ON FATOU-JULIA DECOMPOSITIONS

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ABSTRACT. We propose a Fatou-Julia decomposition for holomorphic pseudosemigroups. It will be shown that the limit sets of finitely generated Kleinian groups, the Julia sets of mapping iterations and Julia sets of complex codimension-one regular foliations can be seen as particular cases of the decomposition. The decomposition is applied in order to introduce a Fatou-Julia decomposition for singular holomorphic foliations. In the well-studied cases, the decomposition behaves as expected.

INTRODUCTION

Iterations of rational mappings and actions of finitely generated Kleinian groups are typical dynamical systems on $\mathbb{C}P^1$. The notion of the Julia sets [15], [16] and the limit sets [14] are significant in their study. Sullivan's dictionary [18] says that they are in a close correspondence (see also [12, pp. 98–99]). More generally, the Julia sets are defined also for actions of semigroups generated by rational maps on $\mathbb{C}P^1$ (cf. [9], [19]). These complex dynamical systems are one-dimensional and on closed manifolds. Transversely holomorphic foliations of complex codimension one yield dynamical systems of a similar kind. Indeed, the holonomy pseudogroups of such foliations act on one-dimensional complex manifolds. If foliations are given on closed manifolds, then the holonomy pseudogroups have certain compactness called 'compact generation'. The notion of the Julia sets is also known for complex codimension-one transversely holomorphic foliations of closed manifolds [6], [8], [1]. One of the aims of this article is to give a unified definition of these Julia sets and limit sets. For this purpose, we will introduce a notion of compactly generated pseudosemigroups and a Fatou-Julia decomposition for them.

The Julia sets are also defined for entire maps on $\mathbb{C}$. In addition, if we consider transversely holomorphic foliations of open manifolds, or the regular parts of singular holomorphic foliations, then their holonomy pseudogroups are no longer compactly generated in general. We will introduce a Fatou-Julia decomposition also for non-compactly generated pseudosemigroups, which coincides with the classical one if iterations of entire maps on $\mathbb{C}$ are considered. The correspondence between typical dynamical systems and pseudo(semi)groups will be as follows.

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	pseudogroups	pseudosemigroups
compactly generated	finitely generated Kleinian Groups transversely holomorphic foliations of compact manifolds	rational functions on $\mathbb{C}P^1$
non-compactly generated	transversely holomorphic foliations of non-compact manifolds holomorphic singular foliations	entire functions on $\mathbb{C}$

In the first section, we will introduce pseudosemigroups (psg for short), which have appeared in a slightly different way, e.g. in [11], [13], [22]. In the second section, a Fatou-Julia decomposition of psg's and pseudogroups are defined and some fundamental properties are shown. Although pseudogroups generate psg's, decompositions for psg's and pseudogroups do not coincide in general. In the third section, compactly generated psg's are introduced. They are a version of compactly generated pseudogroups [8]. In the fourth section, Fatou-Julia decompositions of compactly generated psg's are discussed. It will be shown that Julia sets of compactly generated pseudogroups as psg's and the ones as pseudogroups coincide. It will be also shown that we can find Hermitian metrics adapted to actions of psg's on Fatou sets. In the last section, we will study Fatou-Julia decompositions for one-dimensional singular foliations.

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1. Pseudosemigroups

In order to compare Julia sets for pseudogroups with the Julia sets for mapping iterations, it is convenient to introduce a generalization of pseudogroups.

Definition 1.1. Let T and S be topological spaces. A *local continuous map from T to S* is a continuous map from an open set of T into S. A local continuous map from T to T is also called a local continuous map on T. If f is a local continuous map from T to S, then the *domain* and the *range* of f is denoted by $\operatorname{dom} f$ and $\operatorname{range} f$, respectively. If V is an open subset of $\operatorname{dom} f$, then the restriction of f to V is denoted by $f|_V$. Let f be a local continuous map from T to S.

1) If f is a homeomorphism (resp. diffeomorphism) from $\operatorname{dom} f$ to $\operatorname{range} f$, then f is called a *local homeomorphism* (resp. *local diffeomorphism*).

2) If T is equipped with a complex structure and if f is holomorphic, then f is called a *local holomorphic map*. If moreover f is a diffeomorphism, then f is called a *local biholomorphic diffeomorphism*.
3) Let f be a local map. Suppose that if $x \in \operatorname{dom} f$, then there is a neighborhood U of x such that $f|_U$ is a homeomorphism to the image. Then, f is said to be *étale*.
4) If f is a ramified covering from $\operatorname{dom} f$ to $\operatorname{range} f$, then f is called a *local ramified covering*.
5) Assume that f is a local holomorphic map on $\mathbb{C}$. The *set of singularities* of f is denoted by $\operatorname{Sing} f$, namely, $\operatorname{Sing} f = \{z \in U \mid f'(z) = 0\}$.
6) The germ of a local mapping f at a point $x \in \operatorname{dom} f$ is denoted by f_x.

Definition 1.2. Let T be a topological space and Γ be a family of local continuous mappings on T. Then, Γ is a *pseudosemigroup* (psg for short) if the following conditions are satisfied.

1) $\mathrm{id}_T \in \Gamma$, where id_T denotes the identity map of T.
2) If $\gamma \in \Gamma$, then $\gamma|_U \in \Gamma$ for any open subset U of $\operatorname{dom} \gamma$.
3) If $\gamma_1, \gamma_2 \in \Gamma$ and $\operatorname{range} \gamma_1 \subset \operatorname{dom} \gamma_2$, then $\gamma_2 \circ \gamma_1 \in \Gamma$.
4) Let U be an open subset of T and γ a local continuous mapping defined on U. If for each $x \in U$, there is an open neighborhood, say U_x, of x such that $\gamma|_{U_x}$ belongs to Γ, then $\gamma \in \Gamma$.

If in addition Γ consists of local homeomorphisms, then Γ is a *pseudogroup* if Γ satisfies 1), 2), 3) and the following conditions.

4') Let U be an open subset of T and γ a homeomorphism from U to $\gamma(U)$. If for each $x \in U$, there is an open neighborhood, say U_x, of x such that $\gamma|_{U_x}$ belongs to Γ, then $\gamma \in \Gamma$.

5) If $\gamma \in \Gamma$, then $\gamma^{-1} \in \Gamma$.

If Γ is either a psg or pseudogroup, then we set for $x \in T$

$$\Gamma_x = \{\gamma_x \mid x \in \operatorname{dom} \gamma\}.$$

By abuse of notation, an element of Γ_x is considered as an element of Γ defined on a neighborhood of x.

The terminology 'pseudosemigroup' has appeared in a slightly different way, e.g. in [13], [22], [11].

Definition 1.3. Let T be a topological space and G a set which consists of local continuous mappings on T. The psg *generated by G* is the smallest psg which contains G, and denoted by $\langle G \rangle$. If Γ is a pseudogroup, then we denote Γ_{psg} the psg generated by Γ. If there is a finite number of elements, say $f_1, \ldots, f_r$, of Γ such that $\Gamma = \langle f_1, \ldots, f_r \rangle$, then Γ is said to be *finitely generated*.

In what follows, the n-th iteration of a mapping f, if defined, is denoted by f^n, where $n \in \mathbb{Z}$. If $n = 0$, then f^0 is considered as the identity map.

Remark 1.4. One of differences between pseudo(semi)groups and (semi)groups is illustrated as follows. Let f be a rational mapping on $\mathbb{C}P^1$ and $\mathit{\Gamma}$ the psg generated by f. Let $U = V = \{z \in \mathbb{C} \mid |z| < 1+\varepsilon\}$ and $\varphi(z) = 1/z$. If we set $U_0 = V_0 = \{z \in \mathbb{C} \mid 1/(1+\varepsilon) < |z| < 1+\varepsilon\}$ and identify U_0 and V_0 by φ, then the resulting space is $\mathbb{C}P^1$. Let T be the disjoint union of U and V. Then, $\mathit{\Gamma}$, φ and φ^{-1} generate a psg $\widetilde{\mathit{\Gamma}}$ which acts on T. Let W be a small open subset of U_0 such that $f(W) \subset T$ and $f^2(\varphi(W)) \subset T$. By the condition 4), the mapping g on $W \cup \varphi(W)$ such that $g|_W = f$ and $g|_{\varphi(W)} = f^2$ belongs to $\widetilde{\mathit{\Gamma}}$. The psg $\widetilde{\mathit{\Gamma}}$ is obtained from $\mathit{\Gamma}$, indeed, $(\widetilde{\mathit{\Gamma}}, T)$ is equivalent to $(\mathit{\Gamma}, \mathbb{C}P^1)$ (see Definition 1.20). However, g cannot be realized as a single element of $\mathit{\Gamma}$ although W and $\varphi(W)$ correspond to the same region on $\mathbb{C}P^1$,

Remark 1.5. Let $(\mathit{\Gamma}, T)$ be a pseudogroup. Suppose that U is an open subset of T and that γ is a mapping defined on U. If the restriction of γ to a neighborhood of x belongs to $\mathit{\Gamma}$ for each $x \in U$, then it is always true that $\gamma \in \mathit{\Gamma}_{\mathrm{psg}}$ but $\gamma \in \mathit{\Gamma}$ if and only if γ is a homeomorphism. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and define $\gamma\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ by $\gamma(z) = e^{2\pi\sqrt{-1}\theta}z$, where we regard $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. Let $\mathit{\Gamma}$ be the pseudogroup generated by γ, namely, the smallest pseudogroup which contains γ. If we set $U = \{z \in \mathbb{C} \mid |z-1| < \varepsilon\}$, where ε is a small positive number, then $\gamma|_U \in \mathit{\Gamma}$. We set $V = \{z \in \mathbb{C} \mid \left|z - \sqrt{-1}\right| < \varepsilon\}$. We may assume that $U \cap V = \varnothing$, however, for a suitable choice of n, we have $\gamma^n(V) \cap U \neq \varnothing$. Let γ' be the mapping from $U \amalg V$ to $\mathbb{C}P^1$ by $\gamma'|_U = \gamma$ and $\gamma'|_V = \gamma^{n+1}$. Then $\gamma' \notin \mathit{\Gamma}$ because γ' is not a homeomorphism but $\gamma' \in \mathit{\Gamma}_{\mathrm{psg}}$.

Definition 1.6. Let $(\mathit{\Gamma}, T)$ be a psg. If T is a q-dimensional, possibly non-connected manifold and if $\mathit{\Gamma}$ consists of holomorphic mappings, then $(\mathit{\Gamma}, T)$ is called a *holomorphic pseudosemigroup on a q-dimensional complex manifold*.

Definition 1.7. A pseudosemigroup $\mathit{\Gamma}$ is said to be *étale* if $\mathit{\Gamma}$ consists of étale mappings. A holomorphic pseudosemigroup $\mathit{\Gamma}$ on a one-dimensional complex manifold is said to be *ramified* if $\mathit{\Gamma}$ is generated by local ramified coverings and holomorphic étale mappings.

Note that $\mathit{\Gamma}$ consists of open mappings if $\mathit{\Gamma}$ is étale or ramified.

Although we are interested in holomorphic pseudosemigroups on complex manifolds, we will discuss some more fundamental definitions and properties of psg's. Many of them are borrowed from those of pseudogroups which can be found in [7, §§ 1–2].

Definition 1.8. We denote by $\mathit{\Gamma}_0^\times$ the subset of $\mathit{\Gamma}$ which consists of invertible elements, namely,
$$\mathit{\Gamma}_0^\times = \{\gamma \in \mathit{\Gamma} \mid \gamma^{-1} \in \mathit{\Gamma}\}.$$
We denote by $\mathit{\Gamma}^\times$ the subset of $\mathit{\Gamma}$ which consists of locally invertible elements, namely,
$$\mathit{\Gamma}^\times = \langle \gamma \in \mathit{\Gamma} \mid \gamma^{-1} \in \mathit{\Gamma} \rangle = \langle \mathit{\Gamma}_0^\times \rangle.$$

Note that $\Gamma_0^\times$ is a pseudogroup, and $\Gamma^\times$ is an étale pseudosemigroup.

Definition 1.9. Let (Γ, T) be a psg. If $X \subset T$, then we set

$$\Gamma(X) = \{y \in T \mid \exists x \in X,\ \exists \gamma \in \Gamma \text{ s.t. } y = \gamma(x)\},$$
$$\Gamma^{-1}(X) = \bigcup_{\gamma \in \Gamma} \gamma^{-1}(X).$$

A subset X of T is said to be *forward invariant* if $\Gamma(X) = X$, *backward invariant* if $\Gamma^{-1}(X)$. If X is forward and backward invariant, then X is said to be *completely invariant* or Γ*-invariant*.

Definition 1.10. A subset X of T is said to be Γ*-connected* if X satisfies the following condition: if $X = \coprod_{\lambda \in \Lambda} X_\lambda$ is the decomposition of X into its connected components, then for any $\lambda, \lambda' \in \Lambda$, there exists a sequence $\lambda_0 = \lambda, \lambda_1, \dots, \lambda_r = \lambda'$ such that $\Gamma(X_{\lambda_i}) \cap X_{\lambda_i+1} \neq \varnothing$ holds for $i = 0, \dots, r-1$.

Remark 1.11. T is Γ-connected if and only if $\Gamma \backslash T$ is connected with the quotient topology. If $X \subset T$, then $\Gamma \backslash X \subset \Gamma \backslash T$ is connected if X is Γ-connected. The converse also holds if X is Γ-invariant, and is not always true even if Γ is a pseudogroup. Indeed, let $T = T_1 \amalg T_2$, where $T_1 = T_2 = \mathbb{R}$, and equip T with the natural topology. Let Γ be the pseudogroup generated by $\gamma\colon T_1 \to T_2$ given by $\gamma(x) = x$, $X_1 = (-\infty, 0] \subset T_1$, $X_2 = (0, \infty) \subset T_2$ and $X = X_1 \cup X_2$. Then X is not Γ-connected but $\Gamma \backslash X = \Gamma \backslash T = \mathbb{R}$.

If (Γ, T) is the holonomy pseudogroup of a foliation, then Γ-connected components of Γ-invariant sets correspond to connected components of saturated sets.

The notions of morphisms and equivalences are given as follows.

Definition 1.12. Let (Γ, T) and (Δ, S) be psg's. A *morphism* $\Phi\colon \Gamma \to \Delta$ is a collection Φ of local continuous mappings from T to S with the following properties.

i) $\{\operatorname{dom} \phi \mid \phi \in \Phi\}$ is an open covering of T.
ii) If $\phi \in \Phi$, then any restriction of ϕ to an open set of $\operatorname{dom} \phi$ also belongs to Φ.
iii) Let U be an open subset of T and ϕ a continuous map from U to S. If for any $x \in U$, there exists an open neighborhood U_x of x such that $\phi|_{U_x} \in \Phi$, then $\phi \in \Phi$.
iv) If $\phi \in \Phi$, $\gamma \in \Gamma^\times$ and $\delta \in \Delta^\times$, then $\delta \circ \phi \circ \gamma \in \Phi$,
v) Suppose that $\gamma \in \Gamma$ and $x \in \operatorname{dom} \gamma$. If $x \in \operatorname{dom} \phi$ and $\gamma(x) \in \operatorname{dom} \phi'$, where $\phi, \phi' \in \Phi$, then there is an element $\delta \in \Delta$ such that $\phi(x) \in \operatorname{dom} \delta$, and $\delta \circ \phi = \phi' \circ \gamma$ on a neighborhood of x.

A morphism from (Γ, T) to itself is called an *endomorphism* of (Γ, T).

The properties ii) and iii) are sometimes referred as the 'maximality'.

Definition 1.13 (cf. Definition 1.9)**.** Let $\Phi\colon (\Gamma,T) \to (\Delta,S)$ be a morphism. If $X \subset T$ and $Y \subset S$, then we set

$$\Phi(X) = \{s \in S \,|\, \exists x \in X,\ \exists \phi \in \Phi \text{ s.t. } s = \phi(x)\},$$
$$\Phi^{-1}(Y) = \bigcup_{\phi\in\Phi} \phi^{-1}(Y).$$

Definition 1.14. Let (Γ,T) and (Δ,S) be psg's and Φ a morphism from Γ to Δ.

1) Φ is called an *étale morphism* if Φ consists of étale mappings.
2) If Γ and Δ are holomorphic psg's, and if Φ consists of holomorphic mappings, then Φ is said to be *holomorphic*.
3) Suppose that Γ and Δ are psg's on complex one-dimensional manifolds. A holomorphic morphism is said to be *ramified* if $\phi \in \Phi$ and $x \in \operatorname{dom}\phi$, then there exists an open neighborhood U_x of x such that $\phi|_{U_x}$ is the restriction of the composite of ramified coverings and holomorphic étale mappings.

In what follows, we will consider only holomorphic morphisms if holomorphic psg's are considered.

Definition 1.15. Let (Γ,T) and (Δ,S) be pseudogroups. A collection Φ of local homeomorphisms from T to S is an étale morphism of pseudogroups if Φ satisfies the conditions in Definition 1.12 but 'a continuous map from U to S' in iii) is replaced by 'a local homeomorphism from T to S'.

Definition 1.15 is equivalent to the usual definition of morphisms of pseudogroups [7, 1.4].

Definition 1.16. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of local continuous mappings from T to S. Suppose that $\{\operatorname{dom} f_\lambda\}_{\lambda\in\Lambda}$ is an open covering of T and that if $\gamma \in \Gamma$, $x \in \operatorname{dom}\gamma$, $x \in \operatorname{dom} f_\lambda$ and $\gamma(x) \in \operatorname{dom} f_\mu$, where $\lambda,\mu \in \Lambda$, then there is a $\delta \in \Delta$ such that $f_\mu \circ \gamma = \delta \circ f_\lambda$ on a neighborhood of x. Then, the *morphism generated by* $\{f_\lambda\}_{\lambda\in\Lambda}$ is by definition the smallest morphism which contains $\{f_\lambda\}_{\lambda\in\Lambda}$ and denoted by $\langle f_\lambda\rangle_{\lambda\in\Lambda}$. If every f_λ is étale (resp. holomorphic, ramified), then the étale (resp. holomorphic, ramified) morphism generated by $\{f_\lambda\}_{\lambda\in\Lambda}$ is defined in the same way.

Definition 1.17. Let Γ and Δ be pseudogroups and let Φ be a morphism (resp. étale morphism) of pseudogroups from Γ to Δ. We denote by Φ_{psg} the morphism (resp. étale morphism) of psg's from Γ_{psg} to Δ_{psg} generated by Φ.

If Φ is an étale morphism of pseudogroups, then Φ consists of local homeomorphisms but Φ_{psg} needs not so.

Definition 1.18. Let (Γ,T) be a pseudosemigroup. Assume that there is a covering map $p\colon \widehat{T} \to T$ which satisfies the following *covering property*:

1) For each $\gamma \in \Gamma$, there is a unique mapping $\widehat{\gamma}$ such that $\operatorname{dom}\widehat{\gamma} = p^{-1}(\operatorname{dom}\gamma)$ and that $p \circ \widehat{\gamma} = \gamma \circ p$ holds on $p^{-1}(\operatorname{dom}\gamma)$.

2) If $\gamma_1, \gamma_2 \in \Gamma$, then $\widehat{\gamma_2 \circ \gamma_1} = \widehat{\gamma}_2 \circ \widehat{\gamma}_1$.
3) If U is an open subset of T, then $\widehat{\mathrm{id}_U} = \mathrm{id}_{p^{-1}(U)}$.

The psg $\widehat{\Gamma}$ generated by $\{\widehat{\gamma}\}_{\gamma \in \Gamma}$ together with the morphism generated by p is called the *covering* of Γ associated with p. If p is a Galois covering with Galois group G and the action commutes with $\widehat{\Gamma}$, then $(\widehat{\Gamma}, \widehat{T})$ and p are called *Galois covering with Galois group G*. If (Γ, T) is a holomorphic psg, then we always assume that $(\widehat{\Gamma}, \widehat{T})$ and p are holomorphic. If in addition (Γ, T) is a holomorphic psg on a one-dimensional complex manifold, then we allow p to be a ramified covering. In this case we call $(\widehat{\Gamma}, \widehat{T})$ with the morphism generated by p a *ramified covering*.

Note that the morphism generated by p is an étale or a ramified morphism.

Definition 1.19. If $\Phi_1 \colon \Gamma_1 \to \Gamma_2$ and $\Phi_2 \colon \Gamma_2 \to \Gamma_3$ are morphisms of pseudosemigroups, then the *composite* $\Phi_2 \circ \Phi_1$ is defined by

$$\Phi_2 \circ \Phi_1 = \langle \phi_2 \circ \phi_1 \,|\, \phi_1 \in \Phi_1,\ \phi_2 \in \Phi_2,\ \operatorname{range} \phi_1 \subset \operatorname{dom} \phi_2 \rangle.$$

Definition 1.20. An étale morphism $\Phi \colon \Gamma \to \Delta$ is an *equivalence* if there is an étale morphism $\Psi \colon \Delta \to \Gamma$ such that $\Psi \circ \Phi = \Gamma^\times$ and $\Phi \circ \Psi = \Delta^\times$. Such a Ψ is unique so that it is denoted by Φ^{-1}. We call Φ^{-1} the *inverse morphism* of Φ. An equivalence from (Γ, T) to itself is called *automorphism*.

If Φ_1 and Φ_2 are equivalences, then $\Phi_2 \circ \Phi_1$ is also an equivalence.

Example 1.21. Let f be an endomorphism of $\mathbb{C}P^1$ and ϕ an automorphism of $\mathbb{C}P^1$. Then ϕ naturally induces an equivalence from $\langle f \rangle$ to $\langle \phi \circ f \circ \phi^{-1} \rangle$.

Remark 1.22. If (Γ, T) is a psg, then the identity map on T generates a morphism which is equal to $\Gamma^\times$. In fact, $\Gamma^\times$ is an automorphism of (Γ, T). On the other hand, Γ is an endomorphism of (Γ, T) if and only if $\Gamma = \Gamma^\times$. Indeed, if $\zeta \in \Gamma$, then applying the condition v) to $\phi = \zeta$, $\phi' = \gamma = \mathrm{id}_T$, we see that for any $x \in \operatorname{dom} \zeta$, there exists an open neighborhood U of x and $\delta \in \Gamma$ such that $\delta \circ \zeta = \mathrm{id}_U$ holds. If we set $\phi = \delta$, $\phi' = \gamma = \mathrm{id}_T$, then there exists an open neighborhood V of $\zeta(x)$ and $\delta' \in \Gamma$ such that $\delta' \circ \delta = \mathrm{id}_V$. It follows that $\delta' = \delta' \circ \mathrm{id}_U = \delta' \circ (\delta \circ \zeta) = \zeta$ holds on a neighborhood of x. Therefore $\zeta \in \Gamma^\times$.

Lemma 1.23. *An étale morphism Φ is an equivalence if and only if*

$$\Psi' = \{\text{étale maps from } S \text{ to } T \text{ which are locally of the form } \phi^{-1} \text{ for some } \phi \in \Phi\}$$

is a morphism. Indeed, $\Psi' = \Phi^{-1}$.

Proof. Suppose that Φ is an equivalence and let Ψ be as in Definition 1.20. If $\psi \in \Psi$ and $x \in \operatorname{dom} \psi$, then there is an element $\phi \in \Phi$ such that $\psi(x) \in \operatorname{dom} \phi$. Since $\Phi \circ \Psi = \Delta^\times$, there is an element $\delta \in \Delta^\times$ such that $\phi \circ \psi = \delta$ on a neighborhood of x. We may assume that ϕ and δ are local homeomorphisms by restriction. Since Φ is a morphism, $(\delta)^{-1} \circ \phi \in \Phi$. Therefore, $\psi \in \Psi'$. Conversely, if $\psi' \in \Psi'$ and

$y \in \operatorname{dom}\psi'$, then $\psi' = \phi^{-1}$ holds on a neighborhood of y, where $\phi \in \Phi$. Let $\psi \in \Psi$ such that $y \in \operatorname{dom}\psi$. Since $\Psi \circ \Phi = \Gamma^\times$, we may assume that $\psi \circ \phi = \gamma$ for some $\gamma \in \Gamma^\times$. Hence $\psi' = \phi^{-1} = \gamma^{-1} \circ \psi$ holds on a neighborhood of y. Since Ψ is a morphism, this implies that $\psi' \in \Psi$. It is easy to see that $\Psi' = \Phi^{-1}$ holds if Ψ' is a morphism. □

If we work on pseudogroups, we have $\Phi^{-1} = \{\phi^{-1} \mid \phi \in \Phi\}$. Indeed, an étale morphism Φ of pseudogroups is said to be an equivalence if $\{\phi^{-1} \mid \phi \in \Phi\}$ is an étale morphism of pseudogroups [7].

2. FATOU-JULIA DECOMPOSITION OF PSEUDOSEMIGROUPS

We pose the following assumption in this section.

Assumption 2.1. *(Γ, T) is a holomorphic étale psg on a q-dimensional complex manifold. If $q = 1$, then we allow Γ to be ramified.*

Note that Γ consists of open mappings under the above assumption.

Definition 2.2. Let (Γ, T) be a psg. If $T' \subset T$ be a relatively compact subset, then we denote by $\Gamma_{T'}$ the restriction of Γ to T', that is

$$\Gamma_{T'} = \{\gamma \in \Gamma \mid \operatorname{dom}\gamma \subset T' \text{ and } \operatorname{range}\gamma \subset T'\}.$$

We say that an open connected subset U of T' has the *property (wF)*, or U is a *wF-open set* for short if the following conditions are satisfied:

wF1) If $x \in U$ and $\eta_x \in (\Gamma_{T'})_x$, then there exists an element γ of Γ such that $\operatorname{dom}\gamma = U$ and $\gamma_x = \eta_x$. We call γ an *extension* of η_x to U.

wF2) If we set

$$\begin{aligned}\Gamma^U &= \left\{\gamma \in \Gamma \,\middle|\, \begin{array}{l}\operatorname{dom}\gamma = U, \text{ and } \gamma \text{ is an extension of the germ of} \\ \text{an element of } \Gamma_{T'} \text{ as above}\end{array}\right\} \\ &= \{\gamma \in \Gamma \mid \operatorname{dom}\gamma = U \text{ and } \gamma(U) \cap T' \neq \varnothing\},\end{aligned}$$

then Γ^U is a normal family.

We say that an open connected subset V of T' has the *property (F)*, or V is an *F-open set* for short if the following conditions are satisfied:

F1) V has the property (wF).

F2) If $\gamma \in \Gamma_{T'}$ and $\operatorname{dom}\gamma \subset V$, then $\operatorname{range}\gamma$ is the union of wF-open sets.

Let $F^*(\Gamma_{T'})$ be the union of F-open subsets of T', and $J^*(\Gamma_{T'})$ the complement of $F^*(\Gamma_{T'})$ in T'. We set

$$\begin{aligned}J_0(\Gamma) &= \bigcup_{T' \in \mathscr{T}} J^*(\Gamma_{T'}), \\ J(\Gamma) &= \overline{J_0(\Gamma)},\end{aligned}$$

where $\mathscr{T} = \{T' \subset T \mid T' \text{ is relatively compact}\}$. We call $J(\Gamma)$ the *Julia set* of Γ. The *Fatou set* of Γ is by definition the complement of $J(\Gamma)$ in T. We call Γ-connected components of $F(\Gamma)$ and $J(\Gamma)$ *Fatou components* and *Julia components*, respectively. Fatou sets and Julia sets obtained using by the property (wF) instead of (F) are denoted by adding 'w', e.g. Fatou sets in this sense are denoted by $wF(\Gamma)$.

Needless to say that the 'property (F)' stands for the 'property Fatou'. By '(wF)' we mean 'weak-F'. Note that if U is an F-open set for $(\Gamma_{T'}, T')$ and if $\gamma \in \Gamma_{T'}$ such that $\operatorname{dom}\gamma \subset U$, then $\operatorname{range}\gamma$ is the union of F-open sets. To see this, let $\zeta \in \Gamma_{T'}$ such that $\operatorname{dom}\zeta \subset \operatorname{range}\gamma$. If we set $V = \gamma^{-1}(\operatorname{dom}\zeta)$, then $\zeta \circ \gamma|_V \in \Gamma_{T'}$ and $\operatorname{range}\zeta = \zeta(\gamma(V))$ so that $\operatorname{range}\zeta$ is the union of wF-open sets.

Example 2.3 (see also Example 3.6)**.** Let $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ be a rational map. If we denote by $\langle f \rangle$ the psg generated by f, then $J(\langle f \rangle) = J(f)$, where $J(f)$ denotes the Julia set of f in the usual sense. If $g\colon \mathbb{C} \to \mathbb{C}$ is an entire map, then we can regard g as a local holomorphic map defined on $\mathbb{C}P^1$ with $\operatorname{dom} g = \mathbb{C}$, and $\langle g \rangle$ as a psg which acts on $\mathbb{C}P^1$. If we denote by $J(g)$ the Julia set of g in the usual sense, which is a subset of $\mathbb{C}$, then we have $J(\langle g \rangle) = J(g) \cup \{\infty\}$.

Let $T' \in \mathscr{T}$. If U is an F-open set in T', then U is a wF-open set by definition. If $\gamma \in \Gamma_{T'}$ then $\gamma(U)$ is the union of wF-open sets but $\gamma(U)$ itself is not necessarily a wF-open set.

Example 2.4. Let $T = \mathbb{C}P^1$ and we define $\gamma, \zeta\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ by $\gamma(z) = z^2$, and $\zeta(z) = z^\alpha$, where $\alpha > 1$ and $\alpha \notin \mathbb{Z}$. The mapping ζ is not well-defined on $\mathbb{C}P^1$ so that we regard ζ as local mappings defined on suitable open subsets of $\mathbb{C}P^1 \setminus \{0, \infty\}$ and take all branches. Let Γ be the psg generated by γ and ζ. Then, $F(\Gamma) = \mathbb{C}P^1 \setminus (\{0,\infty\} \cup \{|z| = 1\})$. Let U be a small open disc in $\mathbb{C}P^1 \setminus (\{0,\infty\} \cup \{|z| = 1\})$. If n is large enough, then $\gamma^n(U)$ contains a circle around 0 or ∞. Hence no germ of ζ at a point in $\gamma^n(U)$ is the germ of any element of Γ defined on $\gamma^n(U)$ so that $\gamma^n(U)$ does not have the property (wF). However, if $x \in \gamma^n(U)$, then by choosing a neighborhood of x small enough, we see that the germ of any element of Γ can be extended to an element of Γ.

Some remarks are in order.

Remark 2.5. Let $\widetilde{F}^*(\Gamma_{T'})$ be the complement of $J^*(\Gamma_{T'})$ in T. If we denote by $F_0(\Gamma)$ the complement of $J_0(\Gamma)$ in T, then we have

$$F_0(\Gamma) = \bigcap_{T' \in \mathscr{T}} \widetilde{F}^*(\Gamma_{T'})$$

and $F(\Gamma)$ is the interior of $F_0(\Gamma)$ (see also Lemma 2.16).

Remark 2.6. A related construction for holomorphic correspondences is given in [3].

Remark 2.7. Although the difference between the conditions (F) and (wF) seems quite large, there are several cases where they are equivalent. If Γ is generated by a pseudogroup, then these conditions are equivalent. They are also equivalent if $\Gamma = \langle f \rangle$, where f is an endomorphism of $\mathbb{C}P^1$ or an entire map on $\mathbb{C}$. We will show that if Γ is compactly generated, then the conditions (F) and (wF) are equivalent (Proposition 4.5).

Remark 2.8. As holomorphic mappings are considered, extensions in wF1) of the property (wF) are unique. The extension of γ_x is usually denoted by γ.

Example 2.9. Let T_1, T_2 and T_3 be open unit discs in $\mathbb{C}$ and $T = T_1 \amalg T_2 \amalg T_3$. We denote by z_i the standard coordinates on T_i. We define $\gamma_i\colon T_1 \to T_3$ by $\gamma_i(z_1) = z_1^i$ and $\zeta_i\colon T_2 \to T_3$ by $\zeta_i(z_2) = z_2^i$ but $\operatorname{dom}\zeta_i = \{|z_2| < 1/i\}$, where i is a positive integer. Let $\eta\colon T_1 \to T_2$ be the identity map, and Γ the psg generated by $\{\gamma_i, \zeta_j, \eta\}_{i,j>0}$. Then, $F(\Gamma) = T \setminus (\{0_1, 0_2\} \cup \bigcup_{i=2}^{\infty}\{|z_1| = 1/i\} \cup \bigcup_{i=2}^{\infty}\{|z_2| = 1/i\})$ and $wF(\Gamma) = T \setminus (\{0_2\} \cup \bigcup_{i=2}^{\infty}\{|z_2| = 1/i\})$, where 0_i denotes the origin in T_i. Indeed, ζ_i is not well-defined on a fixed neighborhood of 0_2 if i is large. Note that $\Gamma(F(\Gamma)) = F(\Gamma)$ but $\Gamma(wF(\Gamma)) \supsetneq wF(\Gamma)$.

Definition 2.10. If (Γ, T) is a pseudogroup, then $F_0(\Gamma)$, $J_0(\Gamma)$, $F(\Gamma)$ and $J(\Gamma)$ are defined formally in the same way as in Definition 2.2. Thus obtained Fatou and Julia sets are denoted by $F_{\mathrm{pg},0}(\Gamma)$, $J_{\mathrm{pg},0}(\Gamma)$, $F_{\mathrm{pg}}(\Gamma)$ and $J_{\mathrm{pg}}(\Gamma)$, respectively.

Recall that if Γ is a pseudogroup, then the conditions (wF) and (F) are equivalent. If (Γ, T) is a pseudogroup, then $F_{\mathrm{pg}}(\Gamma) \subset F(\Gamma_{psg})$. The difference between $F_{\mathrm{pg}}(\Gamma)$ and $F(\Gamma_{\mathrm{psg}})$ occurs in wF1) of Definition 2.2.

Example 2.11 (see also Example 4.21)**.** Let $T = \{0 < |z| < 1\} \subset \mathbb{C}$ and set $\gamma(z) = z^2$. Let Γ be the pseudogroup generated by γ and its local inverses, namely, let $\mathscr{U} = \{U \subset T \mid U \text{ is an open subset such that } \gamma\colon U \to \gamma(U) \text{ is a homeomorphism}\}$, and let $\Gamma = \langle \gamma|_U, \gamma^{-1}|_{\gamma(U)} \rangle_{U \in \mathscr{U}}$. Then $F(\Gamma_{\mathrm{psg}}) = F_{\mathrm{pg}}(\Gamma) = T$. On the other hand, let $\widehat{T}$ be the open unit disc and we regard γ as a local mapping defined on $\widehat{T}$ with $\operatorname{dom}\gamma = T$, and let $\widehat{\Gamma}$ be the pseudogroup generated by γ and its local inverses. Then $F(\widehat{\Gamma}_{\mathrm{psg}}) = \widehat{T} \setminus \{0\}$. On the other hand, $F_{\mathrm{pg}}(\widehat{\Gamma}) = \varnothing$. Indeed, once an open subset U of $\widehat{T}$ is fixed, γ^n is not injective on U for large n.

The equality $F_{\mathrm{pg}}(\Gamma) = F(\Gamma_{\mathrm{psg}})$ holds if Γ is compactly generated. See Proposition 4.11.

Remark 2.12. If $q > 1$, then the Julia sets in Definitions 2.2 and 2.10 are tentative. We will need the notion of Green functions for a right definition of them, which we do not discuss in this paper. On the other hand, we can apply Definition 2.2 to rational mappings from $\mathbb{C}P^n$ to $\mathbb{C}P^n$, and obtain the Fatou set in the usual sense. We refer to [4] and [21] for dynamics on $\mathbb{C}P^n$.

In general, $F_0(\Gamma) = F(\Gamma)$ does not hold even if Γ is finitely generated.

Example 2.13. Let $A = \{z \in \mathbb{C} \,|\, 1 < |z| < 2\}$ and define a local mapping α on A by

$$\alpha(z) = \begin{cases} z^2, & \text{if } 1 < |z| < \sqrt{2}, \\ z^2/2, & \text{if } \sqrt{2} < |z| < 2. \end{cases}$$

If we set $\Gamma_0 = \langle \alpha \rangle$, then $J(\Gamma_0) = A$. We regard α as a local mapping on $\mathbb{C}$. For a positive integer i, we set $T_i = \mathbb{C}$, and $T = \coprod_{i=1}^{\infty} T_i$. We define $\gamma_i, \zeta_i \colon T_i \to T_{i+1}$ by $\gamma_i(z) = \alpha(z)$ and $\zeta_i(z) = 4z$. Let γ and ζ be local mappings on T such that $\gamma|_{T_i} = \gamma_i$ and $\zeta|_{T_i} = \zeta_i$, respectively. If we set $\Gamma = \langle \gamma, \zeta \rangle$, then we have

$$J_0(\Gamma) \cap T_i = \bigcup_{i=0}^{\infty} \{z \in \mathbb{C} \,|\, 4^{-i} < |z| < 2 \cdot 4^{-i}\},$$
$$J(\Gamma) \cap T_i = \{0\} \cup J_0(\Gamma).$$

for any i.

Example 2.14 (cf. [1, Example 2.15], see also Theorem 2.19)**.** Let $T_i = \mathbb{C}$, $i = 1, 2, \ldots$, and $T = \coprod_{i=1}^{\infty} T_i$. We define $\gamma_i \colon T_i \to T_{i+1}$ to be the restriction of the identity map to $\{z \in \mathbb{C} \,|\, |z| < 1/i\}$. Let γ be the local diffeomorphism from T to T such that $\gamma|_{T_i} = \gamma_i$. If we denote by Γ the pseudogroup generated by γ, then $J_{\mathrm{pg},0}(\Gamma) \cap T_i = \bigcup_{k=i}^{\infty} \{|z| = 1/k\}$ but $J_{\mathrm{pg}}(\Gamma) \cap T_i = (J_{\mathrm{pg},0}(\Gamma) \cap T_i) \cup \{0\}$. Note that (Γ, T) is not equivalent to the holonomy pseudogroup of the trivial foliation on a foliation chart. On the other hand, if we set $S_1 = \mathbb{C}$, $S_i = \{z \in \mathbb{C} \,|\, |z| < 1/i-1\}$ for $i > 1$ and $S = \coprod_{i=1}^{\infty} S_i$, then γ is a local diffeomorphism on S. If we denote by $\widetilde{\Gamma}$ the pseudogroup generated by γ, then $F_{\mathrm{pg}}(\widetilde{\Gamma}) = S$. Indeed, $(\widetilde{\Gamma}, S)$ is equivalent to the holonomy pseudogroup of the trivial foliation on a foliation chart.

The equality $F_0(\Gamma) = F(\Gamma)$ holds in some important cases. See Theorems 4.1, 5.9 and Corollary 5.8.

Remark 2.15. In what follows, we will discuss Fatou and Julia sets of psg's. However, the results apply to Fatou and Julia sets of pseudogroups without changes.

The following property is frequently used.

Lemma 2.16. *Let (Γ, T) be a psg, and let $T_1,\ T_2 \in \mathscr{T}$. If $T_1 \subset T_2$, then $F^*(\Gamma_{T_1}) \supset F^*(\Gamma_{T_2})$.*

The proof is easy and omitted. Lemma 2.16 implies that it suffices to consider a sequence $\{T_i\}$ in $\mathscr{T}$ such that $\overline{T_i} \subset T_{i+1}$ and that $\bigcup_{i=1}^{\infty} T_i = T$ when defining $J_0(\Gamma)$ and $F_0(\Gamma)$.

Unlike the classical cases, $F(\Gamma)$ and $J(\Gamma)$ need not be completely invariant.

Example 2.17. Let $T_1 = T_2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ and $T = T_1 \amalg T_2$. Let $f \colon T_1 \to T_1$ be such that $f(z) = \sqrt{-1}z$ on $\mathbb{C} \subset \mathbb{C}P^1$, and let $\varphi \colon T_2 \to T_1$ be the identity map. Let $g \colon T_2 \to T_2$ be a rational map such that the classical Julia set $J(g)$ is the whole $\mathbb{C}P^1$, for example, a Lattès map. If we set $\Gamma = \langle f, g, \varphi \rangle$, then $F(\Gamma) = T_1$ and

$J(\Gamma) = T_2$. We have $\Gamma^{-1}(F(\Gamma)) = T$ and $\Gamma^{-1}(J(\Gamma)) = J(\Gamma)$. On the other hand, $\Gamma(F(\Gamma)) = F(\Gamma)$ and $\Gamma(J(\Gamma)) = T$.

Example 2.17 is an example of compactly generated psg's. See Sections 3 and 4. In general, we have the following.

Lemma 2.18. 1) *$F_0(\Gamma)$ and $F(\Gamma)$ are forward Γ-invariant, and we have $F(\Gamma) = \bigcap_{\gamma\in\Gamma}(\gamma^{-1}(F(\Gamma)) \cup (T \setminus (\operatorname{dom}\gamma)))$.*
2) *$J_0(\Gamma)$ and $J(\Gamma)$ are backward Γ-invariant.*
3) *$F_0(\Gamma)$, $J_0(\Gamma)$, $F(\Gamma)$ and $J(\Gamma)$ are $\Gamma^\times$-invariant.*

Proof. If U is an F-open set for $(\Gamma_{T'}, T')$ and if $\gamma \in \Gamma_{T'}$ such that $\operatorname{dom}\gamma \subset U$, then $\gamma(U)$ is the union of F-open sets. Hence we have $\Gamma_{T'}(F^*(\Gamma_{T'})) \subset F^*(\Gamma_{T'})$ for any $T' \in \mathscr{T}$. Hence $\Gamma(F_0(\Gamma)) \subset F_0(\Gamma)$. On the other hand, since the local identity maps belong to Γ, the inclusions are in fact equalities. Since Γ consists of open mappings, we also have $\Gamma(F(\Gamma)) = F(\Gamma)$. If $\gamma \in \Gamma$, then $\gamma(F(\Gamma) \cap (\operatorname{dom}\gamma)) \subset F(\Gamma)$. Hence $F(\Gamma) \cap (\operatorname{dom}\gamma) \subset \gamma^{-1}(F(\Gamma))$. Therefore

$$\begin{aligned} F(\Gamma) &= \bigcap_{\gamma\in\Gamma}((F(\Gamma)\cap(\operatorname{dom}\gamma)) \cup (T \setminus (\operatorname{dom}\gamma))) \\ &\subset \bigcap_{\gamma\in\Gamma}(\gamma^{-1}(F(\Gamma)) \cup (T \setminus (\operatorname{dom}\gamma))). \end{aligned}$$

If we set $\gamma = \mathrm{id}_T$, then $\gamma^{-1}(F(\Gamma)) \cup (T \setminus (\operatorname{dom}\gamma)) = F(\Gamma)$ so that the above inclusion is in fact the equality. The part 2) follows from 1). The part 3) is easy. □

We have the following.

Theorem 2.19 (see also Proposition 4.10)**.** *Let (Γ, T) and (Δ, S) be psg's.*
1) *If $\Phi\colon \Gamma \to \Delta$ is either a covering or ramified covering, then $\Phi^{-1}(F(\Delta)) \subset F(\Gamma)$. If Φ is a Galois covering with a finite Galois group, then $\Phi^{-1}(F(\Delta)) = F(\Gamma)$.*
2) *If $\Phi\colon \Gamma \to \Delta$ is an equivalence, then $\Phi(F(\Gamma)) = F(\Delta)$.*

Proof. We will show 1), because 2) can be shown by similar arguments. Let W be an open subset of S. Then, W is contained in $F(\Delta)$ if and only if $W \subset \widetilde{F}^*(\Delta_{S'})$ for any $S' \in \mathscr{S}$, where $\mathscr{S}$ denotes the set of relatively compact subsets of S. Note that the latter condition is equivalent to $W \cap S' \subset F^*(\Delta_{S'})$ for any $S' \in \mathscr{S}$.

Let U be an open subset of $\Phi^{-1}(F(\Delta))$. Assume that $\phi_1 \in \Phi$ is defined on U and that $\phi_1(U) \subset F(\Delta)$. If $\phi_2 \in \Phi$ and $U \subset \operatorname{dom}\phi_2$, then $\phi_2(U) \subset F(\Delta)$. Indeed, if $x \in U$, then $\phi_2 = \delta \circ \phi_1$ holds for some $\delta \in \Delta$ on a neighborhood of x by v) of Definition 1.12. Hence $\phi_2(x) \in F(\Delta)$ by Lemma 2.18.

Let $x \in \Phi^{-1}(F(\Delta)) \cap T'$ and let $T'_1, \dots, T'_r$ be the connected components of T', where $T' \in \mathscr{T}$. Since T' is relatively compact, we can find a finite number of elements $\phi_1, \dots, \phi_s$ of Φ such that $\{\operatorname{dom}\phi_k\}$ is an open covering of $\overline{T'}$ and that each ϕ_k is the restriction of an element $\widetilde{\phi_k}$ of Φ such that $\operatorname{dom}\widetilde{\phi_k} \supset \overline{\operatorname{dom}\phi_k}$. Moreover, we may assume that each $\widetilde{\phi}_k$ is a local ramified covering with a single singularity, or a local biholomorphic diffeomorphism. If we set $S' = \bigcup_{i=1}^{r}\bigcup_{j=1}^{s}\phi_j(T_i' \cap (\operatorname{dom}\phi_j))$, then $S' \in \mathscr{S}$. We may assume that $x \in \operatorname{dom}\phi_1$. Then $\phi_1(x) \in F(\Delta) \cap S'$ by the above arguments.

Let U be an open connected neighborhood of x which is contained in $(\operatorname{dom}\phi_1) \cap \Phi^{-1}(F(\Delta)) \cap T'$. We may assume that if we set $V = \phi_1(U)$ then V is an F-open set in $F^*(\Delta_{S'})$. We may further assume that if $\delta \in \Delta^V$ and $\delta(V) \cap \operatorname{range}\phi_k \neq \varnothing$, then $\delta(V) \subset \operatorname{range}\widetilde{\phi}_k$. Let $z \in U$ and $\gamma_z \in (\Gamma_{T'})_z$. If $\gamma(z) \in \operatorname{dom}\phi_i$, then there is an element $\delta \in \Delta$ such that $\delta \circ \phi_1 = \phi_i \circ \gamma$ on a neighborhood of z. Since V is an F-open set, δ extends to an element of Δ defined on V. Hence $(\delta \circ \phi_1)_z = (\widetilde{\phi}_i \circ \gamma)_z$. As Φ is a covering or ramified covering, there exists an element ζ of Γ such that $\widetilde{\phi}_i \circ \zeta = \delta \circ \phi_1$ and $\operatorname{dom}\zeta = U$. If $\gamma(z)$ is not a branching point of ϕ_i, then $\zeta_z = \gamma_z$. If $\gamma(z)$ is a branching point of ϕ_i, then we can find a point w which is close enough to z and is not a branching point. We still have $(\widetilde{\phi}_k \circ \gamma)_w = (\phi_k \circ \zeta)_w$ so that $\zeta_w = \gamma_w$. By analyticity, we have $\zeta_z = \gamma_z$. If ϕ_1 is a local biholomorphic diffeomorphism, then for each $\gamma \in \Gamma^U$, $\widetilde{\phi}_{k(\gamma)} \circ \gamma \circ \phi_1^{-1} \in \Delta^V$, where $k(\gamma)$ is determined by γ as above. Since the number of ϕ_i's is finite, this implies that Γ^U is a normal family. If $q = 1$ and ϕ_1 is ramified at $p \in U$, then $\Gamma^U|_{U\setminus\{p\}}$ is a normal family. Since elements of Γ^U are obtained via Δ^V, elements of Γ^U is bounded on a neighborhood of p. Hence Γ^U is a normal family also in this case. Therefore U is a wF-open set. Let $\gamma \in \Gamma_{T'}$ such that $\operatorname{dom}\gamma \subset U$. If $\gamma(x) \in \operatorname{dom}\phi_i$, then there is an open connected set U' of $\operatorname{dom}\gamma$ such that $x \in U'$, $\gamma(U') \subset \operatorname{dom}\phi_i$, and that there is an element δ of Δ such that $\delta \circ \phi_1 = \phi_i \circ \gamma$ holds on U'. Since $\delta(\phi_1(U')) = \phi_i(\gamma(U')) \subset S'$, $\delta \in \Delta_{S'}$ and $\phi_i(\gamma(U'))$ is the union of F-open sets. Let $z \in \gamma(U')$ and $\eta_z \in (\Gamma_{T'})_z$. If $\eta(z) \in \operatorname{dom}\phi_k$, then there is an element $\mu \in \Delta_{S'}$ such that $(\mu \circ \phi_i)_z = (\phi_k \circ \eta)_z$. Since $\phi_i(\gamma(U'))$ is the union of F-open sets, we may assume by shrinking U' that μ is well-defined on $\phi_i(\gamma(U'))$ as an element of Δ. Moreover $\mu(\phi_i(\gamma(U'))) \subset \operatorname{range}\widetilde{\phi}_k$ by the choice of V, because we have $\mu(\phi_i(\gamma(U'))) = \mu(\delta(\phi_1(U'))) \subset \mu \circ \delta(V)$. Now since Φ is a (ramified) covering, there is an element ζ of Γ such that $\mu \circ \phi_i = \widetilde{\phi}_k \circ \zeta$ with $\operatorname{dom}\zeta = \gamma(U')$. We have $(\widetilde{\phi}_k \circ \zeta)_z = (\phi_k \circ \eta)_z$. By similar arguments as above, we can verify that $\zeta_z = \eta_z$ and that $\Gamma^{\gamma(U')}$ is a normal family. Hence $\gamma(U')$ is a wF-open set so that U is an F-open set. Suppose that Φ is a Galois covering with a finite Galois group. Let $\widetilde{U} \subset F_0(\Gamma)$ and assume that $p|_{\widetilde{U}}$ is a homeomorphism. We set $U = p(\widetilde{U})$, where p is the projection which generates Φ. Let $x \in U$ and $S' \in \mathscr{S}$ such that $x \in S'$. If we set $T' = p^{-1}(S')$, then $T' \in \mathscr{T}$ because p is a finite covering. Let $\widetilde{x} \in \widetilde{U}$ such that $p(\widetilde{x}) = x$ and $\widetilde{U}'$ an F-open set for $\Gamma_{T'}$ which contains $\widetilde{x}$. We set $U' = p(\widetilde{U}')$. If $y \in U'$ and $\delta_y \in (\Delta_{S'})_y$, then there is a $\gamma_{\widetilde{y}} \in (\Gamma_{T'})_{\widetilde{y}}$ such that $(p \circ \gamma)_{\widetilde{y}} = (\delta \circ p)_{\widetilde{y}}$, where $\widetilde{y} \in T'$ such that $p(\widetilde{y}) = y$. Then, $\gamma_{\widetilde{y}}$ extends to an element of Γ defined on $\widetilde{U}'$. If $z \in U'$, then $(p \circ \gamma)_{\widetilde{z}} = (\delta' \circ p)_z$ holds for some $\delta' \in \Delta$, where $\widetilde{z}$ the unique element of $\widetilde{U}'$ such that $p(\widetilde{z}) = z$. Since p is a homeomorphism, we have $\delta' = p \circ \gamma \circ p^{-1}$ on a neighborhood of z. Hence $p \circ \gamma \circ p^{-1}$ belongs to Δ, and its domain is U'. As $\Gamma^{\widetilde{U}'}$ is a normal family, $\Delta^{U'}$ is also. Hence U' is a wF-open set for $\Delta_{S'}$. Let $\delta \in \Delta_{S'}$ such that $\operatorname{dom}\delta \subset U'$. We set $V = \operatorname{dom}\delta$ and $\widetilde{V} = p^{-1}(V) \cap \widetilde{U}'$. Then, there is an element $\gamma \in \Gamma$ such that $p \circ \gamma = \delta \circ p$ because Φ is a covering. Moreover, $\gamma \in \Gamma_{T'}$ by the definition of T'. As $\gamma(\widetilde{V})$ is the union of wF-open sets, $\delta(V)$ is also the union of wF-open sets. Hence U' is an F-open set for $\Delta_{S'}$. Therefore U is the union of F-open sets for $\Delta_{S'}$, and $U \subset F_0(\Delta)$. $\square$

Example 2.20. We define $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ by $f(z) = z^2$. Let Γ be the psg generated by f and its local inverses on $\mathbb{C}P^1 \setminus \{0,\infty\}$, then $F(\Gamma) = \mathbb{C}P^1 \setminus (\{0,\infty\} \cup \{|z| = 1\})$. We define $\widetilde{f}\colon \mathbb{C} \to \mathbb{C}$ by $\widetilde{f}(z) = 2z$, and let $\widetilde{\Gamma}$ the psg on $\mathbb{C}$ generated by $\widetilde{f}$ and $\widetilde{f}^{-1}$. Then $F(\widetilde{\Gamma}) = \mathbb{C} \setminus \{0\}$. Let $p\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ be the exponential map. Then p is a morphism from $(\widetilde{\Gamma}, \mathbb{C})$ to $(\Gamma, \mathbb{C}P^1)$, and a covering morphism from $(\widetilde{\Gamma}, \mathbb{C})$ to $(\Gamma', \mathbb{C} \setminus \{0\})$, where Γ' denotes the restriction of Γ to $\mathbb{C} \setminus \{0\}$. We have $F(\Gamma') = F(\Gamma)$ and $p^{-1}(F(\Gamma')) = \mathbb{C} \setminus \sqrt{-1}\mathbb{R} \subsetneq F(\widetilde{\Gamma})$.

Example 2.21. 1) of Theorem 2.19 does not always hold if we simply assume that Φ is a morphism. Let $T_1 = T_2 = \mathbb{C}$ and $T = T_1 \amalg T_2$. We define $\gamma_1\colon T_1 \to T_2$ by $\gamma_1(z) = z$. Let γ_2 be the restriction of γ_1 to the unit disc in T_1. Then, we have $F(\langle\gamma_1\rangle) = T$ and $F(\langle\gamma_2\rangle) = T \setminus \{z \in T_1 \mid |z| = 1\}$. The identity map of T induces a morphism $\Phi\colon (\langle\gamma_2\rangle, T) \to (\langle\gamma_1\rangle, T)$ but $\Phi^{-1}(F(\langle\gamma_1\rangle) \supsetneq F(\langle\gamma_2\rangle)$.

In the next section, we will introduce the notion of compactly generated psg's. Here we present two examples of non-compactly generated psg's in advance. Fatou-Julia decompositions of these psg's are examined under a tentative definition in [1]. The decompositions are as follows under Definition 2.2. Note that these psg's are generated by pseudogroups so that the conditions (wF) and (F) are equivalent. Results are the same as in [1] but we proceed by correcting typographic errors.

Example 2.22 ([1, Examples 8.8 and 8.9])**.** Let $\gamma\colon \mathbb{C} \to \mathbb{C}$ be the mapping given by $\gamma(z) = 2z$, and $\langle\gamma\rangle$ the group generated by γ. Let $T = (\mathbb{C} \setminus \{0\})/\langle\gamma\rangle$ and $S = \{z \in \mathbb{C} \mid |z| < 1 + \varepsilon\}$, where ε is a small positive real number. Let O' be a subset of S defined by $O' = \{z \in \mathbb{C} \mid 1 < |z| < 1 + \varepsilon\}$, and let $\eta\colon O' \to T$ be the mapping induced by the inclusion of O' into $\mathbb{C}$. We define $\xi\colon T \to T$ by $\xi(z) = z^2$, and let Γ be the pseudogroup generated by ξ and η which acts on $T_1 = T \amalg S$. Then $J(\Gamma_{\mathrm{psg}}) = T \amalg \overline{O'}$, where $\overline{O'}$ denotes the closure of O' in S ($J(\Gamma_{\mathrm{psg}})$ is written in [1] as T_1 in error). Although Γ and Γ_{psg} are not compactly generated, we have $J_{\mathrm{pg}}(\Gamma) = T$.

Example 2.23 ([1, Example 8.10])**.** Let $D_{5+\varepsilon}(0)$ be the open disc of radius $5 + \varepsilon$ centered at 0 and let $T = T_1 \amalg T_2$, where $T_1 = T_2 = D_{5+\varepsilon}(0)$. We denote the natural coordinates of T_1 and T_2 by z and w, respectively. Let Γ be the pseudogroup generated by γ_0, γ_1 and γ_2 defined as follows. First set

$$S_i = \{z \in T_i \mid 25/(5+\varepsilon) < |z| < 5 + \varepsilon\},\ i = 1, 2,$$

and define $\gamma_0\colon S_1 \to S_2$ by $\gamma_0(z) = 25/z$. Second, let

$$U_1 = \{re^{\sqrt{-1}t} \in T_1 \mid 1 < r < 2,\ |t| < \delta\},$$

where δ is chosen so small that $\gamma_1\colon U_1 \to T_1$ defined by $\gamma_1(z) = z^2$ is a diffeomorphism onto its image. Finally set

$$V_1 = \{re^{\sqrt{-1}t} \in T_1 \,|\, 2 < r < 4,\ |t| < \delta\},$$

and define $\gamma_2\colon U_1 \to V_1$ by $\gamma_2(z) = 2z$. The action of Γ is essentially on S_1, and S_2 and γ_0 is added in order to be able to consider that Γ is acting on $\mathbb{C}P^1$.

The pseudogroup Γ is not compactly generated. If we set

$$I_k = \{e^{2^{-k+1}\sqrt{-1}\delta}\, t \,|\, 1 \le t \le 4\}, \quad \text{for } k = 0, 1, \ldots,$$
$$A_l = \{2^{i/2^{l-1}} e^{\sqrt{-1}s} \,|\, i = 0, \ldots, 2^l,\ |s| \le 2^{-l+1}\delta\}, \quad \text{for } l = 0, 1, \ldots,$$

(the definitions of I_k and A_l are incorrect in [1]) then

$$J(\Gamma_{\mathrm{psg}}) = J_{\mathrm{pg}}(\Gamma) = [1,4] \cup \bigcup_{k=0}^{\infty} I_k \cup \bigcup_{l=1}^{\infty} A_l.$$

Adding an irrational rotation to Γ as a generator, one can obtain a pseudogroup Γ_1 such that $J((\Gamma_1)_{\mathrm{psg}}) = J_{\mathrm{pg}}(\Gamma_1) = \{z \in T_1 \,|\, 1 \le |z| \le 4\}$. The pseudogroup Γ_1 is not compactly generated, either.

In general, it is almost impossible to tell if a given point of T belongs to $F(\Gamma)$ or not. As in the classical cases, $x \in T$ belongs to $J(\Gamma)$ if, for example,

1) there exists $\gamma \in \Gamma$ such that $\gamma(x) = x$ and $|\gamma'|_x > 1$ (repelling fixed point),
2) there there exists $\gamma \in \Gamma$ such that $\gamma(x) = x$ and $|\gamma'| = 1$ but $\gamma^k \ne \mathrm{id}$ for any positive integer k (parabolic or irrationally indifferent).

The dynamics on $F(\Gamma)$ is expected to be tame. We will later show that if $\Gamma = \Gamma^{\times}$, then $F(\Gamma)$ admits a Γ-invariant Hermitian metric which is locally Lipschitz continuous (Theorem 4.20). If Γ is compactly generated, then $F(\Gamma)$ admits a semi-invariant metric which is locally Lipschitz continuous (Proposition 4.19 and Theorem 4.17).

3. Compactly generated pseudosemigroups

The notion of compactly generated pseudogroups [8] is also valid for pseudogroups.

Definition 3.1. A pseudosemigroup (Γ, T) is *compactly generated* if there is a relatively compact open set T' in T, and a finite collection of elements $\{\gamma_1, \ldots, \gamma_r\}$ of Γ of which the domains and the ranges are contained in T' such that

1) $\{\gamma_1, \ldots, \gamma_r\}$ generates $\Gamma_{T'}$, where $\Gamma_{T'}$ is the restriction of Γ to T',
2) for each γ_i, there exists an element $\widetilde{\gamma}_i$ of Γ such that $\operatorname{dom}\widetilde{\gamma}_i$ contains the closure of $\operatorname{dom}\gamma_i$, $\widetilde{\gamma}_i|_{\operatorname{dom}\gamma_i} = \gamma_i$ and that $\widetilde{\gamma}_i$ is étale on a neighborhood of $\operatorname{dom}\widetilde{\gamma}_i \setminus \operatorname{dom}\gamma_i$,
3) the inclusion of T' into T induces an equivalence from $\Gamma_{T'}$ to Γ.

$(\Gamma_{T'}, T')$ is called a *reduction* of (Γ, T).

A reduction of (Γ, T) is also denoted by (Γ', T').

Remark 3.2. If Γ is a compactly generated psg on a one-dimensional complex manifold, then Γ is étale or ramified. In addition, the last condition in 2) is equivalent to $\operatorname{Sing}\widetilde{\gamma}_i = \operatorname{Sing}\gamma_i$.

Remark 3.3. If pseudogroups are considered, then the condition 3) can be replaced with a much weaker condition that T' meets every orbit of Γ.

Lemma 3.4. *If (Γ', T') is a reduction of (Γ, T), then $\Gamma^\times x$ meets T' for any $x \in T$.*

Proof. Let Φ be the morphism from (Γ', T') to (Γ, T) generated by the inclusion, which is an equivalence. Then $\Psi = \Phi^{-1}$ is an equivalence from (Γ, T) to (Γ', T'). If $x \in T$, then there is an element $\psi \in \Psi$ defined on a neighborhood of x and $\psi(x) \in T'$. We may assume that ψ is a diffeomorphism and $\psi^{-1} \in \Phi$. Since Φ is a morphism, there are elements $\gamma, \zeta \in \Gamma$ such that $(\gamma \circ \psi^{-1})_x = \mathrm{id}_x$ and $(\zeta \circ \mathrm{id})_x = (\psi^{-1})_x$. Therefore $\zeta_x = (\gamma^{-1})_x$ and $\gamma_{\psi^{-1}(x)} = \psi_{\psi^{-1}(x)}$ so that the restriction of ψ to a neighborhood of x belongs to $\Gamma^\times$. □

Lemma 3.5. *If Γ is a compactly generated pseudogroup, then Γ_{psg} is a compactly generated psg.*

Proof. Let (Γ', T') be a reduction of (Γ, T) and suppose that $\Gamma' = \langle \gamma_1, \dots, \gamma_r \rangle$. If $\gamma \in \Gamma$ and if $x \in \operatorname{dom}\gamma$, then there are elements $\gamma' \in \Gamma'$ and $\alpha, \beta \in \Gamma$ such that $\gamma = \beta \circ \gamma' \circ \alpha$ holds on a neighborhood of x. If $\zeta \in \Gamma_{\mathrm{psg}}$ and $y \in \operatorname{dom}\zeta$, then the restriction of ζ to a neighborhood of y belongs to Γ. Hence $\zeta = \beta \circ \zeta' \circ \alpha$ holds for some $\zeta' \in \Gamma'$ and $\alpha, \beta \in \Gamma$. This implies that $(\Gamma'_{\mathrm{psg}}, T')$ is equivalent to $(\Gamma_{\mathrm{psg}}, T)$ because $\Gamma \subset \Gamma^\times_{\mathrm{psg}}$. Since Γ'_{psg} is generated by $\gamma_1, \dots, \gamma_r, {\gamma_1}^{-1}, \dots, {\gamma_r}^{-1}$, Γ_{psg} is compactly generated. □

Example 3.6. Let f be an endomorphism of $\mathbb{C}P^1$, where $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. If we set $\Gamma = \langle f \rangle$, then $(\Gamma, \mathbb{C}P^1)$ is a compactly generated psg. Indeed, $(\Gamma, \mathbb{C}P^1)$ itself is a reduction. Another reduction can be chosen as follows. Let $U = \{z \in \mathbb{C} \mid |z| < 1 + \varepsilon\}$ and $V = \{z \in \mathbb{C} \mid |z| > 1 - \varepsilon\} \cup \{\infty\}$, where $\varepsilon > 0$ is a fixed small number. Let $\Gamma = \langle f, \mathrm{id}_{U \cap V} \rangle$ and $T = U \amalg V$. Then (Γ, T) is equivalent to the psg on $\mathbb{C}P^1$ generated by f. Note that we can embed T into $\mathbb{C}$. Let now $U' = \{z \in \mathbb{C} \mid |z| < 1 + \varepsilon'\}$ and $V' = \{z \in \mathbb{C} \mid |z| > 1 - \varepsilon'\} \cup \{\infty\}$, where $\varepsilon > \varepsilon' > 0$. If we set $T' = U' \amalg V'$ and $\Gamma' = \Gamma|_{T'}$, then (Γ', T') is a reduction of (Γ, T). On the other hand, if f is an entire map on $\mathbb{C}$ and if we regard f as a local mapping on $\mathbb{C}P^1$ with $\operatorname{dom} f = \mathbb{C}$, then $\langle f \rangle$ is not compactly generated.

Example 3.7. Let Γ be the holonomy pseudogroup of a complex codimension-one transversely holomorphic foliation of a closed manifold. Then Γ is a compactly generated pseudogroup, and Γ_{psg} is a compactly generated pseudosemigroup.

Example 3.8. Even if Γ is a compactly generated psg, $\Gamma^\times$ needs not be a compactly generated pseudogroup. Indeed, let Γ be the psg generated by $f \colon z \mapsto z^2$. Then $(\Gamma, \mathbb{C}P^1)$ is compactly generated but $(\Gamma^\times, \mathbb{C}P^1)$ is not.

The following properties are fundamental.

Lemma 3.9. *Let $\Phi\colon \mathit{\Gamma} \to \mathit{\Delta}$ be a morphism which consists of open mappings. If $(\mathit{\Gamma}, T)$ is compactly generated, then Φ is also compactly generated. That is, there is a finite subset $\{\phi_i\}$ of Φ with the following properties*:

1) *For any $\phi \in \Phi$ and $x \in \operatorname{dom}\phi$, there are ϕ_i, $\gamma \in \mathit{\Gamma}^\times$ and $\delta \in \mathit{\Delta}^\times$ such that $\phi = \delta \circ \phi_i \circ \gamma$ on a neighborhood of x.*
2) *For each i, $\operatorname{dom}\phi_i$ is relatively compact, and there is an element $\widetilde{\phi_i} \in \Phi$ such that $\overline{\operatorname{dom}\phi_i} \subset \operatorname{dom}\widetilde{\phi_i}$ and $\phi_i = \widetilde{\phi_i}|_{\operatorname{dom}\phi_i}$.*

Proof. Let $(\mathit{\Gamma}', T')$ be a reduction of $(\mathit{\Gamma}, T)$. Since $\overline{T'}$ is compact, we can find finite subsets $\{\phi_i\}$ and $\{\widetilde{\phi_i}\}$ of Φ such that $\operatorname{dom}\phi_i$ is relatively compact, $\overline{T'} \subset \bigcup \operatorname{dom}\phi_i$, $\overline{\operatorname{dom}\phi_i} \subset \operatorname{dom}\widetilde{\phi_i}$ and $\widetilde{\phi_i}|_{\operatorname{dom}\phi_i} = \phi_i$. Let $x \in T$ and suppose that $\phi \in \Phi$ is defined on a neighborhood of x. Then, there is an element $\gamma \in \mathit{\Gamma}^\times$ such that $\gamma(x) \in T'$, and some ϕ_i is defined on a neighborhood of $\gamma(x)$. By taking a restriction, we may assume that $\gamma \in \mathit{\Gamma}_0^\times$. Since Φ is a morphism, there are elements $\delta, \delta' \in \mathit{\Delta}^\times$ such that $\phi \circ \gamma^{-1} = \delta \circ \phi_i$ and $\delta' \circ \phi = \phi_i \circ \gamma$. As Φ consists of open mappings, $\delta' \circ \delta = \mathrm{id}_{\phi_i \circ \gamma(x)}$ and $\delta \circ \delta' = \mathrm{id}_{\phi(x)}$, where id_y denotes the identity map on a neighborhood of $y \in S$. Hence $\delta \in \mathit{\Delta}^\times$ and $\phi = \delta \circ \phi_i \circ \gamma$ on a neighborhood of x. □

Lemma 3.10. *Let $(\mathit{\Gamma}, T)$, $(\mathit{\Delta}, S)$ be psg's and suppose that $(\mathit{\Gamma}, T)$ is compactly generated.*

1) *If $\Phi\colon \mathit{\Gamma} \to \mathit{\Delta}$ is a covering or ramified covering, then $(\mathit{\Delta}, S)$ is compactly generated.*
2) *If $(\mathit{\Delta}, S)$ is equivalent to $(\mathit{\Gamma}, T)$, then $(\mathit{\Delta}, S)$ is compactly generated.*

Proof. First we show 1). Let $(\mathit{\Gamma}', T')$ be a reduction of $(\mathit{\Gamma}, T)$. Then, Φ is compactly generated with a set of generators $\{\phi_i\}_{i\in I}$ as in Lemma 3.9. We may assume that each ϕ_i is a homeomorphism or a ramified covering with a single singularity. Suppose that $\mathit{\Gamma}' = \langle \gamma_1, \dots, \gamma_r \rangle$. We may assume that domains and ranges of γ_i's are contained in domains of ϕ_k's. Then, for each i, $\phi_j \circ \gamma_i = \delta \circ \phi_k$ holds for some j, k and $\delta \in \mathit{\Delta}$. If we denote by $\mathit{\Delta}'$ the collection of elements of $\mathit{\Delta}$ obtained in this way, then $\mathit{\Delta}'$ is a finite set. If we set $S' = \bigcup_{i\in I} \phi_i(T' \cap (\operatorname{dom}\phi_i))$, then S' is relatively compact and $(\mathit{\Delta}', S')$ is a reduction of $(\mathit{\Delta}, S)$.

The proof of 2) is almost parallel. Let $(\mathit{\Gamma}', T')$ be a reduction of $(\mathit{\Gamma}, T)$ and suppose that $\mathit{\Gamma}' = \langle \gamma_1, \dots, \gamma_r \rangle$. Let Φ be an equivalence from $\mathit{\Gamma}$ to $\mathit{\Delta}$. Then, Φ is compactly generated with a set of generators $\{\phi_i\}$ as in Lemma 3.9. Let $D = \{\phi_i \circ \gamma_j \circ \phi_k^{-1}\}$, where the composition in the right hand side is taken after restrictions if necessary. Then D is a finite set. We set $S' = \bigcup_{i=1}^r \phi_i(T' \cap (\operatorname{dom}\phi_i))$. Then S' is relatively compact. If $\delta \in \mathit{\Delta}$, then we may assume that there are elements $\phi_1, \phi_2 \in \Phi$ such that $\phi_2^{-1} \circ \delta \circ \phi_1 \in \mathit{\Gamma}$ by taking restrictions. Hence $\Phi|_{T'}$ is an equivalence from $(\mathit{\Gamma}', T')$ to $(\langle D \rangle, S')$. Let Ψ be the equivalence from $(\mathit{\Gamma}', T')$ to $(\mathit{\Gamma}, T)$ induced by the inclusion. Then, $\Phi \circ \Psi \circ (\Phi|_{T'})^{-1}$ is equal to the morphism from $(\langle D \rangle, S')$ to $(\mathit{\Delta}, S)$ induced by the inclusion. □

The next lemma is easy.

Lemma 3.11. *Assume that (Γ, T) is compactly generated and let (Γ', T') be a reduction. If $T' \subset V \subset T$ and V is relatively compact, then (Γ_V, V) is also a reduction of (Γ, T).*

4. FATOU SETS OF COMPACTLY GENERATED PSEUDOSEMIGROUPS

We pose the same assumption as Assumption 2.1 in this section.

Let (Γ, T) be a compactly generated pseudosemigroup. Let $(\Gamma_{T'}, T')$ be a reduction and $\Phi\colon \Gamma_{T'} \to \Gamma$ the equivalence induced by the inclusion.

Theorem 4.1. *Let (Γ, T) a compactly generated psg and $(\Gamma_{T'}, T')$ a reduction. Then $F(\Gamma) = \Phi(F^*(\Gamma_{T'}))$ and $J(\Gamma) = \Phi(J^*(\Gamma_{T'}))$. In addition, we have $F_0(\Gamma) = F(\Gamma)$ and $J_0(\Gamma) = J(\Gamma)$.*

Proof. Let $T'' \in \mathscr{T}$. If $T'' \subset T'$, then $F^*(\Gamma_{T'}) \cap T'' \subset F^*(\Gamma_{T''})$ by Lemma 2.16. If $T'' \supset T'$, then Φ induces an equivalence from T' to T'', which we denote by Φ'. We have $\Phi'(F^*(\Gamma_{T'})) = F^*(\Gamma_{T''})$ by Lemma 2.19. Moreover, since Φ is induced by the inclusions, $F^*(\Gamma_{T'}) = F^*(\Gamma_{T''}) \cap T'$. It follows that $F_0(\Gamma) \cap T' = F^*(\Gamma_{T'})$ if $(\Gamma_{T'}, T')$ is a reduction. Therefore, if $T'' \supset T'$, then we have $F_0(\Gamma) \cap T'' = F^*(\Gamma_{T''}) = \Phi'(F^*(\Gamma_{T'}))$. On the other hand, $\Phi'(F^*(\Gamma_{T'})) = \Phi(F^*(\Gamma_{T'})) \cap T''$ by the definition of Φ'. Since we can find an increasing sequence T_i in $\mathscr{T}$ such that $T = \bigcup_{i=1}^{\infty} T_i$, we have $F_0(\Gamma) = \Phi(F^*(\Gamma_{T'}))$. By taking the complement, we have $J_0(\Gamma) = \Phi(J^*(\Gamma_{T'}))$. The above arguments show that $F_0(\Gamma)$ is an open subset of T. Hence $F(\Gamma) = F_0(\Gamma)$ and $J(\Gamma) = J_0(\Gamma)$. □

Remark 4.2. Theorem 4.1 also holds for compactly generated pseudogroups (cf. [1]). The proof is essentially the same and omitted.

Remark 4.3. If (Γ, T) is compactly generated and if (Γ', T') be a reduction, then $F(\Gamma') = F^*(\Gamma_{T'})$ and $J(\Gamma') = J^*(\Gamma_{T'})$.

Remark 4.4. Let (Γ, T) be a psg. Let $\{D_\lambda\}_{\lambda \in \Lambda}$ be an open covering of T by balls in $\mathbb{C}^q$. If we set $D = \coprod_{\lambda \in \Lambda} D_\lambda$, then elements of Γ can be naturally regarded as local mappings on D if their domains and ranges are contained in D. The psg (Γ_D, D) is equivalent to (Γ, T), indeed, the inclusions of D_λ to T induce an equivalence. Hence, if we discuss Fatou-Julia decompositions, we may assume that T is the disjoint union of open balls in $\mathbb{C}^q$, and that the closure of each balls are also disjoint. Suppose now that (Γ, T) is compactly generated and let (Γ', T') be a reduction of (Γ, T). Then, we can find a finite covering of $\overline{T'}$ by open balls $\{D'_i\}_{i=1}^r$ such that for any i, there exists a λ such that $\overline{D'_i} \subset D_\lambda$. If we set $D' = \coprod_{i=1}^r D'_i$, then $(\Gamma_{D'}, D')$ is equivalent to (Γ', T'). Hence we may assume that each connected component of T' is an open ball and its closure is contained in a connected component of T. In what follows, we assume (Γ, T) and (Γ, T') are as above unless otherwise mentioned. Finally note that if $q = 1$ and if U is a wF-open set for (Γ', T'), then the family Γ^U as in wF2) of Definition 2.2 is always normal by virtue of Montel's theorem.

Proposition 4.5. *If Γ is compactly generated, then wF-open sets are F-open sets. Therefore $wF(\Gamma) = F(\Gamma)$ holds and so on.*

Proof. Let (Γ', T') be a reduction of (Γ, T), $\{\gamma_1, \ldots, \gamma_r\}$ a set of generators of Γ', and $\Phi\colon \Gamma \to \Gamma'$ the equivalence which is the inverse of the inclusion. Let d be a positive real number such that any germ of γ_i at a point $z \in T'$ extends to an element of Γ defined on $D_z(d)$. Let V be a wF-open set in T' and let $\gamma \in \Gamma'$ such that $\operatorname{dom}\gamma \subset V$. We set $U = \operatorname{dom}\gamma$. If $x \in U$, then we can find an open subset U' of U such that $x \in U'$ and that the radius of $\gamma(U')$ is less than $d/2$ for any $\gamma \in \Gamma^V$. Let $y \in \gamma(U')$ and assume that an element $\eta_y \in \Gamma'_y$ is given. We denote by $\Gamma'(k)_y$ the set of the germs of elements of Γ'_y which can be represented as the composite of at most k generators. Then $\Gamma'_y = \bigcup \Gamma'(k)_y$. If $\eta_y \in \Gamma'(1)_y$, namely, $\eta_y = (\gamma_i)_y$ for some i, then γ_i is well-defined on $\gamma(U')$ by the choice of d. Moreover, since $\gamma_i \circ \gamma \in \Gamma^V$, the radius of $\gamma_i(\gamma(U'))$ is less than $d/2$. Suppose that if $\eta_y \in \Gamma'(k)_y$, then η_y extends to an element of Γ and $\eta(\gamma(U'))$ is of radius less than $d/2$. If $\eta_y \in \Gamma'(k+1)_y$, then we have $\eta_y = (\gamma_i \circ \zeta)_y$ for some i and $\zeta_y \in \Gamma'(k)_y$. By the assumption, we may assume that ζ is well-defined on $\gamma(U')$, and the radius of $\zeta(\gamma(U'))$ is less than $d/2$. Again by the choice of d, $\gamma_i \circ \zeta$ extends to an element, say θ, which is well-defined on $\gamma(U')$. Since $\theta \circ \gamma \in \Gamma^V$, the radius of $\theta(\gamma(U'))$ is less than $d/2$. By the construction, $\theta_y = (\gamma_i \circ \zeta)_y = \eta_y$. Since Γ^V is a normal family and γ is an open mapping, $\Gamma^{\gamma(U')}$ is also a normal family. Therefore $\gamma(U')$ is a wF-open set. Since y is arbitrary, $\gamma(U)$ is the union of wF-open sets. □

Theorem 4.1 and Proposition 4.5 imply that the definition of Fatou and Julia sets of compactly generated psg's (and pseudogroups) can be quite reduced compared with those of general psg's. Indeed they be defined without taking infinite number of intersections and unions, nor taking interiors and closures. Moreover, it suffices to deal with wF-open sets instead of F-open sets.

Remark 4.6. The technique using $\Gamma'(k)_y$ in the proof of Proposition 4.5 is from [5, Lemme 2.2]. It is frequently used in what follows.

Fatou sets of compactly generated semigroups have a property similar to those of finitely generated semigroups acting on $\mathbb{C}P^1$ [9], [19].

Lemma 4.7. *Suppose that (Γ, T) is compactly generated. Let (Γ', T') be a reduction of (Γ, T) and $\{\gamma_1, \ldots, \gamma_r\}$ a set of generators of Γ'. Then*

$$F(\Gamma') = \bigcap_{i=1}^{r} (\gamma_i^{-1}(F(\Gamma')) \cup (T' \setminus (\operatorname{dom}\gamma_i))).$$

Proof. It suffices to show that $F(\Gamma') \supset \bigcap_{i=1}^{r} (\gamma_i^{-1}(F(\Gamma')) \cup (T' \setminus (\operatorname{dom}\gamma_i)))$ by Lemma 2.18. Suppose that $x \in \bigcap_{i=1}^{r} (\gamma_i^{-1}(F(\Gamma')) \cup (T' \setminus (\operatorname{dom}\gamma_i)))$. If $x \in \operatorname{dom}\gamma_i$, then there is an open neighborhood U_i of x such that $\gamma_i(U_i)$ is an F-open set. We set $U = \bigcap_{x \in \operatorname{dom}\gamma_i} U_i$. If $\gamma_y \in \Gamma'_y$, where $y \in U$, then $\gamma_y = (\zeta \circ \gamma_i)_y$ holds for some i and $\zeta_{\gamma_i(y)} \in \Gamma'_{\gamma_{i(y)}}$

unless $\gamma_y = (\mathrm{id}_T)_y$. Since $\zeta_{\gamma_i(y)}$ extends to an element of Γ defined on $\gamma_i(U)$, γ_y extends to U. Therefore U is an F-open set which contains x. Hence $F(\Gamma') \supset \bigcap_{i=1}^{r}(\gamma_i^{-1}(F(\Gamma')) \cup (T' \setminus (\operatorname{dom}\gamma_i)))$. □

Remark 4.8. If $\Gamma = \langle f \rangle$, where f is an endomorphism of $\mathbb{C}P^1$, then Lemmata 2.18 and 4.7 are reduced to the usual equalities $F(\langle f \rangle) = f(F(\langle f \rangle)) = f^{-1}(F(\langle f \rangle))$ and $J(\langle f \rangle) = f(J(\langle f \rangle)) = f^{-1}(J(\langle f \rangle))$. Similarly, if $f_1, \dots, f_r$ are endomorphisms of $\mathbb{C}P^1$, then $\operatorname{dom} f_i = \mathbb{C}P^1$ for any i so that we have $F(\Gamma') = \bigcap_{i=1}^{r} f_i^{-1}(F(\Gamma'))$, where $\Gamma' = \langle f_1, \dots, f_r \rangle$. This is the case studied in [9] and [19].

Example 4.9. Lemma 4.7 fails if Γ is not compactly generated and if we do not include id_T in the set of generators. Let $T_1 = T_2 = \mathbb{C}P^1$ and define $\gamma_i \colon T_1 \to T_2$ by $\gamma_i(z) = iz$, and $\zeta \colon T_2 \to T_2$ by $\zeta(z) = z^2$. If we set $\Gamma_n = \langle \zeta, \gamma_1, \dots, \gamma_n \rangle$ and $\Gamma = \langle \zeta, \gamma_1, \dots \rangle$, then Γ_n is compactly generated and Γ is not. We have $J(\Gamma_n) = (\bigcup_{i=1}^{n}\{z \in T_1 \mid |z| = 1/i\}) \cup S^1$ and $J(\Gamma) = (\bigcup_{i=1}^{\infty}\{z \in T_1 \mid |z| = 1/i\} \cup \{0_1\}) \cup S^1$, where 0_1 is the origin in T_1 and S^1 is the unit circle in T_2. It is easy to see that $F(\Gamma_n) = (\zeta^{-1}(F(\Gamma_n)) \cup T_1) \cap \bigcap_{i=1}^{n}(\gamma_i^{-1}(F(\Gamma_n)) \cup T_2)$ and $(\zeta^{-1}(F(\Gamma)) \cup T_1) \cap \bigcap_{i=1}^{\infty}(\gamma_i^{-1}(F(\Gamma)) \cup T_2) = (T_1 \setminus (\bigcup_{i=1}^{\infty}\{z \in T_1 \mid |z| = 1/i\})) \cap (T_2 \setminus S^1) \supsetneq F(\Gamma)$.

1) of Theorem 2.19 holds in a strong form for compactly generated psg's.

Proposition 4.10. *Let (Γ, T) and (Δ, S) be psg's and assume that (Γ, T) is compactly generated. If $\Phi \colon \Gamma \to \Delta$ is either an étale morphism or a ramified morphism if $q = 1$, then $\Phi^{-1}(F(\Delta)) \subset F(\Gamma)$.*

Proof. We proceed as in the proof of Theorem 2.19 but it suffices to deal with wF-open sets instead of F-open sets by Proposition 4.5. Let (Γ', T') be a reduction of (Γ, T) with $\Gamma' = \langle \gamma_1, \dots, \gamma_r \rangle$. Let $d_0 > 0$ such that the germ of any γ_i at a point p in T' extends to an element of Γ defined on $D_p(d_0)$. We retain other notations in the proof of Theorem 2.19. Let U be an open subset of $\Phi^{-1}(F(\Delta))$ and W an open subset of $F(\Delta) \cap \operatorname{range}\phi_1$. We assume that W is a wF-open set in $F^*(\Delta_{S'})$. By shrinking W if necessary, we may assume that if $\delta \in \Delta^W$ and if $\delta(W) \cap \operatorname{range}\phi_k \neq \varnothing$, then $\delta(W) \subset \operatorname{range}\widetilde{\phi}_k$ and the radius of $\widetilde{\phi}_k^{-1}(\delta(W))$ is less than $d_0/2$. Finally, let V be a connected open subset of $\phi_1^{-1}(W)$ such that the radius of V is less than d_0.

Let $y \in V$ and $\zeta_y \in \Gamma'_y$. Then, $\zeta_y \in \Gamma'(m)_y$ for some m. If $m = 1$, then ζ is well-defined on V by the choice of d_0. If $\zeta(y) \in \operatorname{dom}\phi_k$, then there is an element $\delta \in \Delta$ such that $(\phi_k \circ \zeta)_y = (\delta \circ \phi_1)_y$. Note that δ is defined on W as an element of Δ. Since $\zeta(V) \subset \widetilde{\phi}_k^{-1}(\delta \circ \phi_1(V))$, the radius of $\zeta(V)$ is less than $d_0/2$. Assume that the same holds for m, and let $\zeta_y \in \Gamma'(m+1)_y$. We have $\zeta_y = (\gamma_i \circ \eta)_y$ for some i and $\eta \in \Gamma'(m)$. By the assumption, η is well-defined on V and the radius of $\eta(V)$ is less than $d_0/2$. Suppose that $\eta(y) \in \operatorname{dom}\phi_j$ and $\gamma_i(\eta(y)) \in \operatorname{dom}\phi_l$. Then there is an element $\delta_1 \in \Delta^W$ such that $\phi_j \circ \eta = \delta_1 \circ \phi_1$. Note that $\operatorname{range}\widetilde{\phi}_j \supset \delta_1(\phi_1(V))$. On the other hand, there is an element $\delta' \in \Delta_{S'}$ such that $(\phi_l \circ \gamma_i)_{\eta(y)} = (\delta' \circ \phi_j)_{\eta(y)}$. Then, $\gamma_i \circ \eta$ is well-defined on V, and $(\phi_l \circ (\gamma_i \circ \eta))_y = (\delta' \circ \phi_j \circ \eta)_y = ((\delta' \circ \delta_1) \circ \phi_1)_y$.

Since $\delta' \circ \delta_1$ is well-defined on W, we have $\widetilde{\phi_l} \circ (\gamma_i \circ \eta) = (\delta' \circ \delta_1) \circ \phi_1$. Therefore, the radius of $(\gamma_i \circ \eta)(V)$ is less than $d_0/2$, and if we set $\delta_2 = \delta' \circ \delta_1$, then $\delta_2 \in \Delta^W$ and $\widetilde{\phi_l} \circ (\gamma_i \circ \eta) = \delta_2 \circ \phi_1$. Finally since Δ^W is a normal family, Γ^V is also a normal family. Hence V is a wF-open set for $\Gamma' = \Gamma_{T'}$. □

Proposition 4.11. *If Γ is a compactly generated pseudogroup, then $F_{\mathrm{pg}}(\Gamma) = F(\Gamma_{\mathrm{psg}})$ and $J_{\mathrm{pg}}(\Gamma) = J(\Gamma_{\mathrm{psg}})$.*

Proof. Let (Γ', T') be a reduction of (Γ, T) in the sense of pseudogroups. Then, $(\Gamma'_{\mathrm{psg}}, T')$ is a reduction of $(\Gamma_{\mathrm{psg}}, T)$. By Theorem 4.1, it suffices to show that $J^*_{\mathrm{pg}}(\Gamma') = J^*(\Gamma'_{\mathrm{psg}})$. Let U be a wF-open subset of $F^*_{\mathrm{pg}}(\Gamma')$ and $x \in U$. If γ is the germ of an element of Γ'_{psg} at x, then γ is the germ of an element of Γ'. Hence γ extends to an element of Γ defined on U, and $(\Gamma_{\mathrm{psg}})^U = \Gamma^U$. Therefore U is a wF-open set for Γ'_{psg}.

Conversely let $U \subset F^*(\Gamma'_{\mathrm{psg}})$ be a wF-open set in the sense of psg's. Then $U \subset F_{\mathrm{pg}}(\Gamma')$. Indeed, let $\{\gamma_1, \dots, \gamma_r\}$ be a set of generators of Γ'. There is a $d_1 > 0$ such that if γ is the germ of one of the γ_i's at a point, say x, in T', then γ is extends to an element of Γ defined on $D_x(2d_1)$. Let $x \in U$ and $V = D_x(d_1)$. By shrinking V if necessary, we may assume that $V \subset U$ and that $\gamma(V)$ is contained in ball of radius d_1 for any $\gamma \in \Gamma^U_{\mathrm{psg}}$. Let $y \in V$ and $\Gamma'(k)_y$ the set of germs of elements of Γ' which can be represented as the composite of at most k generators. Then $\Gamma'_y = \bigcup_{k=0}^\infty \Gamma'(k)_y$. Let $\gamma_y \in \Gamma'(k)_y$. If $k = 1$, then γ_y extends to an element of Γ defined on V. Suppose that germs of elements of $\Gamma'(k)_y$ extends to an element of Γ defined on V, and let γ_y an element of $\Gamma'(k+1)_y$. If we decompose $\gamma_y = (\gamma_i \circ \zeta)_y$, where $\zeta_y \in \Gamma'(k)_y$, then ζ_y extends to an element of Γ defined on V. Since $\zeta(V)$ is contained in a disc of radius d_1 and $\zeta(y) \in T'$, $\gamma_i \circ \zeta$ is well-defined on V. As being the composite of diffeomorphisms, $\gamma_i \circ \zeta$ belongs to Γ. Since $\Gamma^V \subset (\Gamma_{\mathrm{psg}})^U$, Γ^V is a normal family. □

Proposition 4.12. *Let (Γ, T) be a compactly generated pseudogroup, and denote by $F'(\Gamma)$ and $J'(\Gamma)$ its Fatou and Julia sets in the sense of* [1], *respectively. Then $F'(\Gamma) = F_{\mathrm{pg}}(\Gamma) = F(\Gamma_{\mathrm{psg}})$ and $J'(\Gamma) = J_{\mathrm{pg}}(\Gamma) = J(\Gamma_{\mathrm{psg}})$.*

Proof. Let (Γ', T') be a reduction of (Γ, T) and Φ be the equivalence from (Γ', T') to (Γ, T) induced by the inclusion. Then $F'(\Gamma) = \Phi(F^*_{\mathrm{pg}}(\Gamma_{T'}))$ and $J'(\Gamma) = \Phi(J^*_{\mathrm{pg}}(\Gamma_{T'}))$. Hence the claim follows from Theorem 4.1 and Proposition 4.11. □

Example 2.3, Proposition 4.11 and [1, Example 8.3] are summarized as follows.

Theorem 4.13. *The Julia sets of rational mappings on $\mathbb{C}P^1$, the limit sets of finitely generated Kleinian groups acting on $\mathbb{C}P^1$ and the Julia set of compactly generated pseudogroups in the sense of* [1] *can be regarded as Julia sets of compactly generated pseudosemigroups. If we regard entire mappings on $\mathbb{C}$ as local mapping on $\mathbb{C}P^1$, then their Julia sets can be regarded as Julia sets of non-compactly generated pseudosemigroups.*

Proof. If Γ is a finitely generated Kleinian group, then Γ generates a compactly generated pseudogroup on $\mathbb{C}P^1$. If we denote this pseudogroup by Γ_{pg}, then $J_{\mathrm{pg}}(\Gamma_{\mathrm{pg}})$ coincides with the limit set of Γ ([1, Example 8.3]). □

We refer to [15] and [16] for properties of the Julia sets of mapping iterations, to [14] for properties of the limit sets of Kleinian groups.

Remark 4.14. Even if Γ is a Kleinian group but not finitely generated, we can regard $(\Gamma, \mathbb{C}P^1)$ as a pseudogroup or a pseudosemigroup, which are not compactly generated.

Remark 4.15. Let $(\varGamma, T)$ be a compactly generated pseudosemigroup. If $T = \mathbb{C}P^1$, then it is natural to assume that $\varGamma$ is generated by rational mappings and biholomorphic diffeomorphisms defined on $\mathbb{C}P^1$. It is well-known that the Julia sets are infinite set (in fact, perfect) and the limit sets are also infinite unless they consist of at most 2 points. In view of Theorem 4.13, such a property can be seen as one of common properties of Julia sets of groups and semigroups acting on $\mathbb{C}P^1$. On the other hand, if $T \neq \mathbb{C}P^1$, then there are examples of compactly generated pseudogroups of which Julia sets are finite but consist of more than 2 points [1, Examples 8.1 and 8.2].

Dynamics on $F(\varGamma)$ is expected to be tame. For example, on the Julia sets of rational mappings and on the limit sets of finitely generated Kleinian groups, the $\varGamma$-action is contracting or isometric with respect to the hyperbolic metric except elementary cases. We can find a volume form which has a similar property. If $q = 1$, then we can find a metric.

Let $(\varGamma', T')$ be a reduction of $(\varGamma, T)$. We may assume that $T' = \coprod_{i=1}^r T_i'$, where each T_i' is the unit open ball in $\mathbb{C}^q$ (see Remark 4.4). Let η_ε, $0 < \varepsilon < 1$, be a smooth non-negative function on $\mathbb{R}$ such that

1) $\eta_\varepsilon(t) = 1$ on $(-\infty, 1-\varepsilon]$,
2) η_ε is strictly decreasing on $[1-\varepsilon, 1]$,
3) $\eta_\varepsilon(t) = 0$ on $[1, +\infty)$.

Definition 4.16. Let $z_i = (z_i^1, \dots, z_i^q)$ be the standard coordinates on T_i' and set $h_i(z_i) = \eta_\varepsilon(\|z_i\|)$, where $\|\cdot\|$ denotes the standard norm on $\mathbb{C}^q$. The set of functions $\{h_i\}$ is denoted by h and considered as a function on T'. We will represent functions and differential forms on T' in the same way. We define a function f on T' by

$$f(x) = \sup_{\gamma \in (\varGamma')_x} |J\gamma_x| \, h(\gamma(x)),$$

where $|J\gamma_x|$ denotes the absolute value of the Jacobian of γ at x. We set $g = f^2 g_0$ if $q = 1$, $\omega = f^2 \mu_0$ if $q \geq 1$, where g_0 and μ_0 denote the standard Hermitian metric and volume form on $\mathbb{C}^q$, respectively. We denote g_0 also by $dz \otimes d\bar{z}$.

A metric or a volume form as above is said to be lower semicontinuous (resp. locally Lipschitz continuous) if f is lower semicontinuous (resp. locally Lipschitz continuous).

Theorem 4.17 (cf. [1, Lemmata 3.8 and 3.9])**.** *The metric g and volume form ω in Definition 4.16 are lower semicontinuous on T'. Moreover, g and ω are finite and locally Lipschitz continuous on $F(\Gamma')$.*

Proof. The first part is easy. We will show the second part. Let $x \in F(\Gamma')$ and U a wF-open set which contains x. Then Γ^U is a normal family so that $\sup_{\gamma\in\Gamma^U} |J\gamma_x|$ and $f(x)$ are finite. By slightly shrinking U, we may assume that there exists an $m > 0$ such that $|J\gamma_y| \le m$ holds for any $y \in U$ and $\gamma_y \in \Gamma'_y$ because Γ^U is a normal family. We may also assume that $U = D_x(d)$. We will show the following

Claim. There are $\varepsilon_1 > 0$, d_1 and $c > 0$ such that if $y \in D_x(d_1)$ and $h(\gamma(y))\,|J\gamma_y| > f(y)-\varepsilon_1$, then $\gamma \in \Gamma'_y$ induces an element of Γ' defined on $D_x(d_1)$, and $|J\gamma_w| \ge c$ for any $w \in D_x(d_1)$.

Let ε_1 be a positive real number less than $f(x)/2$. Then there is a positive real number d_2 such that $f(y)-f(x) > \varepsilon_1$ for $y \in D_x(d_2)$ by the lower semicontinuity of f. It follows that $f(y)-\varepsilon_1 > f(x) > \frac{f(x)}{2}$ because $f(x) > 0$. Hence, if $|x-y| < \min\{d, d_2\}$ and $h(\gamma(y))\,|J\gamma_y| > f(y)-\varepsilon_1$, then $h(\gamma(y)) \ge \frac{f(x)}{2m} > 0$. It follows that there is a compact subset K' of T' independent of y such that $h(\gamma(y))\,|J\gamma_y| > f(y)-\varepsilon_1$ holds only if $\gamma(y) \in K'$. Note that under the same assumptions, we have $|J\gamma_y| > f(y)-\varepsilon_1 > \frac{f(x)}{2}$. Since Γ^U is a normal family, there is a $d_3 > 0$ such that $|J\gamma_w| \ge \frac{f(x)}{3}$ holds if $|w-y| < d_3$. Let ε_2 be a positive real number such that $D_{K'}(\varepsilon_2) \subset T'$, and d_4 a positive real number such that the radius of $\gamma(D_x(d_4))$ is less than $\varepsilon_2/2$ if $\gamma \in \Gamma^U$. We set $d_1 = \min\{d, d_2, d_3/2, d_4\}$ and $c = f(x)/3$. If $y \in D_x(d_1)$ and $h(\gamma(y))\,|J\gamma_y| > f(y)-\varepsilon_1$, then $\gamma(y) \in K'$. If we denote again by γ the extension of γ_y to an element of Γ^U, then $\gamma(D_x(d_1)) \subset D_{\gamma(y)}(\varepsilon_2) \subset T'$. Hence $\gamma \in \Gamma'$. If $w \in D_x(d_1)$, then $|y-w| < d_3$ so that $|J\gamma_w| \ge c$. This completes the proof of Claim. Note that such a γ belongs to Γ^U.

Let ε_3 be any positive real number less than ε_1 and assume that $y, z \in D_x(d_1)$. Let $\gamma \in \Gamma'_y$ such that $h(\gamma(y))\,|J\gamma_y| > f(y)-\varepsilon_3$. Then $\gamma_z \in \Gamma'_z$ so that $h(\gamma(z))\,|J\gamma_z| \le f(z)$. Hence $f(y)-f(z) < h(\gamma(y))\,|J\gamma_y| - h(\gamma(z))\,|J\gamma_z| + \varepsilon_3$. Since Γ^U is a normal family and each h_i is Lipschitz continuous, there is a Lipschitz constant L for $h\circ\gamma$ independent of γ, namely, $|h(\gamma(y)) - h(\gamma(z))| \le L\,|y-z|$ holds independent of γ (note that it suffices to assume that each h_i is locally Lipschitz continuous if we reduce d_1 if necessary). On the other hand, for each γ, we have

$$|J\gamma_y| - |J\gamma_z| = \frac{|J\gamma_y|^2 - |J\gamma_z|^2}{|J\gamma_y| + |J\gamma_z|} \le \frac{1}{2c} \sup_{w\in D_x(d_1)} 2\,|J\gamma_w|\, q! L_1(\gamma)^{q-1} L_2(\gamma)\,|y-z|,$$

on U, where

$$L_1(\gamma) = \sup_{\substack{1\leq i,j\leq q\\ w\in D_x(d_1)}} \left|\frac{\partial\gamma^i}{\partial z^j}(w)\right|, \quad L_2(\gamma) = \sup_{\substack{1\leq i,j,k\leq q\\ w\in D_x(d_1)}} \left|\frac{\partial^2\gamma^i}{\partial z^j\partial z^k}\right|.$$

Again since Γ^U is a normal family, the above inequality implies that there is a constant L' independent of γ such that $\left|J\gamma_y\right| - |J\gamma_z| \leq L'|y-z|$. Therefore,

$$\begin{aligned} f(y) - f(z) - \varepsilon_3 &< h(\gamma(y))(\left|J\gamma_y\right| - |J\gamma_z|) + (h(\gamma(y)) - h(\gamma(z)))\,|J\gamma_z| \\ &\leq L'|y-z| + L\,|y-z|\,m \\ &= (L' + Lm)\,|y-z|. \end{aligned}$$

Since this estimate is independent of the choice of γ, ε_3 can be arbitrarily small. Hence $f(y) - f(z) \leq (L'+Lm)\,|y-z|$.

By exchanging the role of y and z, we have $f(z) - f(y) \leq (L'+Lm)\,|y-z|$ if $y,z\in D_x(d_1)$. This completes the proof. □

Note that we need only the compactness of $\overline{T'}$ in the construction. The fact that Γ is compactly generated is used only to regard the metric on $F(\Gamma')$ as a metric on $F(\Gamma)$.

Definition 4.18. Let g_1 and g_2 be Hermitian metrics on $F(\Gamma)$. If $z\in F(\Gamma)$, then we denote by $(g_1)_z$ the metric on $T_zF(\Gamma)$. Suppose that we have $g_1 = f_1^2 dz\otimes d\overline{z}$ and $g_2 = f_2^2 dz\otimes d\overline{z}$ on a neighborhood of z. If $f_1(z)\leq f_2(z)$, then we write $(g_1)_z \leq (g_2)_z$. Note that this condition is independent of the choice of charts about z. If $(g_1)_z\leq (g_2)_z$ holds on $F(\Gamma)$, then we write $g_1\leq g_2$. If ω_1 and ω_2 are volume forms on $F(\Gamma)$, then we say $\omega_1\leq\omega_2$ in the same way.

The action of Γ on $F(\Gamma)$ has the following property which we call *semi-invariance*.

Proposition 4.19. *If $x\in F(\Gamma')$ and if $\gamma\in\Gamma'$ is defined on a neighborhood of x, then $\gamma^*g\leq g$ and $\gamma^*\omega\leq\omega$. If $\gamma\in(\Gamma')^\times$, then $\gamma^*g = g$ and $\gamma^*\omega=\omega$.*

Note that if $x\in F(\Gamma')$, $\gamma\in\Gamma'$ and $J\gamma_x = 0$, then $(\gamma^*g)_x = 0$ so that there is no Γ'-invariant metric (nor volume form) on $F(\Gamma')$.

Proof. If $|J\gamma_x| = 0$, then $(\gamma^* g)_x = 0$. Suppose that $|J\gamma_x| \neq 0$. If we set $\Gamma'_{\gamma(x)} \circ \gamma = \{\zeta \circ \gamma \,|\, \zeta \in \Gamma'_{\gamma(x)}\}$, then $\Gamma'_{\gamma(x)} \circ \gamma \subset \Gamma'_x$. It follows that

$$\begin{aligned} f(\gamma(x)) &= \sup_{\zeta \in \Gamma'_{\gamma(x)}} |J\zeta_{\gamma(x)}| h(\zeta(\gamma(x))) \\ &= \frac{1}{|J\gamma_x|} \sup_{\eta \in \Gamma'_{\gamma(x)} \circ \gamma} |J\eta_x| h(\eta(x)) \\ &\leq \frac{1}{|J\gamma_x|} \sup_{\eta \in \Gamma'_x} |J\eta_x| h(\eta(x)) \\ &= \frac{1}{|J\gamma_x|} f(x). \end{aligned}$$

Hence $(\gamma^* g)_x \leq g_x$ and $(\gamma^* \omega)_x \leq \omega_x$. □

Theorem 4.20. *Let (Γ, T) be a psg which is not necessarily compactly generated. Suppose that $\Gamma = \Gamma^\times$. If $q = 1$, then there is an invariant Hermitian metric on $F(\Gamma)$ which is locally Lipschitz continuous. In general, there is an invariant volume form on $F(\Gamma)$ which is locally Lipschitz continuous.*

Note that $\Gamma = \Gamma^\times$ holds if and only if Γ is generated by a pseudogroup. Indeed, $\Gamma^\times = \langle \Gamma_0^\times \rangle$. See Definition 1.8.

Proof. We show the theorem for g because the proof for ω is completely parallel. By replacing (Γ, T) by equivalence we may assume that $T \subset \mathbb{C}$. We will construct a metric on $F(\Gamma)$. Let $\{T_i\}_{i=1}^\infty \subset \mathscr{T}$ such that $\overline{T_i} \subset T_{i+1}$ and $T = \bigcup_{i=1}^\infty T_i$. We have $F(\Gamma) = \mathrm{int}\big(\bigcap_{i=1}^\infty F^*(\Gamma_{T_i})\big)$ by Lemma 2.16, where int denotes the interior. Let h_i, where $i > 1$, be a smooth function on T such that

1) h_i is positive on T_i.
2) $h_i = 1$ on T_{i-1}.
3) If $x, y \in T_i \setminus T_{i-1}$ and if $d(x, \overline{T_{i-1}}) < d(y, \overline{T_{i-1}})$ then $h_i(x) > h_i(y)$.
4) $h_i = 0$ on $T \setminus T_i$,

where d denotes the distance with respect to the standard Hermitian metric on $\mathbb{C}$. We set $F_i = F(\Gamma) \cap T_i$. Let g_2 be the metric on $F^*(\Gamma_{T_2})$ obtained from $h_2^2 dz \otimes d\bar{z}$ as in Definition 4.16, namely, we set $f(z) = \sup_{\gamma \in \Gamma_{T_2}} |J\gamma_x| h_2(\gamma(x))$ and $g_2 = f^2 dz \otimes d\bar{z}$. Then, g_2 is invariant under the Γ_{T_2}-action. We have a metric $\widehat{g}_1$ on F_1 with the following properties with $k = 1$:

1) $\widehat{g}_k$ is invariant under the Γ-action.
2) There are a neighborhood F'_k of $\overline{F_k} \cap F(\Gamma)$ in $F(\Gamma)$ and a locally Lipschitz continuous, Γ-invariant metric $\widehat{g}'_k$ on F'_k such that the restriction of $\widehat{g}'_k$ to F_k is equal to $\widehat{g}_k$ (indeed it suffices to define $\widehat{g}'_k = g_{k+1}|_{F'_k}$).

We call this condition the condition (M$_k$). We extend $\widehat{g}'_1$ to a metric g'_3 on $\Gamma_{T_3}(F'_1)$ by the Γ_{T_3}-action. This is indeed possible. Let $x \in \Gamma_{T_3}(F'_1)$ and let $\gamma_1, \gamma_2 \in \Gamma_{T_3}$ such that $\gamma_1(x), \gamma_2(x) \in F'_1$. If $|(J\gamma_1)_x| \neq |(J\gamma_2)_x|$, then we set $\eta = \gamma_2 \circ (\gamma_1)^{-1}$. The family

$\{\eta^n\}_{n\in\mathbb{Z}}$ cannot be normal on any neighborhood of x. Hence $|(J\gamma_1)_x| = |(J\gamma_2)_x|$ so that the extension exists.

If we denote by G_1 the closure of $\Gamma_{T_3}(F_1)$ in F_3, then $G_1 \subset \Gamma_{T_3}(F_1')$. Indeed, let $x \in G_1$ and U an F-open set for Γ_{T_3} which contains x. We can find a sequence $\{x_i\}$ in F_1 and a sequence $\{\gamma_i\}$ in Γ_{T_3} such that $\{\gamma_i(x_i)\}$ converges to x. We may assume that $\gamma_i(x_i) \in U$. Let $d > 0$ such that $D_x(d) \subset F_1'$ if $x \in F_1$. We may also assume that if $\gamma \in \Gamma^U$, then the radius of $\gamma(U) < d/8$. We regard γ_1^{-1} as an element of Γ^U and set $y_i = \gamma_1^{-1}\gamma_i(x_i)$. As $\{\gamma_i(x_i)\}$ converges to x, $\{y_i\}$ converges to $y = \gamma_1^{-1}(x)$. On the other hand, if we denote by $d(p,q)$ the Euclidean distance between p and q, then $d(y_i,y) \leq d(y_i,y_1) + d(y,y_1) \leq d/4 + d/4 < d$. Therefore $y \in F_1'$ and we have $x = \gamma_1(y) \in \Gamma_{T_3}(F_1')$.

Let $\widehat{f}_1$ be the function on G_1 such that $g_3' = \widehat{f}_1^2 dz \otimes d\bar{z}$, and let $\widetilde{f}_1 = \widehat{f}_1/(1+\widehat{f}_1)$. Then, we can find an extension φ_3 of $\widetilde{f}_1$ to F_3 such that φ_3 is locally Lipschitz continuous and $0 < \varphi_3 < 1$ holds. We set $\psi_3 = h_3\varphi_3/(1-\varphi_3)$ and $\widetilde{g}_3' = \psi_3^2 dz \otimes d\bar{z}$. Let $\widehat{g}_3$ be the metric on F_3 constructed from g_3' as in Definition 4.16, namely, we set $f(z) = \sup_{\gamma\in\Gamma_{T_3}} |J\gamma_x| h_3(\gamma(x))\Psi_3(\gamma(x))$ and $\widehat{g}_3 = f^2 dz \otimes d\bar{z}$. Since $g_3'|_{F_1} = \widehat{g}_1$, $\widetilde{g}_3'|_{\Gamma_{T_3}(F_1)} \leq \widehat{g}_1$ and since g_3' is Γ_{T_3}-invariant, we have $\widehat{g}_3|_{F_1} = \widehat{g}_1$. If we set $\widehat{g}_2 = \widehat{g}_3|_{F_2}$, then $\widehat{g}_2$ satisfies the condition (M_2). By repeating this procedure inductively, we obtain a Hermitian metric on $F(\Gamma)$ which is Γ-invariant and locally Lipschitz continuous. □

Example 4.21 (see also Example 2.11)**.** We define γ: $\mathbb{C}P^1 \to \mathbb{C}P^1$ by $\gamma(z) = z^2$. Then, $J(\gamma) = \{|z| = 1\}$. If we set

$$f(z) = \begin{cases} 1 & \text{if } |z| \leq \frac{1}{2}, \\ 2^k |z|^{2^k-1} & \text{if } 2^{-\frac{1}{2^{k-1}}} \leq |z| \leq 2^{-\frac{1}{2^k}}, \\ 2^k |z|^{-2^k-1} & \text{if } 2^{\frac{1}{2^k}} \leq |z| \leq 2^{\frac{1}{2^{k-1}}}, \\ \frac{1}{|z|^2} & \text{if } |z| \geq 2, \end{cases}$$

then $g = f^2 dz \otimes d\bar{z}$ gives a Hermitian metric on $\mathbb{C}P^1 \setminus \{|z| = 1\}$ which is locally Lipschitz continuous and semi-invariant under the action of Γ, where $\Gamma = \langle\gamma\rangle$. On the other hand, if we consider the Poincaré metric on the unit disc, then γ is contracting by the Schwarz lemma. Hence the Poincaré metrics on the unit disc and $\mathbb{C}P^1 \setminus \{|z| \leq 1\}$ give rise to a Hermitian metric on $\mathbb{C}P^1 \setminus \{|z| = 1\}$ which is of class C^ω and semi-invariant under the action of Γ. On the other hand, there is no Γ-invariant metric on $F(\Gamma)$. Indeed, $0 \in F(\Gamma)$ but $(\gamma^* g)_0 = 0$ for any metric g on $F(\Gamma)$.

Let $\widehat{\Gamma}$ be the psg generated by $\gamma|_{\mathbb{C}P^1\setminus\{0,\infty\}}$ and its local inverses. Then $F(\widehat{\Gamma}) = \mathbb{C} \setminus (S^1 \cup \{0\})$. An invariant metric on $F(\widehat{\Gamma})$ is given by $dz \otimes d\bar{z}/(|z|\log|z|)^2$ on $\{0 < |z| < 1\}$. We can find on $\{1 < |z|\}$ a metric of the same kind.

Remark 4.22. If Γ is a compactly generated pseudogroup, then we can classify Fatou components. By using the classification, we can always find a Γ-invariant metric of class C^ω [1, Theorem 4.21]. See also Theorem 5.9.

Remark 4.23. Let $S_1 = S_2 = \mathbb{C}$ and we denote by $D_i(r)$ the open disc in S_i of radius r and centered at the origin. Let $\gamma\colon S_1 \to S_2$ be the identity map. We set $T = S_1 \amalg S_2$ and $\Gamma = \langle\gamma\rangle$. Then $F(\Gamma) = T$. We define $T_i \in \mathscr{T}$ by setting $T_i = D_1(i) \amalg D_2(i)$. Then the metric obtained from $\{T_i\}$ is equal to the one induced from the standard Hermitian metric on $\mathbb{C}$.

A kind of the converse of Theorem 4.17 holds for compactly generated psg's. A metric g on an open subset U of T is said to be *bounded from below* if there exists $c > 0$ such that $cg_0 \le g$ holds on U, where g_0 is the standard metric on $\mathbb{C}^q$.

Proposition 4.24 (cf. [1, Lemma 2.6])**.** *Let (Γ, T) be a compactly generated psg. If U is forward Γ-invariant and if U admits a continuous Hermitian metric which is semi-invariant and bounded from below, then $U \subset F(\Gamma)$.*

Proof. By Proposition 4.5, it suffices to show that U is contained in $wF(\Gamma)$. Let (Γ', T') be a reduction of (Γ, T) and suppose that $\Gamma' = \langle\gamma_1, \dots, \gamma_r\rangle$. Then, there exists $d > 0$ such that the germ of γ_i at $x \in T'$ extends to an element of Γ defined on $D_x(d)$, where $D_x(d)$ denotes the d-ball centered at x with respect to the standard metric. If $y \in U$, then let $V = D^g_y(cd/4)$, where $D^g_y(cd/4)$ denotes the $(cd/4)$-ball centered at y with respect to g. Since $D^g_y(cd/4) \subset D_y(d/2)$, we may assume that $V \subset U$. Let $z \in V$ and $\gamma_z \in \Gamma'(k)_z$, where $\Gamma'(k)_z$ denotes the set of germs of elements of Γ' which can be represented at most the composition of k generators. If $k = 1$, then γ_z extends to an element, say γ, of Γ defined on V. Moreover, since g is semi-invariant, we have $\gamma(V) = \gamma(D^g_y(cd/4)) \subset D^g_{\gamma(y)}(cd/4) \subset D_{\gamma(y)}(d/2) \subset D_{\gamma(z)}(d)$. Assume that $\gamma_z \in \Gamma'(k)_z$ extends to an element, say γ, of Γ defined on V, and $\gamma(V) \subset D_{\gamma(z)}(d)$. If $\gamma_z \in \Gamma'(k+1)_z$, then we have $\gamma_z = (\gamma_i \circ \zeta)_z$ for some $\zeta_z \in \Gamma'(k)_z$ and γ_i. By the assumption, ζ_z extends to an element, say ζ, of Γ defined on V, and $\zeta(V) \subset D_{\zeta(z)}(d)$. As γ_i also extends to $D_{\zeta(z)}(d)$ because $\zeta(z) \in T'$, $(\gamma_i \circ \zeta)_z$ extends to an element, say η, of Γ defined on V, and we have $\eta(V) \subset D_{\eta(z)}(d)$ by the same argument as above. □

If g is not bounded from below, then the conclusion fails. See Example 5.13. If (Γ, T) is not compactly generated, then there is also a counterexample.

Example 4.25. Let $T_1 = T_2 = \mathbb{C}$ and let $f\colon T_1 \to T_2$ be the inclusion of the open unit disc viewed as a local mapping. Then, the metric on $T_1 \amalg T_2$ induced from the standard metric on $\mathbb{C}$ is invariant under $\langle f\rangle$ but $J(\langle f\rangle) = \{z \in T_1 \mid |z| = 1\}$.

5. Fatou-Julia decomposition for singular holomorphic foliations

For generalities on singular holomorphic foliations we refer to [2] and [20]. Here we follow the latter. Let M be a connected complex manifold and TM the holomorphic tangent bundle of M. We denote by $\mathscr{O}_M$ the tangent sheaf of M. If $\mathscr{S}$ is a coherent sheaf on M, then we set

$$\operatorname{Sing}(\mathscr{S}) = \{x \in M \,|\, \mathscr{S}_x \text{ is not } \mathscr{O}_{M,x}\text{-free}\},$$

where $\mathscr{S}_x$ and $\mathscr{O}_{M,x}$ denote the stalks at x of $\mathscr{S}$ and $\mathscr{O}_M$, respectively. The *rank* of $\mathscr{S}$ is defined to be the rank of the locally free sheaf $S|_{M\setminus\operatorname{Sing}(\mathscr{S})}$, and denoted by $\operatorname{rank}\mathscr{S}$.

Definition 5.1. The *tangent sheaf* $\mathscr{F}$ of a singular foliation of M is an integrable coherent subsheaf of $\mathscr{O}_M$, that is, $\mathscr{F}$ is a coherent subsheaf of $\mathscr{O}_M$ such that

$$[\mathscr{F}_x, \mathscr{F}_x] \subset \mathscr{F}_x \quad \text{for } x \in M \setminus S(\mathscr{F}),$$

where

$$S(\mathscr{F}) = \operatorname{Sing}(\mathscr{O}_W/\mathscr{F}).$$

The set $S(\mathscr{F})$ is called the *singular set of* $\mathscr{F}$. The *dimension of* $\mathscr{F}$ is defined to be $\operatorname{rank}\mathscr{F}$ and denoted by $\dim\mathscr{F}$. The *codimension of* $\mathscr{F}$ is defined to be $\dim M - \operatorname{rank}\mathscr{F}$ and denoted by $\operatorname{codim}\mathscr{F}$.

We call $\mathscr{F}$ a singular foliation by abuse of notation.

Remark 5.2. $S(\mathscr{F})$ is an analytic set which contains $\operatorname{Sing}\mathscr{F}$.

Let M be a complex manifold and $\mathscr{F}$ a singular foliation of M. Then, $\mathscr{F}$ defines a non-singular foliation of codimension $\operatorname{codim}\mathscr{F}$ on $M \setminus S(\mathscr{F})$, which we denote by $\mathscr{F}^{\mathrm{reg}}$.

Let M be a complex manifold and $\mathscr{F}$ a singular foliation of M. We choose a complete transversal T for $\mathscr{F}^{\mathrm{reg}}$, and let Γ be the holonomy pseudogroup of $\mathscr{F}^{\mathrm{reg}}$ with respect to T. Note that $F_{\mathrm{pg}}(\Gamma)$ and $J_{\mathrm{pg}}(\Gamma)$ are Γ-invariant.

Definition 5.3. We set

$$\begin{aligned}
F_0(\mathscr{F}) &= \text{the saturation of } F_{\mathrm{pg},0}(\Gamma) \text{ by leaves of } \mathscr{F}^{reg},\\
J_0(\mathscr{F}) &= M \setminus F_0(\mathscr{F}),\\
F(\mathscr{F}) &= \text{the saturation of } F_{\mathrm{pg}}(\Gamma) \text{ by leaves of } \mathscr{F}^{reg},\\
J(\mathscr{F}) &= M \setminus F(\mathscr{F}).
\end{aligned}$$

If we replace T by another complete transversal T', then the holonomy pseudogroup with respect to T' is equivalent to Γ. Hence $F_0(\mathscr{F})$, $J_0(\mathscr{F})$, $F(\mathscr{F})$ and $J(\mathscr{F})$ are well-defined.

Remark 5.4. Note that $F(\mathscr{F})$ is the interior of $F_0(\mathscr{F})$ and $J(\mathscr{F}) = \overline{J_0(\mathscr{F})}$. Note also that $S(\mathscr{F}) \subset J_0(\mathscr{F}) \subset J(\mathscr{F})$ by the definition. Actually, $J(\mathscr{F}) \setminus S(\mathscr{F})$ is the saturation of $J_{\mathrm{pg}}(\Gamma)$, where Γ is the holonomy pseudogroup of $\mathscr{F}$.

We can find $F(\mathscr{F})$ and $J(\mathscr{F})$ as follows. We denote by p and q the real dimension and complex codimension of $\mathscr{F}^{\mathrm{reg}}$, respectively. Let $\mathscr{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be a foliation atlas for $\mathscr{F}^{\mathrm{reg}}$, namely,

1) each U_λ is homeomorphic to $V_\lambda \times D_\lambda$, where V_λ is an open subset of $\mathbb{R}^p$ and D_λ is an open subset of $\mathbb{C}^q$, and
2) the connected components of the intersection of leaves of $\mathscr{F}^{\mathrm{reg}}$ with U_λ is given by $V_\lambda \times \{p\}$, $p \in D_\lambda$.

We may assume $\{U_\lambda\}_{\lambda\in\Lambda}$ is a refinement of a foliation atlas, and each U_λ is relatively compact. In addition, we assume without loss of generality that each D_λ is an open ball. We set $T = \coprod_{\lambda\in\Lambda} D_\lambda$ and let Γ be the holonomy pseudogroup with respect to T. We assume without generality that Λ is countable, and denote the indices by i. If we set $T_k = \coprod_{i=1}^k D_k$, then $F_{\mathrm{pg},0}(\Gamma) = \bigcap_{k=1}^\infty \widetilde{F}^*_{\mathrm{pg}}(\Gamma_{T_k})$ (see also Lemma 2.16).

The following is a direct consequence of Theorem 4.20.

Theorem 5.5. *If $q = 1$, then $F(\mathscr{F})$ admits a transverse invariant Hermitian metric which is transversely Lipschitz continuous. In general $F(\mathscr{F})$ admits a transverse invariant volume form which is transversely Lipschitz continuous.*

Indeed, if Γ is the holonomy pseudogroup of $\mathscr{F}^{\mathrm{reg}}$ with respect to a complete transversal T, then T admits a Γ-invariant Hermitian metric which is Lipschitz continuous. Transverse invariant volume form can be constructed in the same way.

If M is closed and $S(\mathscr{F}) = \varnothing$, then Γ is compactly generated so that we may assume (Γ, T) is equivalent to (Γ_{T_k}, T_k) for some k. If moreover $\mathscr{F}$ is of codimension one, then we have a transversely holomorphic foliation of complex codimension one, and a Fatou-Julia decomposition of such a foliation is given in [6], [8] and [1]. We denote the Fatou and Julia sets of $\mathscr{F}$ in the sense of [1] by $F_{\mathrm{fol}}(\mathscr{F})$ and $J_{\mathrm{fol}}(\mathscr{F})$, respectively. Then by the definitions, we have the following

Proposition 5.6. *If M is closed and $\mathscr{F}$ is regular, then we have $F_{\mathrm{fol}}(\mathscr{F}) = F(\mathscr{F}) = F_0(\mathscr{F})$ and $J_{\mathrm{fol}}(\mathscr{F}) = J(\mathscr{F}) = J_0(\mathscr{F})$.*

In what follows, we will study holomorphic foliations by curves with isolated singularities. Let $\mathscr{F}$ be such a foliation of a complex $(n+1)$-dimensional manifold M and let $S(\mathscr{F}) = \{p_1, \dots, p_r\}$. The following is well-known.

Lemma 5.7. *Let U_i be an open neighborhood of p_i. Then, no leaf of $\mathscr{F}^{\mathrm{reg}}$ is contained in U_i.*

Proof. We may assume that U_i is the unit open ball in $\mathbb{C}^{n+1}$ and p_i is the origin. Then, it is well-known that there is a holomorphic vector field X on U_i such that $\operatorname{Sing} X = \{x \in U_i \mid X(x) = 0\} = \{0\}$ and that X is tangent to $\mathscr{F}|_{U_i}$. Let $Z(t)$ be an integral curve of X. If we denote by $\|Z(t)\|^2$ the square of distance of $Z(t)$ from the origin with respect to the standard metric, then $\|Z(t)\|^2$ is a subharmonic function. If moreover $\{Z(t)\}$ is entirely contained in U_i, then $\|Z(t)\|^2$ is defined on $\mathbb{C}$ and bounded. Hence $\|Z(t)\|^2$ is constant ([17, Corollary 2.3.4]). If we represent X as $X = \sum_{i=1}^{n+1} f_i \frac{\partial}{\partial z_i}$, where $(z_1, \dots, z_{n+1})$ are the standard coordinates on $\mathbb{C}^{n+1}$, then we have $\sum_{i=1}^{n+1} f_i(Z(t))\overline{Z_i(t)} = 0$, where $Z(t) = (Z_1(t), \dots, Z_{n+1}(t))$. By differentiating with respect to $\overline{t}$, we have $\sum_{i=1}^{n+1} f_i(Z(t))\overline{f_i(Z(t))} = 0$. Hence $Z(t)$ is identically zero by the choice of X. □

Let X be a holomorphic vector field on $\mathbb{C}^{n+1}$ and $\mathscr{F}$ the singular foliation associated with X. Suppose that $\operatorname{Sing} X$ consists of Poincaré type singularities, and let $\operatorname{Sing} X = \{p_1, \dots, p_r\}$. Let U_i be an small round ball at p_i so that $\mathscr{F}$ is transversal to ∂U_i. Then, a foliation is induced on each ∂U_i, which we denote by $\mathscr{F}_i$. Note that $S(\mathscr{F}) = \operatorname{Sing} X$. By removing U_i's from $\mathbb{C}^{n+1}$ and taking the double, we can obtain a non-singular transversely holomorphic foliation of a closed manifold. This kind of examples are studied in [6] when $n = 1$.

Corollary 5.8. 1) *If M is closed, then the holonomy pseudogroup of $\mathscr{F}^{\mathrm{reg}}$ is finitely generated.*

2) *If moreover for each i, there exists an open neighborhood U_i of p_i homeomorphic to a ball such that $\mathscr{F}$ is transversal to ∂U_i, then, the holonomy pseudogroup of $\mathscr{F}^{\mathrm{reg}}$ is compactly generated and $F(\mathscr{F}) = F_0(\mathscr{F})$. We have $J(\mathscr{F}) = J(\mathscr{F}^{\mathrm{reg}}) \cup S(\mathscr{F})$ and $\bigcup_{i=1}^r J(\mathscr{F}_i) \subset J(\mathscr{F}) \cap \bigcup_{i=1}^r \partial U_i$. If $\widetilde{M}$ is the double of M and if $\widetilde{\mathscr{F}}$ is the foliation of $\widetilde{M}$ obtained from $\mathscr{F}^{\mathrm{reg}}$, then $J(\widetilde{\mathscr{F}})$ is the double of $J(\mathscr{F}^{\mathrm{reg}}) \cap M$.*

We do not know any example where the inclusion is strict. On the other hand, if one of ∂U_i's is not transversal to $\mathscr{F}$, then there is an example where $J(\mathscr{F}_j) \subsetneq J(\mathscr{F}) \cap \partial U_j$, where ∂U_j is transversal to $\mathscr{F}$. See Example 5.11.

Proof. Let U_i be an open neighborhood of p_i, where $i = 1, \dots, r$. Let V be an open neighborhood of $M \setminus \bigcup_{i=1}^r U_i$ such that $V \cap S(\mathscr{F}) = \varnothing$. Since $\overline{V}$ is compact, we can find an open covering, say $\mathscr{V}$, of V by a finite number of foliation charts for $\mathscr{F}^{\mathrm{reg}}$. Suppose that $\mathscr{V} = \{V_1, \dots, V_s\}$ and $V_i \cong W_i \times T_i$, where the leaves of $\mathscr{F}^{\mathrm{reg}}|_{V_i}$ are given by $\{W_i \times \{z\}\}$, $z \in T_i$. If we set $T = \coprod_{i=1}^s T_i$, then T is a complete transversal for $\mathscr{F}^{\mathrm{reg}}$ by Lemma 5.7. Therefore the holonomy pseudogroup of $\mathscr{F}^{\mathrm{reg}}$ is finitely generated. If $\mathscr{F}$ is transversal to ∂U_i, then it is shown in [10] that $\mathscr{F}|_{\partial U_i \cup U_i \setminus \{p_i\}}$ is biholomorphically diffeomorphic to $\mathscr{F}|_{\partial U_i} \times (0,1]$. Therefore the holonomy pseudogroup of $\mathscr{F}^{\mathrm{reg}}$ is equivalent to that of $\mathscr{F}^{\mathrm{reg}}|_{M \setminus U_i}$. The last part follows directly from definitions. □

Theorem 5.9. *Suppose that $\dim_{\mathbb{C}} M = 2$ and $S(\mathscr{F}) = \{p_1, \dots, p_r\}$. If for each i, there exists an open neighborhood U_i of p_i homeomorphic to a ball such that $\mathscr{F}$ is transversal to ∂U_i, then the holonomy pseudogroup of $\mathscr{F}^{\mathrm{reg}}$ is compactly generated and we have $F(\mathscr{F}) = F_0(\mathscr{F})$. Moreover, $\mathscr{F}^{\mathrm{reg}}$ admits an invariant transverse Hermitian metric on $F(\mathscr{F})$ which is transversely of class C^ω.*

Proof. Let Γ be the holonomy pseudogroup of $\mathscr{F}^{\mathrm{reg}}$. If $\mathscr{F}$ is transversal to ∂U_i, then Γ is equivalent to the holonomy pseudogroup of $\mathscr{F}^{\mathrm{reg}}|_{M \setminus U_i}$ by Corollary 5.8. Hence, by [1, Theorem 4.21], there exists a Hermitian metric of class C^ω on $F_{\mathrm{pg}}(\Gamma)$ invariant under the action of Γ. □

Remark 5.10. We made use of $F_{\mathrm{pg}}(\Gamma)$ in defining $F(\mathscr{F})$. The same decomposition is obtained even if we replace $F_{\mathrm{pg}}(\Gamma)$ by $F(\Gamma)$ under the assumptions of Theorem 5.9 because Γ is compactly generated.

Example 5.11. Let X be a holomorphic vector field on $\mathbb{C}^2$ defined by

$$X = \lambda z \frac{\partial}{\partial z} + \mu w \frac{\partial}{\partial w},$$

where λ and μ are non-zero complex numbers and (z, w) are the standard coordinates for $\mathbb{C}^2$. Let $\mathscr{F}$ be the singular foliation of $\mathbb{C}P^2$ induced by the integral curves of X. If $\lambda = \mu$, then $J(\mathscr{F}) = S(\mathscr{F}) = \{[1:0:0]\}$ and a transverse invariant Hermitian metric on $F(\mathscr{F})$ is given by

$$g = \frac{|wdz - zdw|^2}{(|z|^2 + |w|^2)^2},$$

where for a 1-form ω, we denote $\omega \otimes \overline{\omega}$ by $|\omega|^2$.

If $\lambda \neq \mu$, then the codimension of $S(\mathscr{F})$ is greater than one. Let $[z_0 : z_1 : z_2]$ be the homogeneous coordinates for $\mathbb{C}P^2$ and consider $\mathbb{C}^2 = \{[1 : z : w]\}$. We set

$$L_0 = \{[\,0 : z_1 : z_2] \in \mathbb{C}P^2\},$$
$$L_1 = \{[z_0 : 0 : z_2] \in \mathbb{C}P^2\},$$
$$L_2 = \{[z_0 : z_1 : 0\,] \in \mathbb{C}P^2\}.$$

Then $S(\mathscr{F}) = \{[1:0:0], [0:1:0], [0:0:1]\}$, and $J(\mathscr{F})$ is described as follows.

1) If $\mu/\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $J(\mathscr{F}) = L_0 \cup L_1 \cup L_2$. An invariant metric on $F(\mathscr{F})$ is given by
$$\frac{|\mu w dz - \lambda z dw|^2}{(|z|\,|w|)^2}.$$
2) If $\mu/\lambda > 1$, then $J(\mathscr{F}) = L_0 \cup L_2$. An invariant metric on $F(\mathscr{F})$ is given by
$$\frac{|\mu w dz - \lambda z dw|^2}{|w|^{2(1+\lambda/\mu)}}.$$
3) If $1 > \mu/\lambda > 0$, then $J(\mathscr{F}) = L_0 \cup L_1$. An invariant metric on $F(\mathscr{F})$ is given by
$$\frac{|\mu w dz - \lambda z dw|^2}{|z|^{2(1+\mu/\lambda)}}.$$
4) If $0 > \mu/\lambda$, then $J(\mathscr{F}) = L_1 \cup L_2$. An invariant metric on $F(\mathscr{F})$ is given by
$$\frac{|\mu w dz - \lambda z dw|^2}{(|z|^{\alpha}\,|w|^{\beta})^2},$$
where $\alpha = (\lambda - 2\mu)/(\lambda - \mu)$ and $\beta = (2\lambda - \mu)/(\lambda - \mu)$. Note that $\alpha > 1$, $\beta > 1$, $\alpha + \beta = 3$ and $\alpha\lambda + \beta\mu = \lambda + \mu$.

If $\mu/\lambda > 0$, then $\mathscr{F}$ is transversal to the unit sphere S^3 in $\mathbb{C}^2$. We denote by $\mathscr{F}'$ the induced foliation on S^3. Then, $\mathscr{F}'$ is transversely Hermitian, namely, it admits a smooth transverse invariant Hermitian metric. It follows that $J(\mathscr{F}') = \varnothing$. Hence, if we denote by ι the inclusion of S^3 to $\mathbb{C}P^2$ via $\mathbb{C}^2$, then $\iota^{-1}(J(\mathscr{F})) \supsetneq J(\mathscr{F}')$.

Example 5.12. Let X be a holomorphic vector field on $\mathbb{C}^2$ defined by $X = (z+\varepsilon w)\dfrac{\partial}{\partial z} + w\dfrac{\partial}{\partial w}$, where (z,w) are the standard coordinates. If we set $Y = z\dfrac{\partial}{\partial z} + w\dfrac{\partial}{\partial w}$, then $[X,Y] = 0$, and $X(z,w)$ and $Y(z,w)$ are linearly independent on $\mathbb{C}^2 \setminus \{w=0\}$. If we denote by $\mathscr{F}$ the foliation of $\mathbb{C}P^2$ induced by X, then Y induces a holonomy invariant trivialization of the normal bundle of $\mathscr{F}^{\mathrm{reg}}$ on $F(\mathscr{F}) = \mathbb{C}P^2 \setminus (L_0 \cup L_2)$, where L_0 and L_2 are as in Example 5.11. Hence we can find a transverse invariant Hermitian metric on $F(\mathscr{F})$. Since X is invariant under homothecies, $\mathscr{F}^{\mathrm{reg}}$ induces a foliation of Hopf manifolds. For example, let $M = (\mathbb{C}^2 \setminus \{0\})/\alpha$, where α is a non-zero complex number and $\alpha(z) = \alpha z$. If we denote by $\mathscr{G}$ the induced foliation of M, then $F(\mathscr{G}) = (\mathbb{C}^2 \setminus \{w=0\})/\alpha$. Since Y is also invariant under α, we can also find a transverse invariant Hermitian metric on the normal bundle of $\mathscr{G}$.

Example 5.13. Let X be a holomorphic vector field on $\mathbb{C}^3$ defined by

$$X = \lambda_1 z_1 \frac{\partial}{\partial z_1} + \lambda_2 z_2 \frac{\partial}{\partial z_2} + \lambda_3 z_3 \frac{\partial}{\partial z_3},$$

where λ_1, λ_2 and λ_3 are non-zero complex numbers. Then, X induces a singular foliation of $\mathbb{C}P^3$ which we denote by $\mathscr{F}$. We set $p_0 = [1:0:0:0]$, $p_1 = [0:1:0:0]$, $p_2 = [0:0:1:0]$ and $p_3 = [0:0:0:1]$. If $\lambda_1 = \lambda_2 = \lambda_3$, then $S(\mathscr{F}) = J(\mathscr{F}) = \{p_0\}$, where we consider $\mathbb{C}^3 = \{[1:z_1:z_2:z_3]\}$. If we set $\omega_{ij} = z_i dz_j - z_j dz_i$, $\left|\omega_{ij}\right|^2 = \omega_{ij} \otimes \overline{\omega_{ij}}$ and

$$g = \frac{|\omega_{12}|^2 + |\omega_{13}|^2 + |\omega_{23}|^2}{(|z_1|^2 + |z_2|^2 + |z_3|^2)^2},$$

then g is a transverse invariant Hermitian metric on $F(\mathscr{F})$. Note that g is bounded from below, and induces an invariant volume form.

In that follows, we assume without generality that $\lambda_1 = 1$. Suppose that λ_2, λ_3 and λ_2/λ_3 do not belong to $\mathbb{R}$. Then $S(\mathscr{F}) = \{p_0, p_1, p_2, p_3\}$, and there are unique real numbers α and β such that $\alpha\lambda_2 + \beta\lambda_3 = 1$. According to Theorem 5.5, there exist invariant volume forms on $F(\mathscr{F})$. In fact, if we set

$$g = \frac{|\lambda_2 z_2 dz_1 - z_1 dz_2|^2}{(|z_2|^{\alpha+1}|z_3|^{\beta})^2} + \frac{|\lambda_3 z_3 dz_2 - \lambda_2 z_2 dz_3|^2}{|z_2 z_3|^2},$$

then g is a transverse invariant Hermitian metric on $\mathbb{C}P^3 \setminus (P_0 \cup P_2 \cup P_3)$, where $P_0 = \{[0:x_1:x_2:x_3] \,|\, x_1,x_2,x_3 \in \mathbb{C}\}$, $P_2 = \{[x_0:x_1:0:x_3]\}$, $P_3 = \{[x_0:x_1:x_2:0]\}$. Note that on the plane $\{[u_0:1:u_2:u_3]\}$, we have

$$g = \frac{|(\lambda_2 - 1)u_2 du_0 + u_0 du_2|^2}{(|u_0|^{(2-\alpha-\beta)}|u_2|^{\alpha+1}|u_3|^{\beta})^2} + \frac{|\lambda_3 u_3(u_0 du_2 - u_2 du_0) - \lambda_2 u_2(u_0 du_3 - u_3 du_0)|^2}{|u_0 u_2 u_3|^2}.$$

Let Δ be the closed triangle formed by 0, λ_2 and λ_3. If 1 is contained in Δ, then $\alpha \geq 0$, $\beta \geq 0$ and $0 \leq \alpha + \beta \leq 1$. This condition is equivalent to that g is bounded from below on $\mathbb{C}P^3 \setminus (P_0 \cup P_2 \cup P_3)$. Indeed in this case we have $F(\mathscr{F}) = \mathbb{C}P^3 \setminus (P_0 \cup P_2 \cup P_3)$. If λ_2 and λ_3 do not satisfy the condition, then $F(\mathscr{F}) = \mathbb{C}P^3 \setminus (P_0 \cup P_1 \cup P_2 \cup P_3)$, where $P_1 = \{[x_0 : 0 : x_2 : x_3]\}$. Even in this case, the above metric is an invariant metric on $\mathbb{C}P^3 \setminus (P_0 \cup P_2 \cup P_3)$ but not bounded from below. A bounded one on $F(\mathscr{F})$ is given by

$$\frac{|\lambda_2 z_2 dz_1 - z_1 dz_2|^2}{(|z_1|\,|z_2|)^2} + \frac{|\lambda_3 z_3 dz_1 - z_1 dz_3|^2}{(|z_1|\,|z_3|)^2} + \frac{|\lambda_3 z_3 dz_2 - \lambda_2 z_2 dz_3|^2}{(|z_2|\,|z_3|)^2}.$$

If in addition the convex hull of 1, λ_2 and λ_3 does not contain 0, then $\mathscr{F}$ is transversal to the unit sphere S^5. Hence $\mathscr{F}$ induces a transversely holomorphic, non-singular foliation of S^5. If we denote this foliation by $\mathscr{F}'$, then $F(\mathscr{F}') = F(\mathscr{F}) \cap S^5$ and $J(\mathscr{F}') = J(\mathscr{F}) \cap S^5$. Since the holonomy pseudogroups of $\mathscr{F}'$ is compactly generated, we see that the conclusion of Proposition 4.24 fails if the metric is not bounded from below.

Instead of exhausting all cases, we will examine the case where $\lambda_2 \in \mathbb{R}$ and $\lambda_3 \notin \mathbb{R}$. If $\lambda_2 > 1$, then $S(\mathscr{F}) = \{p_0, p_1, p_2, p_3\}$ and $J(\mathscr{F}) = P_0 \cup P_2 \cup P_3$. An invariant metric on $F(\mathscr{F})$ is given by

$$\frac{|\lambda_2 z_2 dz_1 - z_1 dz_2|^2}{(|z_2|^{1+1/\lambda_2})^2} + \frac{|\lambda_3 z_3 dz_2 - \lambda_2 z_2 dz_3|^2}{|z_2 z_3|^2}.$$

If $\lambda_2 = 1$, then $S(\mathscr{F}) = \{[0 : x_1 : x_2 : 0]\} \cup \{p_0, p_3\}$ and $J(\mathscr{F}) = \{[x_0 : 0 : 0 : x_3]\} \cup P_0 \cup P_3$. Note that $\{[0 : x_1 : x_2 : 0]\} = P_0 \cap P_3$. An invariant metric on $F(\mathscr{F})$ is given by

$$\frac{|\lambda_2 z_2 dz_1 - z_1 dz_2|^2}{(|z_1|^{1+\lambda_2} + |z_2|^{1+1/\lambda_2})^2} + \frac{|\lambda_3 z_3 dz_2 - \lambda_2 z_2 dz_3|^2}{(|z_1|^{2\lambda_2} + |z_2|^2)\,|z_3|^2}.$$

If $0 < \lambda_2 < 1$, then $S(\mathscr{F}) = \{p_0, p_1, p_2, p_3\}$ and $J(\mathscr{F}) = P_0 \cup P_1 \cup P_3$. If $\lambda_2 < 0$, then $S(\mathscr{F}) = \{p_0, p_1, p_2, p_3\}$ and $J(\mathscr{F}) = P_1 \cup P_2 \cup P_3$. In these cases, invariant metrics can be constructed as in the case where $\lambda_2 > 1$.

Remark 5.14. Note that L_0, L_1 and L_2 are separatrices for X in Example 5.11, and that L_0 is also a separatrix for X in Example 5.12. Example 5.13 also suggests that $J(\mathscr{F})$ has something to do with separatrices.

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