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TWO METHODS OF DETERMINING LOCAL INVARIANTS IN THE SZEGÖ KERNEL

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Introduction.

In this note, we shall express invariantly the asymptotic expansion of the Szegö kernel of a bounded strictly pseudoconvex domain in $\mathbb{C}^2$ with C^∞ smooth boundary equipped with an invariant surface element. The corresponding problem for the Bergman kernel was investigated by Robin Graham [G1]. Adopting Fefferman's approximate solution to the Dirichlet problem for the complex Monge-Ampère equation as a defining function of the domain, Graham determined all the three coefficients of the pole-type singularity of the Bergman kernel and the first two coefficients of the logarithmic singularity, up to the identification of a universal non-vanishing constant which is independent of the choice of a domain. By following his argument, it is not difficult to obtain a similar expression for the Szegö kernel – as will be observed in Section 1 – and the problem is reduced to determining two universal constants. Our aim is to present two different methods of computing these constants.

The first method presented in Section 2 has its origin in Kashiwara's observation [Ka] which was later developed by Boutet de Monvel [B1], [B2]. The domains under consideration are assumed to have analytic boundary. Kashiwara pointed out that the Bergman kernel is the unique solution (modulo real analytic error) of a simple holonomic system. His argument applies also to the Szegö kernel case, and we are reduced to constructing an asymptotic solution of the simple holonomic system which characterizes the Szegö kernel. We shall follow a computation by Boutet de Monvel [B2] of the Bergman kernel case.

The second method uses an explicit asymptotic expansion for the class of complete Reinhardt domains. The third named author obtained in [N2] the corresponding formula for the Bergman kernel by improving the analysis of Boichu and Coeuré [BC]. A fairly simple expression was obtained in [N2] by introducing new independent and dependent variables – the one dimensional hodograph transformation – and the same idea applies to the Szegö kernel. We shall indicate in Section 3 how we have to modify the argument of the Bergman kernel case, together with the procedure of rewriting that expansion formula to the one adapted to the local boundary invariants in the general setting.

Both methods are still valid for the higher dimensional case, and the results are stated corresponding to those in [G1] for the Bergman kernel. However, our understanding of the higher dimensional invariant theory is still at an early stage. We hope these methods

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when applied to that case will be helpful in the invariant theoretic studies of both the Bergman and the Szegö kernels.

§1. Local boundary invariants in the Szegö kernel.

1.1. Universal constants in the asymptotic expansion. Let Ω be a bounded domain in $\mathbb{C}^n$ with C^∞ smooth boundary. If a surface element σ on $\partial\Omega$ is specified, then the *Szegö kernel* $K^{\mathrm{S}}(z,\overline{w})$ for z, $w \in \Omega$ is defined as the reproducing kernel associated with the Hardy space $H^2_\sigma(\Omega)$ consisting of holomorphic functions in Ω having L^2 boundary values with respect to σ. Namely, the Szegö kernel is characterized by the following three properties: $K^{\mathrm{S}}(\cdot,\overline{w}) \in H^2_\sigma(\Omega)$ for every $w \in \Omega$ fixed; $\overline{K^{\mathrm{S}}(z,\overline{w})} = K^{\mathrm{S}}(w,\overline{z})$ for $z, w \in \Omega$; and

$$f(z) = \int_{\partial\Omega} K^{\mathrm{S}}(z,\overline{w})\, f(w)\, \sigma(w) \quad \text{for any} \quad f \in H^2_\sigma(\Omega) \quad \text{and} \quad z \in \Omega.$$

Assuming that Ω is *strictly pseudoconvex*, let us recall that the boundary behavior of the restriction to the diagonal $z = w \in \Omega$ of the Szegö kernel is analogous to that of the Bergman kernel. It was observed by Fefferman [F1], [F3] (see also Boutet-Sjöstrand [BS]) that if $r \in C^\infty(\overline{\Omega})$ is a defining function in the sense $\Omega = \{r > 0\}$ with $dr \neq 0$ on $\partial\Omega$, then the singularity on the diagonal of the boundary is of the form

$$(1.1) \qquad K^{\mathrm{S}}(z) := K^{\mathrm{S}}(z,\overline{z}) = \frac{(n-1)!}{\pi^n}\left[\frac{\varphi^{\mathrm{S}}(z)}{r(z)^n} + \psi^{\mathrm{S}}(z)\, \log r(z)\right] \qquad \text{near} \quad \partial\Omega,$$

where φ^{S} and ψ^{S} are C^∞ smooth functions up to the boundary. This is compared with the case of the Bergman kernel $K^{\mathrm{B}}(z,\overline{w})$, of which the singularity on the diagonal of the boundary takes the form

$$(1.1\mathrm{B}) \qquad K^{\mathrm{B}}(z) := K^{\mathrm{B}}(z,\overline{z}) = \frac{n!}{\pi^n}\left[\frac{\varphi^{\mathrm{B}}(z)}{r(z)^{n+1}} + \psi^{\mathrm{B}}(z)\, \log r(z)\right] \qquad \text{near} \quad \partial\Omega,$$

where φ^{B} and ψ^{B} are functions C^∞ smooth up to the boundary.

In order to make the Szegö kernel invariant under biholomorphic change of coordinates, let us require the surface element σ to be subject to the normalization

$$(1.2) \qquad \sigma \wedge dr = J[r]^{1/(n+1)}\, dV(z) \quad \text{on} \quad \partial\Omega \quad \text{with} \quad dV(z) := \bigwedge_{j=1}^{n} \frac{dz_j \wedge d\overline{z}_j}{-2\sqrt{-1}}\,,$$

where $J[\cdot]$ denotes the *complex Monge-Ampère operator* defined by

$$(1.3) \qquad J[r] := (-1)^n \det \begin{pmatrix} r & r_{\overline{k}} \\ r_j & r_{j\overline{k}} \end{pmatrix},$$

with the subscripts j, $\overline{k}$ standing for differentiation with respect to z_j, $\overline{z}_k$ for j, $k = 1, \cdots, n$. Thus the condition (1.2) is independent of the choice of r. We next require the defining function $r = r^{\mathrm{F}}$ to satisfy

$$(1.4) \qquad J[r] = 1 + O(r^{n+1}) \quad \text{near} \quad \partial\Omega \qquad (\text{and } r > 0 \text{ in } \Omega, \ r = 0 \text{ on } \partial\Omega);$$

namely, $r^{\mathrm{F}} \in C^\infty(\overline{\Omega})$ is *Fefferman's approximate solution* to the Dirichlet problem (see [F2] or the next Subsection):

$$(1.5) \qquad J[u^{\mathrm{MA}}] = 1 \quad \text{and} \quad u^{\mathrm{MA}} > 0 \quad \text{in} \quad \Omega; \qquad u^{\mathrm{MA}} = 0 \quad \text{on} \quad \partial\Omega.$$

In the case of the Bergman kernel, Graham [G1] showed that if $n = 2$ then

$$(1.6\mathrm{B}) \qquad \varphi^{\mathrm{B}} = 1 + O(r^3), \qquad \psi^{\mathrm{B}} = -3\,\eta_1 + k^{\mathrm{B}}\,|A^0_{2\overline{4}}|^2\, r + O(r^2),$$

with a constant $k^{\mathrm{B}} \neq 0$ independent of the choice of Ω, and that if $n \geqq 3$ then

$$(1.7\mathrm{B}) \qquad \varphi^{\mathrm{B}} = 1 + c^{\mathrm{B}}_n\, \|A^0_{2\overline{2}}\|^2\, r^2 + O(r^3), \qquad \|A^0_{2\overline{2}}\|^2 := \sum_{|\alpha|=|\beta|=2} |A^0_{\alpha\overline{\beta}}|^2,$$

with a constant $c^{\mathrm{B}}_n \neq 0$ depending only on n. Here, $A^0_{\alpha\overline{\beta}}$ with $\alpha,\, \beta \in \mathbb{N}_0^{n-1}$ are coefficients of Moser's normal form of $\partial\Omega$, and η_1 stands for the first coefficient of the asymptotic solution to (1.5); see the next Subsection for the precise definition.

Similar expressions hold also for the Szegö kernel. We shall show at the end of this Section that:

Proposition 1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^∞ smooth boundary. Suppose given a surface element σ on the boundary and a defining function $r = r^{\mathrm{F}} \in C^\infty(\overline{\Omega})$ satisfying* (1.2) *and* (1.4)*, respectively. Then φ^{S} and ψ^{S} in* (1.1) *have the following asymptotic behavior near the boundary: If $n = 2$, then there exist constants k^{S}_1 and k^{S}_2 independent of the choice of Ω such that*

$$(1.6) \qquad \varphi^{\mathrm{S}} = 1 + O(r^2), \qquad \psi^{\mathrm{S}} = k^{\mathrm{S}}_1\,\eta_1\, r + k^{\mathrm{S}}_2\,|A^0_{2\overline{4}}|^2\, r^2 + O(r^3).$$

If $n \geqq 3$, then there exists a constant c^{S}_n depending only on n such that

$$(1.7) \qquad \varphi^{\mathrm{S}} = 1 + c^{\mathrm{S}}_n\, \|A^0_{2\overline{2}}\|^2\, r^2 + O(r^3).$$

In Sections 2 and 3, we shall give two proofs of the following:

Theorem 1. *The universal constants in* (1.6) *and* (1.7) *of Proposition 1 above are given by* $k^{\mathrm{S}}_1 = -2$, $k^{\mathrm{S}}_2 = 8/15$ *and* $(n-1)\,(n-2)\,c^{\mathrm{S}}_n = 2/3$.

Remark 1. Applying our methods to the Bergman kernel, we can show that the constants in (1.6B) and (1.7B) are given by $k^{\mathrm{B}} = 24/5$ and $n\,(n-1)\,c^{\mathrm{B}}_n = 2/3$.

Remark 2. Let us consider the case $\Omega \subset \mathbb{C}^2$. It is known that $\psi^{\mathrm{B}} = O(r^2)$ near a boundary point if and only if $\partial\Omega$ is spherical there, that is, Ω is locally biholomorphic to a ball (cf. [G1]). By virtue of Theorem 1, this condition is equivalent to $\psi^{\mathrm{S}} = O(r^3)$.

1.2. Review of Graham's results on local invariants. In order to obtain the expansions (1.6B) and (1.7B) for the Bergman kernel, Graham [G1] identified explicitly local boundary invariants of low weight, and discussed the invariance of the asymptotic solution to the Monge-Ampère boundary value problem (1.5) (cf. also [G2]). In the proof of Proposition 1, we shall use his results in the same context. Let us here give an overview of Graham's results we need, together with precedent theories of Chern-Moser [CM], Fefferman [F2], and others.

The first observation we need is the *local* dependence on $\partial\Omega$ of φ^{S}, ψ^{S} and φ^{B}, ψ^{B}. Inspection of the proof in [F1] or [BS] shows that every one of the Taylor coefficients of φ^{S}, ψ^{S} and φ^{B}, ψ^{B} at a given boundary point is uniquely determined by a finite number of the Taylor coefficients of r at that point. This fact enables us to work locally near an arbitrarily prescribed boundary point, say, the origin $0 \in \mathbb{C}^n$.

Let us take a small neighborhood M of $0 \in \partial\Omega$, which is a strictly pseudoconvex real hypersurface of $\mathbb{C}^n$. We assume for a moment that M is real analytic. After a biholomorphic change of coordinates about the origin, the domain Ω is locally given by $2\,u > |z'|^2 + F(z', \overline{z'}, v)$ with $(z', u + i\,v) \in \mathbb{C}^{n-1} \times \mathbb{C}$, so that M takes the form $2\,u = |z'|^2 + F(z', \overline{z'}, v)$, where F is a real analytic function having the expansion

$$F(z', \overline{z'}, v) = \sum_{|\alpha|,\,|\beta| \geqq 2} A_{\alpha\overline{\beta}}(v)\, z'_\alpha\, \overline{z'_\beta} \quad \text{with} \quad A_{\alpha\overline{\beta}}(v) := \sum_{\ell=0}^{\infty} A^{\ell}_{\alpha\overline{\beta}}\, v^\ell. \tag{1.8}$$

Here, $z'_\alpha = z_{\alpha_1} \cdots z_{\alpha_a}$ with $a = |\alpha|$ for $z' = (z_1, \cdots, z_{n-1})$ and $\alpha = (\alpha_1, \cdots, \alpha_a) \in \{1, \cdots, n-1\}^a$, and similarly for $\overline{z'_\beta}$. Without loss of generality, we may assume $\overline{A^{\ell}_{\alpha\overline{\beta}}} = A^{\ell}_{\beta\overline{\alpha}}$ and that the coefficients $A^{\ell}_{\alpha\overline{\beta}}$ are unchanged under permutation of α and that of β. Recall M is said to be in *Moser's normal form* if the following three conditions are satisfied:

$$\operatorname{tr} A_{2\overline{2}}(v) = 0, \quad (\operatorname{tr})^2 A_{2\overline{3}}(v) = 0, \quad (\operatorname{tr})^3 A_{3\overline{3}}(v) = 0, \tag{1.9}$$

where tr stands for the usual trace obtained by contraction with respect to $\delta^{j\overline{k}}$. In case M is merely C^∞ smooth, expansions in (1.8) are regarded as formal power series. Abusing notation, we denote by $N(A^{\ell}_{\alpha\overline{\beta}})$ a real hypersurface in normal form (1.8) even when it is not real analytic.

Definition ([F3], [G1])**.** A polynomial P in variables $A^{\ell}_{\alpha\overline{\beta}}$ is said to be an *invariant of weight* $w \in \mathbb{N}_0$ if it satisfies the transformation law

$$P(A^{\ell}_{\alpha\overline{\beta}}) = |\det \Phi'(0)|^{2w/(n+1)}\, P(B^{\ell}_{\alpha\overline{\beta}})$$

for any biholomorphic mapping $\Phi : N(A^{\ell}_{\alpha\overline{\beta}}) \to N(B^{\ell}_{\alpha\overline{\beta}})$ which preserves the origin.

Given an invariant P of weight w, a function $P_M \in C^\infty(M)$ is defined by setting

$$P_M(o) := |\det \Phi'(o)|^{2w/(n+1)}\, P(A^{\ell}_{\alpha\overline{\beta}}) \quad \text{for} \quad o \in M,$$

where $\Phi : M \to N(A^{\ell}_{\alpha\overline{\beta}})$ is biholomorphic and satisfies $\Phi(o) = 0$. Conversely, if an assignment $M \mapsto P_M \in C^\infty(M)$ is specified in such a way that the transformation law

$$P_M = |\det \Phi'|^{-2w/(n+1)} P_{\widetilde{M}} \circ \Phi \qquad (\Phi : M \to \widetilde{M} \text{ biholomorphic}) \tag{1.10}$$

is valid, then P_M defines an invariant, provided the dependence of P_M on M is local.

All invariants of low weight were identified by Graham ([G1], Theorem 2.1). In order to state his result, let I_w denote the totality of invariants of weight $w \in \mathbb{N}_0$. Clearly, each I_w is a vector space and $I_0 = \mathbb{C}$. Graham [G1] showed that:

Lemma 1.1 ([G1])**.** (i) *Let* $n = 2$. *Then* $I_1 = I_2 = \{0\}$ *and* $\dim I_3 = \dim I_4 = \dim I_5 = 1$. *For* $w = 3, 4, 5$, *the space* I_w *is respectively spanned by* $A^0_{4\overline{4}}$, $|A^0_{2\overline{4}}|^2$, $5|A^0_{2\overline{5}}|^2 + 9|A^0_{3\overline{4}}|^2 - 2\,\mathrm{Im}\,(A^0_{4\overline{2}}\,A^1_{2\overline{4}})$.

(ii) *Let* $n \geqq 3$. *Then* $I_1 = \{0\}$ *and* $\dim I_2 = 1$. *The space* I_2 *is spanned by* $\|A^0_{2\overline{2}}\|^2 = \sum |A^0_{\alpha\overline{\beta}}|^2$, *where the summation runs over* $|\alpha| = |\beta| = 2$.

Let us next turn to the complex Monge-Ampère boundary value problem (1.5), of which the unique existence of a solution was guaranteed by Cheng-Yau [CY]; the solution has a finite degree of smoothness up to the boundary $u := u^{\mathrm{MA}} \in C^\infty(\Omega) \cap C^{n+3/2-\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$ small. Before this was proved, Fefferman [F2] constructed, locally near a given boundary point, a C^∞ smooth approximate solution $r = r^{\mathrm{F}}$ satisfying (1.4), which turns out to be unique modulo $O(r^{n+2})$. His procedure breaks down in the next step, because of the appearance of logarithmic terms. An asymptotic expansion of the solution was obtained by Lee-Melrose [LM]; they proved that

$$u \sim r \sum_{k=0}^{\infty} \eta_k (r^{n+1}\,\log r)^k, \qquad \eta_k \in C^\infty(\overline{\Omega}), \tag{1.11}$$

and that $\eta_k \mod O(r^{n+1})$ is unique. This in particular improves the above mentioned regularity, so that $u \in C^{n+2-\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$ small.

What we need is that the expansion (1.11) can be localized. Given an arbitrary function $a \in C^\infty(M)$, where M is as before a small portion of $\partial\Omega$, Graham [G2] proved the unique existence of an asymptotic solution u of the form (1.11) to the problem

$$J[u] = 1 + O(r^\infty) \quad \text{near } M, \qquad \eta_0 = 1 + a\,r^{n+1} + O(r^{n+2});$$

furthermore, η_k modulo $O(r^{n+1})$ for each $k \geqq 1$ is independent of the choice of $a \in C^\infty(M)$ and that of C^∞ smooth local defining function r of Ω satisfying (1.4).

Let us proceed to the invariant properties of (1.11), hence we shall work locally without any emphasis on it. Let us begin by recalling that the solution to (1.5) transforms like an invariant of weight -1:

$$u = |\det \Phi'|^{-2/(n+1)}\, \widetilde{u} \circ \Phi, \qquad (\Phi : \Omega \to \widetilde{\Omega} \text{ biholomorphic}), \tag{1.12}$$

where $\widetilde{u}$ corresponds to $\widetilde{\Omega}$. More precisely, (1.12) implies $J[u] = J[\widetilde{u}] \circ \Phi$ (cf. [F2]). Since r and η_k are uniquely determined modulo $O(r^{n+2})$ and $O(r^{n+1})$, respectively, it is possible to consider the transformation laws for these; the results are:

$$\widetilde{r} \circ \Phi = |\det \Phi'|^{2/(n+1)}\, r \mod O(r^{n+2}), \tag{1.13}$$

$$\widetilde{\eta}_k \circ \Phi = |\det \Phi'|^{-2k}\, \eta_k \mod O(r^{n+1}). \tag{1.14}$$

In particular, the boundary value $b\eta_k := \eta_k|_{\partial\Omega}$ is an invariant of weight $k\,(n+1)$. The simplest non-trivial case is $k = 1$. Graham [G1], [G2] determined the linear part of $b\eta_1$. If $n = 2$, then $b\eta_1$ itself is linear. More precisely,

Lemma 1.2 ([G1], [G2]). *If $n = 2$ then $\eta_1 = 4A^0_{4\overline{4}} + O(r)$.*

1.3. Proof of Proposition 1. Let us drop the superscript S, because we are only concerned with the Szegö kernel here. It follows from (1.2) and (1.13) that the Szegö kernel satisfies the transformation law

$$K(z) = |\det \Phi'(z)|^{2n/(n+1)}\, \widetilde{K}\big(\Phi(z)\big), \qquad (\Phi : \Omega \to \widetilde{\Omega} \text{ biholomorphic}), \tag{1.15}$$

as far as a branch of $[\det \Phi'(z)]^{n/(n+1)}$ is defined globally (cf. [H] for an intrinsic treatment). In general, (1.15) remains valid for the singularity of the Szegö kernel, for it can be localized modulo C^∞ smooth error to a neighborhood of a small portion M of $\partial\Omega$. Thus, by (1.1) and (1.13),

$$\widetilde{\varphi} \circ \Phi = \varphi \mod O(r^n), \qquad \widetilde{\psi} \circ \Phi = |\det \Phi'|^{-2n/(n+1)}\, \psi \mod O(r^\infty). \tag{1.16}$$

In particular, the boundary values $\varphi|_M$ and $\psi|_M$ transform like invariants of weight 0 and n, respectively, in the sense of (1.10). These are indeed invariant polynomials in Moser's normal form coefficients, since their dependence on the boundary $\partial\Omega$ is local as noted before. More can be said by using (1.16) as follows.

Let us first extract invariants from φ with the aid of Lemma 1.1. We first observe that $\varphi|_M$ is constant, for it is an invariant of weight 0. Evaluating it in the case of the ball, we find $\varphi|_M = 1$. Next, if we write $\varphi = 1 + P_1\, r + O(r^2)$ with $P_1 \in C^\infty(M)$, then P_1 is an invariant of weight 1, so that $P_1 = 0$. Thus in case $n = 2$ we are done. In case $n \geqq 3$, we further expand as follows: $\varphi = 1 + P_2\, r^2 + O(r^3)$ with $P_2 \in C^\infty(M)$. Then P_2 is an invariant of weight 2, so that $P_2 = c_n\, \|A^0_{2\overline{2}}\|^2$ with a constant c_n. This completes the proof of (1.6) and (1.7) for φ.

It remains to consider ψ for $n = 2$, again by using Lemma 1.1 but this time with the aid of Lemma 1.2. We have observed that $\psi|_M$ is an invariant of weight 2. It then follows from Lemma 1.1 that $\psi|_M = 0$, so that we can write $\psi = Q_1\, r + O(r^2)$ with $Q_1 \in C^\infty(M)$. Then Q_1 is an invariant of weight 3, so that, by Lemma 1.1, it is a multiple of $A^0_{4\overline{4}}$. By virtue of Lemma 1.2, $A^0_{4\overline{4}}$ has an extension η_1, so that we may set $\psi = k_1\, \eta_1\, r + Q_2\, r^2 + O(r^3)$ with $Q_2 \in C^\infty(M)$. By the transformation law (1.14) for η_1, we see that Q_2 is an invariant of weight 4, which is a multiple of $|A^0_{2\overline{4}}|^2$ by Lemma 1.1. Therefore, we obtain (1.6) for $n = 2$, and the proof is complete.

§2. Application of microlocal analysis of Kashiwara and Boutet de Monvel.

In this Section, we shall give a proof of Theorem 1 by using the theory of micro-differential systems. Assuming that the boundary $\partial\Omega$ is real analytic, we shall construct an asymptotic series which actually converges and gives the Szegö kernel. It should be mentioned that the same construction works even in the C^∞ smooth category, and in that case we obtain a formal series giving rise to an asymptotic expansion of the Szegö kernel. This and the next Sections are self-contained, and can be read independently.

2.1. Reformulation of Theorem 1. In order to prove Theorem 1, we shall compute the asymptotic expansion of the Szegö kernel in the following form.

Theorem 2. *Assume that $M := \partial\Omega \subset \mathbb{C}^n$ is in normal form. Consider the behavior of the invariant Szegö kernel at points $\gamma_t = (0, \dots, 0, t/2)$ as $t \to +0$. If $n = 2$, then the singularity of $\pi^2 K^{\mathrm{S}}(\gamma_t)$ at $t = +0$ is given by $1/t^2 + \big(C_1(t) + O(t^3)\big) \log t$, where*

$$C_1(t) := -8A^0_{4\overline{4}}\, t + \left(40A^0_{5\overline{5}} - \frac{440}{3}\, |A^0_{2\overline{4}}|^2\right) t^2.$$

If $n \geqq 3$ then $$\frac{\pi^n}{(n-1)!}\, K^{\mathrm{S}}(\gamma_t) = \frac{1}{t^n} - \frac{2(n-3)}{(n-2)\,(n^2-1)}\, \|A^0_{2\overline{2}}\|^2\, \frac{1}{t^{n-2}} + O\left(\frac{1}{t^{n-3}}\right).$$

Observe that Theorem 2 together with Lemma 1.2 implies $k^{\mathrm{S}}_1 = -2$. To determine other universal constants, we need to know more about the asymptotic behavior of the solution $u = u^{\mathrm{MA}}$ to the problem (1.5). Such information is given by the following:

Corollary. *If $n = 2$, then $u(\gamma_t) = t + C_2(t)\, t^4 \log t + O(t^6 |\log t|)$, that is, $r^{\mathrm{F}}(\gamma_t) = t + O(t^4)$ and $\eta_1(\gamma_t) = C_2(t) + O(t^2)$, where*

$$C_2(t) := 4\, A^0_{4\overline{4}} + \left(-20\, A^0_{5\overline{5}} + \frac{368}{5}\, |A^0_{2\overline{4}}|^2\right) t.$$

If $n \geqq 3$ then $$u(\gamma_t) = t + \frac{8}{3n\,(n^2-1)}\, \|A^0_{2\overline{2}}\|^2\, t^3 + O(t^4).$$

Let us first observe that Theorem 1 follows from Theorem 2 and its Corollary.

Proof of Theorem 1 via Theorem 2. If $n = 2$, then Proposition 1 and the Corollary above imply $\varphi^{\mathrm{S}}(\gamma_t) = 1 + O(t^2)$ and $\psi^{\mathrm{S}}(\gamma_t) = k^{\mathrm{S}}_1\, C_2(t)\, t + k^{\mathrm{S}}_2\, |A^0_{2\overline{4}}|^2\, t^2 + O(t^3)$. Comparing this with Theorem 2, we obtain the desired conclusion $k^{\mathrm{S}}_2 = 8/15$ (and $k^{\mathrm{S}}_1 = -2$).

In case $n \geqq 3$, we have by Proposition 1 and the Corollary above that $\varphi^{\mathrm{S}}(\gamma_t) = 1 + c^{\mathrm{S}}_n\, \|A^0_{2\overline{2}}\|^2\, t^2 + O(t^3)$, while Theorem 2 and its Corollary permit us to write $c_1\, t^n\, K^{\mathrm{S}}(\gamma_t) = 1 + c_2\, \|A^0_{2\overline{2}}\|^2\, t^2 + O(t^3)$ and $u(\gamma_t)/t = 1 + c_3\, \|A^0_{2\overline{2}}\|^2\, t^2 + O(t^3)$ with explicit constants

$$c_1 := \frac{\pi^n}{(n-1)!}, \quad c_2 := \frac{-2\,(n-3)}{(n-2)\,(n^2-1)}, \quad c_3 := \frac{8}{3\,n\,(n^2-1)},$$

so that $c_1\, u(\gamma_t)^n\, K^{\mathrm{S}}(\gamma_t) = 1 + (n\, c_3 + c_2)\, \|A^0_{2\overline{2}}\|^2\, t^2 + O(t^3)$. Thus $c^{\mathrm{S}}_n = n\, c_3 + c_2$, which yields the desired conclusion $(n-1)\,(n-2)\, c^{\mathrm{S}}_n = 2/3$.

We shall here present a proof of the Corollary for $n = 2$; the case $n \geqq 3$ will be discussed in Subsection 2.4.

Proof of Corollary for $n = 2$. Following the algorithm given in [F2], we first compute the smooth part. Then we see that Fefferman's approximate solution satisfies $r^{\mathrm{F}}(\gamma_t) = t + O(t^4)$. This, together with Proposition 1, implies $\psi^{\mathrm{S}}(\gamma_t) = k_1^{\mathrm{S}}\, \eta_1(\gamma_t)\, t + k_2^{\mathrm{S}}\, |A_{24}^0|^2\, t^2 + O(t^3)$, with $k_1^{\mathrm{S}} = -2$ via Lemma 1.2 as noted before. On the other hand, Theorem 2 yields $\psi^{\mathrm{S}}(\gamma_t) = C_1(t) + O(t^3)$, so that $\eta_1(\gamma_t) = 4A_{4\overline{4}}^0 + (-20A_{5\overline{5}}^0 + c\, |A_{2\overline{4}}^0|^2)\, t + O(t^2)$ with a constant c. It remains to determine this c. It was observed by Graham [G1] that $\eta_1 = 0$ whenever $A_{4\overline{4}}^0 = 0$ identically. Thus, the constant c will be identified by computing $A_{5\overline{5}}^0$ and $A_{2\overline{4}}^0$ for such a hypersurface M. Taking M defined by $(z_1 + \overline{z}_1)^2\, (z_2 + \overline{z}_2) = 1$, we make a holomorphic change of coordinates and get

$$2\,\mathrm{Re}\, z_2 = |z_1|^2 + \frac{1}{6} \sum_{p+q \geqq 3,\ p,q \geqq 1} (-1)^{p+q}\, \frac{(p+q+1)!}{p!\, q!}\, z_1^p\, \overline{z}_1^q.$$

Then by the procedure in Chern-Moser [CM] starting from the expression above, we can find a normal form at each point such that $A_{4\overline{4}}^0 = 0$, $A_{5\overline{5}}^0 = 23/9$ and $A_{2\overline{4}}^0 = 5/6$. This gives $c = 368/5$, which is the desired conclusion.

2.2. A microdifferential system characterizing the Szegö kernel, and the construction of an asymptotic solution. The proof of Theorem 2 will be done by formulating the Szegö kernel analogue of Microlocal Analysis of the Bergman kernel which was carried out by Kashiwara [Ka] and Boutet de Monvel [B1], [B2]. In what follows, we shall restrict ourselves to the case where Ω is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ having (real) analytic boundary. Note that if an analytic defining function $\rho = \rho(z, \overline{z})$ (> 0 in Ω) is specified near a boundary point, then the coefficients $\varphi = \varphi^{\mathrm{S}}$, φ^{B} and $\psi = \psi^{\mathrm{S}}$, ψ^{B} in (1.1), (1.1B) are analytic there. This fact is contained implicitly in Boutet-Sjöstrand [BS].

Denoting by $\overline{X}$ the complex conjugate of $X := \mathbb{C}^n$, we regard $X \times \overline{X}$ as the complexification of the diagonal $X^{\mathbb{R}} := \{(z, \overline{w});\ z = w\}$, which is identified with X. Recall that a *microdifferential operator* on X is an analytic pseudodifferential operator $P = P(z, \overline{z}, D_z, \overline{D}_z)$ with $D_z := \partial/\partial z$ which is classical in the sense that the symbol admits a homogeneous expansion of integral degree. Each operand, called a *microfunction*, is the microlocal singularity of a hyperfunction decomposed into holomorphic extensions to conic sets in the complexification $X \times \overline{X}$. More generally, we also consider microdifferential operators $P = P(z, \overline{w}, D_z, \overline{D}_w)$ which are sesqui-holomorphic in $X \times X$. By abusing notation, we write $\overline{z}$ in place of $\overline{w}$, and identify a (hyper-) function on $X^{\mathbb{R}} \simeq X$ with its complex extension to $X \times \overline{X}$.

We are concerned with singularities of the form $\varphi\, \rho^{-N} + \psi\, \log \rho$ with $N \in \mathbb{N}$. Typical ones are $\log \rho$ and $1/\rho$, which respectively represent, up to a multiple, the Heaviside function $Y = Y(\rho)$ of Ω and the Delta measure $\delta = \delta(\rho)$ on $\partial\Omega$ with surface element σ satisfying $\sigma \wedge d\rho = dV(z)$. Let us now recall Kashiwara's observation [Ka] on the Bergman kernel:

(a) *Given a holomorphic microdifferential operator $P = P(z, D_z)$, there exists a unique antiholomorphic microdifferential operator $Q^{\mathrm{B}} = Q^{\mathrm{B}}(\overline{z}, \overline{D}_z)$ such that $(P -$*

$Q^{\mathrm{B}})Y = 0$. *In this case, the Bergman kernel* $K^{\mathrm{B}} = K^{\mathrm{B}}(z, \overline{z})$ *satisfies* $(P - Q^{\mathrm{B}})^* K^{\mathrm{B}} = 0$, *where* $P^* = P^*(z, D_z)$ *and* $(Q^{\mathrm{B}})^* = (Q^{\mathrm{B}})^*(\overline{z}, \overline{D}_z)$ *are formal adjoints of* P *and* Q^{B}, *respectively.*

(b) *If holomorphic microdifferential operators* $P_1, \cdots, P_{2n}$ *are chosen appropriately, e.g.,* $P_j = z_j$ *and* $P_{n+j} = D_j = \partial/\partial z_j$ *for* $j = 1, \cdots, n$, *then the system* $(P_k - Q_k^{\mathrm{B}})Y = 0$ *for* $k = 1, \cdots, 2n$, *with* Q_k^{B} *corresponding to* P_k *as in* (a) *above, characterizes* Y *up to a multiple. Furthermore, the system of formal adjoints characterizes the Bergman kernel. Namely, the (analytic) singularity of* K^{B} *is given by a microfunction solution which exists uniquely up to a multiple.*

The proof of the above facts (a) and (b) applies equally to the Szegö kernel with an arbitrary surface element, and we obtain:

Lemma 2.1. *Assume that a Delta measure* $\delta = \delta(\rho)$ *on* $\partial\Omega$ *is uniquely determined by a microdifferential system* $(P_k - Q_k^{\mathrm{S}})\delta = 0$ *for* $k = 1, \cdots, 2n$ *with* $P_k = P_k(z, D_z)$ *and* $Q_k^{\mathrm{B}} = Q_k^{\mathrm{B}}(\overline{z}, \overline{D}_z)$. *Then, the Szegö kernel* $K^{\mathrm{S}} = K^{\mathrm{S}}(z, \overline{z})$ *associated with* σ *satisfying* $\sigma \wedge d\rho = dV(z)$ *is characterized by the system* $(P_k - Q_k^{\mathrm{S}})^* K^{\mathrm{S}} = 0$ *for* $k = 1, \cdots, 2n$. *In particular, the invariant Szegö kernel is obtained by the choice* $\rho = J[r]^{-1/(n+1)}\, r$, *where* r *is an arbitrary defining function, so that* $\delta(\rho) = J[r]^{1/(n+1)}\, \delta(r)$.

In order to compute the Szegö kernel asymptotically by using Lemma 2.1 above, we follow the argument in [B1], [B2] for the Bergman kernel. This begins with the following observation, where (d) has been previously used implicitly in [Ka]:

(c) *Let* $Y_0 = Y(U_0)$, *where* $U_0(z, \overline{z}) := z_n + \overline{z}_n - z' \cdot \overline{z'}$, *so that* $\Omega_0 := \{U_0 > 0\}$ *is biholomorphically equivalent to a ball. Then, it is possible to construct explicitly an invertible holomorphic microdifferential operator* $A^{\mathrm{B}} = A^{\mathrm{B}}(z, D_z)$ *of infinite order such that* $Y = A^{\mathrm{B}}\, Y_0$.

(d) *Let* $K_0^{\mathrm{B}} = K_0^{\mathrm{B}}(z, \overline{z})$ *denote the Bergman kernel associated with* Ω_0, *so that* K_0^{B} *is a multiple of* U_0^{-n-1}. *Then* $K^{\mathrm{B}} = (A^{\mathrm{B}})^{*-1}\, K_0^{\mathrm{B}}$, *where* $(A^{\mathrm{B}})^* = (A^{\mathrm{B}})^*(z, D_z)$ *is a formal adjoint of* A^{B}.

Note that $(A^{\mathrm{B}})^{*-1}$ is a variant of the Fourier integral operator which is regarded as a pseudodifferential operator of infinite order. A crucial point is that $(A^{\mathrm{B}})^{*-1}$ is a holomorphic operator. To realize the step (c) above, it is convenient to introduce a *complex normal form* as in [B1], [B2], more precisely, a defining function $U := U_0 - H$ of $\partial\Omega$ in $X \times \overline{X}$, where H is defined by setting

$$(2.1) \qquad H(z, \overline{z'}) := \sum_{|\alpha|,\, |\beta| \geqq 2} C_{\alpha\overline{\beta}}(z_n)\, z'_\alpha\, \overline{z'_\beta} \quad \text{with} \quad C_{\alpha\overline{\beta}}(z_n) := \sum_{\ell=0}^{\infty} C_{\alpha\overline{\beta}}^{\ell}\, z_n^{\ell},$$

where the coefficients $C_{\alpha\overline{\beta}}(z_n)$ are subject to the trace conditions as in the case of Moser's normal form (1.8) with (1.9), as well as the symmetrization on the permutation of α and that of β. The coefficients $C_{\alpha\overline{\beta}}^{\ell}$ are computed from $A_{\alpha\overline{\beta}}^{\ell}$ via the implicit function theorem (see the next Subsection). Noting that $D_j\, D_n^{-1}\, U_0 = -\overline{z}_j\, U_0$ for $1 \leqq j \leqq n-1$, we see that $Y = A^{\mathrm{B}}\, Y_0$ is satisfied by $A^{\mathrm{B}} = A^{\mathrm{B}}(z, D_z)$ with the symbol

$$(2.2) \qquad A^{\mathrm{B}}(z, \zeta) = \exp[-H(z, -\zeta'/\zeta_n)\, \zeta_n], \quad \text{where} \quad \zeta = (\zeta', \zeta_n).$$

We are concerned with an operator $A^{\mathrm{S}} = A^{\mathrm{S}}(z, D_z)$ satisfying $A^{\mathrm{S}}\,\delta(U_0) = \delta(\rho)$ and thus $(A^{\mathrm{S}})^{*-1}\,K_0^{\mathrm{S}} = K^{\mathrm{S}}$, where $K_0^{\mathrm{S}} = K_0^{\mathrm{S}}(z, \overline{z})$ stands for the (invariant) Szegö kernel associated with the domain Ω_0, so that K_0^{S} is a multiple of U_0^{-n}. Recalling that the invariant Szegö kernel corresponds to the choice $\rho = J[U]^{-1/(n+1)}\,U$, we see that the symbol in this case is given by

$$A^{\mathrm{S}}(z,\zeta) = V(z, -\zeta'/\zeta_n)\,A^{\mathrm{B}}(z,\zeta) \quad \text{with} \quad V(z,\overline{z'}) := J[U](z,\overline{z'})^{1/(n+1)}. \tag{2.3}$$

In the case of an arbitrary surface element on $\partial\Omega$, we have $\delta(\rho) = V\,\delta(U)$, where $V = V(z, \overline{z})$ is nonvanishing. The argument above remains valid by eliminating the variable $\overline{z}_n$ in the factor $V(z,\overline{z})$. This is done by substituting $\overline{z}_n = -z_n + z'\cdot\overline{z'} + H(z,\overline{z'})$, since only the boundary value is concerned. Therefore, we have obtained, as in (c) and (d) for the Bergman kernel, the following:

Lemma 2.2. *Let $A^{\mathrm{S}} = A^{\mathrm{S}}(z, D_z)$ be a holomorphic microdifferential operator of infinite order having the symbol given by* (2.3) *with $U = U_0 - H$, where H and $A^{\mathrm{B}}(z,\zeta)$ are defined by* (2.1) *and* (2.2)*. Then $\delta(\rho) = A^{\mathrm{S}}\,\delta(U_0)$ with $\rho = J[U]^{-1/(n+1)}\,U$, so that the invariant Szegö kernel is given by $K^{\mathrm{S}} = (A^{\mathrm{S}})^{*-1}\,K_0^{\mathrm{S}}$, where K_0^{S} is a multiple of U_0^{-n}. The Szegö kernel corresponding to an arbitrary surface element on $\partial\Omega$ is obtained by a modification of the definition of $V(z,\overline{z'})$ as described above.*

Proof. We have already observed $\delta(\rho) = A^{\mathrm{S}}\delta(U_0)$, and thus it remains only to prove $K^{\mathrm{S}} = (A^{\mathrm{S}})^{*-1}\,K_0^{\mathrm{S}}$. Let us use Lemma 2.1. If $\delta(\rho)$ is characterized by $(P_k - Q_k^{\mathrm{S}})\delta(\rho) = 0$ for $k = 1, \cdots, 2n$, then $(P_k - Q_k^{\mathrm{S}})^* K^{\mathrm{S}} = 0$. Recalling that $\delta(\rho) = A^{\mathrm{S}}\delta(U_0)$, we have

$$\Big(\big(A^{\mathrm{S}}\big)^{-1} P_k A^{\mathrm{S}} - Q_k^{\mathrm{S}}\Big)\,\delta(U_0) = 0, \qquad \text{and thus} \qquad \big(P_k - Q_k^{\mathrm{S}}\big)^*\,\big(A^{\mathrm{S}}\big)^{*-1}\,K_0^{\mathrm{S}} = 0.$$

This is because A^{S} is an invertible holomorphic operator such that $(A^{\mathrm{S}})^{-1}P_k A^{\mathrm{S}}$ are operators of finite order. Hence, $(A^{\mathrm{S}})^{*-1}K_0^{\mathrm{S}}$ satisfies a microdifferential system characterizing K^{S}, so that $K^{\mathrm{S}} = c(A^{\mathrm{S}})^{*-1}K_0^{\mathrm{S}}$ with some constant c. Regarding K^{S} as a perturbation of K_0^{S}, we obtain the desired conclusion $c = 1$.

By virtue of Lemma 2.2 above, one can construct an asymptotic expansion of the *invariant* Szegö kernel by using the Neumann series expansion $A^{*-1} = 1 + (1 - A^*) + \cdots$, where $A = A^{\mathrm{S}}$. To explain this, we follow [B2] and introduce the *biweight* by setting $\mathrm{w}_2(z') = (1,0) = -\mathrm{w}_2(D')$ and $\mathrm{w}_2(z_n) = (1,1) = -\mathrm{w}_2(D_n)$, where $D' := \partial/\partial z'$. This notion extends naturally to holomorphic microdifferential operators of infinite order. We also set $\mathrm{w}_2(\overline{z'}) = (0,1)$, $\mathrm{w}_2(\overline{z}_n) = (1,1)$ and $\mathrm{w}_2(\log U_0) = (0,0)$, so that the notion of biweight extends also to the operands. Then, the *weight* is defined by $\mathrm{w}(\cdot) := p + q$ when $\mathrm{w}_2(\cdot) = (p,q)$. In the next Subsection, we shall observe that $\mathrm{w}(1 - A) \geqq 4$ and thus $\mathrm{w}(1 - A^*) \geqq 4$, since the biweight is unchanged by taking the formal adjoint. Therefore, the Neumann series makes sense as an asymptotic series. The case of a *general* surface element is similar if we note under obvious notation that $A^{\mathrm{S}}_{\mathrm{inv}}(z,\zeta) = E(z, \zeta'/\zeta_n)\,A^{\mathrm{S}}_{\mathrm{gen}}(z,\zeta)$ with a nonvanishing $E(z,\zeta'/\zeta_n)$, so that $\mathrm{w}(1 - E(z, D'\,D_n^{-1})) \geqq 1$.

2.3. Proof of Theorem 2 for $n = 2$. Let us begin by recalling that

$$D_2^{-m}\, U_0^{-2} = \frac{-1}{(m-2)!}\, t^{m-2} \log t \quad \text{evaluated at } z = \overline{z} = \gamma_t \quad \text{for } 2 \leqq m \in \mathbb{N}, \tag{2.4}$$

where we are only concerned with the singularity at $t = +0$. Since $\mathrm{w}_2(U_0^{-2}) = (-2, -2)$ and thus $\mathrm{w}(U_0^{-2}) = -4$, it suffices to compute the Neumann series expansion of $(A^{\mathrm{S}})^{*-1}$ up to terms of weight $\leqq 8$. Furthermore, we can ignore terms of biweight (p, q) satisfying $p \neq q$ and $4 < p + q \leqq 8$, since these terms vanish when evaluated at $z_1 = \overline{z}_1 = 0$.

By direct computation, we have

$$V(z, \overline{z}_1) = 1 + B^0_{1\overline{3}}\, z_1 \overline{z}_1^3 + B^0_{3\overline{1}}\, z_1^3 \overline{z}_1 + \sum\, B^\ell_{\alpha\overline{\beta}}\, z_1^\alpha \overline{z}_1^\beta\, z_2^\ell\,,$$

where the summation is taken over $\ell \in \mathbb{N}_0$ and $\alpha, \beta \geqq 1$ with $\alpha + \beta \geqq 5$. (We shall later write down the coefficients $B^\ell_{\alpha\overline{\beta}}$ we need in terms of $C^\ell_{\alpha\overline{\beta}}$.) By using this, we obtain

$$A^{\mathrm{S}}(z, D_z) = 1 + a_{1\overline{3}} + a_{3\overline{1}} + a_{3\overline{3}} + a_{4\overline{4}} + \cdots\,,$$

where $a_{\alpha\overline{\beta}} = a_{\alpha\overline{\beta}}(z, D_z)$ are terms of biweight (α, β) given by

$$a_{\alpha\overline{\beta}} = a^{\mathrm{B}}_{\alpha\overline{\beta}} - B^0_{\alpha\overline{\beta}}\, z_1^\alpha\, (D_1 D_2^{-1})^\beta \quad \text{with} \quad a^{\mathrm{B}}_{\alpha\overline{\beta}} = -C^0_{\alpha+1\,\overline{\beta+1}}\, z_1^{\alpha+1}\, (D_1 D_2^{-1})^{\beta+1}\, D_2$$

for $(\alpha, \beta) = (1, 3)$, $(3, 1)$, $(3, 3)$, and

$$\begin{aligned} a_{4\overline{4}} = a^{\mathrm{B}}_{4\overline{4}} &+ B^0_{4\overline{4}}\, z_1^4\, (D_1 D_2^{-1})^4 - B^1_{3\overline{3}}\, z_1^3\, z_2\, (D_1 D_2^{-1})^3 \\ &+ (C^0_{2\overline{4}}\, B^0_{3\overline{1}} + C^0_{4\overline{2}}\, B^0_{1\overline{3}}) z_1^5\, (D_1 D_2^{-1})^5\, D_2 \end{aligned}$$

with

$$a^{\mathrm{B}}_{4\overline{4}} = C^0_{5\overline{5}}\, z_1^5\, (D_1 D_2^{-1})^5\, D_2 - C^1_{4\overline{4}}\, z_1^4 z_2\, (D_1 D_2^{-1})^4\, D_2 + C^0_{2\overline{4}}\, C^0_{4\overline{2}}\, z_1^6\, (D_1 D_2^{-1})^6\, D_2^2\,.$$

This implies, in particular, $\mathrm{w}(1 - A^{\mathrm{S}}) \geqq 4$. Furthermore,

Lemma 2.3. *The singularity of $\pi^2\, K^{\mathrm{S}}(\gamma_t)$ at $t = +0$ modulo $O(t^3) \log t$ is given by $(c_1\, D_2^{-3} + c_2\, D_2^{-3}\, z_2 + c_3\, D_2^{-4}) U_0^{-2}$ evaluated at $z = \overline{z} = \gamma_t$, where z_2 in $D_2^{-3}\, z_2$ stands for a multiplication operator, and*

$$\begin{aligned} c_1 := -4!\, C^0_{4\overline{4}} + 3!\, B^0_{3\overline{3}}\,, \qquad c_2 := -4!\, C^1_{4\overline{4}} + 3!\, B^1_{3\overline{3}}\,, \\ c_3 := 5!\, (C^0_{5\overline{5}} + C^0_{2\overline{4}}\, B^0_{3\overline{1}} + C^0_{4\overline{2}}\, B^0_{1\overline{3}}) - 6!\, C^0_{2\overline{4}}\, C^0_{4\overline{2}} - 4!\, B^0_{4\overline{4}} \\ + 2\, (12\, C^0_{2\overline{4}} - 3\, B^0_{1\overline{3}})\, (12\, C^0_{4\overline{2}} - 3\, B^0_{3\overline{1}}). \end{aligned}$$

Proof. Writing $A = A^{\mathrm{S}}$, we have $A^{*-1} = 1 + (1 - A^*) + (1 - A^*)^2 + \cdots$, where $\cdots$ stands for terms which are irrelevant to the singularity modulo $O(t^3) \log t$. Considering biweight, we get $1 - A^* = -a^*_{3\overline{3}} - a^*_{4\overline{4}} + \cdots$ and

$$(1 - A^*)^2 = (a^*_{1\overline{3}} + a^*_{3\overline{1}})^2 + \cdots = a^*_{1\overline{3}}\, a^*_{3\overline{1}} + a^*_{3\overline{1}}\, a^*_{1\overline{3}} + \cdots = a^*_{1\overline{3}}\, a^*_{3\overline{1}} + \cdots\,,$$

where the last equality follows from the expression of $a_{3\overline{1}}$. Then, the desired conclusion is obtained by the explicit form of $a_{1\overline{3}}$, $a_{3\overline{1}}$, $a_{3\overline{3}}$ and $a_{4\overline{4}}$.

By virtue of (2.4) and Lemma 2.3 above together with the commutation relation $[D_2^{-m}, z_2] = -m\, D_2^{-m-1}$ for $m \in \mathbb{N}$, we get an expression of the singularity of $\pi^2\, K^{\mathrm{S}}(\gamma_t)$ in terms of the coefficients $B^{\ell}_{\alpha\overline{\beta}}$ and $C^{\ell}_{\alpha\overline{\beta}}$. The desired conclusion follows from

$$B^0_{1\overline{3}} = \frac{8}{3}\, C^0_{2\overline{4}}\,, \quad B^0_{3\overline{1}} = \frac{8}{3}\, C^0_{4\overline{2}}\,, \quad B^0_{3\overline{3}} = \frac{16}{3}\, C^0_{4\overline{4}}\,, \quad B^1_{3\overline{3}} = \frac{16}{3}\, C^1_{4\overline{4}}\,,$$
$$B^0_{4\overline{4}} = \frac{25}{3}\, C^0_{5\overline{5}} + C^1_{4\overline{4}} - \frac{128}{9}\, C^0_{2\overline{4}}\, C^0_{4\overline{2}}\,,$$
$$C^0_{2\overline{4}} = A^0_{2\overline{4}}\,, \quad C^0_{4\overline{2}} = A^0_{4\overline{2}}\,, \quad C^0_{4\overline{4}} = A^0_{4\overline{4}}\,, \quad C^1_{4\overline{4}} = -i\, A^1_{4\overline{4}}\,, \quad C^0_{5\overline{5}} = A^0_{5\overline{5}} + \frac{i}{2}\, A^1_{4\overline{4}}\,.$$

These are obtained by direct computation which is simple but long.

Remark 3. The formal substitution $B^{\ell}_{\alpha\overline{\beta}} = 0$ yields the Bergman kernel analogue of Theorem 2 for $n = 2$, that is, the singularity of $(\pi^2/2)\, K^{\mathrm{B}}(\gamma_t)$ at $t = +0$ is given by

$$\frac{1}{t^3} + \big(C^{\mathrm{B}}(t) + O(t^2)\big) \log t, \quad \text{where} \quad C^{\mathrm{B}}(t) := -12 A^0_{4\overline{4}} + \Big(60 A^0_{5\overline{5}} - 216 |A^0_{2\overline{4}}|^2\Big)\, t.$$

This first implies the Corollary of Theorem 2 and then the statement of Remark 1 in Subsection 1.1 for $n = 2$, that is, $k^{\mathrm{B}} = 24/5$ in (1.6B).

2.4. The higher dimensional case.

Proof of Theorem 2. By direct computation, we have $u^{\mathrm{MA}}(\gamma_t) = t + O(t^3)$. Thus by Proposition 1, it suffices to determine the coefficient of $\|A^0_{2\overline{2}}\|^2/t^{n-2}$. For this sake, we consider a domain given by $F = 2\,\varepsilon\,\mathrm{Re}\,(z_1^2\, \overline{z}_2^2)$ with $\varepsilon > 0$ small, and thus $\|A^0_{2\overline{2}}\|^2 = 2\,\varepsilon^2$. It is easily seen that $J[U] = 1 - 16\,\varepsilon^2\, |z_1 z_2|^2$, so that

$$A^{\mathrm{S}}(z, D_z) = 1 + \varepsilon\,(z_1^2 D_2^2 + z_2^2 D_1^2)\, D_n^{-1} + \frac{\varepsilon^2}{2}\,(z_1^4 D_2^4 + z_2^4 D_1^4 + 2\, z_1^2 z_2^2\, D_1^2 D_2^2)\, D_n^{-2}$$
$$- \frac{16\,\varepsilon^2}{n+1}\,(z_1 z_2 D_1 D_2)\, D_n^{-2} + (\text{terms of weight} \geqq 6).$$

This yields $(A^{\mathrm{S}})^* = 1 + c\,\varepsilon^2\, D_n^{-2} + \cdots$ and thus $(A^{\mathrm{S}})^{*-1} = 1 - c\,\varepsilon^2\, D_n^{-2} + \cdots$, where $c := 4\,(n-3)\,(n+1)$. Applying this expression of $(A^{\mathrm{S}})^{*-1}$ to U_0^{-n}, we get the desired conclusion.

Proof of Corollary to Theorem 2 for $n \geqq 3$. Recall that Fefferman's approximate solution $r = r^{\mathrm{F}}$ is unique modulo $O(r^{n+2})$. Following the procedure in [F2], we get $r = r^{\mathrm{F}}(\gamma_t)$ which is a formal power series in t of the form $r = t + c_1 t^3 + O(t^4)$. It then follows from Theorem 2 that

$$\varphi^{\mathrm{S}}(\gamma_t) = \big(1 + c_1 t^2\big)^n\, (1 - c_2\, \|A^0_{2\overline{2}}\|^2\, t^2) + O(t^3) = 1 + (n\, c_1 - c_2\, \|A^0_{2\overline{2}}\|^2)\, t^2 + O(t^3),$$

where $c_2 := 2\,(n-3)/\{(n-2)\,(n^2-1)\}$. On the other hand, Proposition 1 implies that $\varphi^{\mathrm{S}}(\gamma_t) = 1 + c_n^{\mathrm{S}}\,\|A^0_{2\overline{2}}\|^2\,t^2 + O(t^3)$, so that $c_1 = c_3\,\|A^0_{2\overline{2}}\|^2$ with a universal constant c_3. We wish to show that $c_3 = 8/\left\{3n\,(n^2-1)\right\}$. In order to prove this, we consider the same domain as in the proof of Theorem 2 above, and thus $J[U] = 1 - 16\,\varepsilon^2\,|z_1 z_2|^2$ with $2\,\varepsilon^2 = \|A^0_{2\overline{2}}\|^2$, where $U = 2\,\mathrm{Re}\,z_n - |z'|^2 - 2\,\varepsilon\,\mathrm{Re}\,(z_1^2\,\overline{z}_2^2)$. We now use the construction in [F2] crucially. Recall that if we define U_k for $k \in \mathbb{N}$ by

$$U_1 = J[U]^{-1/(n+1)}\,U \quad \text{and} \quad U_k/U_{k-1} = 1 + \frac{1 - J[U_{k-1}]}{k\,(n+2-k)} \quad \text{for} \quad 2 \leqq k \leqq n+1,$$

then $J[U_k] = 1 + O(r^k)$ and thus $U_k = (1 + O(r^k))\,U_{k+1} = r + O(r^{k+1})$. We thus need to compute U_3 restricted to $z = \overline{z} = \gamma_t$. By direct computation, we have $J[U_1] \sim 1 - \left\{1 - 16\,\varepsilon^2/(n+1)\right\}\,(|z_1|^2 + |z_2|^2)\,U$, which gives an approximate expression for U_2/U_1. It is then easy to obtain the desired conclusion for U_3 evaluated at $z = \overline{z} = \gamma_t$.

Remark 4. The difference between $A^{\mathrm{S}}(z, D_z)$ and $A^{\mathrm{B}}(z, D_z)$ modulo terms of weight $\geqq 6$ is $-16\,\varepsilon^2\,(n+1)^{-1}\,(z_1 z_2 D_1 D_2)\,D_n^{-2}$, so that $(A^{\mathrm{B}})^{*-1} = 1 + 4\,\varepsilon^2\,D_n^{-2} + \cdots$, and thus

$$\frac{\pi^n}{n!}\,K^{\mathrm{B}}(\gamma_t) = \frac{1}{t^{n+1}}\left(1 - \frac{2}{n\,(n-1)}\,\|A^0_{2\overline{2}}\|^2\,t^2 + O(t^3)\right),$$

which corresponds to Theorem 2 for $n \geqq 3$. This implies the Corollary as above, and we obtain the statement in Remark 1 for $n \geqq 3$, that is, $n\,(n-1)\,c_n^{\mathrm{B}} = 2/3$ in (1.7B).

§3. Analysis on complete Reinhardt domains.

3.1. An asymptotic expansion of the Szegö kernel in Reinhardt domains. As in Section 1, let us consider a bounded strictly pseudoconvex domain Ω in $\mathbb{C}^n$ with C^∞ smooth boundary, together with a surface element σ on the boundary satisfying (1.2). We now further assume that Ω is a *Reinhardt domain*, that is, Ω is invariant under a natural action of the unit torus. Let us denote by B, and call the *base domain* associated with Ω, the minus-signed logarithmic real representative domain $-\log|\Omega| = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R};\ (e^{-x_1}, \cdots, e^{-x_{n-1}}, e^{-y}) \in \Omega\}$. Introducing a system of multi-polar coordinates $z = (z_1, \cdots, z_n) = (e^{-x_1 - i\theta_1}, \cdots, e^{-x_{n-1} - i\theta_{n-1}}, e^{-y - i\theta_n})$ with $\theta_1, \cdots, \theta_n \in [0, 2\pi)$, we see that $\sigma = \sigma_{\partial B} \wedge d\theta_1 \wedge \cdots \wedge d\theta_n$, where $\sigma_{\partial B}$ is a surface element on ∂B. This change of coordinates provides a locally biholomorphic mapping $E_- : T_B \ni w \mapsto z = e^{-w} := (e^{-w_1}, \cdots, e^{-w_n}) \in \Omega \cap \{z_1 \cdots z_n \neq 0\}$, where T_B denotes the corresponding tube domain $B + i\,\mathbb{R}^n \subset \mathbb{C}^n$, and thus $\det E'_-(w) = (-1)^n\,z_1 \cdots z_n$.

For simplicity, let us restrict ourselves to the case $n = 2$ and assume that Ω is a *complete* Reinhardt domain, that is, Ω is invariant under a natural action of the closed unit polydisc. Then $B = \{y > f(x)\}$, where $f \in C^\infty(I)$ satisfies $f'' > 0$ and I is a half line of the form (x_-, ∞). In order to flatten the boundary of B, we make a change of variables $(x, y) \to (\rho, v)$ defined by $\rho := y - f(x)$ and $v := f'(x)$, so that $B = \{\rho > 0,\ v < 0\}$. We now set $p(v) := f''(x)$ —— the one dimensional *hodograph transformation.* Then

$$\sigma_{\partial B} = \frac{1}{16}\,J[\rho]^{-2/3}\,dv, \quad \text{where} \quad J[\rho] = \frac{p(v)}{|4 z_1 z_2|^2} = \frac{p(v)}{16\,|\det E'_-(w)|^2}\,. \tag{3.1}$$

Furthermore, it is possible to express the Szegö kernel asymptotically by using p and its derivatives. In order to state it more precisely, observe that (1.1) takes the form

$$(3.2)\qquad K^{\mathrm{S}}(z)=\frac{1}{\pi^2}J[\rho](z)^{2/3}\left[\frac{\widetilde{\varphi}^{\mathrm{S}}(v,\rho)}{\rho^2}+\widetilde{\psi}^{\mathrm{S}}(v,\rho)\,\log\rho\right]\quad\text{near}\quad\partial\Omega\cap\{z_1z_2\neq0\},$$

where $\widetilde{\varphi}^{\mathrm{S}}$ and $\widetilde{\psi}^{\mathrm{S}}$ are functions of $v<0$ and $\rho>0$ which are C^∞ smooth up to $\rho=0$, so that they have the Taylor expansions

$$(3.3)\qquad \widetilde{\varphi}^{\mathrm{S}}(v,\rho)=\varphi_0^{\mathrm{S}}(v)+\varphi_1^{\mathrm{S}}(v)\,\rho+O(\rho^2),\qquad \widetilde{\psi}^{\mathrm{S}}(v,\rho)\sim\sum_{k=0}^{\infty}\psi_k^{\mathrm{S}}(v)\,\rho^k.$$

We then have the following main result of this Section.

Theorem 3. *In addition to the hypotheses of Proposition 1 in Section 1, assume $n=2$ and that Ω is a complete Reinhardt domain with base $B=\{\rho:=y-f(x)>0\}$. Define $p(v):=f''(x)$ with $v:=f'(x)$, and consider the Szegö kernel* (3.2) *with* (3.3). *Then,*

$$\varphi_0^{\mathrm{S}}=1,\ \ \varphi_1^{\mathrm{S}}=\frac{1}{6}\,e_1,\ \ \psi_0^{\mathrm{S}}=0,\ \ \psi_1^{\mathrm{S}}=\frac{-1}{72}\,e_3,\ \ \psi_2^{\mathrm{S}}=\frac{1}{360}\,e_{42}+\frac{1}{4320}\,(e_{43}-e_{41}),$$

where $e_1:=p''$, $e_3:=(p^2\,p^{(4)})''$, $e_{41}:=e_1e_3$, $e_{42}:=(p\,e_3')'$ and $e_{43}:=(p\,p^{(4)})^2$.

Remark 5. A similar formula for the Bergman kernel is given in [N2], where the normalization is slightly different from ours. In our notation, this is stated as follows:

$$K^{\mathrm{B}}(z)=\frac{2}{\pi^2}J[\rho](z)\left[\frac{\widetilde{\varphi}^{\mathrm{B}}(v,\rho)}{\rho^3}+\widetilde{\psi}^{\mathrm{B}}(v,\rho)\,\log\rho\right]\quad\text{near}\quad\partial\Omega\cap\{z_1z_2\neq0\},$$

where $\widetilde{\varphi}^{\mathrm{B}}(v,\rho)=1+e_1\,\rho/4+e_2\,\rho^2/12+O(\rho^3)$ with $e_2:=(p\,p^{(3)})'$ and

$$\widetilde{\psi}^{\mathrm{B}}(v,\rho)=\frac{-1}{48}\,e_3+(2\,e_{42}+e_{43}-e_{41})\,\frac{\rho}{480}+O(\rho^2).$$

Remark 6. It is easily seen that $\partial\Omega$ is spherical if and only if p is an at most cubic polynomial (cf. [Ko]). Thus, Remark 2 is clear directly from Theorem 3 for our class of Reinhardt domains. Also, a global characterization of the ball is given in [N2] by using the Bergman kernel. This is immediately translated to the Szegö kernel by comparing Theorem 3 and Remark 5 above. That is, Ω is a ball if and only if $\psi_1^{\mathrm{S}}=0$ on the whole portion of the boundary ∂B. Such a characterization of globally spherical boundary extends to general Reinhardt domains which are bounded and strictly pseudoconvex (cf. [Ko]), for the completeness assumption in Theorem 3 can be eliminated after an obvious modification of the statement.

3.2. Proof of Theorem 1 by using Theorem 3. In order to prove Theorem 1 by using Theorem 3 above, we need to express the expansion (1.1) approximately in terms of (v,ρ). This is done by the following:

Lemma 3.1. *Consider the complex Monge-Ampère boundary value problem in the complete Reinhardt domain* $\Omega \subset \mathbb{C}^2$. *Then,* $r^{\mathrm{F}} = J[\rho]^{-1/3}\,\widetilde{r} + O(\rho^4)$ *and* $\eta_1 = J[\rho]\,\widetilde{\eta}_1 + O(\rho^2)$, *where*

$$\widetilde{r} := \rho - \frac{\rho^2}{12}\,e_1 - \frac{\rho^3}{36}\,(e_{42} - \frac{1}{2}e_1^2), \qquad \widetilde{\eta}_1 := \frac{1}{144}\,e_3 - \frac{\rho}{720}\,(e_{42} - \frac{1}{2}e_{41}).$$

Proof. Setting $u = J[\rho]^{-1/3}\widetilde{u}(v,\rho)$, we have $J[u] = M[\widetilde{u}]\,L\widetilde{u} - p\,Q[\widetilde{u}]$, where

$$M[\widetilde{u}] = \widetilde{u}_\rho^2 - \widetilde{u}\widetilde{u}_{\rho\rho}\,, \quad L\widetilde{u} = \widetilde{u}_\rho + \frac{1}{3}\,p''\widetilde{u} - (p\,\widetilde{u}_v)_v\,, \quad Q[\widetilde{u}] = \widetilde{u}_{\rho\rho}\widetilde{u}_v^2 - 2\widetilde{u}_\rho\widetilde{u}_v\widetilde{u}_{\rho v} + \widetilde{u}\widetilde{u}_{\rho v}^2$$

with $\widetilde{u}_\rho = \partial\widetilde{u}/\partial\rho$, $\widetilde{u}_v = \partial\widetilde{u}/\partial v$, and so on. We thus seek a solution u of $J[u] = 1 + O(\rho^4)$ in the form $\widetilde{u} = \widetilde{r} + \widetilde{\eta}_1\widetilde{r}^4\log\widetilde{r}$ with $\widetilde{r} = \rho + r_2(v)\rho^2 + r_3(v)\rho^3$ and $\widetilde{\eta}_1 = \eta_{10}(v) + \eta_{11}(v)\rho$. Noting that $Q[\widetilde{u}] = O(\rho^4)$, we are reduced to solving

$$M[\widetilde{r}]\,L\widetilde{r} = 1 + O(\rho^3), \qquad\qquad M[\widetilde{u}]\,L\widetilde{u} = 1 + O(\rho^4).$$

The first equation is satisfied if and only if $r_2 = -e_1/12$ and $r_3 = -(2e_2 - e_1^2)/72$. In order to solve the second equation, we set $\widetilde{u}_1 = \widetilde{r} + (\gamma_{10} + \gamma_{11}\rho)\rho^4\log\rho$, where $\gamma_{10} = \eta_{10}$ and $\gamma_{11} = \eta_{11} + 4r_2\gamma_{10}$. Then $\widetilde{u} = \widetilde{u}_1 + O(\rho^5)$, so that $M[\widetilde{u}]\,L\widetilde{u} = M[\widetilde{u}_1]\,L\widetilde{u}_1 + O(\rho^4)$. Writing $M[\widetilde{u}_1] = m_1 + m_2\rho^3\log\rho + O(\rho^5)$ and $L\widetilde{u}_1 = \ell_1 + \ell_2\rho^3\log\rho + O(\rho^5)$ with polynomials m_1, m_2, ℓ_1, ℓ_2 of ρ depending on v, we see that the equation for $\widetilde{u}$ is equivalent to $m_1\ell_1 = 1 + O(\rho^4)$ and $m_1\ell_2 + m_2\ell_1 = O(\rho^2)$. Since $\widetilde{r}$ has been identified, the first equation has a unique solution $\gamma_{10} = -(p\,r_3')'/4$. The second equation consists of two relations, one of which is identically satisfied. Another one determines γ_{11}, that is, $\gamma_{11} = -(p\,\gamma_{10}')'/5 - (7/30)p''\gamma_{10}$. Thus η_{10} and η_{11} are identified as in the statement.

Lemma 3.2. $|A^0_{2\overline{4}}|^2 = J[\rho]^{4/3}\alpha_{24}$, *where* $\alpha_{24} := (p\,p^{(4)}/48)^2 = e_{43}/48^2$.

Proof. In order to reduce computations, we use Proposition 1 or the corresponding result (1.6B) for the Bergman kernel. It then follows from Lemma 3.1 and Theorem 3 (or Remark 5) that $J[\rho]^{4/3}\,e_{43}$ is an invariant of weight 4, so that Lemma 1.1 implies the existence of a constant c such that $|A^0_{2\overline{4}}|^2 = c\,J[\rho]^{4/3}\,e_{43}$. In order to identify c, we consider a tube domain $T_B = B + i\,\mathbb{R}^2$ in $\mathbb{C}^2$ such that the boundary of the base domain B is locally given near the origin by $y = f(x)$, where $f(x) = -x + x^2/2 + x^6/6! + \cdots$. Then, $v = -1 + x + x^5/5! + \cdots$ and $p(-1) = p^{(4)}(-1) = 1$, so that $J[\rho] = 1/16$ at $x = y = 0$. By a simple change of variables, the boundary ∂T_B is transformed to

$$z_2 + \overline{z}_2 = |z_1|^2 + \sum_{j=2}^{4}\frac{z_1^j\,\overline{z}_1^{6-j}}{j!\,(6-j)!} + \cdots = |z_1|^2 + \frac{z_1^2\,\overline{z}_1^4}{48} + \frac{z_1^3\,\overline{z}_1^3}{36} + \frac{z_1^4\,\overline{z}_1^2}{48} + \cdots,$$

where the Jacobian factor $1/16$ comes out. It remains to show that this real hypersurface has a normal form such that $A^0_{2\overline{4}} = 1/48$. This is verified by inspecting the procedure of constructing a normal form in Chern-Moser [CM].

Proof of Theorem 1. By virtue of Theorem 3 together with Lemmas 3.1 and 3.2, we can prove Proposition 1 and Theorem 1 for our class of Reinhardt domains. In Lemma

3.1, we identified the coefficients of $\widetilde{r} = \rho + r_2(v)\rho^2 + r_3(v)\rho^3$ and $\widetilde{\eta}_1 = \eta_{10}(v) + \eta_{11}(v)\rho$. Since $\varphi_1^{\mathrm{S}} + 2r_2 = 0$, it follows that $\varphi^{\mathrm{S}} = (1 + \varphi_1^{\mathrm{S}}\rho)(1 + r_2\rho)^2 + O(\rho^2) = 1 + O(r^2)$, which is the first relation of (1.6) in Proposition 1. We next seek constants k_1^{S} and k_2^{S} in such a way that the second relation of (1.6) holds. Comparing the coefficients of ρ and ρ^2, we get $\psi_1^{\mathrm{S}} = k_1^{\mathrm{S}}\eta_{10}$ and $\psi_2^{\mathrm{S}} = k_1^{\mathrm{S}}(\eta_{11} + \eta_{10}r_2) + k_2^{\mathrm{S}}\alpha_{24}$, where α_{24} is given by Lemma 3.2. Applying Theorem 3 together with Lemmas 3.1 and 3.2, we obtain

$$(k_1^{\mathrm{S}} + 2)e_3 = 0, \quad (k_1^{\mathrm{S}} + 2)\Big(\frac{1}{12}\, e_{41} - e_{42}\Big) + \Big(\frac{5}{16}k_2^{\mathrm{S}} - \frac{1}{6}\Big)e_{43} = 0.$$

These relations determine the constants k_1^{S} and k_2^{S} uniquely as in Theorem 1, since there are domains in our class such that $e_3 \neq 0$ and $e_{43} \neq 0$ locally.

Remark 7. A similar argument is applied to the Bergman kernel (see Remarks 1 and 5). The first relation in (1.6B) follows from $\varphi_1^{\mathrm{B}} + 3r_2 = 0$ and $\varphi_2^{\mathrm{B}} + 3\varphi_1^{\mathrm{B}}r_2 + 3(r_2^2 + r_3) = 0$, where the notation will be self-explanatory. Observe that r_3 is involved in φ_2^{B}. The second one, or, rather $\psi^{\mathrm{B}} = k_1^{\mathrm{B}}\eta_1 + k_2^{\mathrm{B}}|A_{2\bar{4}}^0|^2 r + O(r^2)$, which is equivalent to $\psi_0^{\mathrm{B}} = k_1^{\mathrm{B}}\eta_{10}$ and $\psi_1^{\mathrm{B}} = k_1^{\mathrm{B}}\eta_{11} + k_2^{\mathrm{B}}\alpha_{24}$, takes the form

$$(k_1^{\mathrm{B}} + 3)e_3 = 0, \quad (k_1^{\mathrm{B}} + 3)(e_{41} - 2e_{42}) + \Big(\frac{5}{8}k_2^{\mathrm{B}} - 3\Big)e_{43} = 0.$$

3.3. Proof of Theorem 3. Given $(x, \rho) \in I \times \mathbb{R}_+$ with x arbitrarily fixed, we set

$$K_x(\rho) := \pi^2\, J[\rho](z)^{-2/3}\, K^{\mathrm{S}}(z) \quad \text{with} \quad -\log(|z_1|, |z_2|) = (x, \rho + f(x)),$$

and consider the behavior as $\rho \to 0$ (see (3.2)). Since a complete orthonormal system of $H^2_\sigma(\Omega)$ is given by normalizing monomials, it follows that $K^{\mathrm{S}}(z) = \sum |z^\alpha|^2/||\zeta^\alpha||^2$, where the summation runs over $\alpha \in \mathbb{N}_0^2$. Observe by (3.1) that

$$||\zeta^\alpha||^2 = \int_{\partial\Omega} |\zeta^\alpha|^2\, \sigma(\zeta) = (2\pi)^2 \int_I e^{-2\beta\cdot\Xi} \left\{\frac{f''(\xi)}{16}\right\}^{1/3} d\xi,$$

where $\beta = \beta(\alpha) := \alpha + (2/3, 2/3)$ and $\Xi := (\xi, f(\xi))$. We then get

$$K_x(\rho) = \frac{16}{f''(x)} \sum_{\alpha\in\mathbb{N}_0^2} \frac{\exp[-2\beta_2\rho]}{\widetilde{D}_x(2\beta)} = \sum_{a=0}^{\infty} \widetilde{M}_x(2b)\, e^{-2b\rho} \quad \text{with} \;\; b := a + \frac{2}{3}\,,$$

where $\widetilde{D}_x(2\beta) := 4\displaystyle\int_I e^{-2\beta\cdot(\Xi - X)}\big\{f''(\xi)/f''(x)\big\}^{1/3} d\xi$ with $X := (x, f(x))$, and thus $\widetilde{M}_x(2\beta_2)$ is the sum of $16/\{f''(x)\,\widetilde{D}_x(2\beta)\}$ with respect to $\alpha_1 \in \mathbb{N}_0$. Noting that

$$\sum_{a=0}^{\infty} e^{-2b\rho} = \frac{e^{-\rho/3}}{2\sinh\rho} = \frac{1}{2\rho}\cdot\frac{\rho\, e^{-\rho/3}}{\sinh\rho} = \frac{1}{2\rho}\,(1 + \cdots),$$

which admits termwise indefinite integration and differentiation near $\rho = +0$, we have $D_\rho^k \sum e^{-2b\rho} = \sum (-2b)^k e^{-2b\rho}$ for every $k \in \mathbb{Z}$, where $D_\rho := d/d\rho$. One may thus expect

$$K_x(\rho) = M_x(-D_\rho)\Big(-2\, D_\rho \sum_{a=0}^{\infty} e^{-2b\rho}\Big) \quad \text{with} \quad M_x(2b) := \frac{1}{4b}\, \widetilde{M}_x(2b)$$

at least formally. The expression (3.2) suggests that $M_x(-D_\rho)$ is a pseudodifferential operator of the form $1 + m_1(x) D_\rho^{-1} + m_2(x) D_\rho^{-2} + \cdots$, so that the desired asymptotic expansion will be given by $M_x(-D_\rho)$ applied to the singularity $1/\rho^2$ which corresponds to the spherical boundary. Thus, $M_x(-D_\rho)$ will be obtained from the Fourier integral operator which transplants the boundary $\partial\Omega$ near the reference point to a spherical model (cf. Section 2). In what follows, we regard $b = \beta_2 > 0$ as a continuous variable.

The symbol $M_x(2b)$ is computed by constructing a smooth family of base domains B_x^t for $0 \leqq t \leqq 1$ in such a way that $B_x^1 = B$ and that B_x^0 is the quadratic spherical model which osculates B at the reference point $(x, f(x))$. In order to state it more precisely, we first consider a translation $(\xi_x, \eta_{f(x)}) := (\xi - x, \eta - f(x))$, so that the reference point is the origin in the new coordinates. Dropping the subscripts in $(\xi_x, \eta_{f(x)})$, we next set

$$\text{(3.4)} \qquad Q_x(\xi) := f(x) + f'(x)\,\xi + \frac{f''(x)}{2}\,\xi^2, \qquad R_x^t(\xi) := \frac{1}{t^2}\,\{f(x + t\xi) - Q_x(t\xi)\}.$$

Then, $B_x^t := \{\eta > f_x^t(\xi)\}$ with $f_x^t(\xi) := Q_x(\xi) + R_x^t(\xi)$ for $\xi \in I_x^t := (1/t)(I - x)$ has the required properties. Observe that $(f_x^t)''(\xi) = f''(x + t\xi)$, so that the dependence on the parameter t of the surface element is fairly simple. Let $M_x^t(2b)$ and $D_x^t(2\beta)$ denote $M_x(2b)$ and $D_x(2\beta)$ with B_x^t in place of B, where $D_x(2\beta) := 4\beta_2 \widetilde{D}_x(2\beta)$. Then the symbol $M_x(2b)$ as a formal series will be given by the Taylor expansion of $M_x^t(2b)$ about $t = 0$ evaluated at $t = 1$. This idea goes back to Boichu-Coeuré [BC].

In order to compute $M_x^t(2b)$ at $t = 1$ asymptotically, it is convenient to introduce $\mu = \mu_x(2\beta) := \beta_1/\beta_2 + f'(x)$, which measures the deviation of β from the inward normal of ∂B at the reference point. We then have $M_x^t(2\beta_2) = \sum_{\alpha_1=0}^{\infty} \{16/f''(x)\}/D_x^t(2\beta)$ with $D_x^t(2\beta) = 16\,\beta_2 \int_{I_x^t} \exp\left[-2\beta_2\, \Phi_x(\xi,\mu)\right]\, A_x^t(\xi; 2\beta_2)\, d\xi$, where

$$\text{(3.5)} \quad \Phi_x(\xi,\mu) := \mu\xi + \frac{1}{2} f''(x)\,\xi^2, \quad A_x^t(\xi; 2b) := \exp[-2b\, R_x^t(\xi)] \left\{\frac{f''(x + t\xi)}{f''(x)}\right\}^{1/3}.$$

Let us identify $A_x^t(\xi; 2b)$ with its Taylor expansion about $t = 0$. Then, the localization of ∂B will permit us to replace the half line I_x^t by $\mathbb{R}$ in the expression of $D_x^t(2\beta)$. After a change of scale given by $\tau := t/\sqrt{2b}$, $\lambda := \sqrt{2b}\,\mu$ and $\zeta := \sqrt{2b}\,\xi$, we are led to

$$\frac{1/8}{\sqrt{2\beta_2}}\, D_x^t(2\beta) \sim \int_{\mathbb{R}} \exp[-\Phi_x(\zeta,\lambda)]\, A_x^\tau(\zeta; 1)\, d\zeta = A_x^\tau(-D_\lambda; 1) \int_{\mathbb{R}} \exp[-\Phi_x(\zeta,\lambda)]\, d\zeta,$$

where $D_\lambda := \partial/\partial\lambda$. That is, we first replace λ by $\widetilde{\lambda} \in \mathbb{R}$, do the operation with respect to $\widetilde{\lambda}$, and then set $\lambda = \widetilde{\lambda}$. Evaluating the integral in the last expression and recalling that the Taylor expansion of $A_x^\tau(\zeta; 1)$ about $\tau = 0$ takes the form $1 + \cdots$, we see that

$$M_x^t(2\beta_2) \sim \frac{C_1}{\sqrt{2\beta_2}} \sum_{\alpha_1=0}^{\infty} \widetilde{A}_x^\tau(-D_\lambda) \exp[-\lambda^2/C_2] \tag{3.6}$$

with positive constants $C_1 := 2/\sqrt{2\pi f''(x)}$ and $C_2 := 2f''(x)$, where $\widetilde{A}_x^\tau(-D_\lambda)$ defined by $\widetilde{A}_x^\tau(-D_\lambda) \exp[-\lambda^2/C_2] = 1/\left\{A_x^\tau(-D_\lambda; 1) \exp[\lambda^2/C_2]\right\}$ is a formal power series of τ of the form $1+\cdots$ such that the coefficients are polynomials of $-D_\lambda$. Setting $t = 1$ and thus $\tau = 1/\sqrt{2b}$ in (3.6), let us consider the dependence on τ of the right side.

Lemma 3.3. *Given $E \in \mathcal{S}(\mathbb{R})$, $C_3 \in \mathbb{R}$ and $C_4 < 0$, consider the Taylor expansion, about $\tau = +0$, of $\tau \sum_{k=0}^{\infty} E(\lambda)$ with $\lambda := \tau(k + C_3) + C_4/\tau$. Then, all the coefficients vanish except for the constant term given by $\widehat{E}(0)$, where $\widehat{E}(\zeta) := \int_{\mathbb{R}} e^{-i\widetilde{\lambda}\zeta} E(\widetilde{\lambda})\, d\widetilde{\lambda}$.*

Proof. Applying the Fourier inversion formula and then changing scale, we have

$$E_\tau := \tau \sum_{k=0}^{\infty} E(\lambda) = \int_{\mathbb{R}} \Delta_\tau(\zeta)\, \widehat{E}(\zeta/\tau)\, d\zeta, \quad \text{where} \quad \Delta_\tau(\zeta) := \frac{1}{2\pi} \sum_{k=0}^{\infty} e^{i\lambda\zeta/\tau}.$$

It then follows that only the singularity of $\Delta_\tau(\zeta)$ at $\zeta = 0$ is concerned, and thus it suffices to show that $E_\tau \to \widehat{E}(0)$ as $\tau \to 0$. This is obvious, because E_τ is an approximate Riemann sum of the integral $\widehat{E}(0)$.

It follows from Lemma 3.3 that (3.6) yields

$$M_x(2b) \sim C_1 \int_{\mathbb{R}} \widetilde{A}_x^\tau(-D_\lambda) \exp[-\lambda^2/C_2]\, d\lambda = 2\widetilde{A}_x^\tau(0) \quad \text{with} \quad \tau = \frac{1}{\sqrt{2b}}\,. \tag{3.7}$$

We have thus arrived at the Taylor expansion of the symbol $M_x(2b)$ with respect to $\tau^2 = 1/(2b)$. This is because, in the expansion of $\widetilde{A}_x^\tau(-D_\lambda; 1)$ and thus of $\widetilde{A}_x^\tau(-D_\lambda)$, the coefficients of the odd (resp. even) powers of τ are polynomials of odd (resp. even) powers of $-D_\lambda$. Let us summarize the first expression of (3.7) as follows.

Proposition 3. *Under the same assumption as in Theorem 3, the asymptotic expansion of $\widetilde{\varphi}^{\mathrm{S}}/\rho^2 + \widetilde{\psi}^{\mathrm{S}} \log\rho$ about the base reference point $(x, f(x))$ is given by $M_x(-\partial/\partial\rho)$ applied to $1/\rho^2$, where $M_x(1/\tau^2)$ is a formal power series of τ^2 given by*

$$M_x(1/\tau^2) = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \exp[-\Phi_x(\xi, \lambda)]\, A_x^\tau(\xi; 1)\, d\xi \right\}^{-1} d\lambda.$$

Here $\Phi_x(\xi, \lambda)$ and $A_x^\tau(\xi; 1)$ are defined by (3.5) *with* (3.4)*.*

Proof. It remains to justify the heuristic argument above; this will be done elsewhere in a more general setting. Here we shall rather follow the proof in the case of the Bergman kernel given in [BC] and supplemented in [N2]. We first (micro-)localize the index set of summation, $\mathbb{N}_0^2 + (2/3, 2/3)$, to a conic neighborhood of $\mu = \beta_1/\beta_2 + f'(x)$. We then consider the Taylor expansion corresponding to (3.4), where the remainder estimate amounts to the localization of ∂B around the reference point $(x, f(x))$. There is no essential difficulty in translating the argument for the Bergman kernel to the present case, because of the strict pseudoconvexity assumption on $\partial\Omega$ at points (z_1, z_2) satisfying $z_1 z_2 = 0$.

Once Proposition 3 is established, the proof of Theorem 3 is easy, though actual computation is very long. By virtue of the second expression of (3.7), we can write down explicitly the coefficients of $M_x(D_\rho) = m_0(x) + m_1(x)\, D_\rho^{-1} + \cdots$ in terms of $f^{(2+j)}(x)$ with $j \in \mathbb{N}_0$. Using the relation $d/dx = p(v)\, d/dv$ on the boundary ∂B, we get an expression of $m_k(x)$ in terms of $p^{(j)}(v)$ with $0 \leqq j \leqq 2k$. We thus first obtain $\varphi_0^{\mathrm{S}}(v) = m_0(x) = 1$ and $\varphi_1^{\mathrm{S}}(v) = -m_1(x) = p''(v)/6$. Recalling an elementary equality

$$D_\rho^{-1}(\rho^k \log \rho) = \frac{\rho^{k+1}}{k+1}\Big(\log \rho - \frac{1}{k+1}\Big) \sim \frac{\rho^{k+1}}{k+1} \log \rho \qquad \text{for} \quad k \in \mathbb{N}_0 ,$$

we get $\psi_k^{\mathrm{S}}(v) = -m_{k+2}(x)/k!$ for $k \in \mathbb{N}_0$. This yields the desired expression for $\psi_k^{\mathrm{S}}(v)$ with $k = 0, 1, 2$.

3.4. Special domains of higher dimension. Let us describe a method of proving Theorem 1 and Remark 1 for $n \geqq 3$, by using bounded strictly pseudoconvex complete Reinhardt domains Ω in $\mathbb{C}^n$ with C^∞ smooth boundary. For our purpose, it suffices to consider Ω such that the base domain $B = \{\rho := y - f(x) > 0\}$ is locally given near a point $(x^0, y^0) \in \partial B$ by $f(x) = f_1(x_1) + \cdots + f_{n-1}(x_{n-1})$, so that the hodograph variables are given there by $v = (v_1, \cdots, v_{n-1})$ and $p = \operatorname{diag}(p_1, \cdots, p_{n-1})$, where $v_j := f_j'(x_j) < 0$ and $p_j(v_j) := f_j''(x_j) > 0$. Thus,

$$\sigma_{\partial B} = \frac{dv_1 \wedge \cdots \wedge dv_{n-1}}{4^n\, J[\rho]^{n/(n+1)}} \ , \quad \text{where} \quad J[\rho] = \frac{p_1(v) \cdots p_{n-1}(v)}{4^n |z_1 \cdots z_n|^2} = \frac{\det p(v)}{4^n\, |\det E_-'(w)|^2} \, .$$

In order to state results corresponding to Theorem 3 with Lemmas 3.1 and 3.2, it is convenient to introduce

$$e_1 := \sum_{j=1}^{n-1} p_j'' \, , \quad e_{21} := \sum_{j=1}^{n-1} (p_j\, p_j''')', \quad e_{22} := \sum_{j=1}^{n-1} (p_j'')^2, \quad e_{23} := \sum_{j \neq k} p_j''\, p_k'',$$

so that $e_1^2 = e_{22} + e_{23}$. Recalling near $\partial\Omega \cap \{z_1 \cdots z_n \neq 0\}$ that

$$K^{\mathrm{S}}(z) = \frac{(n-1)!}{\pi^n}\, J[\rho](z)^{n/(n+1)} \left[\frac{\widetilde{\varphi}^{\mathrm{S}}(v, \rho)}{\rho^n} + \widetilde{\psi}^{\mathrm{S}}(v, \rho) \log \rho \right] , \quad \widetilde{\varphi}^{\mathrm{S}}, \widetilde{\psi}^{\mathrm{S}} \in C^\infty(\rho \geqq 0),$$

we write $\widetilde{\varphi}^{\mathrm{S}}(v, \rho) = \varphi_0^{\mathrm{S}}(v) + \varphi_1^{\mathrm{S}}(v)\, \rho + \cdots + \varphi_{n-1}^{\mathrm{S}}(v)\, \rho^{n-1} + O(\rho^n)$; then we first have

$$\varphi_0^{\mathrm{S}} = 1, \quad \varphi_1^{\mathrm{S}} = \frac{1}{2\,(n+1)}\, e_1, \quad \varphi_2^{\mathrm{S}} = \frac{1}{6\,(n^2-1)}\, e_{21} + \frac{n-1}{8\,(n+1)^2\,(n-2)}\, e_{23} \, .$$

The corresponding result for the Bergman kernel is, under obvious notation,

$$\varphi_0^{\mathrm{B}} = 1, \quad \varphi_1^{\mathrm{B}} = \frac{1}{2\,n}\,e_1, \quad \varphi_2^{\mathrm{B}} = \frac{1}{n\,(n-1)}\left(\frac{1}{6}\,e_{21} + \frac{1}{8}\,e_{23}\right).$$

Both are valid near the reference point (x^0, y^0). These provide the desired results when combined with $r^{\mathrm{F}} = J[\rho]^{-1/(n+1)}\,\widetilde{r}^{\mathrm{F}}$ and $||A^0_{2\bar{2}}||^2 = J[\rho]^{2/(n+1)}\alpha_{22}$, where

$$\widetilde{r}^{\mathrm{F}} := \rho - \frac{e_1}{2\,n\,(n+1)}\,\rho^2 + \frac{-n\,(n+1)\,e_{21} + (n^2-1)\,e_{22} - e_{23}}{6\,(n-1)\,n^2\,(n+1)^2}\,\rho^3 + O(\rho^4),$$
$$\alpha_{22} := \frac{1}{16\,n\,(n+1)}\,\{(n-2)\,(n-1)\,e_{22} + 2\,e_{23}\}\,.$$

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