ON THE MODULI B-DIVISORS OF LC-TRIVIAL FIBRATIONS

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Abstract. Roughly speaking, by using the semi-stable minimal model program, we prove that the moduli part of an lc-trivial fibration coincides with that of a klt-trivial fibration induced by adjunction after taking a suitable generically finite cover. As an application, we obtain that the moduli part of an lc-trivial fibration is b-nef and abundant by Ambro's result on klt-trivial fibrations.

Contents

1. Introduction

In this paper, we prove the following theorem. More precisely, we reduce Theorem 1.1 to Ambro's result (see [A2, Theorem 3.3]) by using the semi-stable minimal model program (see, for example, [F7]). For a related result, see [Fl, Theorem 1.4].

Theorem 1.1 (cf. [A2, Theorem 3.3]). *Let $f : X \to Y$ be a projective surjective morphism between normal projective varieties with connected fibers. Assume that (X, B) is log canonical and $K_X + B \sim_{\mathbb{Q},Y} 0$. Then the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}$ is b-nef and abundant.*

Let us recall the definition of *b-nef and abundant* $\mathbb{Q}$-b-divisors.

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Definition 1.2 ([A2, Definition 3.2])**.** A $\mathbb{Q}$-b-divisor $\mathbf{M}$ of a normal complete algebraic variety Y is called *b-nef and abundant* if there exists a proper birational morphism $Y' \to Y$ from a normal variety Y', endowed with a proper surjective morphism $h : Y' \to Z$ onto a normal variety Z with connected fibers, such that:

(1) $\mathbf{M}_{Y'} \sim_{\mathbb{Q}} h^*H$, for some nef and big $\mathbb{Q}$-divisor H of Z;
(2) $\mathbf{M} = \overline{\mathbf{M}_{Y'}}$.

Let us quickly explain the idea of the proof of Theorem 1.1. We assume that the pair (X, B) in Theorem 1.1 is dlt for simplicity. Let W be a log canonical center of (X, B) which is dominant onto Y and is minimal over the generic point of Y. We set $K_W + B_W = (K_X + B)|_W$ by adjunction. Then we have $K_W + B_W \sim_{\mathbb{Q},Y} 0$. Let $h : W \to Y'$ be the Stein factorization of $f|_W : W \to Y$. Note that (W, B_W) is klt over the generic point of Y'. We prove that the moduli part $\mathbf{M}$ of $f : (X, B) \to Y$ coincides with the moduli part $\mathbf{M}^{\min}$ of $h : (W, B_W) \to Y'$ after taking a suitable generically finite base change by using the semi-stable minimal model program. We do not need the *mixed* period map nor the infinitesimal *mixed* Torelli theorem (see [A2, Section 2] and [SSU]) for the proof of Theorem 1.1. We just reduce the problem on lc-trivial fibrations to Ambro's result on klt-trivial fibrations, which follows from the theory of period maps. Our proof of Theorem 1.1 partially answers the questions in [Ko, 8.3.8 (Open problems)].

It is conjectured that $\mathbf{M}$ is b-semi-ample (see, for example, [A1, 0. Introduction], [PS, Conjecture 7.13.3], [Fl], and [BC]). The b-semi-ampleness of the moduli part has been proved only for some special cases (see, for example, [Ka], [F2], and [PS, Section 8]). See also Remark 4.1 below.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. We will make use of the standard notation as in [F8].

2. Preliminaries

Throughout this paper, we do not use $\mathbb{R}$-divisors. We only use $\mathbb{Q}$-divisors.

2.1 (Pairs)**.** A pair (X, B) consists of a normal variety X over $\mathbb{C}$ and a $\mathbb{Q}$-divisor B on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. A pair (X, B)

is called *subklt* (resp. *sublc*) if for any projective birational morphism $g : Z \to X$ from a normal variety Z, every coefficient of B_Z is < 1 (resp. ≤ 1) where $K_Z + B_Z := g^*(K_X + B)$. A pair (X, B) is called *klt* (resp. *lc*) if (X, B) is subklt (resp. sublc) and B is effective.

Let (X, B) be a sublc pair and let W be a closed subset of X. Then W is called a *log canonical center* of (X, B) if there are a projective birational morphism $g : Z \to X$ from a normal variety Z and a prime divisor E on Z such that $\mathrm{mult}_E B_Z = 1$ and that $g(E) = W$.

In this paper, we use the notion of *b-divisors* introduced by Shokurov. For details, we refer to [C, 2.3.2] and [F9, Section 3].

2.2 (Canonical b-divisors)**.** Let X be a normal variety and let ω be a top rational differential form of X. Then (ω) defines a b-divisor $\mathbf{K}$. We call $\mathbf{K}$ the *canonical b-divisor* of X.

2.3 ($\mathbf{A}(X, B)$ and $\mathbf{A}^*(X, B)$)**.** The *discrepancy b-divisor* $\mathbf{A} = \mathbf{A}(X, B)$ of a pair (X, B) is the $\mathbb{Q}$-b-divisor of X with the trace $\mathbf{A}_Y$ defined by the formula

$$K_Y = f^*(K_X + B) + \mathbf{A}_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. Similarly, we define $\mathbf{A}^* = \mathbf{A}^*(X, B)$ by

$$\mathbf{A}^*_Y = \sum_{a_i > -1} a_i A_i$$

for

$$K_Y = f^*(K_X + B) + \sum a_i A_i,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. Note that $\mathbf{A}(X, B) = \mathbf{A}^*(X, B)$ when (X, B) is subklt.

By the definition, we have $\mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil) \simeq \mathcal{O}_X$ if (X, B) is lc (see [F9, Lemma 3.19]). We also have $\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil) \simeq \mathcal{O}_X$ when (X, B) is klt.

2.4 (b-nef and b-semi-ample $\mathbb{Q}$-b-divisors)**.** Let X be a normal variety and let $X \to S$ be a proper surjective morphism onto a variety S. A $\mathbb{Q}$-b-divisor $\mathbf{D}$ of X is *b-nef over S* (resp. *b-semi-ample over S*) if there exists a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$ and $\mathbf{D}_{X'}$ is nef (resp. semi-ample) relative to the induced morphism $X' \to S$.

2.5. Let $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor on a normal variety, where D_i is a prime divisor for every i, $D_i \neq D_j$ for $i \neq j$, and $d_i \in \mathbb{Q}$ for every i. Then we set

$$D^{\geq 0} = \sum_{d_i \geq 0} d_i D_i.$$

3. A quick review of lc-trivial fibrations

In this section, we quickly recall some basic definitions and results on *klt-trivial fibrations* and *lc-trivial fibrations*.

Definition 3.1 (Klt-trivial fibrations)**.** A *klt-trivial fibration* $f : (X, B) \to Y$ consists of a proper surjective morphism $f : X \to Y$ between normal varieties with connected fibers and a pair (X, B) satisfying the following properties:

(1) (X, B) is subklt over the generic point of Y;
(2) $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}(X, B)\rceil) = 1$;
(3) There exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that

$$K_X + B \sim_{\mathbb{Q}} f^*D.$$

Note that Definition 3.1 is nothing but [A1, Definition 2.1], where a klt-trivial fibration is called an lc-trivial fibration. So, our definition of lc-trivial fibrations in Definition 3.2 is different from the original one in [A1, Definition 2.1].

Definition 3.2 (Lc-trivial fibrations)**.** An *lc-trivial fibration* $f : (X, B) \to Y$ consists of a proper surjective morphism $f : X \to Y$ between normal varieties with connected fibers and a pair (X, B) satisfying the following properties:

(1) (X, B) is sublc over the generic point of Y;
(2) $\operatorname{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, B)\rceil) = 1$;
(3) There exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that

$$K_X + B \sim_{\mathbb{Q}} f^*D.$$

In Section 4, we sometimes take various base changes and construct the induced lc-trivial fibrations and klt-trivial fibrations. For the details, see [A1, Section 2].

3.3 (Induced lc-trivial fibrations by base changes)**.** Let $f : (X, B) \to Y$ be a klt-trivial (resp. an lc-tirivial) fibration and let $\sigma : Y' \to Y$ be a generically finite morphism. Then we have an induced klt-trivial (resp. lc-trivial) fibration $f' : (X', B_{X'}) \to Y'$, where $B_{X'}$ is defined by $\mu^*(K_X + B) = K_{X'} + B_{X'}$:

$$\begin{CD} (X', B_{X'}) @>{\mu}>> (X, B) \\ @V{f'}VV @VV{f}V \\ Y' @>>{\sigma}> Y, \end{CD}$$

Note that X' is the normalization of the main component of $X \times_Y Y'$. We sometimes replace X' with X'' where X'' is a normal variety such that there is a proper birational morphism $\varphi : X'' \to X'$. In this case, we set $K_{X''} + B_{X''} = \varphi^*(K_{X'} + B_{X'})$.

Let us explain the definitions of the *discriminant* and *moduli* $\mathbb{Q}$-b-divisors.

3.4 (Discriminant $\mathbb{Q}$-b-divisors and moduli $\mathbb{Q}$-b-divisors)**.** Let $f : (X, B) \to Y$ be an lc-trivial fibration as in Definition 3.2. Let P be a prime divisor on Y. By shrinking Y around the generic point of P, we assume that P is Cartier. We set

$$b_P = \max\left\{ t \in \mathbb{Q} \;\middle|\; \begin{array}{l} (X, B + tf^*P) \text{ is sublc over} \\ \text{the generic point of } P \end{array} \right\}$$

and set

$$B_Y = \sum_P (1 - b_P)P,$$

where P runs over prime divisors on Y. Then it is easy to see that B_Y is a well-defined $\mathbb{Q}$-divisor on Y and is called the *discriminant* $\mathbb{Q}$*-divisor* of $f : (X, B) \to Y$. We set

$$M_Y = D - K_Y - B_Y$$

and call M_Y the *moduli* $\mathbb{Q}$*-divisor* of $f : (X, B) \to Y$. Let $\sigma : Y' \to Y$ be a proper birational morphism from a normal variety Y' and let $f' : (X', B_{X'}) \to Y'$ be the induced lc-trivial fibration by $\sigma : Y' \to Y$ (see 3.3). We can define $B_{Y'}$, $K_{Y'}$ and $M_{Y'}$ such that $\sigma^*D = K_{Y'} + B_{Y'} + M_{Y'}$, $\sigma_*B_{Y'} = B_Y$, $\sigma_*K_{Y'} = K_Y$ and $\sigma_*M_{Y'} = M_Y$. Hence there exist a unique $\mathbb{Q}$-b-divisor $\mathbf{B}$ such that $\mathbf{B}_{Y'} = B_{Y'}$ for every $\sigma : Y' \to Y$ and a unique $\mathbb{Q}$-b-divisor $\mathbf{M}$ such that $\mathbf{M}_{Y'} = M_{Y'}$ for every $\sigma : Y' \to Y$. Note that $\mathbf{B}$ is called the *discriminant* $\mathbb{Q}$*-b-divisor* and that $\mathbf{M}$ is called the *moduli* $\mathbb{Q}$*-b-divisor* associated to $f : (X, B) \to Y$. We sometimes simply say that $\mathbf{M}$ is the *moduli part* of $f : (X, B) \to Y$.

For the basic properties of the discriminant and moduli $\mathbb{Q}$-b-divisors, see [A1, Section 2].

Let us recall the main theorem of [A1].

Theorem 3.5 (see [A1, Theorem 2.7])**.** *Let $f : (X, B) \to Y$ be a klt-trivial fibration and let $\pi : Y \to S$ be a proper morphism. Let $\mathbf{B}$ and $\mathbf{M}$ be the induced discriminant and moduli $\mathbb{Q}$-b-divisors of f. Then,*

(1) $\mathbf{K} + \mathbf{B}$ *is $\mathbb{Q}$-b-Cartier, that is, there exists a proper birational morphism $Y' \to Y$ from a normal variety Y' such that $\mathbf{K} + \mathbf{B} = \overline{K_{Y'} + \mathbf{B}_{Y'}}$,*
(2) $\mathbf{M}$ *is b-nef over S.*

Theorem 3.5 has some important applications, see, for example, [F6] and [F9].

By modifying the arguments in [A1, Section 5] suitably with the aid of [F4, Theorems 3.1, 3.4, and 3.9] (see also [FF]), we can generalize Theorem 3.5 as follows.

Theorem 3.6. *Let $f : (X, B) \to Y$ be an lc-trivial fibration and let $\pi : Y \to S$ be a proper morphism. Let $\mathbf{B}$ and $\mathbf{M}$ be the induced discriminant and moduli $\mathbb{Q}$-b-divisors of f. Then,*

1. $\mathbf{K} + \mathbf{B}$ *is $\mathbb{Q}$-b-Cartier,*
2. $\mathbf{M}$ *is b-nef over S.*

Theorem 3.5 is proved by using the theory of variations of Hodge structure. On the other hand, Theorem 3.6 follows from the theory of variations of *mixed* Hodge structure. We do not adopt the formulation in [F3, Section 4] (see also [Ko, 8.5]) because the argument in [A1] suits our purposes better. For the reader's convenience, we contain the main ingredient of the proof of Theorem 3.6, which easily follows from [F4, Theorems 3.1, 3.4, and 3.9] (see also [FF]).

Theorem 3.7 (cf. [A1, Theorem 4.4])**.** *Let $f : X \to Y$ be a projective morphism between algebraic varieties. Let Σ_X (resp. Σ_Y) be a simple normal crossing divisor on X (resp. Y) such that f is smooth over $Y \setminus \Sigma_Y$, Σ_X is relatively normal crossing over $Y \setminus \Sigma_Y$, and $f^{-1}(\Sigma_Y) \subset \Sigma_X$. Assume that f is semi-stable in codimension one. Let D be a simple normal crossing divisor on X such that $\operatorname{Supp} D \subset \Sigma_X$ and that every irreducible component of D is dominant onto Y. Then the following properties hold.*

1. *$R^p f_* \omega_{X/Y}(D)$ is a locally free sheaf on Y for every p.*
2. *$R^p f_* \omega_{X/Y}(D)$ is semi-positive for every p.*
3. *Let $\rho : Y' \to Y$ be a projective morphism from a smooth variety Y' such that $\Sigma_{Y'} = \rho^{-1}(\Sigma_Y)$ is a simple normal crossing divisor on Y'. Let $\pi : X' \to X \times_Y Y'$ be a resolution of the main component of $X \times_Y Y'$ such that π is an isomorphism over $Y' \setminus \Sigma_{Y'}$. Then we obtain the following commutative diagram:*

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\rho"'] & Y. \end{tikzcd}$$

Assume that f' is projective, D' is a simple normal crossing divisor on X' such that D' coincides with $D \times_Y Y'$ over $Y' \setminus \Sigma_{Y'}$, *and every stratum of D' is dominant onto Y'. Then there exists a natural isomorphism*

$$\rho^*(R^p f_*\omega_{X/Y}(D)) \simeq R^p f'_*\omega_{X'/Y'}(D')$$

which extends the base change isomorphism over $Y \setminus \Sigma_Y$ for every p.

Remark 3.8. For the proof of Theorem 3.6, Theorem 3.7 for $p = 0$ is sufficient. Note that all the local monodromies on $R^q(f_0)_*\mathbb{C}_{X_0\setminus D_0}$ around Σ_Y are unipotent for every q because f is semi-stable in codimension one, where $X_0 = f^{-1}(Y \setminus \Sigma_Y)$, $D_0 = D|_{X_0}$, and $f_0 = f|_{X_0\setminus D_0}$.

We add a remark on the proof of Theorem 3.6 for the reader's convenience.

Remark 3.9. We use the notation in [A1, Lemma 5.2]. We only assume that (X, B) is sublc over the generic point of Y in [A1, Lemma 5.2]. We write

$$B = \sum_{i\in I} d_i B_i$$

where B_i is a prime divisor for every i and $B_i \neq B_j$ for $i \neq j$. We set

$$J = \{i \in I \mid B_i \text{ is dominant onto } Y \text{ and } d_i = 1\}$$

and set

$$D = \sum_{i\in J} B_i.$$

In Ambro's original setting in [A1, Lemma 5.3], we have $D = 0$ because (X, B) is subklt over the generic point of Y. In the proof of [A1, Lemma 5.2 (4)], it is sufficient to consider

$$\widetilde{f}_*\omega_{\widetilde{X}/Y}(\pi^*D) = \bigoplus_{i=0}^{b-1} f_*\mathcal{O}_X(\lceil (1-i)K_{X/Y} - iB + D + if^*B_Y + if^*M_Y\rceil)\cdot\psi^i.$$

We leave the details as exercises for the reader.

The following theorem is a part of [A2, Theorem 3.3].

Theorem 3.10 (see [A2, Theorem 3.3])**.** *Let $f : (X, B) \to Y$ be a klt-trivial fibration such that Y is complete, the geometric generic fiber $X_{\overline{\eta}} = X \times \operatorname{Spec}\overline{\mathbb{C}(\eta)}$ is a projective variety, and $B_{\overline{\eta}} = B|_{X_{\overline{\eta}}}$ is effective, where η is the generic point of Y. Then the moduli $\mathbb{Q}$-b-divisor $\mathbf{M}$ is b-nef and abundant.*

4. Proof of Theorem 1.1

Let us give a proof of Theorem 1.1.

Proof of Theorem 1.1. By taking a dlt blow-up, we may assume that the pair (X, B) is $\mathbb{Q}$-factorial and dlt (see, for example, [F7, Section 4]). If (X, B) is klt over the generic point of Y, then Theorem 1.1 follows from [A2, Theorem 3.3] (see Theorem 3.10). Therefore, we may also assume that (X, B) is not klt over the generic point of Y. Let $\sigma_1 : Y_1 \to Y$ be a suitable projective birational morphism such that $\mathbf{M} = \overline{\mathbf{M}_{Y_1}}$ and $\mathbf{M}_{Y_1}$ is nef by Theorem 3.6. Let W be an arbitrary log canonical center of (X, B) which is dominant onto Y and is minimal over the generic point of Y. We set

$$K_W + B_W = (K_X + B)|_W$$

by adjunction (see, for example, [F5, 3.9]). By the construction, we have $K_W + B_W \sim_{\mathbb{Q},Y} 0$. We consider the Stein factorization of $f|_W : W \to Y$ and denote it by $h : W \to Y'$. Then $K_W + B_W \sim_{\mathbb{Q},Y'} 0$. It is obvious that $h : (W, B_W) \to Y'$ is a klt-trivial fibration. Let Y_2 be a suitable resolution of Y' which factors through $\sigma_1 : Y_1 \to Y$. By taking the base change by $\sigma_2 : Y_2 \to Y_1$, we obtain $\mathbf{M}_{Y_2} = \sigma_2^* \mathbf{M}_{Y_1}$ (see [A1, Proposition 5.5]). Note that the proof of [A1, Proposition 5.5] works for lc-trivial fibrations by some suitable modifications. By the construction, on the induced lc-trivial fibration $f_2 : (X_2, B_{X_2}) \to Y_2$, where X_2 is the normalization of the main component of $X \times_Y Y_2$, there is a log canonical center W_2 of (X_2, B_{X_2}) such that $f_2|_{W_2^\nu} : (W_2^\nu, B_{W_2^\nu}) \to Y_2$ is a klt-trivial fibration, which is birationally equivalent to $h : (W, B_W) \to Y'$. Note that $\nu : W_2^\nu \to W_2$ is the normalization, $K_{W_2^\nu} + B_{W_2^\nu} = \nu^*(K_{X_2} + B_{X_2})|_{W_2}$, and $f_2|_{W_2^\nu} = f_2|_{W_2} \circ \nu$. It is easy to see that

$$K_{Y_2} + \mathbf{M}_{Y_2} + \mathbf{B}_{Y_2} \sim_{\mathbb{Q}} K_{Y_2} + \mathbf{M}^{\min}_{Y_2} + \mathbf{B}^{\min}_{Y_2}$$

where $\mathbf{M}^{\min}$ and $\mathbf{B}^{\min}$ are the induced moduli and discriminant $\mathbb{Q}$-b-divisors of $f_2|_{W_2^\nu} : (W_2^\nu, B_{W_2^\nu}) \to Y_2$ such that

$$K_{W_2^\nu} + B_{W_2^\nu} \sim_{\mathbb{Q}} (f_2|_{W_2^\nu})^*(K_{Y_2} + \mathbf{M}^{\min}_{Y_2} + \mathbf{B}^{\min}_{Y_2}).$$

By replacing Y_2 birationally, we may further assume that $\mathbf{M}^{\min} = \overline{\mathbf{M}^{\min}_{Y_2}}$ by Theorem 3.5. By Theorem 3.10, we see that $\mathbf{M}^{\min}_{Y_2}$ is nef and abundant. Let $\sigma_3 : Y_3 \to Y_2$ be a suitable generically finite morphism such that the induced lc-trivial fibration $f_3 : (X_3, B_{X_3}) \to Y_3$ has a semi-stable resolution in codimension one (see, for example, [KKMS], [SSU, (9.1) Theorem], and [A1, Theorem 4.3]). Note that X_3 is the normalization of the main component of $X \times_Y Y_3$. On Y_3, we will see the following claim by using the semi-stable minimal model program.

Claim. *The following equality*

$$\mathbf{B}_{Y_3} = \mathbf{B}^{\min}_{Y_3}$$

holds.

Proof of Claim. By taking general hyperplane cuts, we may assume that Y_3 is a curve. We write

$$\mathbf{B}_{Y_3} = \sum_P (1-b_P)P \quad \text{and} \quad \mathbf{B}^{\min}_{Y_3} = \sum_P (1-b^{\min}_P)P.$$

Let $\varphi : (V, B_V) \to (X_3, B_{X_3})$ be a resolution of (X_3, B_{X_3}) with the following properties:

- $K_V + B_V = \varphi^*(K_{X_3} + B_{X_3})$;
- π^*Q is a reduced simple normal crossing divisor on V for every $Q \in Y_3$, where $\pi : V \to X_3 \to Y_3$;
- $\operatorname{Supp} \pi^*Q \cup \operatorname{Supp} B_V$ is a simple normal crossing divisor on V for every $Q \in Y_3$;
- π is projective.

Let Σ be a reduced divisor on Y_3 such that π is smooth over $Y_3 \setminus \Sigma$ and that $\operatorname{Supp} B_V$ is relatively normal crossing over $Y_3 \setminus \Sigma$. We consider the set of prime divisors $\{E_i\}$ where E_i is a prime divisor on V such that $\pi(E_i) \in \Sigma$ and

$$\operatorname{mult}_{E_i}(B_V + \sum_{P \in \Sigma} b_P \pi^*P)^{\geq 0} < 1.$$

We run a minimal model program with ample scaling with respect to

$$K_V + (B_V + \sum_{P \in \Sigma} b_P \pi^*P)^{\geq 0} + \varepsilon \sum_i E_i$$

over X_3 and Y_3 for some small positive rational number ε. Note that

$$(V, (B_V + \sum_P b_P \pi^*P)^{\geq 0} + \varepsilon \sum_i E_i)$$

is a $\mathbb{Q}$-factorial dlt pair because $0 < \varepsilon \ll 1$. It is easy to see that

$$K_V + (B_V + \sum_P b_P \pi^*P)^{\geq 0} + \varepsilon \sum_i E_i \sim_{\mathbb{Q}, Y_3} E \geq 0$$

for some effective $\mathbb{Q}$-divisor E on V. First we run a minimal model program with ample scaling with respect to

$$K_V + (B_V + \sum_P b_P \pi^*P)^{\geq 0} + \varepsilon \sum_i E_i \sim_{\mathbb{Q}, X_3} E \geq 0$$

over X_3. Note that every irreducible component of E which is dominant onto Y_3 is exceptional over X_3 by the construction. Thus, if E is dominant onto Y_3, then it is not contained in the relative movable cone over X_3. Therefore, after finitely many steps, we may assume that every irreducible component of E is contained in a fiber over Y_3 (see, for example, [F7, Theorem 2.2]). Next we run a minimal model program with ample scaling with respect to

$$K_V + (B_V + \sum_P b_P \pi^* P)^{\geq 0} + \varepsilon \sum_i E_i \sim_{\mathbb{Q}, Y_3} E \geq 0$$

over Y_3. We can easily see that E is not contained in the relative movable cone over Y_3 in the process of the minimal model program. Therefore, the minimal model program terminates at V' (see, for example, [F7, Theorem 2.2]). Note that all the components of $E + \sum_i E_i$ are contracted by the above minimal model programs. Thus, we have

$$K_{V'} + (B_{V'} + \sum_P b_P \pi'^* P)^{\geq 0} \sim_{\mathbb{Q}, Y_3} 0,$$

where $\pi' : V' \to Y_3$ and $B_{V'}$ is the pushforward of B_V on V'. Note that $B_{V'} + \sum_P b_P \pi'^* P$ is effective by the construction. Of course, we see that

$$(V', (B_{V'} + \sum_P b_P \pi'^* P)^{\geq 0}) = (V', B_{V'} + \sum_P b_P \pi'^* P)$$

is a $\mathbb{Q}$-factorial dlt pair. By the construction, the induced proper birational map

$$(V, B_V + \sum_P b_P \pi^* P) \dashrightarrow (V', B_{V'} + \sum_P b_P \pi'^* P)$$

over Y_3 is B-birational (see [F1, Definition 1.5]), that is, we have a common resolution

$$\begin{tikzcd} & Z \arrow[dl, "a"'] \arrow[dr, "b"] & \\ V \arrow[rr, dashed] & & V' \end{tikzcd}$$

over Y_3 such that

$$a^*(K_V + B_V + \sum_{P \in \Sigma} b_P \pi^* P) = b^*(K_{V'} + B_{V'} + \sum_{P \in \Sigma} b_P \pi'^* P).$$

Let S be any log canonical center of $(V', B_{V'} + \sum_P b_P \pi'^* P)$ which is dominant onto Y_3 and is minimal over the generic point of Y_3. Then

(S, B_S), where

$$K_S + B_S = (K_{V'} + B_{V'} + \sum_P b_P \pi'^* P)|_S,$$

is not klt but lc over every $P \in \Sigma$. This is because

$$B_{V'} + \sum_{P \in \Sigma} b_P \pi'^* P \geq \sum_{P \in \Sigma} \pi'^* P$$

by the construction. Let $g_3 : (W_3, B_{W_3}) \to Y_3$ be the induced klt-trivial fibration from $(W_2^\nu, B_{W_2^\nu}) \to Y_2$ by $\sigma_2 : Y_3 \to Y_2$. It is easy to see that there is a log canonical center S_0 of $(V', B_{V'} + \sum_P b_P \pi'^* P)$ which is dominant onto Y_3 and is minimal over the generic point of Y_3 such that there is a B-birational map

$$(W_3, B_{W_3} + \sum_{P \in \Sigma} b_P g_3^* P) \dashrightarrow (S_0, B_{S_0})$$

over Y_3, where

$$K_{S_0} + B_{S_0} = (K_{V'} + B_{V'} + \sum_{P \in \Sigma} b_P \pi'^* P)|_{S_0}$$

(cf. [F1, Claims (A_n) and (B_n) in the proof of Lemma 4.9]). This means that there is a common resolution

$$\begin{tikzcd} & T \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ W_3 \arrow[rr, dashed] & & S_0 \end{tikzcd}$$

over Y_3 such that

$$\alpha^*(K_{W_3} + B_{W_3} + \sum_P b_P g_3^* P) = \beta^*(K_{S_0} + B_{S_0}).$$

This easily implies that $b_P = b_P^{\min}$ for every $P \in \Sigma$. Therefore, we have $\mathbf{B}_{Y_3} = \mathbf{B}_{Y_3}^{\min}$. □

Then we obtain

$$\mathbf{M}_{Y_3} \sim_{\mathbb{Q}} \mathbf{M}_{Y_3}^{\min} = \sigma_3^* \mathbf{M}_{Y_2}^{\min}$$

because

$$K_{Y_3} + \mathbf{M}_{Y_3} + \mathbf{B}_{Y_3} \sim_{\mathbb{Q}} K_{Y_3} + \mathbf{M}_{Y_3}^{\min} + \mathbf{B}_{Y_3}^{\min}.$$

Thus, $\mathbf{M}_{Y_3}$ is nef and abundant. Since

$$\mathbf{M}_{Y_3} = \sigma_3^* \mathbf{M}_{Y_2} = \sigma_3^* \sigma_2^* \mathbf{M}_{Y_1},$$

$\mathbf{M}$ is b-nef and abundant. Moreover, by replacing Y_3 with a suitable generically finite cover, we have that $\mathbf{M}_{Y_3}$ and $\mathbf{M}_{Y_3}^{\min}$ are both Cartier (see [A1, Lemma 5.2 (5), Proposition 5.4, and Proposition 5.5]) and $\mathbf{M}_{Y_3} \sim \mathbf{M}^{\min}_{Y_3}$. □

We close this paper with a remark on the b-semi-ampleness of $\mathbf{M}$.

Remark 4.1 (b-semi-ampleness)**.** Let $f : X \to Y$ be a projective surjective morphism between normal projective varieties with connected fibers. Assume that (X, B) is log canonical and $K_X + B \sim_{\mathbb{Q},Y} 0$. Without loss of generality, we may assume that (X, B) is dlt by taking a dlt blow-up. We set

$$d_f(X,B) = \left\{ \dim W \;\middle|\; \begin{array}{l} W \text{ is a log canonical center of } (X,B) \\ \text{which is dominant onto } Y \end{array} \right\}.$$

If $d_f(X, B) \in \{0, 1\}$, then the b-semiampleness of M follows from [Ka] and [PS] by the proof of Theorem 1.1. Moreover, it is obvious that $M \sim_{\mathbb{Q}} 0$ when $d = 0$. Note that $\mathbf{M} \sim_{\mathbb{Q}} 0$ when $d_f(X, B) = 0$. It is obvious by the proof of Theorem 1.1.

References

[A1] F. Ambro, Shokurov's boundary property, J. Differential Geom. **67** (2004), no. 2, 229–255.

[A2] F. Ambro, The moduli b-divisor of an lc-trivial fibration, Compos. Math. **141** (2005), no. 2, 385–403.

[BC] C. Birkar, Y. Chen, On the moduli part of the Kawamata–Kodaira canonical bundle formula, preprint (2012).

[C] A. Corti, 3-fold flips after Shokurov, *Flips for* 3*-folds and* 4*-folds*, 18–48, Oxford Lecture Ser. Math. Appl., **35**, Oxford Univ. Press, Oxford, 2007.

[Fl] E. Floris, Inductive approach to effective b-semiampleness, preprint (2012).

[F1] O. Fujino, Abundance theorem for semi log canonical threefolds, Duke Math. J. **102** (2000), no. 3, 513–532.

[F2] O. Fujino, A canonical bundle formula for certain algebraic fiber spaces and its applications, Nagoya Math. J. **172** (2003), 129–171.

[F3] O. Fujino, Higher direct images of log canonical divisors and positivity theorems, preprint (2003). arXiv:math/0302073v1

[F4] O. Fujino, Higher direct images of log canonical divisors, J. Differential Geom. **66** (2004), no. 3, 453–479.

[F5] O. Fujino, What is log terminal?, *Flips for* 3*-folds and* 4*-folds*, 49–62, Oxford Lecture Ser. Math. Appl., **35**, Oxford Univ. Press, Oxford, 2007.

[F6] O. Fujino, On Kawamata's theorem, *Classification of algebraic varieties*, 305–315, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011.

[F7] O. Fujino, Semi-stable minimal model program for varieties with trivial canonical divisor, Proc. Japan Acad. Ser. A Math. Sci. **87** (2011), no. 3, 25–30.

[F8] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.

[F9] O. Fujino, Basepoint-free theorems: saturation, b-divisors, and canonical bundle formula, Algebra Number Theory **6** (2012), no. 4, 797–823.

[FF] O. Fujino, T. Fujisawa, Variations of mixed Hodge structure and semipositivity theorems, preprint (2012).

[Ka] Y. Kawamata, Subadjunction of log canonical divisors for a subvariety of codimension 2, *Birational algebraic geometry (Baltimore, MD, 1996)*, 79–88, Contemp. Math., **207**, Amer. Math. Soc., Providence, RI, 1997.

[KKMS] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. **339**. Springer-Verlag, Berlin-New York, 1973.

[Ko] J. Kollár, Kodaira's canonical bundle formula and adjunction, *Flips for* 3*-folds and* 4*-folds*, 134–162, Oxford Lecture Ser. Math. Appl., **35**, Oxford Univ. Press, Oxford, 2007

[PS] Yu. G. Prokhorov, V. V. Shokurov, Towards the second main theorem on complements, J. Algebraic Geom. **18** (2009), no. 1, 151–199.

[SSU] M.-H. Saito, Y. Shimizu, S. Usui, Variation of mixed Hodge structure and the Torelli problem, *Algebraic geometry, Sendai, 1985*, 649–693, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.

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