

Thus the Bergman kernel $K^{\mathrm{B}}_{\Omega}(z,\overline{w})$ is defined for a complex manifold Ω. (This fact will not be used explicitly, since we shall mainly work locally near a boundary point.) Also, the *transformation law* for the Bergman kernel under a biholomorphic mapping $\Phi : \Omega_1 \to \Omega_2$ is given as follows:

$$K^{\mathrm{B}}_{\Omega_1}(z,\overline{z}) = K^{\mathrm{B}}_{\Omega_2}(\Phi(z),\overline{\Phi(z)}\,)\,|\det \Phi'(z)|^2 \quad \text{for} \quad z \in \Omega_1, \tag{1.6}$$

a relation which can be complexified.

Example. If $\Omega \subset \mathbf{C}^n$ is the unit ball, then

$$K^{\mathrm{B}}(z,\overline{w}) = \frac{n!/\pi^n}{(1 - z\cdot\overline{w})^{n+1}}\,, \quad \text{where} \quad z\cdot\overline{w} = \sum_{j=1}^{n} z_j\,\overline{w}_j\,.$$

For our model domain $\Omega_0 = \big\{z = (z', z_n) \in \mathbf{C}^n;\ z_n + \overline{z}_n > |z'|^2\big\}$,

$$K^{\mathrm{B}}_{\Omega_0}(z,\overline{w}) = \frac{n!}{\pi^n}\big(z_n + \overline{w}_n - z'\cdot\overline{w'}\,\big)^{-n-1}. \tag{1.7}$$

Remark 1.2. (1°) If Ω is a domain in $\mathbf{C}$, then

$$K^{\mathrm{B}}(z,\overline{w}) = -\frac{\partial^2 G(z,w)}{\partial z\,\partial\overline{w}} \qquad \text{for} \quad z, w \in \Omega,$$

where $G(z,w)$ denotes the Green function normalized by multiplying a constant (cf. Schiffer [Scr]). An operator version is given by using the $\overline{\partial}$-operator and its L^2 adjoint $\overline{\partial}^*$ as $\mathbf{K}^{\mathrm{B}} = 1 - \overline{\partial}^*\mathbf{G}\,\overline{\partial}$, where $\mathbf{G}$ denotes the Green operator and $\mathbf{K}^{\mathrm{B}}$, called the *Bergman projector*, stands for the orthogonal projector of $L^2(\Omega)$ to the closed subspace $H^{\mathrm{B}}(\Omega)$.

(2°) An analogous formula is available for a domain Ω in $\mathbf{C}^n$ as far as the complex Laplacian $\square = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ for $(0,1)$-forms on Ω has a closed range in L^2. The generalized inverse $\mathbf{N}$, called the $\overline{\partial}$-*Neumann operator*, satisfies $\mathbf{K}^{\mathrm{B}} = 1 - \overline{\partial}^*\mathbf{N}\,\overline{\partial}$. If, for instance, Ω is a strictly pseudoconvex domain with C^∞ boundary, then $\mathbf{N}$ is defined and C^∞ pseudolocal at every point of the closure $\overline{\Omega}$ (cf. Folland-Kohn [FK]). Then, the Bergman kernel $K^{\mathrm{B}}(z,\overline{w})$ as a function of (z,w) is C^∞ on $\overline{\Omega}\times\overline{\Omega}$ off the diagonal of $\partial\Omega\times\partial\Omega$ (cf. Kerzman [Ke]).

From now on, we assume that $\Omega=\{z;\ r(z)>0\}\subset\mathbf{C}^n$ is a strictly pseudoconvex domain, where r is a smooth (C^∞ or real analytic) defining function of the boundary. It has been known that the Bergman kernel $K^{\mathrm{B}}(z)=K^{\mathrm{B}}(z,\overline{z})$ tends to $+\infty$ as z approaches to a boundary point. The magnitude of divergence is measured by virtue of a theorem of Hörmander [Hö] as follows:

$$\lim_{z\to p} r(z)^{n+1}\,K^{\mathrm{B}}(z)=\frac{n!}{\pi^n}\,J[r](p)\quad\text{for}\quad p\in\partial\Omega, \tag{1.8}$$

where $J[r]$ denotes the *Levi determinant* of r given by

$$J[r]=(-1)^n\,\det\begin{pmatrix} r & \partial r/\partial\overline{z}_k \\ \partial r/\partial z_j & \partial^2 r/\partial z_j\,\partial\overline{z}_k\end{pmatrix}. \tag{1.9}$$

We shall rather refer to $J[\,\cdot\,]$ as the (complex) *Monge-Ampère operator*. A far-reaching refinement of (1.8) is given as follows.

Theorem 1.2 ([F1]). *Let $\Omega=\{z\in\mathbf{C}^n;\ r(z)>0\}$ be a strictly pseudoconvex domain, where r is a C^∞ defining function of $\partial\Omega$. Then there exist $\varphi^{\mathrm{B}},\psi^{\mathrm{B}}\in C^\infty(\overline{\Omega})$ such that*

$$K^{\mathrm{B}}(z,\overline{z})=K^{\mathrm{B}}(z)=\frac{\varphi^{\mathrm{B}}(z)}{r(z)^{n+1}}+\psi^{\mathrm{B}}(z)\,\log r(z). \tag{1.10}$$

In particular, $\varphi^{\mathrm{B}}=(n!/\pi^n)J[r]$ on $\partial\Omega$.

Remark 1.3. If $\partial\Omega$ with r is real analytic, then φ^{B} and ψ^{B} are real analytic too, so that (1.10) is complexified (cf. Kashiwara [Kas]):

$$K^{\mathrm{B}}(z,\overline{w})=\frac{\varphi^{\mathrm{B}}(z,\overline{w})}{r(z,\overline{w})^{n+1}}+\psi^{\mathrm{B}}(z,\overline{w})\,\log r(z,\overline{w}). \tag{1.10$'$}$$

Even when $\partial\Omega$ is C^∞, the above equality (1.10)$'$ remains valid with C^∞ functions $r(z,\overline{w})$, $\varphi^{\mathrm{B}}(z,\overline{w})$, $\psi^{\mathrm{B}}(z,\overline{w})$ of $(z,w)\in\overline{\Omega}\times\overline{\Omega}$ which are regarded as almost analytic extensions of $r(z)=r(z,\overline{z})$, $\ldots$ in the sense that $\partial r(z,\overline{w})/\partial\overline{z}$, $\ldots$ and $\partial r(z,\overline{w})/\partial w$, $\ldots$ vanish to infinite order at $z=w$ (cf. Boutet de Monvel-Sjöstrand [BS]).

Remark 1.4. The singularities (1.10) and (1.10)$'$ are localizable to a neighborhood of a boundary point as follows. If Ω_1 and Ω_2 are strictly pseudoconvex domains with smooth (C^∞ or real analytic) boundaries such that $\overline{\Omega}_1 \cap V = \overline{\Omega}_2 \cap V$ for a neighborhood V of a point $p \in \partial\Omega_1 \cap \partial\Omega_2$, then there exists a smaller neighborhood V_0 of p such that the difference $K^{\mathrm{B}}_{\Omega_1}(z, \overline{w}) - K^{\mathrm{B}}_{\Omega_2}(z, \overline{w})$ are smooth for $z, w \in \overline{\Omega}_1 \cap V_0 = \overline{\Omega}_2 \cap V_0$.

Remark 1.5. (1°) An elementary property of the Bergman kernel is the monotonicity with respect to the domain:

$$K^{\mathrm{B}}_{\Omega_1}(z) \geq K^{\mathrm{B}}_{\Omega_2}(z) \quad \text{when} \quad z \in \Omega_1 \subset \Omega_2.$$

In the proof of (1.8), this fact and the model case formula (1.7) are used together with a localization argument, after a scaling of the coordinates (cf. Hörmander [Hö]).

(2°) Fefferman's original proof of Theorem 1.2 requires a more precise approximation of Ω from inside at a boundary point by a domain Ω_{ball} which is locally biholomorphic to a ball. Roughly speaking, starting from an explicit approximation of the decomposition $1 = \mathbf{K}^{\mathrm{B}} + \overline{\partial}^{*} \mathbf{N} \, \overline{\partial}$, the Bergman kernel is obtained as a Neumann series, where successive integrations over a thin domain given locally by $\Omega \setminus \Omega_{\mathrm{ball}}$ are involved. The estimates are extremely hard (cf. [F1]).

(3°) An alternative proof of Theorem 1.2 is given by Boutet de Monvel and Sjöstrand [BS], where the singularity of the Bergman kernel is written as a Fourier integral distribution with complex phase:

$$K^{\mathrm{B}}(z, \overline{w}) \sim \int_0^\infty e^{-t\, r(z, \overline{w})}\, p^{\mathrm{B}}(z, \overline{w}, t)\, dt \qquad \operatorname{mod} C^\infty,$$

where $p^{\mathrm{B}}(z, \overline{w}, t)$ is a symbol admitting an asymptotic expansion

$$p^{\mathrm{B}}(z, \overline{w}, t) \sim \sum_{j=0}^{\infty} t^{n-j} p^{\mathrm{B}}_j(z, \overline{w}), \qquad p^{\mathrm{B}}_j(\,\cdot\,, \overline{\cdot}\,) \in C^\infty(\overline{\Omega} \times \overline{\Omega}).$$

This expression yields $(1.10)'$ via the following formulas for the Laplace transforms, which are valid for $p \in \mathbf{C}$ with $\operatorname{Re} p > 0$:

$$\begin{aligned} \int_0^\infty t^m\, e^{-pt}\, dt &= \frac{m!}{p^{m+1}} && \text{for } \ m \geq 0, \\ \operatorname{pf} \int_0^\infty t^{-m}\, e^{-pt}\, dt &= \frac{(-1)^m\, p^{m-1}}{(m-1)!}\, (\log p + C_m) && \text{for } \ m \geq 1, \end{aligned}$$

where C_m are constants and pf stands for the Hadamard finite part.

(4°) For Kashiwara's proof [Kas] of (1.10) in the real analytic case and its application, see Section 4 below.

The equality (1.10) in Theorem 1.2 is referred to as an asymptotic expansion. A reason is that if the boundary $\partial\Omega$ is locally flattened by a real change of coordinates $z = \Psi(s, r)$ with $s \in \mathbf{R}^{2n-1}$ then

$$K^{\mathrm{B}}(\Psi(s,r)) = \frac{\varphi^{\mathrm{B}}(\Psi(s,r))}{r^{n+1}} + \psi^{\mathrm{B}}(\Psi(s,r)) \log r,$$

and the Taylor expansions about $r = 0$ of $\varphi^{\mathrm{B}}(\Psi(s,r))$ modulo $O^{n+1}(r)$ and $\psi^{\mathrm{B}}(\Psi(s,r))$ provide an asymptotic expansion of $K^{\mathrm{B}}(\Psi(s,r))$. This is analogous to that of the heat kernel. However, the biholomorphic invariance is lost, for the expansion depends on the choices of the real coordinate system (s, r) and the defining function r. Instead, we make the following tentative definition.

Definition. A domain functional $K(z) = K_\Omega(z)$ is said to satisfy a (biholomorphic) transformation law of *weight* $w \in \mathbf{Z}$ if

$$K_{\Omega_1}(z) = K_{\Omega_2}(\Phi(z))\, |\det \Phi'(z)|^{2w/(n+1)} \tag{1.11}$$

for any biholomorphic mapping $\Phi\colon \Omega_1 \to \Omega_2$. This definition extends to *local domain functionals* defined only near a boundary point.

The equality (1.6) means that the Bergman kernel satisfies a transformation law of weight $n+1$. If there would exist a defining function r satisfying a transformation law of weight -1, then we could speak of an invariant expansion of the Bergman kernel given by the expansions

$$\begin{aligned} \varphi^{\mathrm{B}} &= \sum_{j=0}^{n} \varphi_j\, r^j \qquad \bmod O^{n+1}(r), \\ \psi^{\mathrm{B}} &= \sum_{j=0}^{\infty} \varphi_{n+1+j}\, r^j \quad \bmod O^{\infty}(r), \end{aligned} \tag{1.12}$$

with $\varphi_j \in C^\infty(\overline{\Omega})$ for $j \in \mathbf{N}_0$ satisfying transformation laws of weight j. Here, the first relation in (1.12) means that the difference between both sides is smoothly divisible by r^{n+1}, and the second relation means that

$$\psi^{\mathrm{B}} = \sum_{j=0}^{m} \varphi_{n+1+j}\, r^j \quad \bmod O^{m+1}(r) \qquad \text{for any} \quad m \in \mathbf{N}.$$

In fact, the situation is not so simple. Nevertheless, this is approximately the case, as we shall see in the next subsection.

1.3 The Monge-Ampère boundary value problem

Recall the (complex) Monge-Ampère operator $J[\,\cdot\,]$ defined in (1.9). If $\Phi : \Omega_1 \to \Omega_2$ is a biholomorphic mapping, then

$$J[u_1] = J[u_2] \circ \Phi \quad \text{with} \quad u_1 = |\det \Phi'|^{-2/(n+1)}\, u_2 \circ \Phi$$

for any function u_2 in Ω_2 (cf. Fefferman [F2]). In particular, every solution u of the Monge-Ampère equation $J[u] = 1$ satisfies a transformation law of weight -1 in the sense of (1.11). This fact motivates us to consider the zero Dirichlet boundary value problem

$$\text{(1.13)} \qquad J[u^{\mathrm{MA}}] = 1 \ \text{ and } \ u^{\mathrm{MA}} > 0 \ \text{ in } \ \Omega; \qquad u^{\mathrm{MA}} = 0 \ \text{ on } \ \partial\Omega.$$

The problem (1.13) has a unique solution but it has only a finite degree of smoothness up to the boundary (cf. Cheng-Yau [CY]):

$$\text{(1.14)} \qquad u^{\mathrm{MA}} \in C^\infty(\Omega) \cap C^{n+3/2-\varepsilon}(\overline{\Omega}) \qquad \text{for any } \ \varepsilon > 0.$$

The solution u^{MA} admits an asymptotic expansion, with an arbitrary defining function r of $\partial\Omega$ such that $\Omega = \{r > 0\}$ (cf. Lee-Melrose [LM]):

$$\text{(1.15)} \qquad u^{\mathrm{MA}} \sim r \sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1} \log r)^k, \qquad \eta_k \in C^\infty(\overline{\Omega}).$$

In particular, (1.14) is improved as follows: $u^{\mathrm{MA}} \in C^{n+2-\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$ small. In the expansion (1.15) considered near a reference point at the boundary, the function η_0 depends globally on the choice of r, whereas the Taylor expansions of η_k for $k \geq 1$ are determined locally by those of η_0 and r (cf. [LM]).

Though the solution u^{MA} of (1.13) is a defining function of $\partial\Omega$ and satisfies a transformation law of weight -1, it is not C^∞ smooth up to the boundary. Thus we cannot use u^{MA} in an invariant expansion of the Bergman kernel of the form (1.12). Instead, we confine ourselves to a C^∞ defining function $r = r^{\mathrm{F}}$ of $\partial\Omega$ satisfying (1.13) approximately in the sense that

$$\text{(1.16)} \qquad J[r^{\mathrm{F}}] = 1 + O^{n+1}(r) \quad \text{near } \ \partial\Omega \qquad (r = r^{\mathrm{F}} > 0 \ \text{ in } \ \Omega).$$

Fefferman [F2] considered r^{F} precedent to the above stated works of Cheng-Yau [CY] and Lee-Melrose [LM]. In [F2], an explicit algorithm of constructing r^{F} is given locally near a boundary point (cf. Subsection 3.2 below). We refer to r^{F} as a *Fefferman's defining function.* For later use, we summarize properties of r^{F}:

(1^{F}) r^{F} is unique modulo $O^{n+2}(r)$, or the ambiguity of r^{F} is $O^{n+2}(r)$;

($2^{\rm F}$) $r^{\rm F}$ satisfies a transformation law of weight -1 modulo $O^{n+2}(r)$;
($3^{\rm F}$) $r^{\rm F}$ makes sense locally near a reference point at the boundary.

By ($1^{\rm F}$), we mean that if $r_1^{\rm F}$ and $r_2^{\rm F}$ satisfy (1.16) then $r_1^{\rm F} - r_2^{\rm F} = O^{n+2}(r)$ and that if $r_1^{\rm F}$ satisfies (1.16) so does $r_2^{\rm F} = r_1^{\rm F} + \delta$ whenever $\delta = O^{n+2}(r)$. The fact ($1^{\rm F}$) follows from the condition (1.16); and ($1^{\rm F}$) implies ($2^{\rm F}$), because if $\Phi : \Omega_1 \to \Omega_2$ is biholomorphic then

$$J[r_1^{\rm F}] = J[r_2^{\rm F}] \circ \Phi \quad \text{with } r_1^{\rm F} = |\det \Phi'|^{-2/(n+1)}\, r_2^{\rm F} \circ \Phi$$

for any Fefferman's defining function $r_2^{\rm F}$ of Ω_2. By ($3^{\rm F}$), we mean that the properties ($1^{\rm F}$) and ($2^{\rm F}$) are valid locally near a reference point at the boundary.

By continuing Fefferman's construction beyond $r^{\rm F}$, Graham [G2] constructed a local asymptotic solution $u^{\rm G}$ of (1.13) in the form

$$u^{\rm G} = r \sum_{k=0}^{\infty} \eta_k^{\rm G} \cdot (r^{n+1} \log r)^k, \qquad \eta_k^{\rm G} \in C^\infty(\overline{\Omega}). \tag{1.17}$$

Theorem 1.3 ([G2]). *Let $r = r^{\rm F}$ be a Fefferman's defining function of Ω. Then, for any $a \in C^\infty(\partial\Omega)$, there exists a unique asymptotic solution $u = u^{\rm G}$ of the form* (1.17) *to the problem*

$$J[u] = 1 + O^\infty(r) \quad \text{near } \partial\Omega, \quad \eta_0^{\rm G} = 1 + a\, r^{n+1} + O^{n+2}(r). \tag{1.18}$$

Furthermore, $\eta_k^{\rm G}$ for each $k \geq 1$ has the following properties:

($1^{\rm G}$) *$\eta_k^{\rm G}$ modulo $O^{n+1}(r)$ is independent of the choice of a and $r^{\rm F}$;*
($2^{\rm G}$) *$\eta_k^{\rm G}$ has a transformation law of weight $k(n+1)$ modulo $O^{n+1}(r)$;*
($3^{\rm G}$) *$\eta_k^{\rm G}$ modulo $O^{n+1}(r)$ makes sense locally near a boundary point.*

The asymptotic solution $u^{\rm G}$ is a formal series of the form (1.17). The first relation of (1.18) means that $J[u^{\rm G}] - 1$ is formally flat on $\partial\Omega$ in the sense that for any $m \in \mathbf{N}$ there exists a finite sum $u_m^{\rm G}$ corresponding to (1.17) such that $J[u_m^{\rm G}] - 1$ is continuously divisible by r^m. The meanings of ($1^{\rm G}$)–($3^{\rm G}$) are similar to those of ($1^{\rm F}$)–($3^{\rm F}$), except for the fact that $u^{\rm G}$ is uniquely determined by a and $r^{\rm F}$, where a is prescribed in a neighborhood of a reference point at the boundary.

Let us return to the problem mentioned at the end of the previous subsection. We wish to realize an invariant expansion of the Bergman kernel of the form (1.12) with $r = r^{\rm F}$. Because of the ambiguity of $r^{\rm F}$, the invariance becomes approximate and the expansion of $\psi^{\rm B}$, even if possible, only makes sense as a finite sum, say,

$$(1.12)'_N \qquad \psi^{\mathrm{B}} = \varphi_{n+1} + \varphi_{n+2}\, r + \cdots + \varphi_N\, r^{N-n-1} \mod O^{N-n}(r).$$

Suppose we are given vector subspaces $I_j^{\mathrm{W}} \subset C^\infty(\overline{\Omega})$ $(0 \le j \le N)$ with the following properties:

(1^{W}) Elements of I_j^{W} make sense modulo $O^{N-j+1}(r)$ (we regard these as equivalence classes modulo $O^{N-j+1}(r)$);

(2^{W}) Each element of I_j^{W} satisfies the transformation law of weight j modulo $O^{N-j+1}(r)$;

(3^{W}) The boundary value of each element $\varphi \in I_j^{\mathrm{W}}$ is a CR invariant, and the resulting mapping $I_j^{\mathrm{W}} \to I_j^{\mathrm{CR}}$ is surjective. In addition, if the boundary $\partial\Omega$ is in normal form $N(A)$ near the origin, then $\partial_z^\alpha \partial_{\overline{z}}^\beta \varphi(0)$ $(|\alpha| + |\beta| \le N - j)$ for $\varphi \in I_j^{\mathrm{W}}$ are polynomials in A.

The latter condition in (3^{W}) is referred to as the *polynomial dependence* of $\varphi \in I_j^{\mathrm{W}}$ on the boundary. The functions φ^{B}, ψ^{B} and r^{F} have a similar property, as we shall see in Sections 3 and 4. If $N \ge n$, then the conditions (1^{W})–(3^{W}) yield the expansion of φ^{B} in (1.12) as follows. Since the boundary value of φ^{B} is an element of I_0^{CR} being a constant, (3^{W}) implies the existence of $\varphi_0 \in I_0^{\mathrm{W}}$ such that $\varphi^{\mathrm{B}} = \varphi_0 + O^1(r)$. Then the approximate invariance of the smooth function $\widetilde{\varphi}_1 := (\varphi^{\mathrm{B}} - \varphi_0)/r$ makes sense. By virtue of (1^{W})–(3^{W}) and the polynomial dependence of φ^{B} and r^{F}, the boundary value of $\widetilde{\varphi}_1$ belongs to I_1^{CR}, and thus (3^{W}) implies as before the existence of $\varphi_1 \in I_1^{\mathrm{W}}$ such that $\varphi^{\mathrm{B}} = \varphi_0 + \varphi_1\, r + O^2(r)$. Then induction yields the expansion of φ^{B} as in (1.12).

A construction of I_j^{W} for $0 \le j \le n$ is discussed in the next section.

The same argument applies to the expansion of ψ^{B} as in $(1.12)'_N$, but the approximate invariance of the right side of $(1.12)'_N$ only makes sense modulo $O^{n+1}(r)$ by the ambiguity of $r = r^{\mathrm{F}}$. Consequently, we have invariant expressions of φ_j for $0 \le j \le \min(N, 2n+1)$ whenever I_j^{W} for $0 \le j \le N$ are constructed. In Section 3, we consider the case $n = 2$ and realize the optimal case $N = 5$, that is, we express φ_j for $0 \le j \le 5$ explicitly by constructing I_j^{W} for $0 \le j \le 5$.

§2 Weyl invariants

Elements of the spaces I_j^{W} $(0 \le j \le n)$ in Subsection 1.3 are realized by Weyl invariants in the sense of Fefferman [F3]. This notion was introduced in [F3] as an analogy of that in Riemannian Geometry, where the Bergman kernel is compared with the heat kernel. Reviewing quickly the heat kernel asymptotics in Subsection 2.1, we give the definition of Weyl invariants in Subsection 2.2. Then in Subsection 2.3, we state the main results of this section, due to Fefferman [F3] and Bailey-Eastwood-Graham [BEG], on Weyl invariants and the invariant expansion of the Bergman kernel.

2.1 Heat kernel on a Riemannian manifold

Let (M, g) be an n-dimensional compact Riemannian manifold. We denote by Δ_g the (negative) Laplacian acting on functions on M, and consider the initial value problem for the heat equation:

$$\partial u/\partial t - \Delta_g u = 0 \quad \text{on} \quad M \times [0, \infty), \qquad u|_{t=0} = f,$$

where $f \in C^\infty(M)$ is prescribed arbitrarily. Then there exists a unique solution, which has the form $u(x,t) = \int_M H_t(x,y)\, f(y)\, dV(y)$, where dV stands for the volume element on M. The function $H_t(x,y)$ for $x, y \in M$ and $t > 0$ is called the *heat kernel* (for functions) associated with Δ_g. Let us consider the restriction $H_t(x,x)$ to the diagonal of $M \times M$. This is a smooth function as far as $t > 0$, but becomes singular as $t \to +0$. More precisely, the following asymptotic expansion holds:

$$H_t(x,x) \sim t^{-n/2} \sum_{m=0}^{\infty} a_m(x)\, t^m \quad \text{with } \ a_m \in C^\infty(M).$$

The coefficient functions a_m are determined locally by the metric g. In addition, these are Riemannian invariants defined as follows. Let us take a normal coordinate system $x = (x_1, \ldots, x_n)$ about a reference point $p \in M$. The choice of normal coordinate systems has freedom corresponding to the action of the isotropy group $O(n)$, and an action of $O(n)$ is induced on jets of the metric, $g_{jk,ab\cdots c} = \partial_{x_a} \partial_{x_b} \cdots \partial_{x_c} g_{jk}(p)$, where $\partial_{x_a} = \partial/\partial x_a$, etc. A universal polynomial $P_m = P_m(g_{jk,ab\cdots c})$ is called a (local) *Riemannian invariant* if it is invariant under this action of $O(n)$.

For the curvature tensor R of g, we consider its successive covariant derivatives and denote the components by $R_{ijkl,ab\cdots c}$. Then each $g_{jk,ab\cdots c}$ is a polynomial of $(R_{ijkl,ab\cdots c})$, and thus each Riemannian invariant is written as an $O(n)$-invariant polynomial of $(R_{ijkl,ab\cdots c})$, where $O(n)$ acts tensorially on $(R_{ijkl,ab\cdots c})$. According to Weyl's invariant theory, the vector space of all Riemannian invariants is generated by complete contractions of the form

$$\mathrm{contr}\left(\nabla^{p_1} R \otimes \cdots \otimes \nabla^{p_s} R\right),$$

where the contractions are taken over all indices. Consequently, each a_m in the heat kernel is expressed as a linear combination of these complete contractions such that $2m = p_1 + \cdots + p_s + s$. This equality is seen by scaling the metric.

2.2 Definition of Weyl invariants

CR invariants can be compared with Riemannian invariants with Moser's normal coordinates in place of Riemannian normal coordinates. A substitute for the Riemannian curvature is the curvature of the ambient metric, which is defined as follows.

Let $r = r^{\mathrm{F}}$ be a Fefferman's defining function of the domain Ω. Introducing an extra variable $z_0 \in \mathbf{C}^* = \mathbf{C} \setminus \{0\}$, we consider a function $r_\#(z_0, z) = |z_0|^2 r(z)$ on $\mathbf{C}^* \times \overline{\Omega}$. Then, a tensor of $(1,1)$-type

$$g = \sum_{j.k=0}^{n} g_{j\overline{k}}\, dz_j\, d\overline{z}_k = \sum_{j.k=0}^{n} \frac{\partial^2 r_\#}{\partial z_j\, \partial \overline{z}_k}\, dz_j\, d\overline{z}_k$$

defines a Lorentz-Kähler metric in a neighborhood of $\mathbf{C}^* \times \partial\Omega$. This metric g is called an *ambient metric* associated with $\partial\Omega$. Due to the ambiguity of $r = r^{\mathrm{F}}$ modulo $O^{n+2}(r)$, the ambient metric is well-defined only up to the n-th jets along $\mathbf{C}^* \times \partial\Omega$.

As in the Riemannian case, scalar invariants are constructed from the metric g as follows. For the curvature tensor R of g, we consider successive covariant derivatives $R^{(p,q)} = \overline{\nabla}^{q-2}\nabla^{p-2}R$ and complete contractions of the form

$$W_\# = \operatorname{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\right). \tag{2.1}$$

These are functions in a neighborhood of $\mathbf{C}^* \times \partial\Omega \subset \mathbf{C}^* \times \overline{\Omega}$, and the restrictions $W = W_\#|_{z_0=1}$ are defined near $\partial\Omega$. The *weight* of $W_\#$ in (2.1) is defined by $w = \sum_{j=1}^{s}(p_j + q_j)/2 - s$.

Definition. A *Weyl invariant* of *weight* w is a linear combination of complete contractions of the form (2.1) of the weight w.

By definition, a Weyl invariant $W_\#$ is a functional of r. Nevertheless, we also use this terminology for the composite function $(z_0, z) \mapsto W_\#$ or the equivalence class modulo the ambiguity of $r = r^{\mathrm{F}}$. For a Weyl invariant $W_\#$ of weight w, we set $W = W_\#|_{z_0=1}$. Then

$$W_\#(z_0, z) = |z_0|^{-2w}\, W(z).$$

Accordingly, we still call W a *Weyl invariant* of weight w.

Let us recall that r^{F} is a local domain functional of weight -1 in the sense of (1.12) but with error of $O^{n+2}(r)$. Likewise, Weyl invariants W of weight w are local domain functionals of weight w with some error. The argument involving the error is somewhat technical, and we postpone it until the next subsection. Instead, we give here a transformation law under a biholomorphic mapping $\Phi\colon \Omega_1 \to \Omega_2$ for representatives of Weyl invariants defined by a Fefferman's defining function r_2 of Ω_2 and its pull-back $r_1 = |\det \Phi'|^{-2/(n+1)}\, r_2 \circ \Phi$ to Ω_1. To emphasize the dependence on $r = r^{\mathrm{F}}$, we write $g = g[r]$, $W_\# = W_\#[r]$, $W = W[r]$. Then

$$W[r_1] = |\det \Phi'|^{2w/(n+1)} W[r_2] \circ \Phi. \tag{2.2}$$

This is seen as follows. We lift Φ to a bundle map $\Phi_\# : \mathbf{C}^* \times \Omega_1 \to \mathbf{C}^* \times \Omega_2$ defined by

$$\Phi_\#(z_0, z) = (z_0 \cdot (\det \Phi'(z))^{-1/(n+1)}, \Phi(z)). \tag{2.3}$$

Then $(r_1)_\# = (r_2)_\# \circ \Phi_\#$, and $\Phi_\#$ is an isometry with respect to the metrics $g[r_1]$ and $g[r_2]$. Thus $W_\#[r_2] \circ \Phi_\# = W_\#[r_1]$, which implies (2.2).

2.3 Results of Fefferman and Bailey-Eastwood-Graham

We begin with a consideration of the dependence on the choice of Fefferman's defining function.

Proposition 2.1. *If $W[r]$ is a Weyl invariant of weight $w \le n$, then $W[r]$ modulo $O^{n-w+1}(r)$ is independent of the choice of $r = r^{\mathrm{F}}$.*

The proof of Proposition 2.1 is done by using Moser's normal coordinates. If the boundary $\partial\Omega$ is locally in Moser's normal form $N(A)$, then $W[r]$ is written in terms of the coordinate system $(z', \overline{z'}, \rho_A, v)$ as

$$W[r] = \sum_{m=0}^{n-w} \sum_{\alpha,\beta,\ell} P^{\ell m}_{\alpha\overline{\beta}}(A)\, z'_\alpha\, \overline{z'_\beta}\, v^\ell \rho_A^m + O^{n-w+1}(\rho_A),$$

where $P^{\ell m}_{\alpha\overline{\beta}}(A)$ are polynomials in A (cf. the statement (#) in Subsection 3.2, (B) below.) The desired result then follows, since the main part of the expression of $W[r]$ above is independent of the choice of $r = r^{\mathrm{F}}$.

By Proposition 2.1 above and (2.2) in the previous subsection, we have an approximate transformation law corresponding to (2.2), but for arbitrary Fefferman's defining functions $r_j = r_j^{\mathrm{F}}$ of Ω_j $(j = 1, 2)$:

$$W[r_1] = |\det \Phi'|^{2w/(n+1)} W[r_2] \circ \Phi \mod O^{n-w+1}(r_1). \tag{2.4}$$

In particular, the boundary value of a Weyl invariant of weight $w \le n$ gives a CR invariant of weight w. The converse is the first main result of this section.

Theorem 2.1 ([F3], [BEG]). *Every CR invariant of weight $\le n$ is given by the boundary value of a Weyl invariant.*

The statement of Theorem 2.1 was first proved by Fefferman [F3] for CR invariants of weight $\le n - 19$. The weight restriction was removed recently by Bailey-Eastwood-Graham [BEG]. We outline the proof of Theorem 2.1 in Section 5.

Let I_w^{W} denote the totality of Weyl invariants of weight w. By virtue of Proposition 2.1 and Theorem 2.1 above, the spaces I_w^{W} for $0 \le w \le n$ satisfy the conditions $(1^{\mathrm{W}}), (2^{\mathrm{W}})$ and (3^{W}) in Subsection 1.3 with $N = n$. Consequently, the argument given there is valid, and we have:

Theorem 2.2 ([F3],[BEG]). *For φ^{B} in the expression* (1.10) *of the Bergman kernel, the following expansion holds:*

$$\varphi^{\mathrm{B}} = \sum_{k=0}^{n} W_k\, r^k + O^{n+1}(r) \quad \textit{with} \quad W_k \in I_w^{\mathrm{W}}.$$

§3 Explicit computation in the two dimensional case

For domains in $\mathbf{C}^2$, it is possible to refine Theorems 2.1 and 2.2, as we mentioned at the end of Section 1. We also get explicit results, which are stated in Subsection 3.1. These results are obtained with the aid of asymptotic calculi of the Monge-Ampère equation and the Bergman kernel, where explicit algorithms are necessary. Postponing the calculus of the Bergman kernel until the next section, we discuss that of the Monge-Ampère equation in Subsection 3.2.

3.1 The two dimensional case

(A) Results. Consider for a domain Ω in $\mathbf{C}^2$ the approximate invariant expansions of φ^{B} and ψ^{B} expressing the singularity of the Bergman kernel

$$K^{\mathrm{B}} = \varphi^{\mathrm{B}}\, r^{-3} + \psi^{\mathrm{B}} \log r \quad \text{with} \quad r = r^{\mathrm{F}}$$

in terms of Fefferman's defining function r^{F}. To write down explicit results, it is convenient to normalize φ^{B} and ψ^{B} by writing

$$K^{\mathrm{B}} = \frac{2}{\pi^2}\left(\widetilde{\varphi}^{\mathrm{B}}\, r^{-3} + \widetilde{\psi}^{\mathrm{B}} \log r\right),$$

so that $\widetilde{\varphi}^{\mathrm{B}} = 1$ on $\partial\Omega$, cf. (1.8). As we shall see below, we can completely determine $\widetilde{\varphi}^{\mathrm{B}}$ and $\widetilde{\psi}^{\mathrm{B}}$ both modulo $O^3(r)$. The results are optimal and better than those in the higher dimensional case.

As before, let M be a portion of $\partial\Omega$, and assume that M is in Moser's normal form $N(A)$, where $A = (A^{\ell}_{\alpha\overline{\beta}}) \in \mathcal{N}$. Changing notation slightly, we write $A^{\ell}_{p\overline{q}}$ in place of $A^{\ell}_{\alpha\overline{\beta}}$ with $|\alpha| = p$ and $|\beta| = q$, since $(\alpha, \beta) \mapsto (p, q)$ is bijective. Then the trace conditions on A take the form

$$A_{2\overline{2}}(v) = A_{2\overline{3}}(v) = A_{3\overline{3}}(v) = 0,$$

so that $A^{\ell}_{p\overline{q}} = 0$ for $p + q + \ell \leq 5$. That is, M can be approximated by a sphere to order 5, though in the higher dimensional case the third order approximation is optimal. By this fact, the two dimensional case is exceptional in the sense that the Weyl invariants are less ambiguous (cf. Lemma 3.3 and Remark 3.2 below).

To state the main results of this section, we begin by presenting bases of the vector spaces I^{CR}_w of CR invariants of weight $w \leq 5$.

Lemma 3.1. *$I^{\mathrm{CR}}_1 = I^{\mathrm{CR}}_2 = \{0\}$ and*

$$\dim I^{\mathrm{CR}}_3 = \dim I^{\mathrm{CR}}_4 = 1, \quad \dim I^{\mathrm{CR}}_5 = 2.$$

The spaces I^{CR}_3 and I^{CR}_4 are generated by $A^0_{4\overline{4}}$ and $|A^0_{2\overline{4}}|^2$, respectively. The space I^{CR}_5 is spanned by $F^{\mathrm{CR}}_5(1, 0)$ and $F^{\mathrm{CR}}_5(0, 1)$, where

$$F^{\mathrm{CR}}_5(a, b) = F\big(a, b, -2a + (10/9)\, b, -a + b/3\big)$$

with $F(a, b, c, d) = a\,|A^0_{5\overline{2}}|^2 + b\,|A^0_{4\overline{3}}|^2 + \mathrm{Re}\,\big\{\big(c\,A^0_{3\overline{5}} - i\,d\,A^1_{2\overline{4}}\big)\,A^0_{4\overline{2}}\big\}\,.$

For the proof, see [G1] for $w \leq 4$ and [HKN2] for $w = 5$.

As a consequence of Lemma 3.1, the expansion of $\widetilde{\varphi}^{\mathrm{B}}$ is trivial:

$$\widetilde{\varphi}^{\mathrm{B}} = \text{constant} + O^3(r) \qquad (\text{constant} = 1). \tag{3.1}$$

To proceed further, it is necessary to extend $A^0_{4\overline{4}} \in I^{\mathrm{CR}}_3$ approximate invariantly to the domain Ω. This is done by using the first coefficient function η^{G}_1 of the asymptotic series u^{G} in Subsection 1.3. It is proved by Graham [G2] that:

Lemma 3.2. *The boundary value of η^{G}_1 is a CR invariant of weight* 3. *Specifically, $\eta^{\mathrm{G}}_1 = 4\,A^0_{4\overline{4}}$ on M.*

Let us proceed further to describe $\widetilde{\psi}^{\mathrm{B}}$ modulo $O^3(r)$. As we stated in Subsection 1.3, Fefferman's defining function $r = r^{\mathrm{F}}$ makes invariant sense modulo $O^4(r)$, and η_1^{G} modulo $O^3(r)$ is independent of the choices of r^{F} and the data $a \in C^\infty(M)$ determining u^{G}. Consequently, it suffices to extend $|A^0_{2\overline{4}}|^2 \in I_4^{\mathrm{CR}}$ to Ω in such a way that the extension satisfies an approximate transformation law of weight 4 modulo $O^2(r)$. Such an extension is realized by a Weyl invariant. (The Weyl invariant of weight ≥ 3 are subject to a restriction stronger than that in Subsection 2.2, because Proposition 2.1 is irrelevant to the case $n = 2$. See Remark 3.2 below.) Specifically, we consider complete contractions of weight $w = p + q - 2$ of the form:

$$\|R^{(p,q)}\|^2 = \sum g^{\alpha_1\overline{\alpha'_1}} \cdots g^{\alpha_p\overline{\alpha'_p}} g^{\beta'_1\overline{\beta}_1} \cdots g^{\beta'_q\overline{\beta}_q} R_{\alpha\overline{\beta}} R_{\beta'\overline{\alpha'}},$$

where the sum runs over ordered multi-indices α, α', β, β' of lengths $|\alpha| = |\alpha'| = p$, $|\beta| = |\beta'| = q$, e.g. $\alpha = (\alpha_1, \ldots, \alpha_p) \in \{0, 1, 2\}^p$, and

$$R_{\alpha_1\cdots\alpha_q\overline{\beta}_1\cdots\overline{\beta}_q} = R_{\alpha_1\overline{\beta}_1\alpha_2\overline{\beta}_2;\alpha_3\cdots\alpha_q\overline{\beta}_3\cdots\overline{\beta}_q}.$$

As before, we restrict $\|R^{(p,q)}\|^2$ to $z_0 = 1$ and regard it as a function on the base domain Ω. It is shown in [HKN2] that (cf. Remark 3.2 below):

Lemma 3.3. *If $w = p+q-2 = 4, 5$ then $\|R^{(p,q)}\|^2$ modulo $O^{6-w}(r)$ is independent of the ambient metric. The boundary values are given by*

$$\begin{aligned} 3\,\|R^{(4,2)}\|^2\big|_M &= 7\,\|R^{(3,3)}\|^2\big|_M = 2^8 \cdot 21\,|A^0_{4\overline{2}}|^2, \\ \|R^{(5,2)}\|^2\big|_M &= -4 \cdot (5!)^2 F_5^{\mathrm{CR}}(1, 18), \\ \|R^{(4,3)}\|^2\big|_M &= -4 \cdot (5!)^2 F_5^{\mathrm{CR}}(4/3, 57/5). \end{aligned}$$

Using these three lemmas, we get:

Theorem 3.1. *There exist universal constants c_0, c_1, c_2, c_3, c'_1, c'_2, c'_3 independent of $A \in \mathcal{N}$ such that*

$$\begin{aligned} \widetilde{\psi}^{\mathrm{B}} + c_0\,\eta_1^{\mathrm{G}} &= c_1\|R^{(3,3)}\|^2\, r + \Big(c_2\|R^{(5,2)}\|^2 + c_3\|R^{(4,3)}\|^2\Big)\, r^2 + O^3(r) \\ &= c'_1\|R^{(4,2)}\|^2\, r + \Big(c'_2\|R^{(5,2)}\|^2 + c'_3\|R^{(4,3)}\|^2\Big)\, r^2 + O^3(r). \end{aligned}$$

The constant c_0 was determined in Graham [G1], where he proved

$$\widetilde{\psi}^{\mathrm{B}} = -12A^0_{4\overline{4}} \quad \text{on} \quad M, \quad \text{so that} \quad c_0 = 3. \tag{3.2}$$

It is shown in [HKN2] that:

Theorem 3.2. *For other universal constants in Theorem 3.1 above,*

$$c_1 = \frac{1}{160}, \quad c_2 = \frac{1}{20160}, \quad c_3 = \frac{1}{560};$$
$$c_1' = \frac{3}{1120}, \quad c_2' = \frac{61}{141120}, \quad c_3' = \frac{3}{7840}.$$

Theorems 3.1 and 3.2 together with (3.1) and (3.2) are the main results of this section.

Remark 3.1. For the two dimensional analysis of φ^{B} and ψ^{B} stated above, Graham [G1] originally proved (3.1) and

$$\widetilde{\psi}^{\mathrm{B}} + 3\,\eta_1^{\mathrm{G}} = (\text{constant})\,|A^0_{2\overline{4}}|^2\, r + O^2(r) \qquad (\text{constant} = 24/5),$$

where the determination of the constant is due to [HKN1]. This result on $\widetilde{\psi}^{\mathrm{B}}$ is refined one step further in [HKN2] to get Theorems 3.1 and 3.2, where the first statement of Lemma 3.3 concerning the ambiguity of the Weyl invariants is crucial.

Remark 3.2. In the argument above, we have only considered the complete contractions of the form $\|R^{(p,q)}\|^2$, because these generate all Weyl invariants of weight $w \le 5$, $w \ne 3$ (see [HKN2]). To state it more precisely, let I_w^{W} denote the vector space of all Weyl invariants of weight w which are well-defined modulo $O^{6-w}(r)$, and set $\widetilde{I}_w^{\mathrm{W}} = I_w^{\mathrm{W}}/\sim$, where $\sim$ stands for the equivalence relation of having the same boundary value. Then $\dim \widetilde{I}_1^W = \dim \widetilde{I}_2^W = \dim \widetilde{I}_3^W = 0$, $\dim \widetilde{I}_4^W = 1$ and $\dim \widetilde{I}_5^W = 2$. Bases of $\widetilde{I}_4^W$ and $\widetilde{I}_5^W$ are given by the boundary values of

$$\|R^{(4,2)}\|^2 \quad (\text{or} \quad \|R^{(3,3)}\|^2) \quad \text{and} \quad \{\|R^{(5,2)}\|^2,\, \|R^{(4,3)}\|^2\},$$

respectively. Consequently, there are isomorphisms $\widetilde{I}_w^{\mathrm{W}} \cong \widetilde{I}_w^{\mathrm{CR}}$ for $w \le 5$, $w \ne 3$. In the exceptional case $w = 3$, the CR invariant $A^0_{4\overline{4}}$ generating the space I_3^{CR} is realized by the boundary value of a linear complete contraction, but the contraction is defined only up to $O^1(r)$ (see [HKN2]).

(B) Determination of the universal constants. We first write down $\widetilde{\psi}^{\mathrm{B}}$ explicitly in terms of Moser's normal coordinate system $z = (z_1, z_2)$. It is sufficient to consider an expansion of $\widetilde{\psi}^{\mathrm{B}}$ along the half-line $p_t = (0, t/2) \in \mathbf{C}^2$ $(t > 0)$. Let $F(a, b, c, d)$ be as in Lemma 3.1. Using a method which will be explained in Section 4, We have:

Proposition 3.1. *As $t \to +0$ along the half-line $p_t = (0, t/2)$,*

$$\begin{aligned}\widetilde{\psi}^{\mathrm{B}} = &- 12\, A^0_{4\bar{4}} - (216\, |A^0_{2\bar{4}}|^2 + a_1\, A^0_{5\bar{5}} + a_2\, A^1_{4\bar{4}})\, t \\ &+ \big(F(660, 1116, a_3, a_4) + a_5\, A^0_{6\bar{6}} + a_6\, A^1_{5\bar{5}} + a_7\, A^2_{4\bar{4}}\big)\; t^2 + O^3(t),\end{aligned}$$

where a_j for $j = 1, 2, \dots, 7$ are constants independent of $A \in \mathcal{N}$.

We next refine Lemmas 3.2 and 3.3. It is rather easy to see that

$$r^{\mathrm{F}} = t + O^3(t) \quad \text{as} \quad t \to +0 \quad \text{along} \quad p_t = (0, t/2). \tag{3.3}$$

We have the following two propositions.

Proposition 3.2. *As $t \to +0$ along the half-line $p_t = (0, t/2)$,*

$$\begin{aligned}\eta_1^{\mathrm{G}} =& 4\, A^0_{4\bar{4}} + \Big(\frac{368}{5}\, |A^0_{2\bar{4}}|^2 + b_1\, A^0_{5\bar{5}} + b_2\, A^1_{4\bar{4}}\Big)\, t \\ &- \bigg(F\Big(\frac{680}{3}, \frac{1956}{5}, b_3, b_4\Big) + b_5\, A^0_{6\bar{6}} + b_6\, A^1_{5\bar{5}} + b_7\, A^2_{4\bar{4}}\bigg)\; t^2 + O^3(t),\end{aligned}$$

where b_j for $j = 1, 2, \dots, 7$ are constants independent of $A \in \mathcal{N}$.

Proposition 3.3. *As $t \to +0$ along the half-line $p_t = (0, t/2)$,*

$$\begin{aligned}\|R^{(4,2)}\|^2 &= 2^8 \cdot 7\, |A^0_{4\bar{2}}|^2 + 2^8\, F(50, 936, d_1, d_2)\, t + O^2(t), \\ \|R^{(3,3)}\|^2 &= 2^8 \cdot 3\, |A^0_{4\bar{2}}|^2 + 2^8 \cdot 3\, F(25, 243, d_3, d_4)\, t + O^2(t),\end{aligned}$$

where d_1, d_2, d_3, d_4 are constants independent of $A \in \mathcal{N}$.

Using these three propositions together with Lemma 3.3, (3.2) and (3.3), we can determine all universal constants in Theorem 3.1 and get Theorem 3.2.

3.2 The complex Monge-Ampère asymptotics

The proofs of the results stated in Section 2 and Subsection 3.1 require knowledge of the construction and properties of the asymptotic solutions of the complex Monge-Ampère boundary value problem (1.13). In this subsection, we summarize these. In particular, we present the method of proving Proposition 3.2. After reviewing in the part (A) Graham's construction of his asymptotic solutions as in Theorem 1.3, we consider in the part (B) its expansion with respect to Moser's normal form coefficients $A = (A^{\ell}_{\alpha\bar{\beta}})$. We are then required to write down the linearization with respect to A, and this is done finally in the part (C).

(A) Construction of the asymptotic solution. We first recall Fefferman's construction [F2] of his defining functions r^{F} of $M \subset \partial\Omega$, which are locally defined smooth approximate solutions of (1.13). Starting from an arbitrary smooth defining function ρ of M, we define r_s for $s = 1, \dots, n+1$ successively by

$$(3.4) \qquad r_1 = J[\rho]^{-1/(n+1)}\,\rho, \qquad r_s = \big(1 + c_s^{-1}(1 - J[r_{s-1}])\big)\, r_{s-1},$$

where $c_s = s(n+2-s)$. Then r_s are smooth defining functions satisfying

$$(3.5)_s \qquad J[r_s] = 1 + O^s(\rho) \qquad (s = 1, \dots, n+1),$$

and thus we may set $r^{\mathrm{F}} = r_{n+1}$. In fact, $(3.5)_1$ holds, since $J[\phi\,\rho] = \phi^{n+1} J[\rho] + O^1(\rho)$ whenever ϕ is smooth. Furthermore, $(3.5)_s$ implies $(3.5)_{s+1}$ for $1 \le s \le n$, since

$$(3.6) \quad J[r + \phi\, r^{s+1}] = J[r] + c_{s+1}\phi\, r^s + O^{s+1}(\rho) \quad (s = 1, \dots, n+1)$$

whenever r is a smooth defining function of M satisfying $J[r] = 1 + O^s(\rho)$. Note that $c_{n+2} = 0$ and thus r_{n+2} cannot be defined by (3.4). Instead, the above equality (3.6) for $s = n+1$ yields the uniqueness of r^{F} modulo $O^{n+2}(\rho)$.

We next recall Graham's construction [G2] of his asymptotic solutions u^{G} of (1.13), which are formal series of the form

$$r + r \sum_{k=0}^{\infty} \eta_k \cdot (r^{n+1} \log r)^k \qquad \text{with } r = r^{\mathrm{F}},$$

where η_k are functions of $(z, \overline{z})$ smooth up to M. Starting from a Fefferman's defining function $r = r^{\mathrm{F}}$ with the initial defining function ρ arbitrarily chosen, we set $u_{n+1} = r$ and define u_s for $s \ge n+2$ successively in such a way that each u_s is a formal series as above (in fact, we can choose u_s to be a finite sum) and satisfies

$$(3.7)_s \qquad J[u_s] = 1 + O^{s-0}(r) \qquad (s \ge n+2),$$

where $O^{s-0}(r)$ stands for an error term of the form $r^s \sum_{k=0}^{\infty} \eta_k \cdot (\log r)^k$. Obviously, $(3.7)_{n+1}$ follows from $(3.5)_{n+1}$. For the ambient metric $g = (g_{j\overline{k}})$ with potential $r_\#$, we define an approximate Laplacian by

$$\Delta[g] = \sum_{j,k=0}^{n} g^{j\overline{k}} \frac{\partial^2}{\partial z_j \partial \overline{z}_k}, \qquad \text{where } \left(g^{j\overline{k}}\right) = (g_{j\overline{k}})^{-1}.$$

Using this, we define a linear differential operator $L = L[r]$ by

$$L[r]f = \Delta[g]\left(|z_0|^2 f\right)\big|_{z_0=1}.$$

It follows that if u_s satisfies $(3.7)_s$ then

$$J[u_s + \phi_{s+1}\, r^{s+1}] = J[u_s] + L(\phi_{s+1}\, r^{s+1}) + O^{s+1-0}(r),$$

where ϕ_{s+1} is a formal series of the form $\sum_{k=0}^{\infty} \eta_k \cdot (\log r)^k$. Thus $(3.7)_{s+1}$ is satisfied by $u_{s+1} = u_s + \phi_{s+1}\, r^{s+1}$ if ϕ_{s+1} is subject to

$$(3.8)_s \qquad L(\phi_{s+1}\, r^{s+1}) = 1 - J[u_s] + O^{s+1-0}(r) \qquad (s \geq n+1),$$

which is regarded as a linearized equation of $J[u] = 1$. If $(3.8)_s$ is solved for all s, then an asymptotic solution u^{G} is given by the formal limit of u_s as $s \to \infty$.

To solve $(3.8)_s$ for $s \geq n+1$, we use the coordinate system $(z', \overline{z'}, r, v)$ and try to determine successively the coefficients of the expansion

$$\phi_{s+1}\, r^{s+1} = \sum_{j \geq s} \sum_{k \geq 0} c_{j,k}[\phi_{s+1}]\, r^j (\log r)^k,$$

where $c_{j,k}[\phi_{s+1}]$ are smooth functions of $(z', \overline{z'}, v)$. Setting

$$L = I + E \quad \text{with } I = \partial_r\,(r\partial_r - n - 2),$$

we see that E is a tangential operator in the sense that it does not contain differentiation with respect to r. Consequently, if we write $(3.8)_s$ as

$$(3.9)_s \qquad I(\phi_{s+1}\, r^{s+1}) = 1 - J[u_s] + O^{s+1-0}(r) \quad (s \geq n+1),$$

then the right side belongs to $O^{s-0}(r)$. Dropping the error term $O^{s+1-0}(r)$ in $(3.9)_s$ and regarding the result as an ordinary differential equation of the form $If = g$, we can determine all the coefficients $c_{j,k}[\phi_{s+1}]$ uniquely provided $c_{n+2,0}[\phi_{n+2}]$ is prescribed, a condition which exactly corresponds to the ambiguity of u^{G}. Therefore, u^{G} is obtained as desired.

(B) Dependence of the asymptotic solution on the normal form coefficients. For a surface in Moser's normal form $N(A)$, let us use the real coordinate system $(z', \overline{z'}, \rho_A, v)$. If we consider the Taylor expansions with respect to this coordinate system, then

(#) the Taylor coefficients of r^{F} modulo $O^{n+2}(\rho_A)$ and those of η_k^{G} modulo $O^{n+1}(\rho_A)$ are polynomials in A.

This can be seen as follows. Starting from the defining function $\rho = \rho_A$, we construct $r = r^{\mathrm{F}}$ and u^{G} with $c_{n+2}[\phi_{n+1}] = 0$ by the algorithm given in the part (A) above. Then (#) holds without error terms, and thus we may write $r^{\mathrm{F}} = r^{\mathrm{F}}_A$ and $u^{\mathrm{G}} = u^{\mathrm{G}}_A$. The statement (#) for general r^{F} and u^{G} follows from (1^{F}) and (1^{G}) in Subsection 1.3.

To prove Proposition 3.2, we need to know the explicit dependence of r^{F}_A and u^{G}_A on A. We thus expand u^{G}_A in powers of A as follows (the expansion of r^{F}_A in powers of A will be discussed in the part (C) below):

$$u^{\mathrm{G}}_A = \sum_{s=0}^{\infty} \psi_s \quad \text{with} \quad \psi_s = \sum_{j \geq 1} \sum_{k \geq 0} \eta_{j,k}[\psi_s]\, \rho_A^j (\log \rho_A)^k, \tag{3.10}$$

where $\eta_{j,k}[\psi_s] = \eta_{j,k}[\psi_s](z', \overline{z'}, v; A)$ are homogeneous polynomials of degree s in A such that the coefficients are polynomials in $(z', \overline{z'}, v)$. Regarding (3.10) as an asymptotic series in powers of A, we have:

Proposition 3.4. *There exists a unique asymptotic series u^{G}_A of the form* (3.10) *such that $J[u^{\mathrm{G}}_A] = 1$ and $\eta_{n+2,0} := \sum_{s=0}^{\infty} \eta_{n+2,0}[\psi_s] = 0$.*

Proposition 3.4 is proved by constructing $u^{\mathrm{G}}_{A,s} := \sum_{m \leq s} \psi_m$ for $s \in \mathbf{N}_0$, and the algorithm is actually used in the proof of Proposition 3.2 (cf. [HKN2]). The construction is similar to that of u_s in the part (A) above, and done as follows. First, $u^{\mathrm{G}}_{A,0} = \psi_0 = \rho_A$ follows from the condition $\eta_{n+2,0}[\psi_0] = 0$. For $s > 0$, we have by induction on s that

$$J[u^{\mathrm{G}}_A] = J[u^{\mathrm{G}}_{A,s-1}] + L[\rho_A]\, \psi_s + O^{s+1}(A),$$

where $O^{s+1}(A)$ stands for a term which does not contain polynomials of degree $\leq s$ in A. (Here, ρ_A is regarded as an independent variable, and the dependence of ρ_A on A is not taken into account.) The above equality is written as a linear equation for ψ_s (cf. $(3.8)_s$ in the part (A)):

$$L[\rho_A]\, \psi_s = 1 - J[u^{\mathrm{G}}_{A,s-1}] + O^{s+1}(A).$$

Therefore, ψ_s and thus $u^{\mathrm{G}}_{A,s}$ are determined inductively by solving this equation under the condition $\eta_{n+2,0}[\psi_s] = 0$.

(C) First variation of the Monge-Ampère equation. Let us next consider the dependence of r^{F}_A on $A \in \mathcal{N}$. To prove Proposition 3.3 in the previous subsection, we need to know r^{F}_A modulo $O^2(A)$ explicitly. Less precise information is required also in the proof of Theorem 2.1 (see Section 5 below). We thus consider $r^{\mathrm{F}}_{\varepsilon A}$ for a real parameter ε, and seek an approximate boundary value problem which characterizes the first variation $\widetilde{r}^{\mathrm{F}}_{A} = (d/d\varepsilon) r^{\mathrm{F}}_{\varepsilon A}\big|_{\varepsilon=0}$.

We begin with a heuristic argument for an exact asymptotic solution u^{G}_{A} in place of a smooth approximate one r^{F}_{A}, disregarding the difficulty due to the ambiguity of Fefferman's defining functions. Supposing as if $u^{\mathrm{G}}_{\varepsilon A}$ were smoothly depending on $\varepsilon \in \mathbf{R}$ small and had no singularity even on the boundary, we set $\widetilde{u}^{\mathrm{G}}_{A} = (d/d\varepsilon) u^{\mathrm{G}}_{\varepsilon A}\big|_{\varepsilon=0}$. Then, a relation characterizing $\widetilde{u}^{\mathrm{G}}_{A}$ is obtained by taking the first variation of the formal boundary value problem

$$(3.11) \qquad J[u^{\mathrm{G}}_{\varepsilon A}] = 1 \quad \text{in} \ \ \Omega_\varepsilon, \qquad u^{\mathrm{G}}_{\varepsilon A} = 0 \quad \text{on} \ \ \partial\Omega_\varepsilon = N(\varepsilon A),$$

where Ω_ε is a pseudoconvex side of $N(\varepsilon A)$. The first equality yields

$$(3.12) \qquad L[\rho_0]\, \widetilde{u}^{\mathrm{G}}_{A} = 0 \quad \text{in} \ \ \Omega_0.$$

The second equality of (3.11) is written as $u^{\mathrm{G}}_{\varepsilon A}(z', \overline{z}', u, v) = 0$ evaluated at $u = (|z'|^2 + \varepsilon\, F_A(z', \overline{z'}, v))/2$. Differentiating both sides of this equality with respect to ε and evaluating the result at $\varepsilon = 0$, we have

$$\widetilde{u}^{\mathrm{G}}_{A}(z', \overline{z'}, |z'|^2/2, v) = -\frac{1}{2}\, \frac{\partial u^{\mathrm{G}}_{0}}{\partial u}(z', \overline{z'}, |z'|^2/2, v)\, F_A(z', \overline{z'}, v).$$

Recalling that $u^{\mathrm{G}}_{0} = \rho_0 = 2u - |z'|^2$, we get

$$(3.13) \qquad \widetilde{u}^{\mathrm{G}}_{A} = -F_A \qquad \text{on} \ \ \partial\Omega_0 = N(0) = \{\rho_0 = 0\}.$$

The function $\widetilde{u}^{\mathrm{G}}_{A}$ is obtained by solving the linear equation (3.12) under the boundary condition (3.13).

Returning to the original problem of expressing the first variation of r^{F}_{A}, we have:

Proposition 3.5. *The first variation* $\widetilde{r}^{\mathrm{F}}_{A} = (d/d\varepsilon) r^{\mathrm{F}}_{\varepsilon A}\big|_{\varepsilon=0}$ *exists and satisfies the approximate boundary value problem*

$$(3.14) \quad L[\rho_0]\, \widetilde{r}^{\mathrm{F}}_{A} = O^{n+1}(\rho_0) \quad \textit{in} \ \ \Omega_0, \qquad \widetilde{r}^{\mathrm{F}}_{A} = -F_A \quad \textit{on} \ \ \partial\Omega_0 = N(0).$$

The problem (3.14) *has a formal power series solution which is unique modulo* $O^{n+2}(\rho_0)$.

The proof of the latter part of Proposition 3.5 above is done similarly to that of Proposition 3.4.

To give an explicit representation of $\widetilde{r}^{\mathrm{F}}_A$, it is convenient to lift the problem (3.14) to $\mathbf{C}^* \times \Omega_0$. Setting $(\rho_0)_\# = |z_0|^2\rho_0$, $(F_A)_\# = |z_0|^2 F_A$ and $(\widetilde{r}^{\mathrm{F}}_A)_\# = |z_0|^2\widetilde{r}^{\mathrm{F}}_A$, we write (3.14) as

$$\text{(3.15)} \quad \Delta_0(\widetilde{r}^{\mathrm{F}}_A)_\# = O^{n+1}\left((\rho_0)_\#\right), \qquad (\widetilde{r}^{\mathrm{F}}_A)_\# = -(F_A)_\# + O^1\left((\rho_0)_\#\right),$$

where $\Delta_0 = \Delta[g_0]$, which is the (negative) Laplacian with respect to the ambient metric g_0 with potential $(\rho_0)_\#$. Solutions of (3.15) are given by

$$\text{(3.16)} \quad (\widetilde{r}^{\mathrm{F}}_A)_\# = -(F_A)_\# - \sum_{s=1}^{n+1} \frac{(-\rho_0)^s_\# \Delta_0^s (F_A)_\#}{c_1 \cdots c_s} \quad \bmod O^{n+2}\left((\rho_0)_\#\right),$$

where $c_s = s(n+2-s)$, which are the same constants as those in (3.4). To see that the right side of (3.16) gives a solution of (3.15), we use the projective coordinates $z_0 = \zeta_0$, $z_j = \zeta_j/\zeta_0$ $(j = 1, \ldots, n)$. Then

$$(\rho_0)_\# = \zeta_0\,\overline{\zeta}_n + \zeta_n\,\overline{\zeta}_0 - \sum_{j=1}^{n-1} |\zeta_j|^2 \quad \text{and} \quad g_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where I_{n-1} is the identity matrix. Noting that $(g_0)^{-1} = g_0$, we have

$$\Delta_0 = \frac{\partial^2}{\partial\zeta_0\,\partial\overline{\zeta}_n} + \frac{\partial^2}{\partial\zeta_n\,\partial\overline{\zeta}_0} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial\zeta_j\,\partial\overline{\zeta}_j}\,.$$

This expression permits us to compute the commutator

$$[\Delta_0, (\rho_0)^s_\#] = s\,(\rho_0)^{s-1}_\#(Z + \overline{Z} + n + s),$$

where $Z = \sum_{j=0}^n \zeta_j\,\partial/\partial\zeta_j$. Consequently,

$$\Delta_0\left((\rho_0)^s_\#\Delta_0^s(F_A)_\#\right) = (\rho_0)^s_\#\Delta_0^{s+1}(F_A)_\# + c_s(\rho_0)^{s-1}_\#\Delta_0^s(F_A)_\#.$$

Therefore, $(\widetilde{r}^{\mathrm{F}}_A)_\#$ in (3.16) satisfies (3.15).

Remark 3.3. In proving Lemma 3.2 stated in the previous subsection, Graham [G2] uses essentially the same expression for $\widetilde{r}^{\mathrm{F}}_A$ as that for $(\widetilde{r}^{\mathrm{F}}_A)_\#$ given by (3.16).

§4 Microlocal calculus of the Bergman kernel

4.1 Outline

Proposition 3.1 is proved by using a method of Boutet de Monvel [B1]–[B3] of computing explicitly the singularity of the Bergman kernel.

In this section, we briefly explain his method which remains valid in the n dimensional case (cf. Theorem 4.2 below). To get an alternative proof of Theorem 1.2, Boutet de Monvel and Sjöstrand constructed in [BS] a Fourier integral operator $\mathbf{A}^{\mathrm{FIO}}$ with complex phase, which transforms the Bergman kernel of a strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$ to that of a model domain Ω_0 (cf. Example in Subsection 1.2). It might be difficult to derive information we need from $\mathbf{A}^{\mathrm{FIO}}$. It would seem, for a general strictly pseudoconvex domain Ω, that there is no known system of differential equations which characterizes the Bergman kernel, and that this is a reason why the computation of the Bergman kernel was not so easy.

Kashiwara discovered in [Kas] a system of microdifferential equations (i.e. pseudodifferential equations in the real analytic category or its complexification) which characterizes the Bergman kernel $K^{\mathrm{B}}(z, \overline{z})$ up to a multiplicative constant. This system arises as the formal adjoint of a system which characterizes the singularity of the Heaviside function of the domain Ω (i.e. the characteristic function of Ω or its complexification) up to a multiplicative constant (cf. Theorem 4.1 below). The Heaviside function of the model domain Ω_0 is transformed to that of Ω by a shift (or translation) operator $\mathbf{A}^{\mathrm{shift}}(z, \partial_z)$, and consequently, the operator $\mathbf{A}^{\mathrm{B}}(z, \partial_z)$ which transforms the Bergman kernel of Ω_0 to that of Ω is given by

$$(4.1) \qquad \mathbf{A}^{\mathrm{B}} = \mathbf{A}^{*-1} = \sum_{j=0}^{\infty} (1 - \mathbf{A}^*)^j \qquad \text{for } \mathbf{A} = \mathbf{A}^{\mathrm{shift}},$$

where $\mathbf{A}^* = \mathbf{A}^*(z, \partial_z)$ is the formal adjoint of the shift operator $\mathbf{A} = \mathbf{A}(z, \partial_z)$. This formula, due to Boutet de Monvel, remains valid formally in the C^∞ category.

The operator $\mathbf{A}^{\mathrm{B}}$ is much simpler than the Fourier integral operator $\mathbf{A}^{\mathrm{FIO}}$, because the shift operator $\mathbf{A}^{\mathrm{shift}}$ is completely explicit. However, we have to be careful with two points. We are now in a complexified world, so that z and $\overline{z}$ are independent variables. A point in (4.1) is that $\mathbf{A}^{\mathrm{shift}} = \mathbf{A}^{\mathrm{shift}}(z, \partial_z)$ is realized as a holomorphic operator, and it is convenient to regard $\mathbf{A}^{\mathrm{shift}}$ as a (formal) microdifferential operator of infinite order. For such operators, usual definitions of the composition, the formal adjoint and the asymptotic expansion should be modified. Another point in (4.1) is that $\mathbf{A}^{\mathrm{B}}$ acts on functions on Ω_0, while $\mathbf{A}^*$ with $\mathbf{A} = \mathbf{A}^{\mathrm{shift}}$ acts on functions on Ω. Though we only consider as operands special types of functions related to the asymptotic expansion of the Bergman kernel as in the part (B) of Subsection 3.2, we need to expand these functions in powers of A. Then the singularity on $N(A)$ loses its role in the asymptotic expansions of operators and operands. We thus need to introduce the notion of weight for formal operators and operands. The formal setting in this sense is necessary even under the assumption that the boundary is real analytic.

The proof of Proposition 3.1 is done by computing explicitly the necessary terms in the right side of (4.1). In the remaining part of this section, we describe briefly the justification of (4.1) and its application to the proof of Proposition 3.1, after a quick overview of the theory of hyperfunctions.

4.2 Quick review of hyperfunction theory

For a mild function f on $\mathbf{R}$, say, in the Schwartz class, let us consider the Cauchy integrals

$$F^{\pm}(z) := \frac{1}{2\pi i}\int_{\mathbf{R}} \frac{f(t)}{t-z}\,dt \qquad \text{for} \quad z \in \mathbf{C}^{\pm},$$

where $\mathbf{C}^{\pm} = \{z \in \mathbf{C};\ \pm\mathrm{Im}\, z > 0\}$. According to the Plemelj formula, the boundary values $f^{\pm}(x) = F^{\pm}(x \pm i\,0)$ for $x \in \mathbf{R}$ exist and satisfy $2\,f^{\pm} = \pm f + i\,\mathcal{H}[f]$, where $\mathcal{H}$ is the Hilbert transformation. In particular,

$$f^{+} - f^{-} = f \quad \text{in } \mathbf{R}. \tag{4.2}$$

More generally, for a Schwartz distribution $f \in \mathcal{D}'(\mathbf{R})$, there exist $F^{\pm} \in \mathcal{O}(\mathbf{C}^{\pm})$ such that the boundary values $f^{\pm}$ on $\mathbf{R}$ exist in $\mathcal{D}'(\mathbf{R})$ and satisfy (4.2). Consequently, f is realized by the pair $F = (F^{+}, F^{-})$ regarded as a holomorphic function in a disconnected open set $\mathbf{C}^{+} \cup \mathbf{C}^{-}$. We identify $F_1, F_2 \in \mathcal{O}(\mathbf{C}^{+} \cup \mathbf{C}^{-})$ if $F_1 - F_2$ extends holomorphically to $\mathbf{C}$, and denote the quotient space by $\mathcal{B}(\mathbf{R})$. Thus $\mathcal{D}'(\mathbf{R}) \subset \mathcal{B}(\mathbf{R})$. Elements of $\mathcal{B}(\mathbf{R})$ are called *hyperfunctions* on $\mathbf{R}$. For $F \in \mathcal{O}(\mathbf{C}^{+} \cup \mathbf{C}^{-})$, we regard (4.2) as a formal expression and write $f \in \mathcal{B}(\mathbf{R})$. Differentiation of $f \in \mathcal{B}(\mathbf{R})$ is then defined by that of $F \in \mathcal{O}(\mathbf{C}^{+} \cup \mathbf{C}^{-})$, and the definition is compatible with that on $\mathcal{D}'(\mathbf{R})$.

The space $\mathcal{B}(X)$ of hyperfunctions on an arbitrary open set $X \subset \mathbf{R}$ is defined similarly by taking an open set $U \subset \mathbf{C}$ such that $X \subset U$ is relatively closed. Each element $f \in \mathcal{B}(X)$ is realized by a function $F \in \mathcal{O}(U \setminus X)$, and two functions $F_1, F_2 \in \mathcal{O}(U \setminus X)$ are identified when $F_1 - F_2$ extends holomorphically to U. The space $\mathcal{B}(X)$ is independent of the choice of U. Multiplication of $f \in \mathcal{B}(X)$ by a real analytic function g on X is then defined by that on $F \in \mathcal{O}(U \setminus X)$ by the complex extension of g to a suitable U, and the definition is again compatible with that on $\mathcal{D}'(\mathbf{R})$. It is remarkable that the restriction mapping $\mathcal{B}(\mathbf{R}) \to \mathcal{B}(X)$ is surjective.

Example. The Heaviside function $Y \in \mathcal{D}'(\mathbf{R})$ is realized by a function $F \in \mathcal{O}(\mathbf{C} \setminus [0, \infty))$ satisfying $f^+(x) = (-1/2\pi i) \log x$ for $x > 0$. Thus, the Delta measure $\delta \in \mathcal{D}'(\mathbf{R})$ is realized by the function $-1/2\pi i z$. More generally, a class of distributions on an open set $X \subset \mathbf{R}$ containing the origin is given by holomorphic functions on $U \setminus X$, with U as above, of the form

$$\frac{\varphi(z)}{z^\ell} + \psi(z) \log z \quad \text{with} \quad \ell \in \mathbf{N}_0 \quad \text{and} \quad \varphi, \psi \in \mathcal{O}(U).$$

For $f \in \mathcal{B}(X)$, its *support* $\operatorname{supp} f$ is defined by the complement of the largest open subset of X on which $f = 0$. For a compact set $K \subset X$, we denote by $\mathcal{B}_K(X)$ the totality of $f \in \mathcal{B}(X)$ such that $\operatorname{supp} f \subset K$. Then $\mathcal{B}_K(X)$ is identified with the dual of the space $C^\omega(K)$ of real analytic functions near K. Thus, elements of $\mathcal{B}_K(X)$ are regarded as analytic functionals. Each element $f \in \mathcal{B}(X)$ is expressed as a locally finite sum $f = \sum f_j$ such that $\operatorname{supp} f_j \subset X$ are compact. This gives an alternative definition of $\mathcal{B}(X)$, which remains valid in the higher dimensional case.

For a Schwartz distribution f on an open set $X \subset \mathbf{R}^n$, there exist open convex cones $\Gamma_j \subset \mathbf{R}^n$ with vertices at the origin and functions $F_j \in \mathcal{O}(X + i\,\Gamma_j)$ for $j = 1, \dots, N$ such that

$$f(x) = \sum_{j=1}^{N} F_j(x + i\,\Gamma_j 0) \quad \text{for} \quad x \in X, \tag{4.3}$$

where $F_j(x + i\,\Gamma_j 0)$ denote the limits of $F_j(x + i\,y)$ as $y \to 0$ with $y \in \Gamma_j$. Similarly for $f \in \mathcal{B}(X)$, and this property can be used as a definition of $\mathcal{B}(X)$, in which an arbitrary list of holomorphic functions $(F_1, \dots, F_N)$ is considered. Let $WF_A(f)$ denote the analytic wave front set of $f \in \mathcal{D}'(X)$. Then for $(x_0, y) \in T^*X \setminus 0$, we have $(x_0, y) \notin WF_A(f)$ if and only if there exists a representation of f of the form (4.3) for x near x_0 such that $y \notin \bigcup \Gamma_j^\circ$, where Γ_j° denote the (open) dual cones of Γ_j. The *microanalyticity* of $f \in \mathcal{B}(X)$ is defined by this condition, and the *singular spectrum* of f is defined by $\text{S.S.}\, f = \{(x, y) \in T^*X \setminus 0;\ f \notin \mathcal{A}_{(x,y)}\}$, where $\mathcal{A}_{(x,y)}$ denotes the set of germs of hyperfunctions which are microanalytic in the direction (x, y). Thus $\text{S.S.}\, f = WF_A(f)$ for $f \in \mathcal{D}'(X)$.

A microlocal singularity (in the analytic category) of a hyperfunction is called a *microfunction*. That is, for $f \in \mathcal{B}(X)$, a microfunction at $(x, y) \in T^*X \setminus 0$ is defined by f modulo $\mathcal{A}_{(x,y)}$. The equivalence class is denoted by $[f]$, and the totality of such equivalence classes is denoted by $\mathcal{C}_{(x,y)}$. Given a microfunction $[f] \in \mathcal{C}_{(x,y)}$, there exists $F \in \mathcal{O}(X + i\,\Gamma)$

with an open convex cone Γ such that $y \in \Gamma^\circ$ and that $f(x) - F(x + i\,\Gamma 0)$ is microanalytic in the direction (x, y). Thus $[f]$ is identified with the equivalence class $[F]$ of $F(x + i\,\Gamma 0)$.

Differentiation of a microfunction $[f] \in \mathcal{C}_{(x,y)}$ is defined by using a holomorphic function F such that $f(x) - F(x + i\,\Gamma 0)$ is microanalytic, and similarly for multiplication by a real analytic function. These define the action of linear differential operators with analytic coefficients on microfunctions. It is also possible to define indefinite integration of $[f]$ with respect to a variable, say $\partial_{x_1}^{-1}$ at (x, y) with $y_1 \neq 0$. The analogue of pseudodifferential operator in analytic category, acting on microfunction, is called *microdifferential operator*. The symbol of a microdifferential operator of order m is a formal series $P(z, \xi) = \sum_{j=-\infty}^{m} p_j(z, \xi)$ of holomorphic functions on a conic open set $\Omega \subset T^*\mathbf{C}^n \setminus 0$ such that each p_j is homogeneous of degree j in ξ and satisfies

$$|p_j(z, \xi)| \leq C_K^{-j} (-j)! \quad \text{for } j < 0 \tag{4.4}$$

on each compact set $K \subset \Omega$, where $C_K > 0$ is a constant. Near a point $(x, y) \in \Omega \cap T^*\mathbf{R}^n$ with $y_n \neq 0$, each $p_j(z, \xi)$ admits an expansion

$$p_j(z, \xi) = \sum_{k=-\infty}^{j} \sum_{|\alpha|=j-k} a_{k\alpha}(z) {\xi'}^{\alpha} \xi_n^k.$$

Thus replacing ξ by ∂_z we may define $P(z, \partial_z)F(z)$ as a convergent series for each holomorphic function $F(z)$ on a wedge $X + i\Gamma$ such that $X + i\Gamma^\circ \subset \Omega$. In this action the ambiguity of the indefinite integral $\partial_{z_n}^{-1}$ causes only a difference by a function that extends holomorphically to $z = x$. Thus the action of $P(z, \partial_z)$ to $[F(x + i\Gamma 0)] \in \mathcal{C}_{(x,y)}$ can be defined by the modulo class of $P(z, \partial_z)F(z)$.

Remark 4.1. A microdifferential operator of infinite order $P(z, \partial_z)$ is also defined by giving the symbol

$$P(z, \xi) = \sum_{j=-\infty}^{\infty} p_j(z, \xi) \qquad (p_j \in \mathcal{O}(\Omega)),$$

where each p_j is homogeneous of degree j in ξ. In addition to (4.4), it is required that

$$|p_j(z, \xi)| \leq C_{K,\varepsilon}\, \varepsilon^j / j! \qquad (j \in \mathbf{N}_0,\ \varepsilon > 0),$$

where $C_{K,\varepsilon} > 0$ is a constant. Thus $P(z, \partial_z)$ is a local operator. In Subsection 4.4 below, we shall be concerned with a shift operator $\mathbf{A} =$

$\mathbf{A}(z, \partial_z)$. Though $\mathbf{A}$ is not a local operator, we regard it as a formal microdifferential operator of infinite order.

A far more precise description of the matters in this subsection is found in a book by Kaneko [Kan].

4.3 Kashiwara's characterization of the Bergman kernel

Let $\Omega \subset \mathbf{C}^n$ be a strictly pseudoconvex domain with a local defining function ρ which is positive in Ω and real analytic near a point $p \in \partial\Omega$, and let $M \subset \partial\Omega$ be a small neighborhood of p. Setting $X = \mathbf{C}^n$ and denoting by X' the complex conjugate of X, we regard $X \times X'$ as the complexification of X identified with $\mathbf{R}^{2n}$. Then ρ extends holomorphically to a neighborhood $U \subset X \times X'$ of M, and the complexification of M is given by $N = \{\rho(z, \overline{z}) = 0\} \subset U$. We also have $\Omega = \{\rho(z, \overline{z}) > 0\} \subset X \times X'$ and $\Omega \times \Omega' \subset \{\operatorname{Re} \rho(z, \overline{z}) > 0\}$ near M.

The Bergman kernel $K^{\mathrm{B}} = \varphi^{\mathrm{B}} \rho^{-n-1} + \psi^{\mathrm{B}} \log \rho$ near M has a multi-valued holomorphic extension to $U \setminus N$ (cf. Remark 1.3). Thus, setting

$$U^{\pm} = \{(z, \overline{z}) \in U;\ \pm \operatorname{Im} \rho(z, \overline{z}) > 0\},$$

we have $K^{\mathrm{B}} \in \mathcal{O}(U^+)$. Another multi-valued function on $U \setminus N$ is defined by $Y(\rho) = -(1/2\pi i) \log \rho$, and we have $Y(\rho) \in \mathcal{O}(U^+ \cup U^-)$ which represents the characteristic function of Ω near M. Let us regard K^{B} and $Y(\rho)$ as elements of $\mathcal{O}(U^+)$. Then these define hyperfunctions with the same singular spectrum

$$T^*_M X = \{(x, \lambda\, d\rho(x)) \in T^*X;\ x \in M,\ 0 \neq \lambda \in \mathbf{R}\},$$

the conormal bundle of M. Similarly for multi-valued functions on $U \setminus N$ of the form

$$K = \sum_{\ell=1}^{m} \varphi_\ell\, \rho^{-\ell} + \psi \log \rho \quad \text{with } \varphi_\ell, \psi \in \mathcal{O}(U),\ \ m \in \mathbf{N}. \tag{4.5}$$

For $(x, y) = (x, d\rho(x)) \in T^*_M X$, elements $[K] \in \mathcal{C}_{(x,y)}$ defined by K of the form (4.5) are called *holomorphic microfunctions*, and the totality of these is denoted by $(\mathcal{C}_{N|X\times X'})_{(x,y)}$. In what follows, we omit the bracket in $[K]$ and regard K as a holomorphic microfunction.

Action of microdifferential operators on $\mathcal{C}_{(x,y)}$ preserves the subspace $(\mathcal{C}_{N|X\times X'})_{(x,y)}$. Let $K \in (\mathcal{C}_{N|X\times X'})_{(x,y)}$ such that $\varphi \neq 0$ in (4.5). Then, for a microdifferential operator of the form $P(z, \partial_z)$, there exists a unique microdifferential operator of the form $Q(\overline{z}, \partial_{\overline{z}})$ such that

$$P(z, \partial_z) K = Q(\overline{z}, \partial_{\overline{z}}) K. \tag{4.6}$$

Such an operator P is generated by z_j and $\partial/\partial z_j$ for $j = 1, \dots, n$. Using these generators, we get a system of equations of the form (4.6), and this system characterizes K in $(\mathcal{C}_{N|X\times\overline{X}})_{(x,y)}$ up to a constant multiple. For a general theory including these facts, see Sato-Kawai-Kashiwara [SKK] and Schapira [Sca].

A system characterizing the Bergman kernel can be written down explicitly. The following theorem is due to Kashiwara [Kas].

Theorem 4.1 ([Kas]). *The Bergman kernel* K^{B} *satisfies*

$$P^*(z, \partial_z)\, K^{\mathrm{B}} = Q^*(\overline{z}, \partial_{\overline{z}})\, K^{\mathrm{B}}, \tag{4.7}$$

whenever $P(z, \partial_z)\, Y(\rho) = Q(\overline{z}, \partial_{\overline{z}})\, Y(\rho)$ *with* $Y(\rho) = (-1/2\pi i)\log\rho$, *where* $P^* = P^*(z, \partial_z)$ *and* $Q^* = Q^*(\overline{z}, \partial_{\overline{z}})$ *are the formal adjoints of* P *and* Q*, respectively.*

In the next subsection, we shall give a procedure of constructing the solution to this system of equation by using Moser's normal coordinates.

4.4 A formula of Boutet de Monvel

In the previous subsection, we fixed a domain and considered microdifferential operators of finite order. To study the shift operator $\mathbf{A}$ mentioned in Subsection 4.1, we need to define formal microdifferential operators of infinite order. These operators act on holomorphic microfunctions of infinite order defined by setting $m = \infty$ in (4.5).

It is non-trivial to define such operators, via the symbols, carrying the operations of taking composition, formal adjoint and inverse. We need to introduce the notion of *weight* for the variable $z = (z_1, \dots, z_n)$ by setting

$$\mathrm{w}(z_j) = -1/2 \quad (j = 1, \dots, n-1), \qquad \mathrm{w}(z_n) = -1,$$

and extend it to ∂_z and the dual variable $\xi = (\xi_1, \dots, \xi_n)$ of z by

$$\mathrm{w}(\partial/\partial z_j) = \mathrm{w}(\xi_j) = -\mathrm{w}(z_j) \qquad (j = 1, \dots, n).$$

(We do not consider the notion of weight for polynomials in $A \in \mathcal{N}$ in this subsection.) Then we may say polynomials $P_{w/2} = P_{w/2}(z, \xi, \xi_n^{-1})$ to be of homogeneous weight $w/2$. By a formal sum of such polynomials $P_{w/2}$ with respect to $w \in \mathbf{Z}$ bounded above, we define the (total) symbol of a *formal microdifferential operator of infinite order*. In other words, we regard the symbol as an asymptotic series of decreasing weight. For these operators, operations of taking the composition, the formal adjoint and the inverse are defined, as usual, by using weight in place of order. These operations are compatible with those for microdifferential operators of finite order.

We next define holomorphic microfunctions of infinite order. Again, it is necessary to introduce the notion of weight for holomorphic microfunctions with support $N(0) = \{\rho_0 = 0\} \subset X \times \overline{X}$ by setting

$$\mathrm{w}(\overline{z}_j) = \mathrm{w}(z_j), \quad \mathrm{w}(\log \rho_0) = 0, \quad \mathrm{w}(\rho_0^{-\ell}) = \ell.$$

We consider asymptotic series of decreasing weight:

$$K = \sum_{j=-\infty}^{m} K_{j/2} \quad \text{with } \mathrm{w}(K_{2/j}) = 2/j, \tag{4.8}$$

where $K_{j/2} \in (\mathcal{C}_{N(0)|X\times\overline{X}})_{(0,dz_0)}$. Then we can define an action of formal operators of infinite order $P(z,\partial_z) = \sum_{j=-\infty}^{m'} P_{j/2}(z,\partial_z)$ to K of the form (4.8) by setting

$$P(z,\partial_z)K = \sum_{j=-\infty}^{m+m'} K'_{j/2} \quad \text{with } K'_{j/2} = \sum_{k+\ell=j} P_{k/2}(z,\partial_z)K_{\ell/2}.$$

We refer to a series of the form (4.8) as a *holomorphic microfunction of infinite order*.

Let us restrict ourselves to real analytic surfaces in Moser's normal form $N(A)$ $(A \in \mathcal{N})$. To define the shift operator $\mathbf{A}$ by giving its symbol, we need the following:

Lemma 4.1 ([B1]). *There exists a unique complex-valued defining function of $N(A)$ of the form $\rho_B^{\mathrm{BM}}(z,\overline{z}) = \rho_0(z,\overline{z}) - H_B(z,\overline{z'})$, where $H_B(z,\overline{z'})$ are convergent power series of the form*

$$H_B(z,\overline{z'}) = \sum_{|\alpha|,\,|\beta|\geq 2} B_{\alpha\overline{\beta}}(z_n)\, z'_\alpha\, \overline{z'_\beta}, \quad B_{\alpha\overline{\beta}}(z_n) = \sum_{\ell=0}^{\infty} B^{\ell}_{\alpha\overline{\beta}}\, z_n^{\ell}.$$

The coefficients $B = (B^{\ell}_{\alpha\overline{\beta}})$ are polynomials in $A = (A^{\ell}_{\alpha\overline{\beta}})$, and the trace conditions (1.3) *are valid for $B_{\alpha\overline{\beta}}(z_n)$ in place of $A_{\alpha\overline{\beta}}(v)$.*

With the defining function ρ_B^{BM} in Lemma 4.1, any holomorphic microfunction with support $N(A)$ is written as

$$\varphi\,\rho^{-m} + \psi \log \rho \quad \text{with } \rho = \rho_B^{\mathrm{BM}}. \tag{4.9}$$

Let us expand (4.9) by using

$$\left(\rho_B^{\mathrm{BM}}\right)^{-m} = \rho_0^{-m}\left(1-\frac{H_B}{\rho_0}\right)^{-m} = \sum_{\ell=0}^{\infty}\binom{m}{\ell}(-H_B)^{\ell}\rho_0^{-m-\ell},$$

$$\log\rho_B^{\mathrm{BM}} = \log\rho_0 + \log\left(1-\frac{H_B}{\rho_0}\right) = \log\rho_0 - \sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(\frac{H_B}{\rho_0}\right)^{\ell}.$$

The right sides are asymptotic series of decreasing weight, since H_B consists of terms of weight ≤ -2. Consequently, we obtain an expression of (4.9) as a formal sum of holomorphic microfunctions with support $N(0)$. The asymptotic series thus obtained uniquely determines the original holomorphic microfunction (4.9). We thus identify (4.9) with its asymptotic expansion of the form (4.8).

Lemma 4.2 ([B1]). *Let $\mathbf{A}(z,\partial_z)$ be a formal microdifferential operator of infinite order defined by the symbol*

$$\mathbf{A}(z,\xi) = \exp[-H_B(z,-\xi'/\xi_n)\,\xi_n] \quad \textit{with } \xi = (\xi',\xi_n).$$

Then

$$Y(\rho_B^{\mathrm{BM}}) = \mathbf{A}Y(\rho_0).$$

Lemma 4.2 is proved by direct computation using the relations

$$(\partial/\partial z_j)\,(\partial/\partial z_n)^{-1}\log\rho_0 = \overline{z}_j\log\rho_0 \qquad (j=1,\ldots,n-1).$$

Changing the notation slightly, we denote by K_A^{B} the Bergman kernel associated with the domain bounded by $N(A)$. The singularity of its complex extension is again denoted by $K_A^{\mathrm{B}} = K_A^{\mathrm{B}}(z,\overline{z})$. Regarding it as a holomorphic microfunction, we can state the following theorem of Boutet de Monvel [B1], which is used in the proof of Proposition 3.1.

Theorem 4.2 ([B1]). *Let $\mathbf{A}(z,\partial_z)$ be as in Lemma 4.2. Then the formal adjoint $\mathbf{A}^*$ is invertible as a formal microdifferential operator of infinite order, and the following equality holds:*

$$K_A^{\mathrm{B}} = \mathbf{A}^{*-1}K_0^{\mathrm{B}}. \tag{4.10}$$

The invertibility of $\mathbf{A}^*$ is a consequence of the fact that the symbol expansion of $1-\mathbf{A}^*$ consists of terms of negative weight. In fact, the inverse $\mathbf{A}^{*-1}$ is given by (4.1), because the right side of (4.1) makes sense as an asymptotic series of decreasing weight. The formula (4.10) follows from Theorem 4.1, since $K = \mathbf{A}^{*-1}K_0^{\mathrm{B}}$ satisfies the microdifferential equation $P^*K = Q^*K$ whenever $P = P(z, \partial_z)$ and $Q = Q(\overline{z}, \partial_{\overline{z}})$ satisfy

$$PY(\rho_B^{\mathrm{BM}}) = QY(\rho_B^{\mathrm{BM}}). \tag{4.11}$$

In fact, by virtue of Lemma 4.2 and the commutation relation

$$Q(\overline{z}, \partial_{\overline{z}}) \circ \mathbf{A}(z, \partial_z) = \mathbf{A}(z, \partial_z) \circ Q(\overline{z}, \partial_{\overline{z}}),$$

it follows from (4.11) that

$$\mathbf{A}^{-1} \circ P \circ \mathbf{A}Y(\rho_0) = QY(\rho_0). \tag{4.12}$$

Then Theorem 4.1 yields $\mathbf{A}^* \circ P^* \circ \mathbf{A}^{*-1}K_0^{\mathrm{B}} = Q^*K_0^B$, so that $P^*K = Q^*K$. A point is that $\mathbf{A}^{-1} \circ P \circ \mathbf{A}$ is an operator of finite order. This fact automatically follows from the relation (4.12).

Let us next sketch the proof of Proposition 3.1 by using the formula (4.10) in Theorem 4.2. We consider the expansion of $\widetilde{\psi}^{\mathrm{B}}$ along the half-line $p_t = (0, t/2) \in \mathbf{C}^2$ $(t > 0)$:

$$\widetilde{\psi}^{\mathrm{B}}(p_t) = F_3(A) + F_4(A)\,t + F_5(A)\,t^2 + \cdots.$$

Then each F_j depends only on the terms in $\mathbf{A}^{*-1}$ of the form

$$F_{jk}(A)\, z_2^k\, (\partial/\partial z_2)^{k-j} \qquad (k = 0, 1, \ldots, j-3). \tag{4.13}$$

In addition, if we write $\mathbf{A}^{*-1} = 1 + \sum_{j=-\infty}^{-1} Q_{j/2}$ the expansion of $\mathbf{A}^{*-1}$ of decreasing weight, then F_j is determined by Q_{-j}. On the other hand, if we set $\mathbf{A} = 1 + \sum_{j=-\infty}^{-1} P_{j/2}$, then the trace conditions (1.3) for H_B yield $P_{j/2} = 0$ for $j \geq -4$, and thus $\mathbf{A} = 1 + \sum_{j=-\infty}^{-5} P_{j/2}$. Using these facts, we can show that Q_{-3}, Q_{-4} and Q_{-5} are written as

$$Q_{-3} = -P_{-3}^*, \qquad Q_{-4} = -P_{-4}^* + P_{-2}^* \circ P_{-2}^*$$
$$Q_{-5} = -P_{-5}^* + P_{-5/2}^* \circ P_{-5/2}^* + P_{-2}^* \circ P_{-3}^* + P_{-3}^* \circ P_{-2}^*.$$

The identification of $F_j(A)$ given in Proposition 3.1 is done by computing explicitly the terms of the form (4.13) in each Q_{-j}.

§5 Parabolic invariant theory

In this section, we outline the proof of Theorem 2.1. This amounts to reviewing the invariant theory of Fefferman [F3] supplemented by Bailey-Eastwood-Graham [BEG].

5.1 $H_\#$-invariants of the curvature

Recall that CR invariants are H-invariants of $A \in \mathcal{N}$ (cf. (1.5)). To compare the boundary values of Weyl invariants with CR invariants, it is convenient to represent linear fractional transformations $h \in H$ by matrices $h_\#$ with respect to the projective coordinates $\zeta = (\zeta_0, \ldots, \zeta_n)$ used in Subsection 3.2, (C). Let $H_\#$ denote the (parabolic) subgroup of $\mathrm{SU}(g_0)$ given by

$$H_\# = \{h_\# \in \mathrm{SU}(g_0);\ h_\# e_0 = \lambda e_0 \text{ with some } \lambda \in \mathbf{C}^*\},$$

where $e_0 = {}^t(1, 0, \ldots, 0) \in \mathbf{C}^{n+1}$. Then each element $h_\# \in H_\#$ defines $h \in H$ such that $\lambda = \big(\det h'(0)\big)^{-1/(n+1)}$. An $H_\#$-action on $\mathcal{N}$ is given by $h_\#.A = h.A$, and the definition of CR invariants (1.5) is written as

$$P(h_\#.A) = |\lambda|^{2w} P(A) \qquad \text{for } h_\# \in H_\#,\ \ A \in \mathcal{N}.$$

To regard the boundary values of Weyl invariants as $H_\#$-invariants, we need to define $H_\#$-invariants of the curvature R of the ambient metric by using the $H_\#$-action on $A \in \mathcal{N}$. We first identify R with its Taylor expansion about e_0 with respect to the coordinates $\zeta = (\zeta_0, \ldots, \zeta_n)$. That is, given a domain with boundary in Moser's normal form $N(A)$, we write $R = (R_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 2}$ for the components

$$R_{\alpha\overline{\beta}} = R_{\alpha_1\overline{\beta}_1\alpha_2\overline{\beta}_2;\alpha_3\cdots\alpha_p\overline{\beta}_3\cdots\overline{\beta}_q}$$

of the covariant derivatives of the curvature R evaluated at e_0, where $\alpha = \alpha_1 \cdots \alpha_p$ and $\beta = \beta_1 \cdots \beta_q$ are lists of holomorphic indices $0, 1, \ldots, n$. We now introduce the notion of *weight* for the components $R_{\alpha\overline{\beta}}$, as a generalization of that for Weyl invariants, by setting

$$\mathrm{w}(R_{\alpha\overline{\beta}}) = \mathrm{w}(\alpha\overline{\beta}) = \frac{\|\alpha\overline{\beta}\|}{2} - 1 \quad \text{with } \|\alpha\overline{\beta}\| = \sum_{j=1}^{p} \|\alpha_j\| + \sum_{j=1}^{q} \|\beta_j\|,$$

where $\|0\| = 0$, $\|j\| = 1$ for $j = 1, \ldots, n-1$ and $\|n\| = 2$.

Let us next restrict ourselves to the components $R_{\alpha\overline{\beta}}$ of weight $\leq n$. We then see, as in the proof of Proposition 2.1, that $R_{\alpha\overline{\beta}}$ is a polynomial in A, so that we may write $R_{\alpha\overline{\beta}}$ as $R_{\alpha\overline{\beta}}(A)$. Furthermore,

$(5.1)_1$ $\quad R_{\alpha\overline{\beta}}(A)$ is a polynomial in A of homogeneous weight $\mathrm{w}(\alpha\overline{\beta})$,

$(5.1)_2$ $$R_{\alpha\overline{\beta}}(A) = 0 \qquad (-1 \leq \mathrm{w}(\alpha\overline{\beta}) < 1).$$

These are seen as follows. Given $h_\# = (h_i{}^j) \in H_\#$, we consider the curvature corresponding to $h_\#.A \in \mathcal{N}$. Then the components of weight $\leq n$ and of type (p,q) are transformed by

$$(5.2) \qquad R_{\alpha\overline{\beta}}(h_\#.A) = \lambda^{p-1}\,\overline{\lambda}^{q-1} \sum_{|\alpha'|=p,|\beta'|=q} h_\alpha{}^{\alpha'}\overline{h_\beta{}^{\beta'}}\, R_{\alpha'\overline{\beta'}}(A)$$

with $\lambda \in \mathbf{C}$ defined by $h_\# e_0 = \lambda\, e_0$, where $h_\alpha{}^{\alpha'} = h_{\alpha_1}{}^{\alpha'_1}\cdots h_{\alpha_p}{}^{\alpha'_p}$ and $h_\beta{}^{\beta'} = h_{\beta_1}{}^{\beta'_1}\cdots h_{\beta_q}{}^{\beta'_q}$. The transformation law is thus weighted by the factor $\lambda^{p-1}\overline{\lambda}^{q-1}$. If in particular $h_\#$ corresponds to a dilation ϕ_r, then $R_{\alpha\overline{\beta}}(h_\#.A) = r^{-2\mathrm{w}(\alpha\overline{\beta})}R_{\alpha\overline{\beta}}(A)$. Thus $(5.1)_1$ is obtained. The proof of $(5.1)_2$ is simple. Since components of $A \in \mathcal{N}$ satisfy $\mathrm{w}(A^\ell_{\alpha\overline{\beta}}) \geq 1$, it follows that each $R_{\alpha\overline{\beta}}(A)$ with $\mathrm{w}(\alpha\overline{\beta}) < 1$ is a constant, which is 0 because $R_{\alpha\overline{\beta}}(0) = 0$.

Regarding $R_{\alpha\overline{\beta}} \in \mathbf{C} = \mathbf{R} + i\mathbf{R}$ with $|\alpha|, |\beta| \geq 2$ as independent variables, we denote by $\mathcal{R}^{\mathrm{aux}}$ the totality of the points $R = (R_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 2}$ satisfying

$$(5.3) \qquad R_{\alpha\overline{\beta}} = 0 \qquad (-1 \leq \mathrm{w}(\alpha\overline{\beta}) < 1).$$

Thus $\mathcal{R}^{\mathrm{aux}}$ is a countable dimensional real vector space. Truncating components of $R = (R_{\alpha\overline{\beta}}) \in \mathcal{R}^{\mathrm{aux}}$ by $\mathrm{w}(\alpha\overline{\beta}) \leq n$, we obtain an infinite dimensional vector space $\mathcal{R}^{\mathrm{aux}}_n$ as the quotient space of $\mathcal{R}^{\mathrm{aux}}$. This space $\mathcal{R}^{\mathrm{aux}}_n$ admits an $H_\#$-action

$$H_\# \times \mathcal{R}^{\mathrm{aux}}_n \ni (h_\#, R_n) \mapsto h_\#.R_n \in \mathcal{R}^{\mathrm{aux}}_n$$

given by the right side of (5.2) with $R_{\alpha'\overline{\beta'}}$ in place of $R_{\alpha'\overline{\beta'}}(A)$. In fact, since $h_i{}^j = 0$ for $\|i\| < \|j\|$, it follows that the $H_\#$-action on $\mathcal{R}^{\mathrm{aux}}_n$ above is well-defined.

Returning to the components of the curvature $R = (R_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 2}$, we write $R_n(A) = (R_{\alpha\overline{\beta}}(A))_{\mathrm{w}(\alpha\overline{\beta})\leq n}$ and denote by $\mathcal{R}_n$ the image of the map $\mathcal{N} \ni A \mapsto R_n(A) \in \mathcal{R}^{\mathrm{aux}}_n$. It then follows from (5.2) and the definition of the $H_\#$-action on $\mathcal{R}^{\mathrm{aux}}_n$ that

$$h_\#.(R_n(A)) = R_n(h_\#.A) \in \mathcal{R}^{\mathrm{aux}}_n.$$

That is, the map $A \mapsto R_n(A)$ is $H_\#$-equivariant and $\mathcal{R}_n$ is an $H_\#$-invariant subset of $\mathcal{R}^{\mathrm{aux}}_n$. In what follows, we sometimes abbreviate the variable $R_n \in \mathcal{R}^{\mathrm{aux}}_n$ by writing R.

Definition. A polynomial $P = P(R)$ in $R \in \mathcal{R}_n^{\mathrm{aux}}$ is called an $H_\#$-*invariant* of $\mathcal{R}_n$ of *weight* $w \le n$ if

$$P(h_\#.R) = |\lambda|^{2w} P(R) \quad \text{for any} \quad (h_\#, R) \in H_\# \times \mathcal{R}_n.$$

Two $H_\#$-invariants are identified if these are identical as functions on $\mathcal{R}_n$. The totality of $H_\#$-invariants of $\mathcal{R}_n$ is denoted by $I_w(\mathcal{R}_n)$.

For $R \in \mathcal{R}^{\mathrm{aux}}$, let us consider complete contractions

$$W(R) = \mathrm{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\right) \tag{5.4}$$

of the tensors $R^{(p,q)} = (R_{\alpha\overline{\beta}})_{|\alpha|=p,|\beta|=q}$ with respect to the flat metric g_0. Then $W(R)$ is a polynomial in $R \in \mathcal{R}^{\mathrm{aux}}$ of homogeneous weight. If $\mathrm{w}(W(R)) \le n$, then $W(R)$ depends only on $R \in \mathcal{R}_n^{\mathrm{aux}}$ because of (5.3), and thus $W(R)$ gives an $H_\#$-invariant of $\mathcal{R}_n$. We define *Weyl invariants of* $\mathcal{R}_n$ as linear combinations of the complete contractions of the form (5.4) which are of homogeneous weight $\le n$. Denoting by $I_w^{\mathrm{W}}(\mathcal{R}_n)$ the totality of Weyl invariants of weight w, we have $I_w^{\mathrm{W}}(\mathcal{R}_n) \subset I_w(\mathcal{R}_n)$ for $w \le n$.

The surjection $\mathcal{N} \ni A \mapsto R(A) \in \mathcal{R}_n$ induces a map

$$I_w(\mathcal{R}_n) \ni P(R) \mapsto P(R(A)) \in I_w^{\mathrm{CR}} \qquad (w \le n). \tag{5.5}$$

Therefore, Theorem 2.1 follows from:

Theorem 2.1′. (I) *The map* (5.5) *is surjective (and thus bijective).*

(II) $I_w^{\mathrm{W}}(\mathcal{R}_n) = I_w(\mathcal{R}_n)$ *for* $w \le n$.

We outline the proofs of (I) and (II) in Subsections 5.2 and 5.3, respectively.

5.2 Bijectivity of (5.5)

The proof of the part (I) in Theorem 2.1′ is done by giving the inverse of the map (5.5). We first note by $w \le n$ that any $Q(A) \in I_w^{\mathrm{CR}}$ depends only on

$$A_n = (A^\ell_{\alpha\overline{\beta}})_{\mathrm{w}(\alpha\overline{\beta}\ell) \le n} \quad \text{for} \ A = (A^\ell_{\alpha\overline{\beta}}) \in \mathcal{N},$$

so that one may write $Q(A) = Q(A_n)$. Let $\mathcal{N}_n$ denote the totality of such A_n, that is, $\mathcal{N}_n = \{A_n;\ A \in \mathcal{N}\}$. Then, $R(A) \in \mathcal{R}_n$ for $A \in \mathcal{N}$

depends only on $A_n \in \mathcal{N}_n$, and thus the map $\mathcal{N} \ni A \mapsto R(A) \in \mathcal{R}_n$ induces a surjection

$$\mathcal{F} : \mathcal{N}_n \ni A_n \mapsto R(A_n) \in \mathcal{R}_n, \quad \text{where } R(A_n) = R(A). \tag{5.6}$$

This surjection is $H_\#$-equivariant, where the $H_\#$-action

$$H_\# \times \mathcal{N}_n \ni (h_\#, A_n) \mapsto h_\#.A_n \in \mathcal{N}_n$$

is well-defined from the $H_\#$-action on $\mathcal{N}$. We have:

Theorem 5.1. *The surjection $\mathcal{F}$ in* (5.6) *is bijective and the inverse $\mathcal{G} = \mathcal{F}^{-1}$ extends to a polynomial map $\mathcal{R}_n^{\mathrm{aux}} \to \mathcal{N}_n$, in the sense that the components are polynomials in $R \in \mathcal{R}_n^{\mathrm{aux}}$. (The map $\mathcal{G}$ is automatically $H_\#$-equivariant.)*

Assuming for a while the validity of Theorem 5.1, let us first prove the bijectivity of the map (5.5). Given $Q(A_n) \in I_w^{\mathrm{CR}}$ arbitrarily, we set $P(R) = Q(\mathcal{G}(R))$ for $R \in \mathcal{R}_n$. Then

$$P(\mathcal{F}(A_n)) = Q(\mathcal{G} \circ \mathcal{F}(A_n)) = Q(A_n),$$

and the $H_\#$-equivariance of $\mathcal{G}$ implies $P(R) \in I_w(\mathcal{R}_n)$. Conversely, given $P(R) \in I_w(\mathcal{R}_n)$ arbitrarily, we set $Q(A_n) = P(\mathcal{F}(A_n))$ for $A_n \in \mathcal{N}_n$. Then

$$Q(\mathcal{G}(R)) = P(\mathcal{F} \circ \mathcal{G}(R)) = P(R),$$

and the $H_\#$-equivariance of $\mathcal{F}$ implies $Q(A_n) \in I_w^{\mathrm{CR}}$. Consequently, the pull-back by $\mathcal{G}$ gives the inverse map of (5.5), and thus (I) in Theorem 2.1′ is proved.

To prove Theorem 5.1, we extend the target space $\mathcal{R}_n$ of the map $\mathcal{F}$ in (5.6) to $\mathcal{R}_n^{\mathrm{aux}}$. That is, if we denote this new map again by $\mathcal{F}$,

$$\mathcal{F} : \mathcal{N}_n \to \mathcal{R}_n^{\mathrm{aux}} \qquad (\text{and } \ \mathcal{F}(\mathcal{N}_n) = \mathcal{R}_n)\,. \tag{5.7}$$

Now note that $\mathcal{F}$ is finite dimensional in the sense that $\mathcal{N}_n$ is a finite dimensional vector space. Then the injectivity of $\mathcal{F}$ follows from the following proposition.

Proposition 5.1. *The differential $\mathcal{F}'(0) : \mathcal{N}_n \to \mathcal{R}_n^{\mathrm{aux}}$ of $\mathcal{F}$ in* (5.7) *at the origin is injective. Consequently, $\mathcal{F}$ is an embedding and $\mathcal{R}_n \subset \mathcal{R}_n^{\mathrm{aux}}$ is a finite dimensional manifold. (We are always working near the origin.)*

To complete the proof of Theorem 5.1, it remains to show that $\mathcal{G}$ extends to a polynomial map. By Proposition 5.1, we get an extension of $\mathcal{G}$,

$$\mathcal{R}_n^{\mathrm{aux}} \ni R \mapsto A(R) = (A^{\ell}_{\alpha\overline{\beta}}(R)) \in \mathcal{N}_n,$$

such that each component $A^{\ell}_{\alpha\overline{\beta}}(R)$ is a formal power series in R of homogeneous weight $\mathrm{w}(\alpha\overline{\beta}\ell)$. In addition, the series $A^{\ell}_{\alpha\overline{\beta}}(R)$ depends only on a finite number of components of R and is convergent near the origin. Using $(5.1)_2$, we can remove monomials of degree $> \mathrm{w}(\alpha\overline{\beta}\ell)$ from $A^{\ell}_{\alpha\overline{\beta}}(R)$ without changing the value on $\mathcal{R}_n$. The resulting polynomials give a polynomial extension of $\mathcal{G}$.

We conclude this subsection by sketching the proof of Proposition 5.1. Setting $R_n = \mathcal{F}'(0)A_n$, we wish to show that $R_n = 0$ implies $A_n = 0$. To express $R_n = (R_{\alpha\overline{\beta}})$ explicitly, we take Fefferman's defining functions $r_{\varepsilon A}$ of $N(\varepsilon A)$ given in Subsection 3.2, (B), and denote by $\widetilde{r}^{\mathrm{F}}_A$ the first variation at $\varepsilon = 0$. Then

$$R_{\alpha\overline{\beta}} = \partial^{\alpha}_{\zeta}\partial^{\beta}_{\overline{\zeta}}(\widetilde{r}^{\mathrm{F}}_A)_{\#}|_{e_0}, \quad \text{where } (\widetilde{r}^{\mathrm{F}}_A)_{\#}(\zeta,\overline{\zeta}) = |z_0|^2\widetilde{r}^{\mathrm{F}}_A(z,\overline{z}). \tag{5.8}$$

Turning from ∂_ζ and $\partial_{\overline{\zeta}}$ to ∂_z and $\partial_{\overline{z}}$, we see that the assumption $R_n = 0$ is equivalent to

$$\partial^{\alpha}_{z}\partial^{\beta}_{\overline{z}}(\widetilde{r}^{\mathrm{F}}_A)_{\#}(0,0) = 0 \qquad (\mathrm{w}(\alpha\overline{\beta}) \le n,\ |\alpha|,|\beta| \ge 2). \tag{5.9}$$

On the other hand, we have seen in Subsection 3.2, (C) that $\widetilde{r}^{\mathrm{F}}_A$ is uniquely determined modulo $O^{n+2}(\rho_0)$ as a solution of the linear equation

$$L_{\rho_0}(\widetilde{r}^{\mathrm{F}}_A)_{\#} = O^{n+1}(\rho_0), \quad \widetilde{r}^{\mathrm{F}}_A|_{2u=|z'|^2} = -\sum_{\mathrm{w}(\alpha\overline{\beta}\ell)\le n} A^{\ell}_{\alpha\overline{\beta}}\, z'_{\alpha}\,\overline{z'_{\beta}}\, v^{\ell}. \tag{5.10}$$

Now $A_n = 0$ follows from (5.9) via (5.10). The proof is similar to that of the uniqueness of Moser's normal form, where the trace conditions (1.3) are used crucially.

5.3 $H_{\#}$-invariants of $\mathcal{R}_n$ are Weyl invariants.

Let $T_0\mathcal{R}_n$ denote the tangent space of $\mathcal{R}_n$ at the origin, and thus

$$T_0\mathcal{R}_n = \mathcal{F}'(0)\,T_0\,\mathcal{N}_n \subset \mathcal{R}_n^{\mathrm{aux}} \qquad (T_0\mathcal{N}_n = \mathcal{N}_n \text{ as a set}).$$

Then the $H_{\#}$-action on $\mathcal{R}_n$ induces an $H_{\#}$-action on $T_0\mathcal{R}_n$, which agrees with that on $\mathcal{R}_n^{\mathrm{aux}}$ restricted to $T_0\mathcal{R}_n$. The $H_{\#}$-invariants of $T_0\mathcal{R}_n$ is defined as in the definition of those of $\mathcal{R}_n$, in which $\mathcal{R}_n$ is literally replaced by $T_0\mathcal{R}_n$. Now the proof of the part (II) in Theorem 2.1$'$ is reduced to:

Theorem 5.2. *Every $H_\#$-invariant of $T_0\mathcal{R}_n$ of weight $\leq n$ is a Weyl invariant.*

Assuming the validity of Theorem 5.2 above for a moment, we first prove the statement (II). Let $P(R)$ be an $H_\#$-invariant of $\mathcal{R}_n$ of weight $\leq n$. We denote by $p(R)$ the lowest degree part of $P(R)$. Then $p(R)$ is an $H_\#$-invariant on $T_0\mathcal{R}_n$. It then follows from Theorem 5.2 that there exists a Weyl invariant $W(R)$ such that $p(R) = W(R)$ on $T_0\mathcal{R}_n$. Though $W(R)$ differs $p(R)$ from on $\mathcal{R}_n$, the difference consists of terms of higher degree. Thus we can repeat this procedure and write $P(R)$ as a sum of Weyl invariants. This proves (II).

The proof of Theorem 5.2 requires a defining system of equations of $T_0\mathcal{R}_n$. In view of (5.8), we have

$$T_0\mathcal{R}_n = \{(R_{\alpha\overline{\beta}}) \in \mathcal{R}_n^{\mathrm{aux}}\,;\ R_{\alpha\overline{\beta}} = \partial_\zeta^\alpha \partial_{\overline{\zeta}}^\beta (\widetilde{r}_A^{\mathrm{F}})_\#|_{e_0},\ A \in \mathcal{N}_n\}.$$

From this expression, we obtain a defining system of $T_0\mathcal{R}_n$ in terms of the variables $(R_{\alpha\overline{\beta}}) \in \mathcal{R}_n^{\mathrm{aux}}$:

$$(5.11)_1 \qquad R_{\alpha\overline{\beta}} = \overline{R_{\beta'\overline{\alpha'}}} \quad \text{(for any permutation } \alpha'\,\overline{\beta'} \text{ of } \alpha\,\overline{\beta}\,),$$

$$(5.11)_2 \qquad R_{0\alpha\overline{\beta}} = (1-|\alpha|)R_{\alpha\overline{\beta}}, \qquad R_{\alpha\overline{0\beta}} = (1-|\beta|)R_{\alpha\overline{\beta}},$$

$$(5.11)_3 \qquad \sum_{j,k=0}^{n} g_0^{j\overline{k}} R_{j\alpha\overline{k\beta}} = R_{0\alpha\overline{n\beta}} + R_{n\alpha\overline{0\beta}} - \sum_{j=1}^{n-1} R_{j\alpha\overline{j\beta}} = 0.$$

Here $(g_0^{j\overline{k}}) = (g_0)^{-1} = g_0$. The Hermitian symmetry $(5.11)_1$ is equivalent to the fact that $(\widetilde{r}_A^{\mathrm{F}})_\#$ is real. The reduction rule $(5.11)_2$ is a consequence of the homogeneity of $(\widetilde{r}_A^{\mathrm{F}})_\#$ in ζ and $\overline{\zeta}$. (Here we have set $R_{\alpha\overline{\beta}} = 0$ if $|\alpha| \leq 1$ or $|\beta| \leq 1$.) The relation $(5.11)_3$ comes from $\Delta_0(\widetilde{r}_A^{\mathrm{F}})_\# = O^{n+1}((\rho_0)_\#)$ of (3.15).

Disregarding the weight restriction, we consider $H_\#$-invariants of

$$\mathcal{H} = \big\{(R_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 2};\ R_{\alpha\overline{\beta}} \text{ satisfy } (5.11)_1,\ (5.11)_2 \text{ and } (5.11)_3\big\}.$$

As far as $H_\#$-invariants of weight $\leq n$ are concerned, an invariant of $T_0\mathcal{R}_n$ is an invariant of $\mathcal{H}$, and vice versa. Consequently, Theorem 5.2 is contained in a more general:

Theorem 5.3. *Every $H_\#$-invariant of $\mathcal{H}$ is a Weyl invariant.*

Fefferman [F3] proved this result for invariants of weight $\leq n-19$. The weight restriction was later removed by Bailey-Eastwood-Graham

[BEG]. The proof in [BEG] is constructive and gives an algorithm of writing the given $H_\#$-invariant as a linear combination of complete contractions. In what follows, we explain this algorithm.

Let $R = (R_{\alpha\overline{\beta}}) \in \mathcal{H}$. Then each $R_{\alpha\overline{\beta}}$ is written as a linear combination of

$$\widetilde{A}^{\ell}_{\alpha'\overline{\beta'}} = R_{\alpha' n \cdots n \overline{\beta'}}$$

where α' and β' are lists of $\{1, \ldots, n-1\}$ of length ≥ 2, and the number of n is ℓ. In fact, indices 0, $\overline{0}$, $\overline{n}$ in $R_{\alpha\overline{\beta}}$ can be deleted by repeated use of $(5.11)_2$ and $(5.11)_3$. Setting $\widetilde{A}^{\ell}_{p\overline{q}} = (\widetilde{A}^{\ell}_{\alpha\overline{\beta}})_{|\alpha|=p, |\beta|=q}$, we regard it as a symmetric tensor on $\mathbf{C}^{n-1}$ of type (p, q). Then the $U(n-1)$-action on $\widetilde{A}^{\ell}_{p\overline{q}}$ is the restriction of the $H_\#$-action to $U(n-1) \subset H_\#$. Thus an $H_\#$-invariant $P(R)$ can be regarded as a $U(n-1)$-invariant $P(\widetilde{A})$ of $\widetilde{A}^{\ell}_{\alpha\overline{\beta}}$. (This procedure amounts to rewriting polynomials in R as those in $A^{\ell}_{\alpha\overline{\beta}}$.) Using Weyl's invariant theory for $U(n-1)$, we can write $P(\widetilde{A})$ as a linear combination of complete contractions on $\mathbf{C}^{n-1}$, that is, those with respect to $(\delta^{j\overline{k}})$:

$$\operatorname{contr}\left(\widetilde{A}^{\ell_1}_{p_1\overline{q}_1} \otimes \cdots \otimes \widetilde{A}^{\ell_d}_{p_d\overline{q}_d}\right). \tag{5.12}$$

In addition, we can make so that these contractions do not contain

$$\operatorname{tr}\widetilde{A}^{\ell}_{2\overline{2}},\ (\operatorname{tr})^2\widetilde{A}^{\ell}_{2\overline{3}},\ (\operatorname{tr})^2\widetilde{A}^{\ell}_{3\overline{2}},\ (\operatorname{tr})^3\widetilde{A}^{\ell}_{3\overline{3}}.$$

From these contractions on $\mathbf{C}^{n-1}$, we manufacture complete contractions with respect to the ambient metric g_0, depending on the degree d of the polynomial $P(\widetilde{A})$, as follows.

At first, let $d < n$. From the linear combination $P(\widetilde{A})$ of complete contractions of the form (5.12), we make a partial sum consisting of terms corresponding to $\ell_1 = \cdots = \ell_d = 0$, and replace the complete contractions there formally by those with respect to g_0:

$$\operatorname{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_d,q_d)}\right).$$

Then we get a Weyl invariant, which agrees with the given $H_\#$-invariant $P(R)$. The proof of this fact requires careful examination of the $H_\#$-action on complete contractions of the form (5.12).

When $d \geq n$, we cannot expect this. In fact, if for instance an $H_\#$-invariant $P(R)$ of degree $d \geq n$ contains an alternating sum of n indices, then $P(R)$ is not manufactured by the procedure above. We thus proceed as follows. Let us first recall that, in the case $d < n$, we have formally replaced complete contractions with respect to $(\delta^{j\overline{k}})$ by those with respect to g_0. That is, we have ignored the right side of

$$\sum_{j,k=1}^{n-1} \delta^{j\overline{k}} T_{j\overline{k}} + \sum_{j,k=0}^{n} g_0^{j\overline{k}} T_{j\overline{k}} = T_{0\overline{n}} + T_{n\overline{0}}$$

for an arbitrary tensor $(T_{j\overline{k}})$ of type $(1,1)$. Taking account of the right side, we now express complete contractions of the form (5.12) in terms of partial contractions with respect to g_0. That is, we manufacture tensors $(T_{\alpha\overline{\beta}})$ by partially contracting $R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}$ in such a way that (5.12) is given by a linear combination of components of $(T_{\alpha\overline{\beta}})$ of the form $T_{0\cdots 0 n\cdots n\,\overline{0}\cdots\overline{0}\overline{n}\cdots\overline{n}}$. The indices 0 and $\overline{0}$ can be eliminated by repeated use of $(5.11)_2$. We then get an expression of the given $H_\#$-invariant $P(R)$ as a linear combination of components $T_{n\cdots n\,\overline{n}\cdots\overline{n}}$ of $(T_{\alpha\overline{\beta}})$. Let us recall by the definition of $T_0\mathcal{R}_n$ that $R_{\alpha\overline{\beta}} = \partial_\zeta^\alpha \partial_{\overline{\zeta}}^\beta (\widetilde{r}_A^{\mathrm{F}})_\#|_{e_0}$. Likewise, $T_{\alpha\overline{\beta}}$ are given by the "values" of formal power series at e_0. Then $T_{\alpha\overline{\beta}}$ are extended to jets at e_0, and the partial derivatives of the extensions of $T_{\alpha\overline{\beta}}$ make sense. If these are used as substitutes for the covariant derivatives, then a scalar is obtained by the complete contraction. We do this procedure after some algebraic manipulations which are technical. Then the scalar above is a Weyl invariant, which coincides with the original $H_\#$-invariant $P(R)$ up to a multiple. It turns out that components of $(T_{\alpha\overline{\beta}})$ other than $T_{n\cdots n\,\overline{n}\cdots\overline{n}}$ do not contribute to the resulting Weyl invariant.

§6 Full invariant expansion of the Bergman kernel

So far, we have considered an invariant expression of the singularity of the Bergman kernel $K^{\mathrm{B}} = \varphi^{\mathrm{B}} r^{-n-1} + \psi^{\mathrm{B}} \log r$ by using Fefferman's defining function $r = r^{\mathrm{F}}$. Because of the ambiguity of r^{F}, it was only possible to express φ^{B} modulo $O^{n+1}(r)$ in general (Theorem 2.2), and ψ^{B} modulo $O^2(r)$ even in the case $n = 2$ (Theorems 3.1 and 3.2). In this section, we express ψ^{B} modulo $O^\infty(r)$ invariantly by using a special family of Fefferman's defining functions. (The details are found in [Hi].) That family, which we denote by $\mathcal{R}_{\partial\Omega}^{\mathrm{F}}$, is parametrized by $C^\infty(\partial\Omega)$ and satisfies

$$(6.1) \qquad r_1 := |\det \Phi'|^{-2/(n+1)} r_2 \circ \Phi \in \mathcal{R}_{\partial\Omega_1}^{\mathrm{F}} \quad \text{for} \quad r_2 \in \mathcal{R}_{\partial\Omega_2}^{\mathrm{F}}$$

for biholomorphic mappings $\Phi : \Omega_1 \to \Omega_2$. This can be regarded as an exact transformation law of weight -1 without error.

In Subsection 6.1 below, we lift the Monge-Ampère boundary value problem (1.12) to a $\mathbf{C}^*$-bundle over Ω. Then the lifted problem admits asymptotic solutions which are similar to those of the original problem (1.12) in Theorem 1.3. Elements of $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ are obtained as the "smooth parts" of these asymptotic solutions. Using $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, we define as before Weyl invariants, which inherit the ambiguity measured by $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$. These Weyl invariants with ambiguity, together with r, are used in expressing a full expansion of ψ^{B} in the Bergman kernel.

6.1 A special family of Fefferman's defining functions

Given a strictly pseudoconvex domain Ω with C^∞ boundary, we take a thin one-sided neighborhood $V \subset \overline{\Omega}$ of $\partial\Omega$ and consider the following equation for a function $U = U(z_0, z)$ on $\mathbf{C}^* \times V$:

$$(-1)^n \det \left(\partial^2 U/\partial z_j \partial \overline{z}_k\right)_{j,k=0,\dots,n} = |z_0|^{2n}. \tag{6.2}$$

In terms of differential forms, (6.2) is intrinsically written as

$$(-1)^n (\partial\overline{\partial} U)^{n+1} = dv, \tag{6.3}$$

where $dv = (n+1)!|z_0|^{2n} dz_0 \wedge d\overline{z}_0 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$. If U is of the form $U(z_0, z) = |z_0|^2 u(z)$, then (6.2) is reduced to the equation $J[u] = 1$. That is, (6.2) is a lift of the complex Monge-Ampère equation to the $\mathbf{C}^*$-bundle $\mathbf{C}^* \times V$. The bundle structure on $\mathbf{C}^* \times V$ is given by $\Phi_\#$ in (2.3), where Φ is a local (or formal) biholomorphic change of coordinates near a point of $\partial\Omega$. The transition function $\Phi_\#$ preserves dv. Thus $\Phi_\#$ preserves the equation (6.3) in the sense that if U_2 satisfies (6.3) so does $U_1 = U_2 \circ \Phi_\#$.

We consider asymptotic solutions to (6.2) of the form

$$U = r_\# + r_\# \sum_{k=1}^{\infty} \eta_k \cdot (r^{n+1} \log r_\#)^k \quad \text{with } r_\# = |z_0|^2 r, \tag{6.4}$$

where $\eta_k \in C^\infty(V)$, and r is a defining function of Ω, $r > 0$ in Ω. Let us identify two such formal series if the corresponding r and η_k agree modulo $O^\infty(\partial\Omega)$. Then the totality of such asymptotic solutions is parametrized by $C^\infty(\partial\Omega)$ as follows.

Proposition 6.1. *Let X be a C^∞ vector field on V which is transversal to $\partial\Omega$. Then, for any $f \in C^\infty(\partial\Omega)$, there exists a unique asymptotic solution U of the form* (6.4) *to the equation* (6.2) *such that*

$$X^{n+2} r|_{\partial\Omega} = f.$$

If U_2 is an asymptotic solution of the form (6.4) in $V_2 \supset \partial\Omega_2$ so is $U_1 = U_2 \circ \Phi_\#$ in $V_1 \supset \partial\Omega_1$, where $\Phi : V_1 \to V_2$ is a biholomorphic mapping satisfying $\Phi(\partial\Omega_1) = \partial\Omega_2$. This transformation law is rewritten as $r_1 = |\det \Phi'|^{-2/(n+1)} r_2 \circ \Phi$ and $\eta_{1,k} = |\det \Phi'|^{2k} \eta_{2,k} \circ \Phi$, where

$$U_j = (r_j)_\# + (r_j)_\# \sum_{k=1}^{\infty} \eta_{j,k} \cdot \left(r_j^{n+1} \log(r_j)_\#\right)^k \qquad (j = 1, 2).$$

For an asymptotic solution U to (6.2) of the form (6.4), we call r in (6.4) the *smooth part* of U, and denote by $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ the totality of the smooth parts. Then the transformation law (6.1) for $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ is valid. In addition, each smooth part r is a Fefferman's defining function, that is, r satisfies $J[r] = 1 + O^{n+1}(r)$.

Remark 6.1. If we drop the subscript # from $r_\#$ in (6.4), then we get Graham's asymptotic solutions (1.17) with $\eta_0^{\mathrm{G}} = 1$. However, the transformation law (6.1) breaks down. Similarly, if we add the subscript # to r in $r^{n+1} \log r_\#$, then again (6.1) breaks down.

6.2 A refinement of Theorem 2.2

Starting from a Fefferman's defining function $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, we construct Weyl invariants as in Subsection 2.2. That is, for the Lorentz-Kähler metric g with potential $r_\#$ in a thin neighborhood $\mathbf{C}^* \times V \subset \mathbf{C}^* \times \overline{\Omega}$ of $\mathbf{C}^* \times \partial\Omega$, we consider the curvature R of g and successive covariant derivatives $R^{(p,q)} = \overline{\nabla}^{q-2} \nabla^{p-2} R$. Then a *Weyl invariant* of weight w is defined as a linear combination of the complete contractions of the form

$$\mathrm{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\right) \quad \text{with} \quad \frac{1}{2} \sum_{j=1}^{s} (p_j + q_j) - s = w.$$

By definition, a Weyl invariant $W_\#$ is a functional of $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, and thus we write $W_\# = W_\#[r]$. As in Section 2, we also use this terminology for the composite function $(z_0, z) \mapsto W_\#[r]$. We denote the restriction of $W_\#[r]$ to $z_0 = 1$ by $W[r]$, and still call it a *Weyl invariant*. It follows from the construction that the transformation law (6.1) for $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ implies

$$W[r_1] = |\det \Phi'|^{2w/(n+1)} W[r_2] \circ \Phi$$

for a Weyl invariant of weight w, cf. (2.4) in Subsection 2.3.

With $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, let us consider the expression (1.10) in Theorem 1.2 for the Bergman kernel. Observe that ψ^{B} is uniquely determined modulo $O^\infty(r)$ and independent of the choice of r. Nevertheless, we regard ψ^{B} as a functional of $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ and write $\psi^{\mathrm{B}} = \psi^{\mathrm{B}}[r]$. Then we have:

Theorem 6.1. *For each $j \geq n+1$, there exists a Weyl invariant W_j of weight j such that if $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ then*

$$\psi^{\mathrm{B}}[r] = \sum_{k=0}^{\infty} W_{k+n+1}[r]\, r^k \mod O^{\infty}(r).$$

That is, for each $m > 0$, $\psi^{\mathrm{B}}[r] = \sum_{k=0}^{m} W_{k+n+1}[r]\, r^k \mod O^{m+1}(r)$.

This theorem refines Theorem 2.2 (cf. Remark 6.2 below).

6.3 Generalization of the CR invariant

Recall that Theorem 2.2 follows from Theorem 2.1. In order to refine Theorem 2.1, we need to generalize the notion of CR invariant taking account of the ambiguity described by $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$. Let us begin by recalling that Proposition 6.1 gives a bijection $C^{\infty}(\partial\Omega) \to \mathcal{R}^{\mathrm{F}}_{\partial\Omega}$ as far as a vector field X is specified. For a reference point $p \in \partial\Omega$, this parametrization is localizable to a neighborhood of p, but we rather consider formally. We have a bijection $C^{\infty}_{\partial\Omega,p} \to \mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$, where $C^{\infty}_{\partial\Omega,p}$ and $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ denote the spaces of all Taylor expansions about p of elements of $C^{\infty}(\partial\Omega)$ and $\mathcal{R}^{\mathrm{F}}_{\partial\Omega}$, respectively. Thus $C^{\infty}_{\partial\Omega,p}$ and $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ consist of formal power series, though the notation $C^{\infty}_{\partial\Omega,p}$ might be somewhat confusing.

The family $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ satisfies a formal transformation law corresponding to (6.1), and this transformation law is transplanted to $C^{\infty}_{\partial\Omega,p}$. To write it down explicitly, we assume that $\partial\Omega$ near p is in Moser's normal form $N(A)$, and take $X = \partial/\partial\rho_A$ with respect to the coordinate system $(z', \overline{z'}, \rho_A, v)$. Each element $f \in C^{\infty}_{\partial\Omega,p}$ is written in the form

$$f(z', \overline{z'}, v) = \sum_{\alpha,\beta,\ell} C^{\ell}_{\alpha\overline{\beta}}\, z'_{\alpha}\, \overline{z'_{\beta}}\, v^{\ell}.$$

We denote by $\mathcal{C}$ the totality of collections of the coefficients $C = (C^{\ell}_{\alpha\overline{\beta}})$. Thus $C^{\infty}_{\partial\Omega,p}$ is identified with $\mathcal{C}$. If $r \in \mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ is in the image of f under the bijection $C^{\infty}_{\partial\Omega,p} \to \mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$, then

$$r = \sum_{\alpha,\beta,\ell,m} P^{\ell m}_{\alpha\overline{\beta}}(A, C)\, z'_{\alpha}\, \overline{z'_{\beta}}\, v^{\ell} \rho_A^m,$$

where $P^{\ell m}_{\alpha\overline{\beta}}(A, C)$ are polynomials in $(A, C) \in \mathcal{N} \times \mathcal{C}$. We thus write $r = r(A, C)$, and use the notation $\mathcal{R}^{\mathrm{F}}_{N(A)}$ for the totality of $r = r(A, C)$ with $(A, C) \in \mathcal{N} \times \mathcal{C}$. Thus $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$ is identified with $\mathcal{R}^{\mathrm{F}}_{N(A)}$, and we have a bijection $\mathcal{C} \to \mathcal{R}^{\mathrm{F}}_{N(A)}$ as far as $A \in \mathcal{N}$ is specified.

The H-action (1.4) on $\mathcal{N}$ extends to that on $\mathcal{N} \times \mathcal{C}$ as follows. For $(A, C) \in \mathcal{N} \times \mathcal{C}$ and $h \in H$, we define $(\widetilde{A}, \widetilde{C}) = h.(A, C)$ by $\widetilde{A} = h.A$ and $r(\widetilde{A}, \widetilde{C}) = |\det E'_{h,A}|^{-2/(n+1)}\, r(A, C) \circ h$, where $E_{h,A}$ is defined by $(1.4)'$. Then we have, as a generalization of (1.4), a group action

$$H \times \mathcal{N} \times \mathcal{C} \ni (h, A, C) \mapsto h.(A, C) \in \mathcal{N} \times \mathcal{C}, \tag{6.5}$$

which is regarded as a transformation law for $C^{\infty}_{\partial\Omega,p}$ parametrizing $\mathcal{R}^{\mathrm{F}}_{\partial\Omega,p}$.

We now recall that CR invariants are defined by (1.5). This notion is generalized as follows. Let $I^{\mathrm{CR}}_w(\mathcal{C})$ denote the totality of polynomials P in $(A, C) \in \mathcal{N} \times \mathcal{C}$ such that

$$P(A, C) = |\det h'(0)|^{2w/(n+1)}\, P(h.(A, C)) \quad \text{for any} \quad h \in H.$$

Then $I^{\mathrm{CR}}_w = I_w(\mathcal{N}) \subset I_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{CR}}_w(\mathcal{C})$, where $I_w(\mathcal{N} \times \mathcal{C})$ stands for the space of H-*invariants* of $\mathcal{N} \times \mathcal{C}$, and similarly for $I_w(\mathcal{N})$. As in the case of CR invariants, elements of $I^{\mathrm{CR}}_w(\mathcal{C})$ can be identified with smooth (C^{∞} or real analytic) functions on $\partial\Omega$.

6.4 Boundary values of $\mathcal{C}$-dependent Weyl invariants

We want to refine Theorem 2.1 in such a way that the refinement implies Theorem 6.1. As in the previous subsection, let us consider a surface in Moser's normal form $N(A)$, and take $X = \partial/\partial\rho_A$ with respect to the coordinate system $(z', \overline{z'}, \rho_A, v)$. For a Weyl invariant $W = W[r]$ of weight w, the value at the origin is a polynomial in (A, C). We thus write it as $W(A, C)$, and denote the totality of these polynomials by $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C})$. Let $I^{\mathrm{W}}_w(\mathcal{N})$ denote the totality of $W(A, C) \in I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C})$ which do not contain the variable $C \in \mathcal{C}$. Then Proposition 2.1 implies $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N})$ for $w \le n$, and Theorem 2.2 is restated as

$$I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N}) = I^{\mathrm{CR}}_w \qquad \text{for} \quad w \le n.$$

Improving this, we have:

Theorem 6.2. *For any $w \in \mathbf{N}_0$, $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{CR}}_w(\mathcal{C})$ and thus $I^{\mathrm{W}}_w(\mathcal{N}) = I^{\mathrm{CR}}_w$.*

Theorem 6.3. *If $n \ge 3$, then $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N})$ for $w \le n+2$ and $I^{\mathrm{W}}_{n+3}(\mathcal{N} \times \mathcal{C}) \ne I^{\mathrm{W}}_{n+3}(\mathcal{N})$. If $n = 2$, then $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N})$ for $w \le 5$ and $I^{\mathrm{W}}_6(\mathcal{N} \times \mathcal{C}) \ne I^{\mathrm{W}}_6(\mathcal{N})$.*

In the case $n = 2$, these theorems imply $I^{\mathrm{W}}_w(\mathcal{N} \times \mathcal{C}) = I^{\mathrm{W}}_w(\mathcal{N}) = I^{\mathrm{CR}}_w$ for $w \le 5$ (cf. Remark 3.2).

Remark 6.2. By direct computation, we can show that if $n = 2$ then $W_6(A, C) \notin I_6^{\mathrm{W}}(\mathcal{N})$ for the Weyl invariant W_6 in Theorem 6.1. This fact will be published elsewhere.

Theorem 6.1 is proved by using Theorem 6.2 if we recall the proof of Theorem 2.2 which uses Theorem 2.1. In the next subsection, we outline the proof of Theorem 6.2, which is analogous to that of Theorems 2.1. We omit the proof of Theorem 6.3, which is technical and consists of careful inspection of the proof of Theorem 6.2.

6.5 $\mathcal{C}$-dependent invariant theory

Recalling that Theorem 2.1 follows from Theorem 2.1′ at the end of Subsection 5.1, let us first formulate a substitute for Theorem 2.1′. We have to remove the weight restriction by using $\mathcal{N} \times \mathcal{C}$ in place of $\mathcal{N}$.

For a surface in Moser's normal form $N(A)$ with $X = \partial/\partial\rho_A$, we take $r = r(A, C) \in \mathcal{R}^{\mathrm{F}}_{N(A)}$ and consider the curvature R of the Lorentz-Kähler metric g with potential $r_\#$. As in Subsection 5.1, we identify R with the collection of the components $R_{\alpha\overline{\beta}}$ of the covariant derivatives at e_0, and write $R = (R_{\alpha\overline{\beta}})$. Then each $R_{\alpha\overline{\beta}}$ is a polynomial in $(A, C) \in \mathcal{N} \times \mathcal{C}$. We thus write $R_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}}(A, C)$ and define a map

$$\mathcal{F} : \mathcal{N} \times \mathcal{C} \ni (A, C) \mapsto R(A, C) \in \mathcal{R}^{\mathrm{aux}},$$

where $R(A, C) = (R_{\alpha\overline{\beta}}(A, C))$, and set $\mathcal{R} = \mathcal{F}(\mathcal{N} \times \mathcal{C})$. This map $\mathcal{F}$ and $\mathcal{R}$ are refinements of the map in (5.7) and $\mathcal{R}_n$.

Let us recall that the $H_\#$-action on $\mathcal{N}$ induces that on $\mathcal{R}_n$ via (5.2). Likewise, the $H_\#$-action on $\mathcal{N} \times \mathcal{C}$, defined by $h_\#.(A, C) = h.(A, C)$, induces that on $\mathcal{R}$. Thus we can define *$H_\#$-invariants* of weight w on $\mathcal{R}$, and we denote the totality of these by $I_w(\mathcal{R})$. The map $\mathcal{F}$ is $H_\#$-equivariant and induces an injection

$$\mathcal{F}^*\colon I_w(\mathcal{R}) \ni P(R) \mapsto P(\mathcal{F}(A, C)) \in I_w(\mathcal{N} \times \mathcal{C}) = I_w^{\mathrm{CR}}(\mathcal{C}),$$

which corresponds to the map in (5.5). Let $I_w^{\mathrm{W}}(\mathcal{R})$ denote the subspace of $I_w(\mathcal{R})$ consisting of elements which are given by linear combinations of complete contractions of the form (5.4) of weight w. Then we can state a substitute for Theorem 2.1′ as follows.

Theorem 6.2′. (I) *The map $\mathcal{F}^*$ is bijective.*
(II) $I_w(\mathcal{R}) = I_w^{\mathrm{W}}(\mathcal{R})$ *for each* $w \in \mathbf{N}_0$.

Theorem 6.2 follows from Theorem 6.2′.

As in the case of Theorem 2.1′, the proof of (I) is reduced to proving the injectivity of $\mathcal{F}'(0)$.

The statement (II) for $w \leq n$ is equivalent to that in Theorem 2.1′, and most parts of the proof work as well for the case $w > n$. The point is to show

$$I_w^{\mathrm{W}}(T_0\mathcal{R}) = I_w(T_0\mathcal{R}), \tag{6.6}$$

where $T_0\mathcal{R} \subset \mathcal{R}^{\mathrm{aux}}$ is the tangent space of $\mathcal{R}$ at 0. In Subsection 5.3, we outlined the proof of (6.6) for $w \leq n$, where $(5.11)_3$ was used crucially. The equality $(5.11)_3$, stating that $(R_{\alpha\overline{\beta}})$ is trace-free, follows from the equation

$$\Delta_0(\widetilde{r}_A^{\mathrm{F}})_{\#} = O^{n+1}((\rho_0)_{\#}). \tag{6.7}$$

where Δ_0 and $(\rho_0)_{\#}$ are those in (3.15). To prove (6.6) in the case $w > n$, we need to compute explicitly the error term $O^{n+1}((\rho_0)_{\#})$ of (6.7) when $\widetilde{r}_A^{\mathrm{F}}$ is replaced by

$$\widetilde{r}_{A,C} = \frac{d}{d\varepsilon} r(\varepsilon A, \varepsilon C)\bigg|_{\varepsilon=0}.$$

The result is:

$$\Delta_0 \widetilde{r}_{A,C} = c_n\, \mu^{n+1} \Delta_0^{n+2} \widetilde{r}_{A,C}, \quad \text{where } c_n = \frac{(-1)^{n+1}}{(n+1)!^2}.$$

Using this equality in place of (6.7), we can remove the restriction $w \leq n$ in the argument of Subsection 5.3, and obtain (6.6) with the aid of the invariant theory of [BEG].

Remark 6.3. In general, the Weyl invariants W_k in Theorem 6.1 are not uniquely determined, since there are linear relations among the boundary values of complete contractions of the form (2.1). For instance, in the case $n = 2$, the boundary values of $\|R^{(4,2)}\|^2$ and $\|R^{(3,3)}\|^2$ are linearly dependent (and, accordingly, Theorem 3.1 includes two expressions for W_4 and W_5). Under the terminology of this section, $\|R^{(3,3)}\|^2$ and $\|R^{(2,4)}\|^2$ are polynomials on $\mathcal{R}^{\mathrm{aux}}$ such that the restrictions to the submanifold $\mathcal{R}$ are linearly dependent functions.

The situation is similar for the Weyl invariants W_k in Theorem 2.2, though we do not know specific examples of non-uniqueness. (Note that $\|R^{(3,3)}\|^2$ and $\|R^{(2,4)}\|^2$ for $n = 2$ are irrelevant to Theorem 2.2 because of the weight restriction.) It should be mentioned that a basis of Weyl invariants of degree $d < n$ is given in [BEG]; in particular, it is shown that, if $d = 2 < n$, then $\|R^{(p,q)}\|^2$ $(p \geq q \geq 2)$ form a basis of quadratic Weyl invariants.

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