# ON EXISTENCE AND QUASICONFORMAL DEFORMATIONS OF TRANSVERSELY HOLOMORPHIC FOLIATIONS

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ABSTRACT. We review our recent work on transversely holomorphic foliations of complex codimension one. Some remarks from a viewpoint of characteristic classes are also given.

There are many sources of transversely holomorphic foliations, e.g., holomorphic vector fields, group actions, etc. They are of their own interest even simply viewed as foliations of real codimension two [5]. It seems however difficult to construct examples. It is also difficult to tell if a given foliation of real codimension two admits a transverse holomorphic structure. As an attempt to answer these problems, we considered transversely quasiconformal foliations in [3]. A foliation is said to be transversely quasiconformal if there is a real number $K$ for which the holonomy pseudogroup consists of $K$-quasiconformal local homeomorphisms. If a foliation is $K$-quasiconformal, any infinitesimal circle on the normal bundle will be deformed in a bounded way when moved along the leaves. This is in contrast to the dynamics of Anosov or projectively Anosov flows.

The main theorem in [3] is as follows. Throughout this article, foliations are assumed to be transversely oriented for simplicity. We refer to [3] for precise definitions and statements.

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**Theorem 1** (Theorem 3.3 [3]).

1) *Let $\mathcal{F}$ be a real codimension two foliation of a manifold $M$. If $\partial M \neq \phi$, then assume that $\mathcal{F}$ is transversal to $\partial M$. If $\mathcal{F}$ is $K$-quasiconformal, then $\mathcal{F}$ admits a transverse holomorphic structure after taking a transverse $K$-quasiconformal conjugate of $\mathcal{F}$.*
2) *Let $W$ be a codimension zero submanifold of $M$ and assume that $\partial W$ is transversal to $\mathcal{F}$. Assume that $\mathcal{F}$ is $K$-quasiconformal and that a transverse holomorphic structure is given to $\mathcal{F}|_W$. Suppose that the transverse holomorphic structure on $W$ satisfies the 'compatibility condition' explained below, then the transverse holomorphic structure of $\mathcal{F}|_W$ extends to a transverse holomorphic structure of $\mathcal{F}$ on the whole $M$ after taking a transverse $K$-quasiconformal conjugate of $\mathcal{F}$ which is transversely holomorphic on $W$.*

The part 1) of Theorem 1 is shown as a straightforward adaptation to foliations of Tukia's method in [7] for group actions. Indeed, he showed that a group action on $\boldsymbol{C}$ can be made to be holomorphic by replacing the holomorphic structure on $\boldsymbol{C}$ if the action is $K$-quasiconformal. In the proof, he gave a method to construct a Beltrami coefficient invariant under the group action by using some elementary hyperbolic geometry on the Poincaré disc. His method is so natural as to be also valid for foliations.

In order to show the part 2), one has to obtain an invariant Beltrami coefficient which is trivial on $W$. The compatibility condition is needed for this purpose. To illustrate this condition, consider the case where $M = N \times [0,1]$ for some manifold $N$ and the leaves of $\mathcal{F}$ is the product of leaves of $\mathcal{F}|_N$ and $[0,1]$. Set $W = N \times [0,\epsilon) \cup N \times (1-\epsilon, 1]$ for small $\epsilon$ and assume that a transverse holomorphic structure is given to $\mathcal{F}|_W$. If this structure can be extended to the whole $M$, the holomorphic structures on $N \times [0,\epsilon)$ and on $N \times (1-\epsilon, 1]$ should obviously coincide. In other words, any holonomies (which are trivial in this example) associated with a leaf path from $N \times \{0\}$ to $N \times \{1\}$ should be holomorphic with respect to the given transverse holomorphic structure. The compatible condition in 2) is a generalization of this condition.

The part 2) of the above theorem allows us to glue two transversely holomorphic foliations under certain conditions. For example, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two transversely holomorphic flows on a manifold $M$ with compact boundary. Assume that the leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ are transversal to $\partial M$ and that they meet $\partial M$ at most once, then $(M, \mathcal{F}_1)$ and $(M, \mathcal{F}_2)$ can be glued after changing the transverse structure on one piece (Corollary 3.6 in [3]). A typical example of this kind is the classical example of Bott [4], namely, let $\lambda$ be an element of $\boldsymbol{C}\setminus\boldsymbol{R}$ and let $X_\lambda$ be the holomorphic vector field on $\boldsymbol{C}^2$ defined by the formula $X_\lambda = z\dfrac{\partial}{\partial z}+\lambda w\dfrac{\partial}{\partial w}$, where $(z,w)$ is the standard coordinate of $\boldsymbol{C}^2$. It is easy to see that $X_\lambda$ is transversal to the unit sphere $S^3$. Let $\mathcal{F}_\lambda$ be the induced foliation of $S^3$, then $\mathcal{F}_\lambda$ has two closed orbits with simple repelling-contracting dynamics. It is easy to see that $\mathcal{F}_\lambda$ can be obtained by gluing two solid tori equipped with a transversely holomorphic flow whose unique closed leaf (orbit) is the core. These flows on the solid torus is seen to be parametrized by $\lambda$ appeared in the definition of $X_\lambda$. In fact, this $\lambda$ can be detected by means of the Bott class. The Bott class is the most fundamental secondary characteristic class for transversely holomorphic foliations, and the following is known:

**Theorem 2** (Bott [4], cf. [2])**.** *Denote by* $\mathrm{Bott}(\mathcal{F}_\lambda)$ *be the Bott class, which is an element of* $H^3(S^3;\boldsymbol{C}/\boldsymbol{Z})$*, of the foliation of* $S^3$ *as above. Let* $[S^3]$ *be the fundamental cycle of* $S^3$*, then* $\mathrm{Bott}(\mathcal{F}_\lambda)[S^3]=\lambda+(1/\lambda) \mod \boldsymbol{Z}$*. In particular,* $\mathrm{Bott}(\mathcal{F}_\lambda)$ *varies continuously as* $\lambda$ *varies.*

By changing the gluing map, examples on the Lens spaces can be obtained. One can also obtain a transversely holomorphic flow, say $\mathcal{G}_\lambda$ on $S^3$ which can be extended to the vector field $z\dfrac{\partial}{\partial z}+\lambda\bar{w}\dfrac{\partial}{\partial\bar{w}}$ on $\boldsymbol{C}^2$ (more precisely, we consider the plane field spanned by its real and imaginary parts). It is easily verified that the flow $\mathcal{G}_\lambda$ is indeed transversely holomorphic, and there is an automorphism of $S^3$ which maps the orbits of $\mathcal{F}_\lambda$ to the orbits of $\mathcal{G}_\lambda$ in a transversely holomorphic way. However, it is evident that $\mathcal{G}_\lambda$ cannot be extended to any holomorphic vector field on $\boldsymbol{C}^2$. This is because the embedding (or realization) of $S^3$ in $\boldsymbol{C}^2$ is not appropriate. We refer the section 4 of [3] for more details.

On the other hand, it is shown in [3] that $\mathcal{F}_\lambda$ and $\mathcal{F}_\mu$ are quasiconformally equivalent, namely, there is a foliation preserving homeomorphism from $(S^3,\mathcal{F}_\lambda)$ to $(S^3,\mathcal{F}_\mu)$ which is transversely quasiconformal. By explicitly constructing such a equivalence, one obtains

**Corollary 3.**

1) *There is a transversely holomorphic foliation and a continuous family of quasiconformal conjugations of it such that the Bott class vary continuously.*
2) *The Bott class is* not *invariant under transversely quasiconformal homeomorphisms. Thus the Bott class is* not *well-defined in the category of transversely quasiconformal foliations.*

It is straightforward from the definition that the Bott class is invariant under transversely holomorphic foliation preserving diffeomorphisms. It seems unknown if the Bott class is invariant under foliation preserving diffeomorphisms which are not necessarily transversely holomorphic. The part 2) of Corollary 3 implies that the invariance fails if the regularity is insufficient.

Finally, we explain the above results in terms of the classifying spaces. Let $\Gamma_1^{\boldsymbol{C}}$ and $\Gamma_2^{\mathrm{qc}}$ be the pseudogroups generated by local biholomorphic diffeomorphisms of $\boldsymbol{C}$ and by orientation preserving local quasiconformal homeomorphisms of $\boldsymbol{R}^2$, respectively. Let $B\Gamma_1^{\boldsymbol{C}}$ and $B\Gamma_2^{\mathrm{qc}}$ be the classifying spaces for $\Gamma_1^{\boldsymbol{C}}$-structures and $\Gamma_2^{\mathrm{qc}}$-structures, respectively. Then, there is a natural mapping $\pi : B\Gamma_1^{\boldsymbol{C}} \to B\Gamma_2^{\mathrm{qc}}$.

**Definition 4.** A mapping $f : M \to B\Gamma_2^{\mathrm{qc}}$ is said to be bounded if there is a real number $K \geq 1$ such that the corresponding pseudogroup consists of $K$-quasiconformal mappings.

Note that if the mapping $f$ as above admits a lift to $B\Gamma_1^{\boldsymbol{C}}$, it is bounded. Given a transversely ($K$-)quasiconformal foliation of a manifold $M$, there is a classifying map from $M$ to $B\Gamma_2^{\mathrm{qc}}$. Such a classifying mapping is also bounded.

The following is a reformulation of Theorem 1, 1) and Corollary 3, 2).

**Theorem 5.**

1) *A mapping $f : M \to B\Gamma_2^{\mathrm{qc}}$ admits a lift to $B\Gamma_1^{\boldsymbol{C}}$ if the mapping $f$ is bounded.*
2) *The Bott class does not belong to the image of the mapping*
$$\pi^* : H^3(B\Gamma_2^{\mathrm{qc}}; \boldsymbol{C}/\boldsymbol{Z}) \to H^3(B\Gamma_1^{\boldsymbol{C}}; \boldsymbol{C}/\boldsymbol{Z}).$$

*Remark 6.*

1) In considering classifying space, the homotopy classes of mappings are relevant. Thus an important question is find good criteria for mappings to $B\Gamma_2^{\mathrm{qc}}$ being homotopic to bounded ones.
2) Concerning 2) of Theorem 5, a much stronger result is known. Let $B\Gamma_2^1$ and $B\overline{\Gamma}_2^1$ be the classifying spaces for real codimension two $C^1$-foliations and for real codimension two $C^1$-foliations with trivialized normal bundles, respectively. Then the mapping $B\Gamma_1^{\boldsymbol{C}} \to B\Gamma_2^{\mathrm{qc}}$ is decomposed as $B\Gamma_1^{\boldsymbol{C}} \to B\Gamma_2^1 \to B\Gamma_2^{\mathrm{qc}}$. Hence the mapping $\pi^*$ in Theorem 5, 2) is also decomposed as $H^3(B\Gamma_2^{\mathrm{qc}}; \boldsymbol{C}/\boldsymbol{Z}) \to H^3(B\Gamma_2^1; \boldsymbol{C}/\boldsymbol{Z}) \to H^3(B\Gamma_1^{\boldsymbol{C}}; \boldsymbol{C}/\boldsymbol{Z})$. On the other hand, it is shown in [6] that $B\overline{\Gamma}_2^1$ is contractible, which implies that $H^3(B\Gamma_2^1; \boldsymbol{C}/\boldsymbol{Z}) = 0$. Hence the image of $\pi^*$ is in fact zero.

*Remark 7.* Most of the statements are valid also for the imaginary part of the Bott class, which is an element of $H^3(B\Gamma_1^{\boldsymbol{C}}; \boldsymbol{R})$.

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