INFINITESIMAL DERIVATIVE OF THE BOTT CLASS AND THE SCHWARZIAN DERIVATIVES

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Abstract. An infinitesimal derivative of the Bott class is defined by generalizing Heitsch'es construction. We prove a formula relating the infinitesimal derivative to the Schwarzian derivatives, which gives a generalization of the Maszczyk formula for the Godbillon-Vey class of real codimension-one foliations. As an application, a residue of infinitesimal derivatives with respect to the Julia set in the sense of Ghys, Gomez-Mont and Saludes is introduced.

Introduction

The Bott class is a secondary characteristic class of transversally holomorphic foliations defined in a similar manner to the Godbillon-Vey class. It is significant that the Bott class varies continuously under deformations of foliations. The derivative of the Bott class can be defined if the family is smooth. Moreover, the derivatives with respect to infinitesimal deformations are also defined, which we call *infinitesimal derivatives* in this article. An explicit construction of them was presented by Heitsch [14], [15], where the infinitesimal derivative of the Bott class was given if normal bundles are trivial. If the normal bundle is not necessarily trivial, then the derivative of the imaginary part was given. The real part of the Bott class proves useful in the study of Fatou-Julia decompositions of foliations [13] as well as of the Futaki invariant [11], [12]. For this reason, it would be worthwhile if the infinitesimal derivative of the Bott class is defined without additional assumptions. In this paper, by modifying Heitsch'es construction, we define the infinitesimal derivatives of the Bott class in full generality. Some applications concerning the Fatou-Julia decomposition in the sense of Ghys, Gomez-Mont and Saludes will be also discussed.

It is shown by Maszczyk [19] that the infinitesimal derivative of the Godbillon-Vey class of real codimension-one foliation is described in terms of classical Schwarzian derivative. The formula is easily seen to be valid also for the Bott class of complex codimension-one foliations. It will be shown that the same is

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also true for higher codimensional cases if we replace the classical Schwarzian derivative with the projective Schwarzian derivatives. In particular, we show that the infinitesimal derivatives of the Bott class of transversally complex projective foliations vanish. It is in analogy with the fact that the imaginary part of the Bott class is trivial if the foliation is transversally Hermitian or transversally complex affine. Examples are given in the final section.

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1. Relevant definitions

In this paper, manifolds are smooth and without boundary unless otherwise stated.

Definition 1.1. A foliation $\mathcal{F}$ of a manifold M is said to be transversally holomorphic of complex codimension q if there is an open covering $\mathcal{U} = \{U_i\}$ of M with the following properties:

1) Each U_i is homeomorphic to $V_i \times D^{2q}$, where V_i is an open subset of $\mathbb{R}^p$ and D^{2q} is an open ball in $\mathbb{C}^q$ ($p + 2q = \dim M$).
2) The foliation restricted to U_i is given by $\{V_i \times \{z\}\}$, $z \in D^{2q}$.
3) Under the identification in 1), the transition function φ_{ji} from U_i to U_j is of the form $\varphi_{ji}(x, z) = (\psi_{ji}(x, z), \gamma_{ji}(z))$, where γ_{ji} is a local biholomorphic diffeomorphism.

Such an atlas $\{\mathcal{U}, \{\varphi_{ji}\}\}$ is called a *foliation atlas*. An open covering of M is *adapted* if it is simple and gives a refinement of a foliation atlas for $\mathcal{F}$.

Definition 1.2. Let $\mathcal{F}$ be a transversally holomorphic foliation. Denote by $E = E(\mathcal{F})$ the complex subbundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ locally spanned by $\dfrac{\partial}{\partial x_k^i}$ and $\dfrac{\partial}{\partial \bar{z}_k^j}$, where $(x_k, z_k) = (x_k^1, \cdots, x_k^p, z_k^1, \cdots, z_k^q)$ are local coordinates as in Definition 1.1. The *complex normal bundle* $Q(\mathcal{F})$ of $\mathcal{F}$ is by definition $T_{\mathbb{C}}M/E$. The line bundle $K_{\mathcal{F}} = \bigwedge^q Q(\mathcal{F})^*$ is called the *canonical bundle*, and $-K_{\mathcal{F}} = \bigwedge^q Q(\mathcal{F})$ is the anti-canonical bundle.

Notation 1.3. We denote by $I_{(1)}(U)$ the ideal of $\mathbb{C}$-valued differential forms $\Omega^*(U)$ on U, locally generated by $dz^1, \cdots, dz^q$. Set $I_{(k)}(U) = I_{(1)}(U)^k$, $I^p_{(k)}(U) = I_{(k)}(U) \cap \Omega^p(U)$, and denote the sheaves generated by these ideals by $I_{(k)}$ and $I^p_{(k)}$. We set $I^p_{(k,l)} = I^p_{(k)}/I^p_{(l)}$, namely, an element of $I^p_{(k,l)}(U)$ is a family $\{\omega_i\}$, for which $\omega_i \in I^p_{(k)}$ is defined on an open subset V_i of U, where $\bigcup V_i = U$, such that $\omega_j - \omega_i \in I^p_{(l)}(V_i \cap V_j)$ if $V_i \cap V_j \neq \varnothing$. Finally, we set $I_{(k,l)} = \bigoplus_p I^p_{(k,l)}$.

Note that $I_{(k)} = \{0\}$ for $k > q$. If $p < l$, then $I^p_{(k,l)} = I^p_{(k)}$ because $I^p_{(l)} = \{0\}$. In what follows, the sheaf of germs of sections of a vector bundle V is also denoted by V by abuse of notation. Then, $E^* \cong I^1_{(0,1)}$.

Notation 1.4. Let $\mathcal{S}$ be a presheaf on M and $\mathcal{U}$ an open covering of M. The set of Čech r-cochains with values in $\mathcal{S}$ is denoted by $\check{C}^r(\mathcal{U};\mathcal{S})$, or by $\check{C}^r(\mathcal{S})$ if $\mathcal{U}$ is obvious. Components of Čech cochains are represented by attaching or removing indices, e.g., a cochain $\{\omega_i\}$ is denoted by ω and vice versa.

Elements of $\check{C}^r(\mathcal{U};\Omega^s)$ are called Čech-de Rham (r,s)-cochains. $\check{C}^r(\mathcal{U};\Omega^s)$ is also denoted by $\mathcal{A}^{r,s}(\mathcal{U})$. If $c \in \mathcal{A}^{r,s}(\mathcal{U})$ and $c' \in \mathcal{A}^{t,u}(\mathcal{U})$, then the product $c \cup c' \in \mathcal{A}^{r+t,s+u}(\mathcal{U})$ is defined by $(c \cup c')_{i_0 \cdots i_{r+t}} = (-1)^{st} c_{i_0 \cdots i_r} \wedge c'_{i_r \cdots i_{r+t}}$. The Čech differential and the de Rham differential are denoted by δ and d, respectively. The Čech-de Rham differential $\mathcal{D}$ is defined by $\mathcal{D} = \delta + (-1)^r d$.

Definition 1.5. Let $\check{C}^*(\mathcal{U};\mathbb{Z})$ be the Čech complex with coefficients in $\mathbb{Z}$. Then $\check{C}^r(\mathcal{U};\mathbb{Z}) \subset \mathcal{A}^{r,0}(\mathcal{U})$. The quotient $\mathcal{A}^{*,*}(U)$-module equipped with the natural differential $\mathcal{A}^{*,*}(U)/\check{C}^*(U;\mathbb{Z})$ is called the modified Čech-de Rham complex.

Let $\mathcal{U} = \{U_i\}$ be an adapted covering. Then $-K_\mathcal{F}$ is trivial when restricted to each U_i. Let e_i be a trivialization of $-K_\mathcal{F}|_{U_i}$ and $\{J_{ij}\}$ a family of non-zero functions such that $e_j = e_i J_{ij}$. Noticing that $\log J_{ij}$ is well-defined, since the covering is adapted, we set $\Theta = (2\pi\sqrt{-1})^{-1}\delta \log J$. It is classical that Θ represents $c_1(Q(\mathcal{F}))$ in $\check{H}^2(M;\mathbb{Z})$. Let ∇_i be a Bott connection defined on U_i, and let θ_i be its connection form with respect to e_i.

Definition 1.6. We set $\beta_{ij} = \theta_j - \theta_i - d\log J_{ij}$ and call $\beta = \{\beta_{ij}\}$ the *difference cochain* of $\{\nabla_i\}$.

Note that $\beta_{ij} \in I_{(1)}(U_i \cap U_j)$.

The Bott class is represented in terms of the following cochains in the modified Čech-de Rham complex:

Definition 1.7. Set

$$u_1(\nabla,e) = \frac{-1}{2\pi\sqrt{-1}}(\theta + \log J),\ \bar{u}_1(\nabla,e) = \frac{1}{2\pi\sqrt{-1}}(\overline{\theta} + \overline{\log J}),$$
$$v_1(\nabla,e) = \frac{-1}{2\pi\sqrt{-1}}(d\theta + \beta) \text{ and } \bar{v}_1(\nabla,e) = \frac{1}{2\pi\sqrt{-1}}(\overline{d\theta} + \overline{\beta}).$$

If ∇ and e are clear, then they will be omitted.

We have $\mathcal{D}u_1 = v_1 - \Theta$ and $\mathcal{D}\bar{u}_1 = \bar{v}_1 - \Theta$.

Theorem 1.8. [3] *Let $B_q(\mathcal{F})$ be the Bott class of $\mathcal{F}$. Then $B_q(\mathcal{F})$ is represented by the cochain $B_q(\nabla,e)$ in the modified Čech-de Rham complex defined* *by the formula*

$$B_q(\nabla, e) = u_1 \cup v_1^q + \Theta \cup u_1 \cup v_1^{q-1} + \cdots + \Theta^q \cup u_1,$$

which is independent of the choice of $\mathcal{U}$, local trivializations e of $-K_{\mathcal{F}}$, and the family of Bott connections ∇.

Definition 1.9. Let $\{\mathcal{F}_s\}_{s\in S}$ be a family of transversally holomorphic foliations, of a fixed codimension, of a fixed manifold. Then $\{\mathcal{F}_s\}$ is said to be a *continuous deformation* of $\mathcal{F}_0$ if $\{\mathcal{F}_s\}$ is a continuous family as plane fields, and the transversal holomorphic structures also vary continuously, where $0 \in S$ is the base point. A smooth family $\{\mathcal{F}_s\}_{s\in S}$ is said to be *smooth* if it is a smooth family of plane fields and the transversal holomorphic structures vary smoothly.

Given a smooth family $\{\mathcal{F}_s\}$ of transversally holomorphic foliations, set $-K_s = \bigwedge^q Q(\mathcal{F}_s)$. We may assume that there is a family $\{e_{s,i}\}$ of local trivializations of $-K_s$ such that each $e_{s,i}$ is defined on U_i. Let $\{J_{s,ij}\}$ be functions such that $e_{s,j} = e_{s,i}J_{s,ij}$. Then we may further assume that $J_{s,ij}$ is independent of s. We denote $J_{s,ij}$ by J_{ij}. The cocycle $\Theta_s = (2\pi\sqrt{-1})^{-1}\delta \log J_s$ is also independent of s and denoted by Θ. Choose then a smooth family $\{\nabla_s\}$ of local Bott connections and denote by $\{\theta_{s,i}\}$ the connection forms of ∇_s with respect to $\{e_{s,i}\}$. Let $\{\beta_{s,ij}\}$ be the difference cochain of ∇_s. Then by definition $\theta_{s,j} - \theta_{s,i} = d\log J_{ij} + \beta_{s,ij}$. Finally, for any cochain ω_s, we denote by $\dot{\omega}_s$ the partial derivative of ω_s with respect to s.

Under these choices of cochains, we have the following

Proposition 1.10. *Let $u_1(s) = u_1(\nabla_s, e_s)$ and $v_1(s) = v_1(\nabla_s, e_s)$, respectively, and let $\dot{u}_1(s) = \dfrac{-1}{2\pi\sqrt{-1}}\dot{\theta}_s$. Then, $\dfrac{\partial B_q(\mathcal{F}_s)}{\partial s}$ naturally determines an element of $H^{2q+1}(M;\mathbb{C})$, which is represented by $\displaystyle\sum_{k=0}^{q} v_1(s)^k \cup \dot{u}_1(s) \cup v_1(s)^{q-k}$.*

Proof. First, note that $\dot{u}_1(s)$ is the partial derivative of $u_1(s)$ with respect to s. Set $\dot{v}_1(s) = \dfrac{-1}{2\pi\sqrt{-1}}(d\dot{\theta}_s + \dot{\beta}_s)$. Then $\mathcal{D}\dot{u}_1(s) = \dot{v}_1(s)$. Note that we have

$$\begin{aligned}\frac{\partial}{\partial s}B_q(\nabla_s, e_s) =& \sum_{k=0}^{q} \Theta^k \cup \dot{u}_1(s) \cup v_1(s)^{q-k} \\ &+ \sum_{k=0}^{q-1}\sum_{l=0}^{q-k-1} \Theta^k \cup u_1(s) \cup v_1(s)^l \cup \dot{v}_1(s) \cup v_1(s)^{q-k-l-1}.\end{aligned}$$

Set $\rho_k = \sum_{l=0}^{q-k} \Theta^{k-1} \cup u_1(s) \cup v_1(s)^l \cup \dot{u}_1(s) \cup v_1(s)^{q-k-l}$ for $k = 1, \dots, q$. Then we have

$$\frac{\partial}{\partial s} B_q(\nabla_s, e_s) + \mathcal{D}\rho_1 + \cdots + \mathcal{D}\rho_q = \sum_{k=0}^{q} v_1(s)^k \cup \dot{u}_1(s) \cup v_1(s)^{q-k}.$$

□

Corollary 1.11. *Assume that each ∇_s is a global connection. Then $\dfrac{\partial B_q(\mathcal{F}_s)}{\partial s}$ is represented by a global $(2q+1)$-form $(-2\pi\sqrt{-1})^{-(q+1)}(q+1)\dot{\theta}_s \wedge (d\theta_s)^q$.*

The above representative is the same as the one given by Heitsch [15] when normal bundles are trivial.

The imaginary part of the Bott class is an element of $H^{2q+1}(M;\mathbb{R})$. Indeed, it can be described without using the cocycle Θ as follows.

Theorem 1.12 (cf. [2])**.** *Let $\xi_q(\nabla, e)$ be the cocycle in the Čech-de Rham complex defined by the formula*

$$\xi_q(\nabla, e) = \frac{1}{2}\sqrt{-1} \sum_{k=0}^{q} \left(\bar{v}_1^k \cup (u_1 - \bar{u}_1) \cup v_1^q + v_1^k \cup (u_1 - \bar{u}_1) \cup \bar{v}_1^q \right).$$

Then $\xi_q(\nabla, e)$ represents $\xi_q(\mathcal{F}) = \sqrt{-1}(B_q(\mathcal{F}) - \overline{B_q(\mathcal{F})})$, which is independent of the choice of ∇ and e.

Proof. Set $\alpha_k = \sum_{r=0}^{q-k-1} \Theta^k \cup \bar{u}_1 \cup \bar{v}_1^r \cup (u_1 - \bar{u}_1) \cup v_1^{q-k-r-1}$. Then

$$\sum_{k=0}^{q} \bar{v}_1^k \cup (u_1 - \bar{u}_1) \cup v_1^q - \mathcal{D}(\alpha_0 + \cdots + \alpha_{q-1}) = B_q(\nabla, e) - \overline{B_q(\nabla, e)}.$$

The claim follows from this equation and its complex conjugate. □

If $\log J$ takes values in $\sqrt{-1}\mathbb{R}$ and if $\beta = 0$, then $\widetilde{u}_1 = u_1 - \bar{u}_1$, v_1 and $\bar{v}_1$ are globally well-defined differential forms, and the representative of ξ_q in Theorem 1.12 coincides with the standard one.

2. Infinitesimal derivatives of the Bott class

We will introduce the infinitesimal derivative by following Heitsch [14]. In what follows, tensors are usually represented in the form of matrices and the multiplications are considered under the usual multiplication laws together with the tensor or wedge products.

Let $\{\underline{e}_i = (\underline{e}_{i,1}, \cdots, \underline{e}_{i,q})\}$ be a family of local trivializations of $Q(\mathcal{F})$ and $\{\underline{\omega}_i = {}^t(\underline{\omega}_i^1, \cdots, \underline{\omega}_i^q)\}$ its dual. Let $\underline{A}_{ji}$ be the matrix valued function such that $(\underline{e}_{i,1}, \cdots, \underline{e}_{i,q}) = (\underline{e}_{j,1}, \cdots, \underline{e}_{j,q})\underline{A}_{ji}$, then $\underline{A}_{ji}\underline{\omega}_i = \underline{\omega}_j$. Let $\underline{\nabla} = (\{\underline{\theta}_i\}, \{\underline{\beta}_{ij}\})$ be a pair of a family of local Bott connection forms and the difference cochain with respect to $\{\underline{e}_i\}$. That is, $\underline{\theta}_i$ is the connection form with respect to $\underline{e}_i$ of a Bott connection $\underline{\nabla}_i$ on U_i so that $\underline{\nabla}_i\underline{e}_i = (\underline{\nabla e}_{i,1}, \cdots, \underline{\nabla e}_{i,q}) = (\underline{e}_{i,1}, \cdots, \underline{e}_{i,q})\underline{\theta}_i$, and $\underline{\beta}_{ij} = \underline{A}_{ji}^{-1} d\underline{A}_{ji} + \underline{A}_{ji}^{-1}\underline{\theta}_j\underline{A}_{ji} - \underline{\theta}_i$, where $\underline{e}_{i,k}(\underline{\theta}_i)^k{}_l = (\underline{\theta}_i)^k{}_l \otimes \underline{e}_{i,k}$. One has then $d\underline{\omega}_i + \underline{\theta}_i \wedge \underline{\omega}_i = 0$, $\underline{\beta}_{ji} = -\underline{A}_{ji}\underline{\beta}_{ij}\underline{A}_{ji}^{-1}$ and $\underline{\beta}_{ij} \in I_{(1)}(U_{ij})$, where $U_{ij} = U_i \cap U_j$.

Definition 2.1. Set $E^s \otimes Q(\mathcal{F}) = \bigwedge^s E^* \otimes Q(\mathcal{F})$. Let $\mathcal{U} = \{U_i\}$ and $s \in (E^s \otimes Q(\mathcal{F}))(U)$, where U is an open subset of M contained in U_i. Define a mapping $d_{\underline{\nabla},i} : (E^s \otimes Q(\mathcal{F}))(U) \to (E^{s+1} \otimes Q(\mathcal{F}))(U)$ by

$$d_{\underline{\nabla},i}(s) = \underline{e}_i(d\varphi + \underline{\theta}_i \wedge \varphi),$$

where $\varphi = \underline{\omega}_i(s)$ and φ is considered as an s-form by arbitrarily extending it. We equip $\{\check{C}^t(E^s \otimes Q(\mathcal{F}))\}$ with the Čech differential δ and the differential $d_{\underline{\nabla}}$. The total complex with differential $\delta + (-1)^t d_{\underline{\nabla}}$ is denoted by $\mathcal{E}^*(Q(\mathcal{F}))$.

Lemma 2.2. *$d_{\underline{\nabla},i}$ is independent of i, and the family $\{d_{\underline{\nabla},i}\}$ induces a well-defined mapping $d_{\underline{\nabla}} : E^s \otimes Q(\mathcal{F}) \to E^{s+1} \otimes Q(\mathcal{F})$.*

Proof. If s is a section of $(E^s \otimes Q(\mathcal{F}))(U_i \cap U_j)$, then $d_{\underline{\nabla},j}(\underline{e}_j\underline{\omega}_j(s)) = d_{\underline{\nabla},i}(\underline{e}_i\underline{\omega}_i(s)) + \underline{e}_i(\underline{\beta}_{ij} \wedge \underline{\omega}_i(s))$. The right hand side is equal to $d_{\underline{\nabla},i}(\underline{e}_i\underline{\omega}_i(s))$ as a section of $(E^{s+1} \otimes Q(\mathcal{F}))(U_i \cap U_j)$ because $E^* \cong I^1_{(0,1)}$. □

Definition 2.3. Let $H^*(M; \Theta_{\mathcal{F}})$ be the cohomology of $((E^s \otimes Q(\mathcal{F}))(M), d_{\underline{\nabla}})$, and $\mathcal{H}^*(M; \Theta_{\mathcal{F}})$ the cohomology of the total complex $(\mathcal{E}^*(Q(\mathcal{F})), \delta + (-1)^s d_{\underline{\nabla}})$.

The first definition is justified by the fact that $((E^* \otimes Q(\mathcal{F}))(M), d_{\underline{\nabla}})$ is a resolution of $\Theta_{\mathcal{F}}$ if $\underline{\nabla}$ is a global Bott connection ([9]) and by Lemma 2.2. It is easy to see that the natural mapping $H^p(M; \Theta_{\mathcal{F}}) \to \mathcal{H}^p(M; \Theta_{\mathcal{F}})$ is injective if $p = 1$. Indeed, an isomorphism between $H^p(M; \Theta_{\mathcal{F}})$ and $\mathcal{H}^p(M; \Theta_{\mathcal{F}})$ can be constructed by using a partition of unity. However, we distinguish them because a certain difference will occur when defining infinitesimal derivatives (cf. Definitions 2.16 and 4.11).

Definition 2.4 (cf. [15])**.** An element $\underline{\mu}$ of $\mathcal{H}^1(M; \Theta_{\mathcal{F}})$ is called an *infinitesimal deformation* of $\mathcal{F}$. If $(\{\underline{\sigma}_i\}, \{\underline{s}_{ij}\}) \in \mathcal{E}^1(Q(\mathcal{F}))$ is a representative of $\underline{\mu}$, then the pair $(\{-\underline{\sigma}_i\}, \{-\underline{s}_{ij}\})$ is called the *infinitesimal derivative* of $\underline{\omega} = \{\underline{\omega}_i\}$.

Since $E^* \cong I^1_{(0,1)}$, an infinitesimal derivative $(\{-\underline{\sigma}_i\}, \{-\underline{s}_{ij}\})$ satisfies the following relations for some $\mathfrak{gl}(q; \mathbb{C})$-valued function $\underline{g}_{ij}$ on U_{ij} and some $\mathfrak{gl}(q; \mathbb{C})$-valued 1-form $\underline{\theta}'_i$ on U_i:

(2.5.a) $$\underline{e}_i\left(d(\underline{\omega}_i(\underline{\sigma}_i)) + \underline{\theta}_i \wedge (\underline{\omega}_i(\underline{\sigma}_i))\right) = \underline{e}_i\underline{\theta}'_i \wedge \underline{\omega}_i,$$

(2.5.b) $$(\underline{\sigma}_j - \underline{\sigma}_i) - \underline{e}_j\left(d(\underline{\omega}_j(\underline{s}_{ij})) + \underline{\theta}_j\underline{\omega}_j(\underline{s}_{ij})\right) = \underline{e}_j\underline{g}_{ij}\underline{\omega}_j,$$

(2.5.c) $$(\underline{\delta s})_{ijk} = 0,$$

where each σ_i is arbitrarily extended to a $Q(\mathcal{F})$-valued differential form. Note that $g_{ij} = -g_{ji}$ need not hold in general.

Infinitesimal derivatives of Bott connections can be defined as follows if $\underline{\mu}$ is represented by an element of $\check{C}^0(E^* \otimes Q(\mathcal{F}))$. Note that cocycles in $\check{C}^0(E^* \otimes Q(\mathcal{F}))$ are elements of $(E^1 \otimes Q(\mathcal{F}))(M)$ closed under $d_{\underline{\nabla}}$.

Definition 2.6. Suppose that $\underline{\mu} \in H^1(M; \Theta_{\mathcal{F}})$ and let $\underline{\sigma} = \{\underline{\sigma}_i\} \in \check{C}^0(E^* \otimes Q(\mathcal{F}))$ be a representative of μ. Then any pair $\underline{\nabla}' = (\{\underline{\theta}'_i\}, \{\underline{g}_{ij}\})$ satisfying (2.5.a) and (2.5.b) with $\underline{s} = 0$ is called an *infinitesimal derivative* of the Bott connection $\underline{\nabla} = (\{\underline{\theta}_i\}, \{\underline{\beta}_{ij}\})$ with respect to $\underline{\sigma}$.

The infinitesimal derivative of the Bott class is defined as follows.

Definition 2.7. Let $\underline{\mu} \in H^1(M; \Theta_{\mathcal{F}})$ and let $\sigma \in (E^1 \otimes Q(\mathcal{F}))(M)$ be a representative. Set

$$\theta' = \operatorname{tr} \underline{\theta}',\ \theta = \operatorname{tr} \underline{\theta},\ \beta = \operatorname{tr} \underline{\beta},\ g = \operatorname{tr} \underline{g} \text{ and } u'_1 = \frac{-1}{2\pi\sqrt{-1}}(\theta' + g).$$

The cohomology class in $H^{2q+1}(M; \mathbb{C})$ represented by

$$D_\sigma B_q(\underline{\nabla}, \underline{\nabla}') = \sum_{k=0}^{q} v_1^k \cup u'_1 \cup v_1^{q-k}$$

is called the *infinitesimal derivative* of the Bott class with respect to $\underline{\mu}$, and is denoted by $D_{\underline{\mu}} B_q(\mathcal{F})$.

The independence of the infinitesimal derivatives from the choice of $\underline{\sigma}$, $\underline{\nabla}$, $\underline{\nabla}'$ and local trivializations will be shown in Theorems 2.14 and 2.17.

Since the Bott class can be defined in terms of $K_{\mathcal{F}}$ alone, it is natural to expect that so is its infinitesimal derivative. Indeed, it can be done as follows. Let $\{e_i\}$ be a family of local trivializations of $-K_{\mathcal{F}}$, where e_i is defined on U_i. Let $\{J_{ij}\}$ be a family of smooth functions such that $e_j = e_i J_{ij}$. A Bott connection on $Q(\mathcal{F})|_{U_i}$ naturally induces a connection on $-K_{\mathcal{F}}|_{U_i}$, which is also called a Bott connection. Then, a family of local Bott connections on $-K_{\mathcal{F}}$ is a pair $(\{\theta_i\}, \{\beta_{ij}\})$ satisfying $\theta_j - \theta_i = d \log J_{ij} + \beta_{ij}$, where θ_i is the connection form of a Bott connection on $-K_{\mathcal{F}}|_{U_i}$ with respect to e_i. Finally, let $\{\omega_i\}$ be the family of local trivializations of $K_{\mathcal{F}}$ dual to $\{e_i\}$.

Recalling that $E^* \cong I^1_{(0,1)}$, we introduce the following

Definition 2.8. We denote $(E^* \otimes Q(\mathcal{F}))(U)$ also by $I^1_{(0,1)}(U; Q(\mathcal{F}))$, and set $I^*_{(q-1,q)}(U; -K_{\mathcal{F}}) = I^*_{(q-1,q)}(U) \otimes (-K_{\mathcal{F}}|_U)$.

Let $\varphi \in I^p_{(q-1,q)}(U_i \cap U_j; -K_{\mathcal{F}})$. Then φ can be written as $\varphi = \varphi_i \otimes e_i$ on U_i, where $\varphi_i \in I^p_{(q-1,q)}(U_i)$. Set then $d_{\nabla,i}\varphi = e_i(d\varphi_i + \theta_i \wedge \varphi_i)$. Since $\beta_{ij} \in I_{(1)}(U_{ij})$, the identity $d_{\nabla,j}\varphi = d_{\nabla,i}\varphi$ holds. Hence $\{d_{\nabla,i}\}$ induces a globally well-defined map, which is denoted by d_∇. One has $d_\nabla \circ d_\nabla = 0$. Indeed, the identity $d_\nabla(d_\nabla(e_i\varphi_i)) = e_i(d\theta_i \wedge \varphi_i)$ holds on U_i. The equation $d\theta_i \wedge \varphi_i = 0$ holds in $I^p_{(q-1,q)}(U_i)$, since $\varphi_i \in I^p_{(q-1,q)}(U_i)$ and $d\theta_i \in I_{(1)}(U_i)$.

Definition 2.9. Set $\mathcal{K}^{r,s} = \check{C}^r(I^{s+q-1}_{(q-1,q)}(\mathcal{U}; -K_\mathcal{F}))$ and equip it with the differentials δ and d_∇. Let $\mathcal{K}^*$ be the total complex with the differential $\delta + (-1)^r d_\nabla$, and $\mathcal{H}^*(M; -K_\mathcal{F})$ the cohomology of $\mathcal{K}^*$. We regard the complex $(I^{*+q-1}_{(q-1,q)}(M; -K_\mathcal{F}), d_\nabla)$ as a subcomplex of $(\mathcal{K}^*, \delta + (-1)^r d_\nabla)$, and denote its cohomology by $H^*(M; -K_\mathcal{F})$.

The natural mapping $H^1(M; -K_\mathcal{F}) \to \mathcal{H}^1(M; -K_\mathcal{F})$ is injective, and one can construct an isomorphism by using a partition of unity.

A version of infinitesimal deformations of $-K_\mathcal{F}$ is defined as follows.

Definition 2.10. An element μ of $\mathcal{H}^1(M; -K_\mathcal{F})$ is called an *infinitesimal deformation* of $-K_\mathcal{F}$. If $(\{\sigma_i\}, \{s_{ij}\}) \in \mathcal{K}^1$ is a representative of μ, then the cocycle $(\{-\sigma_i\}, \{-s_{ij}\})$ is called the *infinitesimal derivative* of $\omega = \{\omega_i\}$ with respect to (σ, s).

If $(\{-\sigma_i\}, \{-s_{ij}\})$ is an infinitesimal derivative, then the following identities hold:

$$e_i\bigl(d(\omega_i(\sigma_i)) + \theta_i \wedge (\omega_i(\sigma_i))\bigr) = e_i\theta'_i \wedge \omega_i, \tag{2.11.a}$$

$$(\sigma_j - \sigma_i) - e_j\bigl(d(\omega_j(s_{ij})) + \theta_j \wedge \omega_j(s_{ij})\bigr) = e_j g_{ij}\omega_j, \tag{2.11.b}$$

$$s_{jk} - s_{ik} + s_{ij} = 0. \tag{2.11.c}$$

Suppose that local trivializations and local connections of $Q(\mathcal{F})$ are given. Then those of $-K_\mathcal{F}$ are induced in the following way. Let $\{\underline{e}_i\}$ be a family of local trivializations of $Q(\mathcal{F})$ and $\{e_i\}$ a family of local trivializations of $-K_\mathcal{F}$ defined by $e_i = \underline{e}_{i,1} \wedge \cdots \wedge \underline{e}_{i,q}$. We locally trivialize $K_\mathcal{F}$ by the dual $\{\omega_i = \underline{\omega}_i^1 \wedge \cdots \wedge \underline{\omega}_i^q\}$ of $\{e_i\}$. Then $\{\theta_i = \operatorname{tr}\underline{\theta}_i\}$ is a family of local Bott connection forms with respect to $\{e_i\}$. They satisfy the equations $d\omega_i + \theta_i \wedge \omega_i = 0$ and $\theta_j - \theta_i = d\log J_{ij} + \beta_{ij}$, where $J_{ij} = \det \underline{A}_{ij}$ and $\beta_{ij} = \operatorname{tr}\underline{\beta}_{ij}$.

Lemma 2.12. *Let $\underline{\mu} \in \mathcal{H}^1(M; \Theta_\mathcal{F})$ and let $\underline{m} = (\{\underline{\sigma}_i\}, \{\underline{s}_{ij}\}) \in \mathcal{E}^1(\mathcal{U}; Q(\mathcal{F}))$ be its representative. Set*

$$r_0(\underline{m})_i = \sum_{k=1}^q \underline{\omega}_i^1 \wedge \cdots \wedge \underline{\omega}_i^{k-1} \wedge \underline{\omega}_i^k(\underline{\sigma}_i) \wedge \underline{\omega}_i^{k+1} \wedge \cdots \wedge \underline{\omega}_i^q \otimes e_i,$$

$$r_1(\underline{m})_{ij} = \sum_{k=1}^q (-1)^{k-1}\underline{\omega}_j^1 \wedge \cdots \wedge \underline{\omega}_j^{k-1} \wedge \underline{\omega}_j^k(\underline{s}_{ij}) \wedge \underline{\omega}_j^{k+1} \wedge \cdots \wedge \underline{\omega}_j^q \otimes e_j.$$

Then $r = r_0 \oplus r_1$ induces isomorphisms $r\colon \mathcal{H}^1(M; \Theta_\mathcal{F}) \to \mathcal{H}^1(M; -K_\mathcal{F})$ and $r\colon H^1(M; \Theta_\mathcal{F}) \to H^1(M; -K_\mathcal{F})$, where the induced mappings are denoted by r by abuse of notation. Moreover, if $\underline{m}$ satisfies (2.5.a), (2.5.b) *and* (2.5.c), *then $r(\underline{m})$ satisfies* (2.11.a), (2.11.b) *and* (2.11.c) *with $\theta' = \operatorname{tr}\underline{\theta}'$ and $g = \operatorname{tr}\underline{g}$.*

Proof. It is clear that $r(\underline{m}) \in \mathcal{K}^*$. By (2.5.a), $d(\underline{\omega}_i(\underline{\sigma}_i)) + \underline{\theta}_i \wedge (\underline{\omega}_i(\underline{\sigma}_i)) = \underline{\theta}'_i \wedge \underline{\omega}_i$ for some $\mathfrak{gl}(q;\mathbb{C})$-valued 1-form $\underline{\theta}'_i$. Since $\theta = \operatorname{tr}\underline{\theta}$ and $\beta = \operatorname{tr}\underline{\beta}$, we have

$dr_0(\underline{m})_i + \theta_i \wedge r_0(\underline{m})_i = (\operatorname{tr}\underline{\theta}')_i \wedge \omega_i \otimes e_i$ and $r_0(\underline{m})_j - r_0(\underline{m})_i = d_\nabla r_1(\underline{m})_{ij} + (\operatorname{tr}\underline{g}_{ij})\,\omega_j \otimes e_j$. It is easy to see that $\delta r_1(\underline{m})_{ijk} = 0$. Hence $r(\underline{m})$ is closed under $\delta + (-1)^r d_\nabla$ and the last part of the lemma follows.

Assume that $\underline{m}$ is exact. Then $\sigma_i = \underline{e}_i\,(df_i + \underline{\theta}_i f_i)$ and $s_{ij} = \underline{e}_j f_j - \underline{e}_i f_i$ for some collection $\{\underline{e}_i f_i\}$ of local sections of $Q(\mathcal{F})$. Set $\rho_i = \sum_{k=1}^{q} (-1)^{k-1}\underline{\omega}_i^1 \wedge \cdots \wedge \underline{\omega}_i^{k-1} \wedge f_i^k \wedge \underline{\omega}_i^{k+1} \wedge \cdots \wedge \underline{\omega}_i^q \otimes e_i$. Then $d_\nabla \rho_i = r_0(\underline{m})_i$ and $\rho_j - \rho_i = r_1(\underline{m})_{ij}$. Conversely, let $m = (\{\sigma_i\}, \{s_{ij}\})$ be a cocycle in $\mathcal{K}^1$. Then (2.11.a), (2.11.b) and (2.11.c) hold. Let $\{\underline{\sigma}_i^k\}$ be a family of 1-forms and $\{\underline{s}_{ij}^k\}$ a family of functions such that $\underline{\omega}_i^k \wedge (\omega_i(\sigma_i)) = -\underline{\sigma}_i^k \wedge \omega_i$ and $\underline{\omega}_j^k \wedge \omega_j(s_{ij}) = \underline{s}_{ij}^k \omega_j$. Set $\underline{\sigma}_i = \sum_{k=1}^{q} \underline{\sigma}_i^k \otimes \underline{e}_{i,k}$ and $\underline{s}_{ij} = \sum_{k=1}^{q} \underline{s}_{ij}^k \otimes \underline{e}_{j,k}$. Then $(\underline{\sigma}, \underline{s})$ is well-defined as an element of $\mathcal{E}^1(Q(\mathcal{F}))$ and independent of the choice of $\{\underline{\sigma}_i^k\}$ and $\{\underline{s}_{ij}^k\}$. We have $d(\underline{\omega}_i(\underline{\sigma}_i) \wedge \omega_i) = d(\underline{\omega}_i(\underline{\sigma}_i)) \wedge \omega_i + \underline{\omega}_i(\underline{\sigma}_i) \wedge \theta_i \wedge \omega_i$ and $d(\underline{\omega}_i(\underline{\sigma}_i) \wedge \omega_i) = \underline{\theta}_i \wedge \underline{\omega}_i \wedge (\omega_i(\sigma_i)) + \underline{\omega}_i \wedge d(\omega_i(\sigma_i)) = -\underline{\theta}_i \wedge \underline{\omega}_i(\underline{\sigma}_i) \wedge \omega_i - \underline{\omega}_i \wedge \theta_i \wedge \omega_i(\sigma_i)$. Hence $d(\underline{\omega}_i(\underline{\sigma}_i)) + \underline{\theta}_i \wedge \underline{\omega}_i(\underline{\sigma}_i) = 0$ in $I^1_{(0,1)}$.

On the other hand, $(d(\underline{\omega}_j(\underline{s}_{ij})) + \underline{\theta}_j \underline{\omega}_j(\underline{s}_{ij})) \wedge \omega_j = -\underline{\theta}_j \wedge \underline{\omega}_j \wedge \omega_j(s_{ij}) - \underline{\omega}_j \wedge (-g_{ij}\omega_j - \theta_j \wedge \omega_j(s_{ij}) + \omega_j(\sigma_j) - \omega_j(\sigma_i)) + \underline{\omega}_j(\underline{s}_{ij})\theta_j \wedge \omega_j + \underline{\theta}_j \underline{\omega}_j(\underline{s}_{ij}) \wedge \omega_j$. It follows that $\underline{e}_j(d(\underline{\omega}_j(\underline{s}_{ij})) + \underline{\theta}_j \underline{\omega}_j(\underline{s}_{ij})) = \underline{\sigma}_j - \underline{\sigma}_i$. We also have $\delta \underline{s} = 0$. Therefore, if we set $r'(m) = (\underline{\sigma}, \underline{s})$, then $r'(m)$ is closed and it induces a mapping of the cohomology which is equal to r^{-1}. Finally, the construction shows that $H^1(M; \Theta_{\mathcal{F}})$ is mapped to $H^1(M; -K_{\mathcal{F}})$ under the mapping r. This completes the proof. □

Infinitesimal derivatives of the Bott class are determined by infinitesimal deformations of $-K_{\mathcal{F}}$ as follows.

Definition 2.13. Let $\mu \in H^1(M; -K_{\mathcal{F}})$ and $\sigma = \{\sigma_i\} \in I^q_{(q-1,q)}(M; -K_{\mathcal{F}})$ a representative of μ. Then any pair $\nabla' = (\{\theta_i'\}, \{g_{ij}\})$ satisfying (2.11.a) and (2.11.b) is called an *infinitesimal derivative* of the Bott connection $\nabla = (\{\theta_i\}, \{\beta_{ij}\})$ with respect to σ.

Theorem 2.14. *Let $\mu \in H^1(M; -K_{\mathcal{F}})$ be an infinitesimal deformation and $\sigma = \{\sigma_i\} \in I^q_{(q-1,q)}(M; -K_{\mathcal{F}})$ a representative of μ. Let $\nabla' = (\{\theta_i'\}, \{g_{ij}\})$ be the infinitesimal derivative of ∇ with respect to σ. Set*

$$D_\sigma B_q(\nabla, \nabla') = \sum_{k=0}^{q} v_1^k \cup u_1' \cup v_1^{q-k},$$

where $u_1' = \dfrac{-1}{2\pi\sqrt{-1}}(\theta' + g)$. Then $D_\sigma B_q(\nabla, \nabla')$ represents a class in $H^{2q+1}(M; \mathbb{C})$, which is independent of the choice of cochains and connections.

Proof. Note that σ is globally well-defined and that if we define $\underline{\sigma}_i$ as in the proof of Lemma 2.12, then $\{\underline{\sigma}_i\}$ induces a globally well-defined element of

$I^1_{(1)}(M;Q(\mathcal{F}))$, which we denote by $\underline{\sigma}$. Under these settings, the following lemma holds.

Lemma 2.15. *Let φ_k be (r_k, s_k)-cochains, where $0 \leq k \leq q$. Suppose that $(\varphi_k)_{i_0\cdots i_{r_k}} \in I^{s_k}_{(1)}$ for any k and any $i_0, \cdots, i_{r_k}$. Write $(\varphi_k)_{i_0\cdots i_{r_k}} = \sum_m (\alpha_{k;i_0\cdots i_{r_k}})_m \wedge \underline{\omega}_i^m$ and set $\langle \varphi_k | \underline{\sigma}_i \rangle_{i_0\cdots i_{r_k}} = \sum_m (\alpha_{k;i_0\cdots i_{r_k}})_m \wedge \underline{\sigma}_i^m$. Then $\langle \varphi_k | \underline{\sigma}_i \rangle \wedge \omega_i = -\varphi_k \wedge \omega_i(\sigma_i)$ and $\sum_{k=0}^{q} \varphi_0 \cup \cdots \cup \varphi_{k-1} \cup \langle \varphi_k | \underline{\sigma}_i \rangle \cup \varphi_{k+1} \cup \cdots \cup \varphi_q = 0$. Moreover, $\varphi_0 \cup \cdots \cup \varphi_{k-1} \cup \langle \varphi_k | \underline{\sigma}_i \rangle \cup \varphi_{k+1} \cup \cdots \cup \varphi_q$ is independent of i.*

Proof of Lemma 2.15. First note that we can obtain $\langle \varphi_k | \underline{\sigma}_i \rangle$ by taking a contraction from $\varphi_k \otimes \underline{\sigma}_i$ and then a reduction to a differential form. Note also that the last part of the lemma follows from the assumption $s = 0$ and the identity (2.11.a). If $\omega \in I^*_{(q)}$, then we can also consider the contraction of $\omega \otimes \underline{\sigma}_i$. Let $\rho_1, \cdots, \rho_n \in I^*_{(q)}$. Then $\langle \rho_1 \wedge \cdots \wedge \rho_n | \underline{\sigma} \rangle = \langle \rho_1 | \underline{\sigma}_i \rangle \wedge \rho_2 \wedge \cdots \wedge \rho_n + \cdots + \rho_1 \wedge \cdots \wedge \rho_{n-1} \wedge \langle \rho_n | \underline{\sigma}_i \rangle$. Under the assumption, $\varphi_0 \cup \cdots \cup \varphi_q = 0$, so that the lemma holds. □

We now return to the proof of the theorem.

Claim 1. $D_\sigma B_q(\nabla, \nabla')$ is closed.

We have $\mathcal{D}(D_\sigma B_q(\nabla, \nabla')) = (-2\pi\sqrt{-1})^{-(q+1)} \sum_{k=0}^{q} (d\theta+\beta)^k \cup \mathcal{D}(\theta'+g) \cup (d\theta+\beta)^{q-k}$ since $\mathcal{D}(d\theta+\beta) = 0$. We will show that $(d\theta+\beta)^k \cup \mathcal{D}(\theta'+g) \cup (d\theta+\beta)^{q-k} = -(d\theta+\beta)^k \cup (\langle d\theta | \underline{\sigma} \rangle + \langle \beta | \underline{\sigma} \rangle) \cup (d\theta+\beta)^{q-k}$. Then the claim follows from Lemma 2.15. We have $d\theta'_i \wedge \omega_i = d\theta_i \wedge (\omega_i(\sigma)) = -\langle d\theta_i | \underline{\sigma} \rangle$ by (2.11.a). On the other hand, $e_j\theta'_j \wedge \omega_j = e_i\theta'_i \wedge \omega_i + e_i\beta_{ij} \wedge (\omega_i(\sigma)) + e_i dg_{ij} \wedge \omega_i$ by (2.11.a) and (2.11.b). Then $(\delta\theta' - dg)_{ij} \wedge \omega_i = \beta_{ij} \wedge (\omega_i(\sigma)) = -\langle \beta_{ij} | \underline{\sigma} \rangle \wedge \omega_i$. Finally, $e_i(\delta g_{ijk})\omega_i = 0$ by (2.11.b) and by the assumption $s = \{s_{ij}\} = 0$.

Claim 2. $D_\sigma B_q(\nabla, \nabla')$ is independent of the choice of ∇' once σ is fixed.

Let $(\{\tilde{\theta}'_i\}, \{\tilde{g}_{ij}\})$ be another infinitesimal derivative of ∇ with respect to σ. Then $e_i(\tilde{\theta}'_i - \theta'_i) \wedge \omega_i = 0$ and $\tilde{g}_{ij} = g_{ij}$. Hence $(d\theta+\beta)^k \cup (\tilde{\theta}'+\tilde{g}) \cup (d\theta+\beta)^{q-k} = (d\theta+\beta)^k \cup (\theta'+g) \cup (d\theta+\beta)^{q-k}$ for each k.

Claim 3. The class $[D_\sigma B_q(\nabla, \nabla')]$ is independent of the choice of σ.

Let $\{\tilde{\sigma}_i\}$ be another representative of μ and $\widetilde{\nabla}' = (\{\tilde{\theta}'_i\}, \{\tilde{g}_{ij}\})$ an infinitesimal derivative of ∇ with respect to $\{\tilde{\sigma}_i\}$. It suffices to show that $\sum_{k=0}^{q} (d\theta+\beta)^k \cup (\tilde{\theta}' + \tilde{g} - \theta' - g) \cup (d\theta+\beta)^{q-k}$ is exact. Set $\psi = \tilde{\sigma} - \sigma$. Then there is an element $\tau = \{\tau_i\} \in I^{q-1}_{(q-1,q)}(\mathcal{U}; -K_\mathcal{F})$ and a family $\{h_i\}$ of functions on U_i such that $e_i\omega_i(\psi_i) = e_i\big(d(\omega_i(\tau_i)) + \theta_i \wedge (\omega_i(\tau_i)) + h_i\omega_i\big)$ and $\tau_j - \tau_i = 0$. Let $\underline{\tau}_i^m$, $m = 1, \ldots, q$, be 1-forms such that $\underline{\omega}_i^m \wedge \omega_i(\tau_i) = -\underline{\tau}_i^m \wedge \omega_i$. Then $e_i(\tilde{\theta}'_i - \theta'_i) \wedge \omega_i = -e_i(\langle d\theta_i | \underline{\tau}_i \rangle + dh_i) \wedge \omega_i$ and $e_jh_j\omega_j - e_ih_i\omega_i = e_i(\langle \beta_{ij} | \underline{\tau}_i \rangle +$

$(\tilde{g}_{ij} - g_{ij})) \wedge \omega_i$. The identity $\sum\limits_{k=0}^{q} (d\theta + \beta)^k \cup (\tilde{\theta}' + \tilde{g} - \theta' - g) \cup (d\theta + \beta)^{q-k} = \mathcal{D}\left(\sum\limits_{k=0}^{q} (d\theta + \beta)^k \cup\, h \,\cup (d\theta + \beta)^{q-k}\right)$ follows from Lemma 2.15.

Claim 4. The class $[D_\sigma B_q(\nabla, \nabla')]$ is independent of the choice of ∇.

Let $\widetilde{\nabla} = (\{\varphi_i\}, \{\rho_{ij}\})$ be another Bott connection and set $\psi_i = \varphi_i - \theta_i$. Then $\psi_i \in I^1_{(1)}(U_i)$. Assume that $\{\sigma_i\}$ satisfies (2.11.a), (2.11.b) and (2.11.c). Then $d(\omega_i(\sigma_i)) + \varphi_i \wedge (\omega_i(\sigma_i)) = \varphi'_i \wedge \omega_i$ for some 1-form φ'_i. Since (2.11.b) for $\widetilde{\nabla}$ is the same as (2.11.b) for ∇ because $s = \{s_{ij}\} = 0$, $(\{\varphi'_i\}, \{g_{ij}\})$ is an infinitesimal derivative of $\widetilde{\nabla}$. If we denote $\{\psi_i\}$ by ψ, then $\mathcal{D}\psi = (d\varphi + \rho) - (d\theta + \beta)$. Setting $\psi' = \varphi' - \theta'$, one has $\psi_i \wedge (\omega_i(\sigma_i)) = \psi'_i \wedge \omega_i$. It then follows that

$$\begin{aligned}
&\text{(2.15.b)}\quad (-2\pi\sqrt{-1})^{q+1} \left(D_\sigma B_q(\widetilde{\nabla}, \widetilde{\nabla}') - D_\sigma B_q(\nabla, \nabla')\right) \\
&= \sum_{k=1}^{q} \sum_{l=0}^{k-1} (d\theta + \beta)^l \cup \mathcal{D}\psi \cup (d\varphi + \rho)^{k-l-1} \cup (\varphi' + g) \cup (d\varphi + \rho)^{q-k} \\
&\quad + \sum_{k=0}^{q} (d\theta + \beta)^k \cup \psi' \cup (d\varphi + \rho)^{q-k} \\
&\quad + \sum_{k=0}^{q-1} \sum_{l=0}^{q-k-1} (d\theta + \beta)^k \cup (\theta' + g) \cup (d\theta + \beta)^l \cup \mathcal{D}\psi \cup (d\varphi + \rho)^{q-k-l-1}.
\end{aligned}$$

Since $\psi \in I^1_{(1)}(\mathcal{U})$, one has

$$\begin{aligned}
&\text{(2.15.c)}\quad \mathcal{D}\big((d\theta + \beta)^m \cup (\theta' + g) \cup (d\theta + \beta)^k \cup \psi \cup (d\varphi + \rho)^l\big) \\
&= -(d\theta + \beta)^m \cup (\langle d\theta |\, \underline{\sigma}\rangle + \underline{\beta} |\, \underline{\sigma}\rangle) \cup (d\theta + \beta)^k \cup \psi \cup (d\varphi + \rho)^l \\
&\quad - (d\theta + \beta)^m \cup (\theta' + g) \cup (d\theta + \beta)^k \cup \mathcal{D}\psi \cup (d\varphi + \rho)^l,
\end{aligned}$$

and

$$\begin{aligned}
&\text{(2.15.d)}\quad \mathcal{D}\big(-(d\theta + \beta)^m \cup \psi \cup (d\varphi + \rho)^k \cup (\varphi' + g) \cup (d\varphi + \rho)^l\big) \\
&= -(d\theta + \beta)^m \cup \mathcal{D}\psi \cup (d\varphi + \rho)^k \cup (\varphi' + g) \cup (d\varphi + \rho)^l \\
&\quad - (d\theta + \beta)^m \cup \psi \cup (d\varphi + \rho)^k \cup (\langle d\varphi |\, \underline{\sigma}\rangle + \langle \rho |\, \underline{\sigma}\rangle) \cup (d\varphi + \rho)^l,
\end{aligned}$$

where $m + k + l = q - 1$. Adding (2.15.c) and (2.15.d) to the right hand side of (2.15.b), varying m, k, l and by using Lemma 2.15, we see that $D_\sigma B_q(\widetilde{\nabla}, \widetilde{\nabla}') - D_\sigma B_q(\nabla, \nabla')$ is exact.

Claim 5. $D_\sigma B_q(\nabla, \nabla')$ is independent of the choice of the family of local trivializations $\{e_i\}$.

We fix σ and $\nabla = (\{\theta_i\}, \{\beta_{ij}\})$, and let $\{e'_i\}$ be another family of local trivializations. Then we may assume that $e'_i = e_i u_i$ for some $\mathbb{C}^*$-valued function u_i. Hence $\omega'_i = u_i^{-1}\omega_i$ and $e'_j = u_j u_i^{-1} \alpha_{ij} e'_i$. The connection form of ∇ with respect to $\{e'_i\}$ is $(\{\theta_i + u_i^{-1}du_i\}, \{\beta_{ij}\})$ so that $(\{\theta'_i\}, \{g_{ij}\})$ is also an infinitesimal derivative of ∇. This completes the proof of Claim 5 and the theorem follows. □

Definition 2.16. If $\mu \in H^1(M; -K_{\mathcal{F}})$, then we denote by $D_\mu B_q(\mathcal{F})$ the cohomology class in $H^{2q+1}(M; \mathbb{C})$ represented by $D_\sigma B_q(\nabla, \nabla')$ in Theorem 2.14.

It follows from Lemma 2.12 and Theorem 2.14 that Definition 2.16 is an alternative definition of the infinitesimal derivative of the Bott class.

Theorem 2.17. *If* $\underline{\mu} \in H^1(M; \Theta_{\mathcal{F}})$, *then* $D_{\underline{\mu}} B_q(\mathcal{F}) = D_{r(\underline{\mu})} B_q(\mathcal{F})$.

It is known that a smooth family of transversally holomorphic foliations induces an element of $H^1(M; \Theta_{\mathcal{F}}) \cong H^1(M; -K_{\mathcal{F}})$ ([14]).

Theorem 2.18. *If* $\mu \in H^1(M; -K_{\mathcal{F}})$ *is induced by a smooth family* $\{\mathcal{F}_s\}$, *then*

$$D_\mu B_q(\mathcal{F}) = \left.\frac{\partial}{\partial s} B_q(\mathcal{F}_s)\right|_{s=0}.$$

Proof. Let $\dot{\theta}_s$ be the one defined after Definition 1.9 and let $\dot{\theta} = \dot{\theta}_s|_{s=0}$. Then $\dot{\theta}$ is an infinitesimal derivative of θ [15, Theorem 2.23]. Hence $D_{r(\underline{\mu})} B_q(\mathcal{F}) = \left.\frac{\partial}{\partial s} B_q(\mathcal{F}_s)\right|_{s=0}$ by Proposition 1.10. □

The infinitesimal derivative of the Bott class constructed above is related with the previously constructed infinitesimal derivatives as follows.

Theorem 2.19. *Let* $\underline{\mu} \in H^1(M; \Theta_{\mathcal{F}})$.

1) *If* $-K_{\mathcal{F}}$ *is trivial, then* $D_{\underline{\mu}} B_q(\mathcal{F})$ *coincides with the infinitesimal derivative of the Bott class in* [15].
2) *Let* $D_{\underline{\mu}} \xi_q(\mathcal{F})$ *be the infinitesimal derivative of the imaginary part of the Bott class defined in* [15]. *Then* $D_{\underline{\mu}} \xi_q(\mathcal{F}) = -2 \operatorname{Im} D_{\underline{\mu}} B_q(\mathcal{F})$.

Proof. These infinitesimal derivatives are constructed under the assumption that $\beta = 0$ and $g = 0$. Hence $D_{\underline{\mu}} B_q(\mathcal{F})$ is represented by a global $(2q+1)$-form $(-2\pi\sqrt{-1})^{-(q+1)}(q+1)\theta' \wedge (d\theta)^q$. The claims are now obvious. □

3. Schwarzian Derivatives

In what follows, the natural coordinates of $\mathbb{C}^q$ will be denoted by $z = {}^t(z^1, \cdots, z^q)$ unless otherwise stated.

Definition 3.1 ([18], [22], etc.)**.** Let γ be a biholomorphic local diffeomorphism of $\mathbb{C}^q$. Let $u = {}^t(u^1, \cdots, u^q)$ be the natural coordinates of the target and set $\gamma^k = u^k \circ \gamma$. The *projective Schwarzian derivative* (the *Schwarzian derivatives* or the *Schwarzians* for short) Σ_γ of γ is given as follows:

$$\Sigma_\gamma = \sum_{k,l,t,s} \frac{\partial z^l}{\partial u^k}\frac{\partial^2\gamma^k}{\partial z^t \partial z^s}\frac{\partial}{\partial z^l} \otimes dz^t \otimes dz^s$$
$$+ \sum_{l,t,s} \frac{-1}{q+1}\left(\frac{\partial \log J_\gamma}{\partial z^t}\delta_{l,s}\frac{\partial}{\partial z^l} \otimes dz^t \otimes dz^s + \frac{\partial \log J_\gamma}{\partial z^s}\delta_{l,t}\frac{\partial}{\partial z^l} \otimes dz^t \otimes dz^s\right),$$

where $D\gamma$ denotes the differential of γ, $J_\gamma = \det D\gamma$ is the Jacobian and $\delta_{l,t}$ is the Kronecker delta. If $q > 1$, then let $\Sigma^l_{t,s}$ be the coefficient of $\dfrac{\partial}{\partial z^l} \otimes dz^t \otimes dz^s$ in Σ_γ and define a tensor Λ_γ by the formula

$$\Lambda_\gamma = \frac{-1}{q-1}\sum_{l=1}^{q}\left(\frac{\partial \Sigma^l_{t,s}}{\partial z^l} - \sum_{u=1}^{q}\Sigma^l_{t,u}\Sigma^u_{s,l}\right) dz^t \otimes dz^s.$$

We have

$$\Lambda_\gamma = \sum_{t,s}\frac{-1}{q+1}\frac{\partial^2 \log J_\gamma}{\partial z^t \partial z^s} dz^t \otimes dz^s - \sum_{t,s}\frac{-1}{q+1}\frac{\partial \log J_\gamma}{\partial z^t}\frac{-1}{q+1}\frac{\partial \log J_\gamma}{\partial z^s} dz^t_i \otimes dz^s$$
$$- \sum_{l,t,s}\frac{-1}{q+1}\frac{\partial \log J_\gamma}{\partial z^l}\frac{\partial z^l}{\partial u^k}\frac{\partial^2 \gamma^k}{\partial z^t \partial z^s} dz^t \otimes dz^s.$$

If $q = 1$, then Λ_γ is defined by the above formula because $\Sigma_\gamma = 0$ and coincides with the classical Schwarzian derivative. Indeed,

$$\Lambda_\gamma = -\frac{1}{2}\left(\frac{\gamma'''}{\gamma'} - \frac{3}{2}\left(\frac{\gamma''}{\gamma'}\right)^2\right) dz \otimes dz$$

holds, where $\gamma' = \dfrac{d\gamma}{dz}$, $\gamma'' = \dfrac{d^2\gamma}{dz^2}$ and $\gamma''' = \dfrac{d^3\gamma}{dz^3}$.

It is classical that γ is a restriction of a projective transformation if and only if $\Lambda_\gamma = 0$ if $q = 1$. If $q > 1$, then γ is a restriction of a projective transformation if and only if $\Sigma_\gamma = 0$. It is also known that Σ_γ is symmetric and trace-free in the sense that $\Sigma^l_{t,s} = \Sigma^l_{s,t}$ and $\sum\limits_{l=1}^{q} \Sigma^l_{l,s} = 0$. One of the significant properties of Σ_γ is that it is a cocycle, namely, $\Sigma_{\gamma\circ\zeta} = \zeta^*\Sigma_\gamma + \Sigma_\zeta$ holds for any local biholomorphic mapping γ and ζ ([20],[21]). On the other hand, Λ_γ is a kind of the curvature tensor for Σ_γ ([10], [20]), but is not a cocycle if $q > 1$. We refer to [18], [21], [23], [20], [7] and [22] for more details of the Schwarzians.

In terms of matrix valued differential forms, the above tensors are expressed as follows.

Lemma 3.2. *Set* $\partial \log J_\gamma = \begin{pmatrix} \dfrac{\partial \log J_\gamma}{\partial z^1} & \cdots & \dfrac{\partial \log J_\gamma}{\partial z^q} \end{pmatrix}$. *Then*

$$\begin{aligned}\Sigma_\gamma =& \frac{\partial}{\partial z} \otimes D\gamma^{-1} \cdot dD\gamma \otimes dz \\ &+ \sum_{k=1}^{q} \frac{-1}{q+1} \left(\frac{\partial}{\partial z^k} \otimes (\partial \log J_\gamma \cdot dz) \otimes dz^k + \frac{\partial}{\partial z^k} \otimes dz^k \otimes (\partial \log J_\gamma \cdot dz) \right),\end{aligned}$$

and

$$\begin{aligned}\Lambda_\gamma =& \frac{-1}{q+1} d\partial \log J_\gamma \otimes dz - \frac{-1}{q+1} \partial \log J_\gamma D\gamma^{-1} \cdot dD\gamma \otimes dz \\ &- \frac{-1}{q+1} (\partial \log J_\gamma \cdot dz) \otimes \frac{-1}{q+1} (\partial \log J_\gamma \cdot dz).\end{aligned}$$

4. Relation between the infinitesimal derivative of the Bott class and the Schwarzian derivatives

Let $\omega = \{\omega_i\}$ be a family of local trivializations of $-K_\mathcal{F}$ and ∇ a family of local Bott connections on $-K_\mathcal{F}$ induced by a family of Bott connections on $Q(\mathcal{F})$. For each i, let $z_i = {}^t(z_i^1, \cdots, z_i^q)$ be the local coordinates in the transversal direction and $\{\gamma_{ji}\}$ the transition functions in the transversal direction so that $z_j = \gamma_{ji}(z_i)$. Finally let μ be an element of $H^1(M; -K_\mathcal{F})$, then μ can be regarded as an element of $H^1(M; \Theta_\mathcal{F})$ by Lemma 2.12. Let $\sigma = \{\sigma_i\}$ be a representative of μ as an element of $H^1(M; \Theta_\mathcal{F})$. If V is a vector bundle, then $(\bigwedge^l T^*M) \wedge (T^*M \otimes V) \wedge (\bigwedge^{q-l-1} T^*M)$ is identified with $\bigwedge^q T^*M \otimes V$.

Definition 4.1. Let V be a vector bundle over M. Sections of V are said to be foliated if they are locally constant along the leaves and if they are transversally holomorphic. Let $\varGamma_\mathcal{F}(V)$ be the sheaf of germs of foliated sections of V. The Čech complex with coefficients in $\varGamma_\mathcal{F}(V)$ is denoted by $\check{C}^*_\mathcal{F}(\mathcal{U}; V)$, and its cohomology group is denoted by $\check{H}^*_\mathcal{F}(M; V)$.

Definition 4.2. Let φ be a (r, s)-cochain. For $0 \le k \le r$, define a family $\partial_{(k)}\varphi = \{(\partial_{(k)}\varphi)_{i_0 \cdots i_r}\}$ of $Q(\mathcal{F})^*$-valued s-forms on $U_{i_0 \cdots i_r}$ by setting $(\partial_{(k)}\varphi)_{i_0 i_1 \cdots i_r} = \sum_{l=1}^{q} \dfrac{\partial \varphi_{i_0 i_1 \cdots i_r}}{\partial z^l_{i_k}} \otimes dz^l_{i_k}$, where $\dfrac{\partial}{\partial z^l_{i_k}} h dz^{l_1}_{i_k} \wedge \cdots \wedge dz^{l_s}_{i_k} = \dfrac{\partial h}{\partial z^l_{i_k}} dz^{l_1}_{i_k} \wedge \cdots \wedge dz^{l_s}_{i_k}$ for any function h. Set then $\widehat{\partial} = \sum_{k=0}^{r} \partial_{(k)} \colon \check{C}^r_\mathcal{F}(\mathcal{U}; K_\mathcal{F}) \to \check{C}^r_\mathcal{F}(\mathcal{U}; K_\mathcal{F} \otimes Q(\mathcal{F})^*)$.

Lemma 4.3. *The mapping* $\widehat{\partial}$ *induces a homomorphism on the cohomology and the induced homomorphism is independent of the choice of the foliation atlas.*

The proof is straightforward and omitted. We denote again by this homomorphism by $\widehat{\partial}$.

Definition 4.4. Let $[d\log J]$ be the class in $\check{H}^1_{\mathcal{F}}(M;Q(\mathcal{F})^*)$ represented by $d\log J$. Set $\mathcal{L} = -(2\pi\sqrt{-1})^{-(q+1)}\widehat{\partial}((d\log J)^q) \in \check{C}^q_{\mathcal{F}}(\mathcal{U};K_{\mathcal{F}}\otimes Q(\mathcal{F})^*)$ and $\mathcal{L}(\mathcal{F}) = [\mathcal{L}] \in \check{H}^q_{\mathcal{F}}(M;K_{\mathcal{F}}\otimes Q(\mathcal{F})^*)$.

The class $[d\log J]$ is independent of the choice of the foliation atlas.

Definition 4.5. Let X be a vector field on an open set U of $\mathbb{C}^q$ and ω a p-form. Set $\iota_X\omega = \omega(\,\cdot\,,\cdots,\,\cdot\,,X)$ and define a $Q(\mathcal{F})^*$-valued p-form $\langle\omega\|\,\Sigma_\gamma\rangle$ by the formula $\langle\omega\|\,\Sigma_\gamma\rangle = \sum\limits_{i,t,s}(\iota_{\partial_i}\omega)\Sigma^i_{t,s}\wedge dz^t\otimes dz^s$, where $\iota_{\partial_i} = \iota_{\frac{\partial}{\partial z^i}}$. If η is a $Q^*(\mathcal{F})$-valued p-form and $\sigma = \sum\limits_i \dfrac{\partial}{\partial z^i}\otimes\sigma^i$ is a $Q(\mathcal{F})$-valued 1-form, then define a $(p+1)$-form $\langle\eta\,|\,\sigma\rangle$ by setting $\langle\eta\,|\,\sigma\rangle = \sum\limits_i(\iota_{\partial_i}\eta)\wedge\sigma^i$.

The next lemma is easy.

Lemma 4.6. *Let* $\eta = \{\eta_{i_0\cdots i_q}\} \in \check{C}^q_{\mathcal{F}}(\mathcal{U};K_{\mathcal{F}}\otimes Q(\mathcal{F})^*)$ *and* $(a,b) = (\{a_i\},\{b_{ij}\}) \in \mathcal{E}^1(Q(\mathcal{F}))$. *Define then an element* $\langle\eta\,|(a,b)\rangle$ *of* $\mathcal{A}^{q,q+1}(\mathcal{U})\oplus\mathcal{A}^{q+1,q}(\mathcal{U}) \subset \mathcal{A}^{2q+1}(\mathcal{U})$ *by setting* $\langle\eta\,|(a,b)\rangle_{i_0\cdots i_q,i_0\cdots i_{q+1}} = \left\langle\eta_{i_0\cdots i_q}|\,a_{i_q}\right\rangle \oplus (-1)^q\left\langle\eta_{i_0\cdots i_q}|\,b_{i_qi_{q+1}}\right\rangle$. *Then* $\langle\,\cdot\,|\,\cdot\,\rangle$ *induces a well-defined pairing*

$$\langle\,\cdot\,|\,\cdot\,\rangle : \check{H}^q_{\mathcal{F}}(M;K_{\mathcal{F}}\otimes Q(\mathcal{F})^*)\times\mathcal{H}^1(M;\Theta_{\mathcal{F}}) \to H^{2q+1}(M;\mathbb{C}).$$

Proposition 4.7. *If* $\mu \in H^1(M;-K_{\mathcal{F}})$, *then* $D_\mu B_q(\mathcal{F})$ *is equal to* $\langle\mathcal{L}(\mathcal{F})|\,\mu\rangle$.

Proof. In this proof, the index i_k is denoted by k. Since $D_\mu B_q(\mathcal{F})$ is independent of the choice of connections and representatives, we may choose $\omega_i = dz_i^1\wedge\cdots\wedge dz_i^q$ and assume that $\theta_i = 0$, then $\beta_{ij} = -d\log J_{ij}$. Let $\{\sigma_i\}\in\mathcal{K}^1$ be a representative of μ and $\underline{\sigma} = \{\underline{\sigma}_i\}$ a representative of μ as an element of $H^1(M;\Theta_{\mathcal{F}})$, where $\underline{e}_i = \left(\dfrac{\partial}{\partial z_i^1},\cdots,\dfrac{\partial}{\partial z_i^q}\right)$. We may assume that $\sigma_k = \underline{\sigma}_k^1\wedge dz_k^2\wedge\cdots\wedge dz_k^q + \cdots + dz_k^1\wedge\cdots\wedge dz_k^{q-1}\wedge\underline{\sigma}_k^q$ by Lemma 2.12. We set $\Delta_k = \det\begin{pmatrix}\partial_k\log J_{01}\\ \vdots\\ \partial_k\log J_{q-1,q}\end{pmatrix}$, then $\left((d\log J)^k\cup\theta'\cup(d\log J)^{q-k}\right)_{0\cdots q} = (-1)^{\frac{q(q+1)}{2}}d\omega_k(\sigma_k)\Delta_k$ because $c_1\cup\cdots\cup c_k = (-1)^{\frac{k(k-1)}{2}}c_1\wedge\cdots\wedge c_k$ if each c_i is a $(1,1)$-cochain. Let $\rho(k)$ be the (q,q)-cochain given by $\rho(k)_{0\cdots q} = (-1)^{\frac{q(q-1)}{2}}\langle d\log J_{01}\wedge\cdots\wedge d\log J_{q-1,q}|\,\underline{\sigma}_k\rangle$, where $0\le k\le q$ means the k-th index of $0,\cdots,q$. Then $\rho(k) = (-1)^{\frac{q(q-1)}{2}}\langle\omega_k|\,\underline{\sigma}_k\rangle\Delta_k = (-1)^{\frac{q(q-1)}{2}}\omega_k(\sigma_k)\Delta_k$. Hence $(\mathcal{D}''\rho(k))_{0\cdots q} = (-1)^{\frac{q(q+1)}{2}}d\omega_k(\sigma_k)\Delta_k - \langle(d\log J)^q\,|\,\underline{\sigma}_k\rangle$. On the other hand we can show that $(\mathcal{D}'\rho(k))_{0\cdots q+1} = \left((d\log J)^k\cup g\cup(d\log J)^{q-k}\right)_{0\cdots q+1}$ by direct calculations. Thus $D_\mu B_q(\mathcal{F})$ is cohomologous to $-(2\pi\sqrt{-1})^{q+1}\langle(d\log J)^q|\,\underline{\sigma}\rangle$. $\square$

We will need the explicit form of coboundaries in proving Proposition 5.10.

The cocycle $\mathcal{L}$ is calculated as follows.

Lemma 4.8. *Let* Λ *be the foliated Čech* 1*-cochain valued in* $Q^*(\mathcal{F})$ *defined by* $\Lambda_{ij} = \Lambda_{\gamma_{ij}}$. *Then*

$$\mathcal{L}_{i_0\cdots i_q} = \frac{q+1}{(2\pi\sqrt{-1})^{q+1}(q-1)!} \sum_{\tau\in\mathfrak{S}_{q+1}} (\operatorname{sgn}\tau)((d\log J)^{q-1}\cup\Lambda)_{i_{\tau(0)}\cdots i_{\tau(q)}}.$$

Proof. We denote the indices $i_0, \cdots, i_q$ by $0, \cdots, q$. If we set $(\bigwedge d\log J)_{n\cdots m;q} = d\log J_{nq}\wedge d\log J_{n+1,q}\wedge\cdots\wedge d\log J_{mq}$, then $d\log J_{01}\wedge d\log J_{12}\wedge\cdots\wedge d\log J_{q-1,q} = (\bigwedge d\log J)_{0\cdots q-1;q}$. Since $\partial_{(j)}h dz_i = d\partial_i h\otimes dz_i - hD\gamma_{ji}^{-1}dD\gamma_{ji}\otimes dz_i$,

$$\begin{aligned}
&\partial_{(k)}(\textstyle\bigwedge^q d\log J)_{0\cdots q}\\
&=\sum_{l=0}^{q-1}(\textstyle\bigwedge d\log J)_{0\cdots l-1;q}\wedge d\partial_q\log J_{lq}\wedge(\bigwedge d\log J)_{l+1\cdots q-1;q}\otimes dz_q\\
&\quad-\sum_{l=0}^{q-1}(\textstyle\bigwedge d\log J)_{0\cdots l-1;q}\wedge(\partial_q\log J_{lq}D\gamma_{kq}^{-1}dD\gamma_{kq})\wedge(\bigwedge d\log J)_{l+1\cdots q-1;q}\otimes dz_q.
\end{aligned}$$

On the other hand, the following equation holds by Lemma 3.2, namely,

$$\begin{aligned}
\text{(4.8.a)}\quad &-\sum_{\substack{0\le l\le q-1\\ l\ne k}}(\textstyle\bigwedge d\log J)_{0\cdots l-1;q}\wedge\langle d\log J_{lq}\|\,\Sigma_{kq}\rangle\wedge(\bigwedge d\log J)_{l+1\cdots q-1;q}\\
&-(q+1)^2(\textstyle\bigwedge d\log J)_{0\cdots k-1;q}\wedge\Lambda_{kq}\wedge(\bigwedge d\log J)_{k+1\cdots q-1;q}\\
&+q(\textstyle\bigwedge d\log J)_{0\cdots k-1;q}\wedge\langle d\log J_{kq}\|\,\Sigma_{kq}\rangle\wedge(\bigwedge d\log J)_{k+1\cdots q-1;q}\\
&=(q+1)(\textstyle\bigwedge d\log J)_{0\cdots k-1;q}\wedge d\partial_q\log J_{kq}\wedge(\bigwedge d\log J)_{k+1\cdots q-1;q}\otimes dz_q\\
&\quad-\sum_{l=0}^{q-1}(\textstyle\bigwedge d\log J)_{0\cdots l-1;q}\wedge(\partial_q\log J_{lq}D\gamma_{kq}^{-1}dD\gamma_{kq})\wedge(\bigwedge d\log J)_{l+1\cdots q-1;q}\otimes dz_q,
\end{aligned}$$

where $\Sigma_{ij} = \Sigma_{\gamma_{ij}}$. As we have $(q+1)(\Lambda_{ij}-\Lambda_{ik}+\Lambda_{jk}) = \langle d\log J_{kj}\|\,\Sigma_{ij}\rangle$, the left hand side of (4.8.a) is equal to

$$\begin{aligned}
&-(q+1)\sum_{l=0}^{q-1}(\textstyle\bigwedge d\log J)_{0\cdots l-1;q}\wedge(\Lambda_{ql}-\Lambda_{kl}+\Lambda_{kq})\wedge(\bigwedge d\log J)_{l+1\cdots q-1;q}\\
&+(q+1)^2(\textstyle\bigwedge d\log J)_{0\cdots k-1;q}\wedge\Lambda_{qk}\wedge(\bigwedge d\log J)_{k+1\cdots q-1;q}\\
=&-(q+1)\sum_{l=0}^{q-1}(-1)^{q-l-1}(\textstyle\bigwedge d\log J)_{0\cdots l-1,l+1\cdots q-1;q}\wedge(\Lambda_{ql}-\Lambda_{kl})\\
&+(q+1)(\textstyle\bigwedge d\log J)_{0\cdots k-1,q-1,k+1\cdots q-2;k}\wedge\Lambda_{kq}\\
&+(q+1)^2(-1)^{q-k-1}(\textstyle\bigwedge d\log J)_{0\cdots k-1;q}\wedge(\bigwedge d\log J)_{k+1\cdots q-1;q}\wedge\Lambda_{qk}.
\end{aligned}$$

Noticing that $(\bigwedge d\log J)_{0\cdots l-1,l+1\cdots q-1;q} = -(\bigwedge d\log J)_{0\cdots k-1,q,k+1,\cdots l-1,l+1\cdots q-1;k}$ if $k \neq l$ and taking the sum of the above equality with respect to k, we obtain

$$\begin{aligned}\widehat{\partial}(\textstyle\bigwedge^q d\log J)_{0\cdots q} &= (q+1)\sum_{k=0}^{q}(-1)^{q-k-1}(\textstyle\bigwedge d\log J)_{0\cdots k-1,k+1\cdots q-1;q} \wedge \Lambda_{qk}\\ &\quad - (q+1)\sum_{l=0}^{q-1}\sum_{k\neq l}(-1)^{q-l-1}(\textstyle\bigwedge d\log J)_{0\cdots k-1,q,k+1\cdots l-1,l+1\cdots q-1;k} \wedge \Lambda_{kl}\\ &\quad + (q+1)\sum_{k=0}^{q-1}(\textstyle\bigwedge d\log J)_{0\cdots k-1,q-1,k+1,\cdots q-2;k} \wedge \Lambda_{kq},\end{aligned}$$

from which the lemma follows. □

Definition 4.9. Given $\mu \in H^1(M; -K_{\mathcal{F}})$ and $\sigma \in I^1_{(0,1)}(M; Q(\mathcal{F}))$ a representative of μ as an element of $H^1(M; \Theta_{\mathcal{F}})$, we define a Čech-de Rham $(1,2)$-cochain $L(\mu)$ by setting $L(\mu)_{ij} = \langle \Lambda_{ij} | \, \sigma_j \rangle$.

A generalization of the Maszczyk formula [19] for arbitrary transversally holomorphic foliations now follows from Proposition 4.7 and Lemma 4.8.

Theorem 4.10. *If $\mu \in H^1(M; -K_{\mathcal{F}})$, then $D_\mu B_q(\mathcal{F})$ is represented by the Čech-de Rham $(q, q+1)$-cocycle, whose value on $U_{i_0\cdots i_q}$ is given by*

$$\frac{(q+1)}{(2\pi\sqrt{-1})^{q+1}(q-1)!}\sum_{\tau\in\mathfrak{S}_{q+1}}(\operatorname{sgn}\tau)((d\log J)^{q-1} \cup L(\mu))_{i_{\tau(0)}\cdots i_{\tau(q)}}.$$

If $q = 1$, then the infinitesimal derivative of the Bott class is represented by the Čech-de Rham $(1,2)$-cocycle

$$\frac{1}{4\pi^2}\left(\frac{\gamma'''}{\gamma'} - \frac{3}{2}\left(\frac{\gamma''}{\gamma'}^2\right)\right) dz \wedge \sigma,$$

where σ is a representative of μ.

Note that $\langle \mathcal{L}(\mathcal{F}) | \, \mu \rangle$ is well-defined for any $\mu \in \mathcal{H}^1(M; -K_{\mathcal{F}})$. Hence we can extend Definitions 2.7 and 2.16 as follows.

Definition 4.11. Let $\mu \in \mathcal{H}^1(M; -K_{\mathcal{F}})$. The *infinitesimal derivative* of the Bott class with respect to μ is defined to be $\langle \mathcal{L}(\mathcal{F}) | \, \mu \rangle$.

Definition 4.12. The Bott class of a transversally holomorphic foliation $\mathcal{F}$ is said to be *infinitesimally rigid* if $\langle \mathcal{L}(\mathcal{F}) | \, \mu \rangle = 0$ for any $\mu \in \mathcal{H}^1(M; \Theta_{\mathcal{F}})$.

Definition 4.13. A transversally holomorphic foliation $\mathcal{F}$ is said to be *transversally complex projective* on U if $\mathcal{F}$ admits a structure of a $(\mathrm{PSL}(q+1; \mathbb{C}), \mathbb{C}P^q)$-foliation on U whose underlying transversal holomorphic structure coincides with the original one. If $U = M$, then $\mathcal{F}$ is said to be transversally complex projective. A transversal complex projective structure is also called a transversal projective structure for short. If a transversal complex projective structure

$\mathcal{P}$ is given on an open subset U, then a foliation atlas is said to be adapted to $\mathcal{P}$ if the atlas gives the structure $\mathcal{P}$ on U.

By Lemma 4.8, $\mathcal{L}(\mathcal{F})$ is the obstruction for $\mathcal{F}$ to admit a transversal projective structure if $q = 1$. If $q > 1$, it remains true that $\mathcal{L}(\mathcal{F})$ vanishes if $\mathcal{F}$ admits a transversal projective structure. However, it will be an obstruction for existence of certain reduced structures because Λ is a kind of the curvature tensor of the Schwarzian derivative Σ.

It is well-known that if a foliation admits a first-order transversal geometric structure such as Hermitian metrics or complex affine structures, then the imaginary part of the Bott class vanishes. There is an infinitesimal version of this fact involving complex projective structures, which are of second order.

Theorem 4.14. *The Bott class of transversally projective foliations is infinitesimally rigid.*

Indeed, the Bott class is infinitesimally rigid if $\mathcal{L}(\mathcal{F}) = 0$. Note that there are transversally projective foliations with non-trivial Bott classes (Example 7.2, see also [5]). On the other hand, it is classical that the Bott class admits continuous deformations ([8], see also Example 7.1). Note also that the imaginary part of Theorem 4.14 follows from [6].

Remark 4.15. There is an obvious analogue of above constructions for the Godbillon-Vey class of real foliations, and the infinitesimal derivative of the Godbillon-Vey class is represented in terms of the Schwarzians. The codimension-one case is exactly the Maszczyk formula [19]. Theorem 4.14 for real foliations and the Godbillon-Vey class is highly non-trivial, because it is well-known that the Godbillon-Vey class admits continuous deformations (cf. [16]).

In contrast to real foliations, the Godbillon-Vey class of transversally holomorphic foliations is known to be infinitesimally rigid [5]. The proof of Theorem 4.10 is independent of results in [5] and we have another proof the rigidity as follows.

Corollary 4.16. *The Godbillon-Vey class of transversally holomorphic foliations is rigid under both actual and infinitesimal deformations, where infinitesimal deformations and actual deformations mean elements of* $H^1(M; -K_{\mathcal{F}})$ *and smooth deformations as in Definition 1.9, respectively.*

Proof. We give a proof of the rigidity under infinitesimal deformations, from which the rigidity under actual deformations easily follows. Let $c_1(\mathcal{F})$ be the first Chern class of $Q(\mathcal{F})$ and $\mathrm{GV}_{2q}(\mathcal{F})$ the Godbillon-Vey class of $\mathcal{F}$. It is known that $\mathrm{GV}_{2q}(\mathcal{F}) = c\sqrt{-1}\xi_q(\mathcal{F})c_1(\mathcal{F})^q$, where c is a non-zero real constant [1, Theorem A]. We denote by $D_\mu \mathrm{GV}_{2q}(\mathcal{F})$ the infinitesimal derivative of $\mathrm{GV}_{2q}(\mathcal{F})$ with respect to μ (see [15]), where $\mu \in H^1(M; -K_{\mathcal{F}})$. Since $c_1(\mathcal{F})$ is rigid under deformations, we have $D_\mu \mathrm{GV}_{2q}(\mathcal{F}) = c\sqrt{-1}(D_\mu \xi_q)c_1(\mathcal{F})^q =$

$-c(D_\mu B_q(\mathcal{F})-\overline{D_\mu B_q(\mathcal{F})})c_1(\mathcal{F})^q$ by Theorem 2.19. By Theorem 4.10, $D_\mu B_q(\mathcal{F})$ is represented by a Čech-de Rham $(q,q+1)$-cocycle whose value on $U_{i_0\cdots i_q}$ belongs to $I_{(q)}(U_{i_0\cdots i_q})$ (see Notation 1.3). On the other hand, it is well-known that $c_1(\mathcal{F})$ is represented by an element of $I_{(1)}(M)$ (cf. [8]). It follows that $D_\mu B_q(\mathcal{F})c_1(\mathcal{F})^q$ is trivial as a cohomology class. Since $c_1(\mathcal{F})$ is a real class, $\overline{D_\mu B_q(\mathcal{F})}c_1(\mathcal{F})^q$ is also trivial. □

5. Localization

Definition 5.1. Let $\mathcal{U}=\{U_i\}_{i\in I}$ be an open covering of M and $\omega=\{\omega_{i_0,\cdots,i_p}\}$ a Čech-de Rham (r,s)-cochain. Set $I_\omega=\{i\in I\,|\,\exists(i_1,\cdots,i_r)\in I^r \text{ s.t. } \omega_{i,i_1,\cdots,i_r}\neq 0\}$ and define the *support of* ω by $\operatorname{supp}\omega=\bigcup_{i\in I_\omega}U_i$. If $\operatorname{supp}\omega$ is relatively compact, then ω is said to be of compact support.

Let ω be a globally defined differential form and denote by $s(\omega)$ the support of ω in the usual sense. If V is an open set containing $s(\omega)$, then, taking refinements of coverings, we may assume that $s(\omega)\subset\operatorname{supp}\omega\subset V$.

The localization of $D_\mu B_q(\mathcal{F})$ is defined by means of Γ-vector fields. The notion of Γ-vector fields and basic X-connections below are originally due to Heitsch [16]. The following definitions are slight modifications of those in [16].

Definition 5.2 ([3])**.** A vector field X defined on an open set O_X of M is said to be a *Γ-vector field* for $\mathcal{F}$ if $[E,X]\subset E$ on O_X. Set $Z_X=\{X\in E\}\cup(M\setminus O_X)$. Then $\mathcal{F}$ and X form a transversally holomorphic foliation $\mathcal{F}_X$ on the open set $M\setminus Z_X$. If X is a Γ-vector field on O_X, then X induces a foliated section of $Q(\mathcal{F})$ on O_X, which is denoted by X_Q.

Note that Z_X is saturated by leaves of $\mathcal{F}$ if O_X is saturated. Given a Γ-vector field X, we denote by U_X an open neighborhood (which is not necessarily saturated) of Z_X and by V_X an open neighborhood of $M\setminus U_X$. We will choose U_X arbitrarily small.

Definition 5.3. Let X be a Γ-vector field for $\mathcal{F}$ on O_X, and let U_X and V_X be as above. A Bott connection $\nabla^X=\{\nabla^X_i\}$ of $-K_{\mathcal{F}}$ is said to be a basic X-connection for $\mathcal{F}$ supported off V_X if $(\nabla^X_i)_X s=\mathcal{L}_X s$ provided $U_i\subset U_X$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

Note that basic X-connections depend only on X_Q.

One can always obtain a globally well-defined basic X-connection from a family of local basic-X connections by using a partition of unity. Thus obtained connection is a basic X-connection for $\mathcal{F}$ supported off V_X in the sense of Heitsch. Once an isomorphism $Q(\mathcal{F})\cong\mathbb{C}X_Q\oplus Q(\mathcal{F}_X)$ is fixed, a basic X-connection induces a Bott connection for $\mathcal{F}_X$ on V_X.

Let W be an open subset of M. We denote by $H^1_c(W;\Theta_{\mathcal{F}}|_W)$ the cohomology of elements of $I^q_{(q-1,q)}(W;-K_{\mathcal{F}})$ with compact support. Elements of

$H^1_c(W;\Theta_{\mathcal{F}}|_W)$ can be regarded as infinitesimal deformations of $\mathcal{F}$ whose support is compact and is contained in W.

Definition 5.4. Let X be a Γ-vector field for $\mathcal{F}$ on O_X, and let U_X and V_X be as above. Let W be an open subset of M and $\mu \in H^1_c(W;\Theta_{\mathcal{F}}|_W)$. Then, denote by $\operatorname{res} D_\mu B_q(\mathcal{F},X)$ an element of $H^{2q+1}_c(U_X\cap W;\mathbb{C})$ represented by $D_\sigma B_q(\nabla^X,(\nabla^X)')$, where ∇^X is a basic X-connection supported off V_X, and $(\nabla^X)'$ is the infinitesimal derivative of ∇^X with respect to σ.

It is clear that $\operatorname{res} D_\mu B_q(\mathcal{F},X)$ depends on X_Q but not on X itself, so that the residue is also denoted by $\operatorname{res} D_\mu B_q(\mathcal{F},X_Q)$.

Theorem 5.5. *$\operatorname{res} D_\mu B_q(\mathcal{F},X)$ is well-defined. Let $\iota : U_X\cap W \to M$ be the inclusion and $\iota_* : H^{2q+1}_c(U_X\cap W;\mathbb{C}) \to H^{2q+1}(M;\mathbb{C})$ the natural mapping. Then $\iota_* \operatorname{res} D_\mu B_q(\mathcal{F},X) = D_\mu B_q(\mathcal{F})$. Moreover, if Z_X is decomposed into connected components $Z_1,\cdots,Z_r$, then the residue is naturally decomposed into elements of $H^{2q+1}(U_i\cap W;\mathbb{C})$ as well, where U_i, $i=1,\ldots,r$, are mutually disjoint open neighborhoods of Z_i.*

Proof. By the assumption, μ is represented by a cocycle compactly supported in W. It follows from (2.11.a) and (2.11.b) that the support of the infinitesimal derivative of any Bott connection is compact and contained in W when taken the wedge product with elements of $I_{(q)}(M)$. On the other hand, if basic-X connections are used in calculation, cochains such as $(d\theta+\beta)^q$ vanish on V_X thanks to the Bott vanishing for $\mathcal{F}_X$. It follows that the supports of the coboundaries constructed in Claims 3 and 4 in the proof of Theorem 2.14 are compact and contained in $U_X\cap W$. The last claim also follows from similar arguments. □

Let X_0 be a Γ-vector field for $\mathcal{F}_0$ on O_{X_0}, and let U_{X_0} and V_{X_0} be as above. If there is a trivialization $e_{V_{X_0}}$ of $-K_{\mathcal{F}}|_{V_{X_0}}$, then the residue of the Bott class is defined as an element of $H^{2q+1}_c(U_{X_0};\mathbb{C}/\mathbb{Z})$ [3]. When residues are considered, a version of Theorem 2.18 holds under some additional conditions.

Theorem 5.6. *Let $\{(\mathcal{F}_s,X_s,e_s)\}$ be a smooth family of triples with the following properties:*

1) *$\{\mathcal{F}_s\}$ is a smooth family of transversally holomorphic foliations.*
2) *$\{X_s\}$ is a smooth family such that each X_s is a Γ-vector field for $\mathcal{F}_s$ and that Z_{X_s} is independent of s. We denote Z_{X_s} by Z_X.*
3) *There are open neighborhoods U_X of Z_X and V_X of $M\setminus U_X$ such that $\{e_s\}$ restricted to V_X is a smooth family of trivializations of $-K_{\mathcal{F}_s}|_{V_X}$.*

Assume that e_0 is foliated and that $\mathcal{L}_X e_0 = 0$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to X. Let $\mu \in H^1(M;-K_{\mathcal{F}})$ be the infinitesimal deformation induced by $\{\mathcal{F}_s\}$, then $\operatorname{res} D_\mu B_q(\mathcal{F},X) = \left.\dfrac{\partial}{\partial s}\operatorname{res} B_q(\mathcal{F}_s,X_s,e_s)\right|_{s=0}.$

Proof. Under the assumptions, we can repeat the proof of Theorem 2.18 in a compactly supported manner. Indeed, since there is a trivialization of $-K_{\mathcal{F}}$ on V_X, the cocycle Θ in Proposition 1.10 is zero on V_X. Moreover, since e_0 is foliated with respect to $\mathcal{F}_X$, the cochains u_1 and v_1 belong to $I_{(1)}(\mathcal{F}_X)$. Hence the cochains ρ_k are equal to zero on V_X. Therefore, the equality in the statement holds in $H^{2q+1}_c(M;\mathbb{C})$. □

Remark 5.7. It is natural to choose X_Q as e_0 if $q = 1$. Theorem 5.6 fails if the assumption on e_0 is dropped as shown in Example 7.3, although the left hand side is independent of e_0. The assumption is needed in order that Proposition 1.10 works in a compactly supported manner.

Localization using $\mathcal{L}$ is given as follows. Let $\mathcal{H}^1_c(W;\Theta_{\mathcal{F}}|_W)$ be the cohomology of $\mathcal{K}^*$ (Definition 2.9) with compact support.

Theorem 5.8. *Let $\mathcal{F}$ be a transversally holomorphic foliation of M. Suppose that $\mathcal{F}$ admits on an open set V of M, possibly $V = \varnothing$, a transversal complex projective structure $\mathcal{P}$. Let U be an open neighborhood of $M \setminus V$. Finally, let $\mu \in \mathcal{H}^1_c(W;\Theta_{\mathcal{F}}|_W)$, where W is an open subset of M, and let σ be a representative of μ. Then $\langle \mathcal{L} \,|\, \sigma\rangle$ represents an element of $H^{2q+1}_c(U \cap W;\mathbb{C})$, which is independent of the choice of representatives, where the foliation atlas is always chosen to be adapted to $\mathcal{P}$ on V.*

Proof. By the choice of the foliation atlas, the support of $\mathcal{L}$ is contained in U. Hence the support of $\langle \mathcal{L} \,|\, \sigma\rangle$ is contained in $U \cap W$. If we choose another foliation atlas adapted to $\mathcal{P}$ and obtain $\mathcal{L}'$, then $\mathcal{L}$ and $\mathcal{L}'$ are cohomologous as cocycles supported on U. It is not difficult to show that $\langle \mathcal{L} \,|\, \sigma\rangle$ and $\langle \mathcal{L} \,|\, \sigma'\rangle$ represent the same cohomology class if σ and σ' are representatives of μ. □

Definition 5.9. An element of $H^{2q+1}_c(U \cap W;\mathbb{C})$ obtained in Theorem 5.8 is denoted by $\operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P})|\,\mu\rangle$.

Proposition 5.10. *Let W be an open subset of M and let $\mu \in H^1_c(W;-K_{\mathcal{F}}|_W)$. Then $\operatorname{res} D_\mu B_q(\mathcal{F},X) = \operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P})\,|\,\mu\rangle \in H^{2q+1}_c(W;\mathbb{C})$ holds for any Γ-vector field X and any transversal projective structure $\mathcal{P}$.*

This follows from the fact that the support of the coboundaries constructed in Proposition 4.7 are compact.

6. Relation to the Fatou-Julia decomposition

If the complex codimension is equal to one, the localization in terms of the (classical) Schwarzian $\mathcal{L}(\mathcal{F})$ and the Fatou-Julia decomposition in the sense of Ghys, Gomez-Mont and Saludes [13] are related as follows. Let $\mathcal{B}_{\mathcal{F}}$ be the sheaf of germs of locally L^∞-foliated sections of $\overline{Q}(\mathcal{F})^* \otimes Q(\mathcal{F})$, where $\overline{Q}(\mathcal{F})$ denotes the complex conjugate of $Q(\mathcal{F})$. Then $H^0(M;\mathcal{B}_{\mathcal{F}})$ is the space of locally L^∞-foliated sections of $\overline{Q}(\mathcal{F})^* \otimes Q(\mathcal{F})$. The space $H^0(M;\mathcal{B}_\mathcal{F})$ is a Banach space with the essential supremum norm, and there is a natural mapping $\delta : H^0(M;\mathcal{B}_\mathcal{F}) \to H^1(M;\Theta_\mathcal{F})$. The image of δ consists of infinitesimal deformations preserving $\mathcal{F}_\mathbb{R}$, where $\mathcal{F}_\mathbb{R}$ denotes the underlying real foliation.

Lemma 6.1. *Let M be a closed manifold and let $\sigma \in H^0(M;\mathcal{B}_\mathcal{F})$. Set $\mu = \delta(\sigma)$, then $\langle \mathcal{L} \,|\, \sigma\rangle$ is well-defined as an integrable 3-form. It is equal to $\langle \mathcal{L}(\mathcal{F}) | \,\mu\rangle$ as an element of $\mathrm{Hom}_\mathbb{C}(H^{\dim M-3}(M;\mathbb{C}),\mathbb{C}) \cong H^3(M;\mathbb{C})$.*

Proof. After extending σ as a section of $E^* \otimes Q(\mathcal{F})$ by requiring $\sigma|_{T\mathcal{F}} = 0$, we approximate it by differential forms of class C^∞. The lemma follows by the Lebesgue convergence theorem. □

More detailed information on $H^0(M;\mathcal{B}_\mathcal{F})$ was obtained in [13]. Let F be the Fatou set and J the Julia set. The Julia set is measurably decomposed into the recurrent component J_0 and the ergodic components $J_1, \cdots, J_r$. There is a corresponding decomposition $H^0(M;\mathcal{B}_\mathcal{F}) = \bigoplus_{k=0}^{r} H^0(J_k;\mathcal{B}_\mathcal{F}) \oplus H^0(F;\mathcal{B}_\mathcal{F})$. It is almost by definition that the mapping δ restricted to $\bigoplus_{k=1}^{r} H^0(J_k;\mathcal{B}_\mathcal{F})$ is injective [13, p. 307]. Moreover, $\delta|_{J_0}$ is equal to zero and the image of $\delta|_{J_k}$ is one-dimensional for $k \neq 0$. Recalling that $H^0(M;\mathcal{B}_\mathcal{F})$ is a Banach space, choose a basis σ_k of unit length of $H^0(J_k;\mathcal{B}_\mathcal{F})$ for each $k > 0$. By choosing a section, we fix an isomorphism $\varphi\colon H^1(M;\Theta_\mathcal{F}) \cong H^0(J;\mathcal{B}_\mathcal{F}) \oplus \mathcal{H}_I \oplus \mathcal{H}_O$, where $H^0(J;\mathcal{B}_\mathcal{F}) \oplus \mathcal{H}_I = \mathrm{Image}\,\delta$ and $\mathcal{H}_O \cong \mathrm{coker}\,\delta$. Elements of $\mathcal{H}_I$ correspond to infinitesimal deformations preserving $\mathcal{F}_\mathbb{R}$, which cannot be induced by infinitesimal deformations supported on J, and elements of $\mathcal{H}_O$ correspond to infinitesimal deformations which do not preserve $\mathcal{F}_\mathbb{R}$.

We normalize the volume of M to be 1 and denote by $|J_k|$ the volume of J_k. Note that $|J_k| > 0$ for $k > 0$. We propose the following

Definition 6.2. The *infinitesimal derivative of the Bott class with respect to the ergodic component J_k, $k > 0$,* is the element of $H^3(M;\mathbb{C})$ determined by $|J_k|\,\langle \mathcal{L} \,|\, \sigma_k\rangle$ and denoted by $\partial_{J_k} B_1(\mathcal{F})$.

It is easy to see that $\partial_{J_k} B_1(\mathcal{F})$ is independent of the choice of σ_k.

Proposition 6.3. *Let $\mu \in H^1(M;\Theta_\mathcal{F})$ and let $\mu = \mu_J + \mu_I + \mu_O$ be the decomposition given by the isomorphism φ. Decompose further μ_J as $\sum_{k=1}^{r} a_k(|J_k|\,\sigma_k)$. Then there is a decomposition of $\langle \mathcal{L}(\mathcal{F}) | \,\mu\rangle$ in $H^3(M;\mathbb{C})$ as*

$$\langle \mathcal{L}(\mathcal{F}) | \,\mu\rangle = \sum_{k=1}^{r} a_k \partial_{J_k} B_1(\mathcal{F}) + \langle \mathcal{L}(\mathcal{F}) | \,\mu_I\rangle + \langle \mathcal{L}(\mathcal{F}) | \,\mu_O\rangle.$$

It follows from the classification of the Fatou components [13] that each Fatou component admits transversal projective structures.

Definition 6.4. Let U be a neighborhood of the Julia set J. Fix a transversal projective structure $\mathcal{P}$ on the Fatou sets. Given $\mu \in H^1(M;\Theta_{\mathcal{F}})$ and a smooth representative σ of μ, we denote by $\operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P}) \,|\, \mu\rangle$ an element of $H^3_c(U;\mathbb{C})$ represented by $\langle \mathcal{L} \,|\, \sigma\rangle$.

The class $\operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P}) \,|\, \mu\rangle$ is independent of the choice of a foliation atlas adapted to $\mathcal{P}$ as well as the representative σ. If J can be decomposed into connected components, then the residue admits a natural decomposition.

Remark 6.5. The Julia set affects $\langle \mathcal{L}(\mathcal{F}) \,|\, \mu_I\rangle$ so that it need not vanish. In addition, the image of $\operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P}) \,|\, \mu\rangle$ in $H^3(M;\mathbb{C})$ and $\sum_{k=1}^{r} a_k \partial_{J_k} B_1(\mathcal{F})$ are distinct in general. See Example 7.1.

Let X be a Γ-vector field on O_X. If $\{O_i\}$ is a foliation atlas for O_X, then there are projections $\pi_i : O_i \to \mathbb{C}$ which give the transversal holomorphic structure. The $(1,0)$-part X_i of $\pi_{i*}X|_{O_i}$ is well-defined and holomorphic, since X is a Γ-vector field. By integrating $2\operatorname{Re} X_i$, we can find a refinement $\{V_i\}$ of $\{O_i\}$ such that the transversal direction of transition functions is the restriction of translations in $\mathbb{C}$ and that $X_i = \dfrac{\partial}{\partial z_i}$ on V_i, where z_i denotes the transversal coordinates on V_i. Hence a projective structure is determined, and is denoted by $\mathcal{P}_X$. It is clear that $\mathcal{P}_X$ depends only on X_Q. Note that the flat connection with respect to the local trivializations $\{X_i\}$ of $Q(\mathcal{F})$ is a unique basic X-connection.

Definition 6.6. The transversal projective structure $\mathcal{P}_X$ as above is called the transversal projective structure associated with X.

There are foliated trivializations of $Q(\mathcal{F})$ on the most of the Fatou components. Indeed, wandering Fatou components are locally trivial fibrations over finite Riemann surfaces, and each restriction of $\mathcal{F}$ to semi-wandering and dense components is a G-Lie foliation [13]. Let F' be the union of wandering Fatou components of which the base spaces are closed surfaces of genus $g \neq 1$, and let U be an arbitrarily small neighborhood of $J \cup F'$. There always exists a foliated trivialization X_Q of $Q(\mathcal{F})$ on a neighborhood O of $M \setminus U$, and an element $\operatorname{res} D_\mu B_1(\mathcal{F}, X_Q)$ of $H^3_c(U;\mathbb{C})$.

Proposition 6.7. *Let X be any lift of X_Q to a Γ-vector field. Then* $\operatorname{res} D_\mu B_1(\mathcal{F}, X) = \operatorname{res}\langle \mathcal{L}(\mathcal{F},\mathcal{P}_{X_Q}) \,|\, \mu\rangle \in H^3_c(U_X;\mathbb{C})$.

This corresponds to the following fact, where a version of residues $\operatorname{res}^*_W B_1(\mathcal{F}, e)$ is defined by using transversal invariant Hermitian metric and trivialization of $Q(\mathcal{F})$ ([3, Definition 5.1]).

Proposition 6.8 (cf. [3, Corollary 5.4])**.** *Let F', W and X be as above. Then, there is a well-defined element* $\operatorname{res} B_1(\mathcal{F}, X, e)$ *of $H^3_c(W;\mathbb{C})$, where $e = \{e_i\}$ is* *a family of local trivializations of* $Q(\mathcal{F})$ *such that* $e_i = X$ *if* $U_i \subset M \setminus (J \cup F')$. *Moreover,* $\operatorname{res} B_1(\mathcal{F}, X, e) = \operatorname{res}^*_W B_1(\mathcal{F}, e)$.

Proof. Since there is a foliated trivialization of $Q(\mathcal{F})(= -K_{\mathcal{F}})$ on $F_0 \setminus F'$, the arguments in [3] remain valid even if F_0 is replaced with F'. □

7. Examples

Example 7.1. Let $X = \lambda_0 z^0 \dfrac{\partial}{\partial z^0} + \lambda_1 z^1 \dfrac{\partial}{\partial z^1}$ be a holomorphic vector field on $\mathbb{C}^2$, where (z^0, z^1) are the natural coordinates. Assume that $\lambda_0\lambda_1 \neq 0$ and that $\lambda = \lambda_0/\lambda_1$ is not a negative real number. Then X induces a transversally holomorphic foliation $\mathcal{F}_\lambda$ of S^3. The family $\{\mathcal{F}_\lambda\}$ is a smooth family of transversally holomorphic foliations, and $B_1(\mathcal{F}_\lambda)$ is the natural image of $(\lambda+\lambda^{-1})[S^3]$, where $[S^3]$ is the generator of $H^3(S^3;\mathbb{Z})$ [8]. Let Y be the Γ-vector field for $\mathcal{F}_\lambda$ induced by $\nu z^1 \dfrac{\partial}{\partial z^1}$. Then Z_Y consists of two circles C_0 and C_1. Let $\mu \in H^1(M; \mathcal{F}_\alpha)$ be the infinitesimal deformation induced by the family $\{\mathcal{F}_\lambda\}$. Let U_i be a tubular neighborhood of C_i and identify $H^3_c(U_i;\mathbb{C})$ with $H^1(C_i;\mathbb{C})$ by integration along the fiber. The residue $\operatorname{res} D_\mu B_1(\mathcal{F}_\alpha, Y)$ is naturally decomposed into the sum of $\operatorname{res}_{C_i} D_\mu B_1(\mathcal{F}_\alpha, Y) \in H^1(C_i;\mathbb{C})$ for $i = 0, 1$. By Theorem 5.6, $\operatorname{res}_{C_0} D_\mu B_1(\mathcal{F}_\alpha, Y) = [C_0]$ and $\operatorname{res}_{C_1} D_\mu B_1(\mathcal{F}_\alpha, Y) = -\dfrac{1}{\alpha^2}[C_1]$. Let $\mathcal{P}_Y$ be the projective structure on a neighborhood of $S^3 \setminus (U_0 \cup U_1)$ associated with Y. Then $\operatorname{res}\langle L(\mathcal{F}_\alpha, \mathcal{P}_Y) |\, \mu\rangle$ is the sum of $\operatorname{res}_i \langle L(\mathcal{F}_\alpha, \mathcal{P}_Y) |\, \mu\rangle \in H^1(C_i;\mathbb{C})$ for $i = 0, 1$. We have $\operatorname{res}_i \langle L(\mathcal{F}_\alpha, \mathcal{P}_Y) |\, \mu\rangle = \operatorname{res}_{C_i} D_\mu B_1(\mathcal{F}_\alpha, Y)$ by Proposition 6.7.

If $\alpha = 1$, then $\mathcal{F}_1$ is the Hopf fibration and is transversally projective. Hence $\mathcal{L}(\mathcal{F}_1) = 0$ and $B_1(\mathcal{F}_1)$ is infinitesimally rigid. However, $\operatorname{res}_i \langle L(\mathcal{F}_1, \mathcal{P}_Y) |\, \mu\rangle$ can be non-trivial because $\mathcal{P}_Y$ might not be extended to the whole S^3. On the other hand, the Julia set is empty so that the localization given in Section 6 is trivial. If $\alpha \neq 1$, then $\mathcal{L}(\mathcal{F}_\alpha)$ is non-trivial. Indeed, $B_1(\mathcal{F}_\alpha)$ is not infinitesimally rigid and $\mathcal{F}_\alpha$ cannot admit any transversal projective structures. The Julia set is equal to $C_0 \cup C_1$ and is of Lebesgue measure zero so that $\partial_J B_1(\mathcal{F}_\alpha) = 0$. It follows that $\langle \mathcal{L}(\mathcal{F}) |\, \mu'_F\rangle + \langle \mathcal{L}(\mathcal{F}) |\, \mu_O\rangle \neq 0$. An example of non-trivial μ_O is given in [4].

There are similar foliations on S^{2q+1} obtained from the vector field $\sum_{i=0}^{q} \lambda_i z^i \dfrac{\partial}{\partial z^i}$ on $\mathbb{C}^{q+1}$. If the convex hull of $\lambda_0, \cdots, \lambda_q$ does not contain the origin, a foliation $\mathcal{F}_\lambda$ of S^{2q+1} is induced and

$$B_q(\mathcal{F}_\lambda) = \frac{(\lambda_0 + \cdots + \lambda_q)^q}{\lambda_0 \cdots \lambda_q}[S^{2q+1}].$$

It follows from Theorem 2.18 that the most of $\mathcal{F}_\lambda$ does not admit any transversal projective structures.

Example 7.2. Let $H = \left\{ (a^i_j)_{0\le i,j\le q} \;\middle|\; a^i_0 = 0 \text{ if } i > 0 \right\}$ be the subgroup of $G = \mathrm{SL}(q+1;\mathbb{C})$ and Γ a discrete subgroup of G such that $M = \Gamma\backslash G/\mathrm{U}(q)$ is a closed manifold, where $\mathrm{U}(q)$ is considered as a subgroup of $\mathrm{SL}(q+1;\mathbb{C})$ by the mapping $A \in \mathrm{U}(q) \mapsto (\det A)^{-1} \oplus A \in \mathrm{SL}(q+1;\mathbb{C})$. The cosets $\{gH\}_{g\in G}$ induce a transversally holomorphic foliation $\mathcal{F}$ of M. The Bott class of $\mathcal{F}$ is non-trivial ([5]) and is infinitesimally rigid because $\mathcal{F}$ is transversally projective.

Example 7.3 ([17])**.** Let $Y_\lambda = \sum_{j=1}^{q} \lambda_j z^j \dfrac{\partial}{\partial z^j}$ be a holomorphic vector field on $\mathbb{C}^q$, and let $\mathcal{F}_\lambda$ be the foliation of $S^1 \times \mathbb{C}^q$ induced by the vector field $\dfrac{\partial}{\partial t} + Y_\lambda$, where S^1 is identified with $\mathbb{R}/\mathbb{Z}$, and $(z^1, \cdots, z^q)$ and t denote the standard coordinates of $\mathbb{C}^q$ and $\mathbb{R}$, respectively. Suppose that δ_j, $j = 1, \cdots, q$, are non-zero complex numbers. Then Y_δ is a Γ-vector field for any λ. Let $e = \dfrac{\partial}{\partial z^1} \wedge \cdots \wedge \dfrac{\partial}{\partial z^q}$ be a trivialization of $-K_{\mathcal{F}_\lambda}$. Then

$$B_q(\mathcal{F}_\lambda, Y_\delta, e) = \frac{1}{2\pi\sqrt{-1}}(\lambda_1 + \cdots + \lambda_q)\frac{(\delta_1 + \cdots + \delta_q)^q}{\delta_1 \cdots \delta_q}[S^1],$$

where $[S^1]$ denotes the natural generator of $H^1(S^1;\mathbb{Z})$. Hence the residue $D_\mu B_q(\mathcal{F}_\lambda, Y_\delta)$ can vary if Γ-vector fields are deformed even if $\mathcal{F}_\lambda$ is fixed. On the other hand, e is foliated with respect to $\mathcal{F}_{Y_\delta}$ if $\lambda_1 + \cdots + \lambda_q = \delta_1 + \cdots + \delta_q = 0$. The derivative of $B_q(\mathcal{F}_\lambda, Y_\delta, e)$ is trivial in this case.

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