

# ON INDEPENDENT RIGID CLASSES IN $H^*(WU_q)$

TARO ASUKE

ABSTRACT. We introduce a family of rigid, linearly independent classes of  $H^*(WU_q)$ . The family is different from the one studied by Hurder in [7], and some of the classes are decomposed into products of elements of  $H^*(WU_q)$ . We will show the independence by examining of a complexification of Baker's example in [2].

## INTRODUCTION

One of classical questions in the study of secondary characteristic classes of foliations is that to determine the kernel and the image of the characteristic homomorphism. For example, if there are linearly dependent classes, then the relations belong to the kernel. If transversely holomorphic foliations are studied, these questions concern the mappings  $H^*(WU_q) \rightarrow H^*(B\Gamma_q^{\mathbb{C}})$  or  $H^*(W_q^{\mathbb{C}}) \rightarrow H^*(\overline{B\Gamma_q^{\mathbb{C}}})$ . Some of related results are as follows. It is shown by Baum and Bott that all classes in  $H^{2q+1}(WU_q)$  and  $H^{2q+1}(W_q^{\mathbb{C}})$  admit continuous deformations [3]. These classes are said to be variable classes, which are also studied by Hurder [5]. On the other hand, classes rigid under deformations are called rigid classes. Hurder studied such classes in  $H^*(W_q^{\mathbb{C}})$  and gave a family of linearly independent classes [7]. In this paper, we give a family of rigid classes in  $H^*(WU_q)$  which is linearly independent and is different from Hurder's one as elements of  $H^*(W_q^{\mathbb{C}})$ . Some of these classes are shown to be products of a variable class and a rigid class. We will show the independence by studying a complexification of Baker's example [2] which will be constructed by using  $SL(k+n; \mathbb{C})$ . The example is classical if  $k = n = 1$ . Also, if  $k = 1$ , then the non-triviality of the Godbillon-Vey class is shown in [1]. The case where  $k = 2$  and  $n = 3$  is studied in [4] by means of more direct calculations of differential forms.

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*Date:* December 26, 2011.

*2010 Mathematics Subject Classification.* Primary 58H10; Secondary 53C12, 58H15, 57R32.

*Key words and phrases.* characteristic classes, holomorphic foliations, rigid classes.

The author is supported in part by Grant for Basic Science Research Projects from the Sumitomo Foundation.

## 1. PRELIMINARIES

Throughout the paper, the Lie algebra of a Lie group is denoted by the corresponding fraktur letter. For example,  $\mathfrak{g}$  denotes the Lie algebra of  $G$ . If  $\mathfrak{g}$  is a complex Lie algebra, then the Lie algebra  $\mathfrak{g}$  viewed as a real Lie algebra is denoted by  $\mathfrak{g}_{\mathbb{R}}$ . The coefficients of cohomology groups are chosen in  $\mathbb{C}$  unless otherwise mentioned.

First we introduce secondary characteristic classes.

**Definition 1.1.** Let  $\mathbb{C}[v_1, \dots, v_q]$  be the polynomial ring generated by  $v_1, \dots, v_q$ , where the degree of  $v_j$  is set to be  $2j$ . We denote by  $I_q$  the ideal generated by monomials of degree greater than  $2q$ , and set  $\mathbb{C}_q[v_1, \dots, v_q] = \mathbb{C}[v_1, \dots, v_q]/I_q$ . We also define  $\mathbb{C}_q[\bar{v}_1, \dots, \bar{v}_q]$  by replacing  $v_j$  by  $\bar{v}_j$ . We set

$$\begin{aligned} \mathrm{WU}_q &= \bigwedge[\tilde{u}_1, \dots, \tilde{u}_q] \otimes \mathbb{C}_q[v_1, \dots, v_q] \otimes \mathbb{C}_q[\bar{v}_1, \dots, \bar{v}_q], \\ \mathrm{W}_q &= \bigwedge[u_1, \dots, u_q] \otimes \mathbb{C}_q[v_1, \dots, v_q], \\ \mathrm{W}_q^{\mathbb{C}} &= \bigwedge[u_1, \dots, u_q, \bar{u}_1, \dots, \bar{u}_q] \otimes \mathbb{C}_q[v_1, \dots, v_q] \otimes \mathbb{C}_q[\bar{v}_1, \dots, \bar{v}_q]. \end{aligned}$$

These algebras are equipped with derivatives such that  $d\tilde{u}_i = v_i - \bar{v}_i$ ,  $du_i = v_i$  and  $dv_i = d\bar{v}_i = 0$ . The degree of  $\tilde{u}_i$ ,  $u_i$  and  $\bar{u}_i$  are set to be  $2i - 1$ . We define  $\overline{\mathrm{W}}_q$  by replacing  $u_i$  and  $v_j$  by  $\bar{u}_i$  and  $\bar{v}_j$  in  $\mathrm{W}_q$ . Then,  $\mathrm{W}_q^{\mathbb{C}} = \mathrm{W}_q \wedge \overline{\mathrm{W}}_q$ .

Since  $H^*(\mathrm{W}_q^{\mathbb{C}})$  is isomorphic to  $H^*(\mathrm{W}_q) \otimes H^*(\overline{\mathrm{W}}_q)$ , there is a natural inclusion from  $H^*(\mathrm{W}_q)$  to  $H^*(\mathrm{W}_q^{\mathbb{C}})$ . There is also a natural mapping from  $H^*(\mathrm{WU}_q)$  to  $H^*(\mathrm{W}_q^{\mathbb{C}})$  which maps  $\tilde{u}_i$  to  $u_i - \bar{u}_i$ ,  $v_i$  to  $v_i$  and  $\bar{v}_i$  to  $\bar{v}_i$ , respectively. Indeed, this mapping corresponds to the natural mapping from  $\overline{B\Gamma}_q^{\mathbb{C}}$  to  $B\Gamma_q^{\mathbb{C}}$ , which is a part of the homotopy fibration  $\overline{B\Gamma}_q^{\mathbb{C}} \rightarrow B\Gamma_q^{\mathbb{C}} \rightarrow \mathrm{BGL}(q; \mathbb{C})$  and also is the classifying map of the  $\Gamma_q^{\mathbb{C}}$ -structure of  $\overline{B\Gamma}_q^{\mathbb{C}}$ , namely, the map which forgets the triviality of the complex normal bundle.

The following result is classical.

**Theorem 1.2.** *Let  $B\Gamma_q^{\mathbb{C}}$  be the classifying space of transversely holomorphic foliations of complex codimension  $q$ , and let  $\overline{B\Gamma}_q^{\mathbb{C}}$  the classifying space of transversely holomorphic foliations of complex codimension  $q$  with trivialized complex normal bundles. Then, there is a well-defined homomorphism  $H^*(\mathrm{WU}_q) \rightarrow H^*(B\Gamma_q^{\mathbb{C}})$  and  $H^*(\mathrm{W}_q^{\mathbb{C}}) \rightarrow H^*(\overline{B\Gamma}_q^{\mathbb{C}})$ .*

The above homomorphisms are called the characteristic homomorphisms. Let  $M$  be a manifold and  $\mathcal{F}$  a transversely holomorphic foliation of  $M$ , of complex codimension  $q$ . Then, the classifying map induces a homomorphism from  $H^*(\mathrm{WU}_q)$  to  $H^*(M)$ . If the complex normal bundle of  $\mathcal{F}$  is trivial, then by fixing the homotopy type of a trivialization, we obtain a homomorphism from  $H^*(\mathrm{W}_q^{\mathbb{C}})$  to  $H^*(M)$ .

If smooth (real) foliations with trivialized normal bundles are considered, then we can choose  $\mathbb{R}$  as coefficients. We denote by  $W_q^{\mathbb{R}}$  the algebra  $W_q$  but the coefficients are chosen in  $\mathbb{R}$ . In order to distinguish  $W_q^{\mathbb{R}}$  from  $W_q$ , we denote  $u_i$  by  $h_i$  and  $v_j$  by  $c_j$  when we consider elements of  $W_q^{\mathbb{R}}$ . Then, the characteristic homomorphism is defined as a mapping from  $H^*(W_q^{\mathbb{R}})$  to  $H^*(\overline{BT}_q; \mathbb{R})$ , where  $\overline{BT}_q$  denotes the classifying space of foliations of real codimension  $q$  with trivialized normal bundles. Elements of  $H^*(WU_q)$ ,  $H^*(W_q^{\mathbb{C}})$ ,  $H^*(W_q)$  and  $H^*(W_q^{\mathbb{R}})$  or their image under the characteristic homomorphisms which involve  $\tilde{u}_i$ ,  $u_i$ ,  $\bar{u}_i$  or  $h_i$  are called secondary characteristic classes.

There is a natural mapping, say  $\rho$ , from  $WU_{q+1}$  to  $WU_q$  such that if  $i \leq q$  then  $\rho(\tilde{u}_i) = \tilde{u}_i$ ,  $\rho(v_i) = v_i$ ,  $\rho(\bar{v}_i) = \bar{v}_i$  and if  $i = q + 1$  then  $\rho(\tilde{u}_i) = 0$ ,  $\rho(v_i) = 0$  and  $\rho(\bar{v}_i) = 0$ , respectively. The mapping  $\rho$  induces a homomorphism  $\rho_*: H^*(WU_{q+1}) \rightarrow H^*(WU_q)$ . The classes of  $H^*(WU_q)$  in the image of  $\rho_*$  are called rigid classes. Indeed, it is known that rigid classes are rigid under deformations of foliations. Similar homomorphisms from  $W_{q+1}$  to  $W_q$ , from  $W_{q+1}^{\mathbb{C}}$  to  $W_q^{\mathbb{C}}$  and from  $W_{q+1}^{\mathbb{R}}$  to  $W_q^{\mathbb{R}}$  are also defined. The classes in the image are also called rigid classes and are rigid under deformations. On the other hand, a class  $\omega \in H^*(WU_q)$  is said to be variable if  $\omega$  varies continuously with respect to a family of foliations.

Let  $G$  be a Lie group. Let  $H$  and  $K$  be Lie subgroups of  $G$  such that  $H \supset K$ . We assume that  $G/H$  admits a complex structure invariant under the left  $G$ -action. We also assume that there are an  $\text{Ad}_K$ -invariant splitting of  $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$  and an  $\text{Ad}_K$ -invariant Hermitian metric on  $\mathfrak{g}/\mathfrak{h}$ . We assume in addition that there is a discrete subgroup  $\Gamma$  of  $G$  such that  $M = M_K = \Gamma \backslash G/K$  is a closed manifold. We denote by  $\mathcal{F}_K$  the transversely holomorphic foliation of  $M$  induced by the left cosets of  $H$ . Note that if  $K \subset K' \subset H$  then there is a natural projection, say  $\pi$ , from  $M_K$  to  $M_{K'}$  and that  $\mathcal{F}_K = \pi^* \mathcal{F}_{K'}$ .

If  $M_K$  and  $\mathcal{F}_K$  are as above, then the characteristic homomorphism  $\chi: H^*(WU_q) \rightarrow H^*(M_K)$  is well-studied. We state some results without proofs (see [1] and references therein).

**Theorem 1.3.** *Suppose that  $K$  is connected. If we denote by  $\varphi: H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}) \rightarrow H^*(M_K)$  the natural mapping, then there is a natural mapping  $\chi_K: H^*(WU_q) \rightarrow H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k})$  such that  $\chi = \varphi \circ \chi_K$  holds.*

**Theorem 1.4.** *Let  $G$  be a complex semisimple group. Let  $K$  be a compact connected subgroup and  $H$  a complex, closed, connected subgroup of  $G$ . Let  $\mathfrak{g}_0$  be a compact real form of  $\mathfrak{g}$  and  $G_0$  be the compact real form of  $G$  with Lie algebra  $\mathfrak{g}_0$ . Then, there is an isomorphism between  $H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k})$  and  $H^*(G_0) \otimes H^*(G_0/K)$ .*

Theorem 1.4 is shown by constructing an isomorphism of Lie algebras  $\kappa: \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$  and a diffeomorphism  $\tau: G_0 \times (G_0/K) \rightarrow (G_0 \times G_0)/K$ . We set  $f_K = \tau^* \circ (\kappa^*)^{-1}: C^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}) \rightarrow \Omega^*(G_0 \times (G_0/K))$ , where  $C^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k})$  denotes the set of cochains on  $\mathfrak{g}_{\mathbb{R}}$  relative to  $\mathfrak{k}$  with coefficients in  $\mathbb{C}$ , and  $\Omega(G_0 \times (G_0/K))$  denotes the set of  $\mathbb{C}$ -valued differential forms on  $G_0 \times (G_0/K)$ , respectively. Then,  $f_K$  induces an isomorphism  $f_{K*}: H^*(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}) \rightarrow H^*(\mathfrak{g}_0) \otimes H^*(\mathfrak{g}_0, \mathfrak{k})$ .

Let  $\theta$  be the Bott connection as in [1, Lemma 3.1.11] or the unitary connection as in [1, Lemma 3.1.13], which is a  $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ -valued 1-form on  $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ . Let  $\theta^c$  be the complex conjugate of  $\theta$  with respect to  $\mathfrak{g}_0$ , namely,  $\theta^c$  is a  $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ -valued 1-form on  $G \times \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$  defined as follows. First let  $\theta|_{\mathfrak{g}_0 \oplus \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})} = \alpha + \sqrt{-1}\beta$  be the decomposition of  $\theta|_{\mathfrak{g}_0 \oplus \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})}$  into real and imaginary parts, namely, the both  $\alpha$  and  $\beta$  are  $\mathbb{R}$ -valued. Then, we set  $\theta^c = \alpha - \sqrt{-1}\beta$ . By [1, Lemma 3.2.6] and [1, Proposition 3.2.7], we have the following.

**Proposition 1.5.** *Let  $\theta$  and  $\theta^c$  be as above. If we denote by  $p_i$  the projection from  $\mathfrak{g} \oplus \mathfrak{g}$  to  $\mathfrak{g}$  as the  $i$ -th factor, then  $f_K\theta = p_1^*\theta + p_2^*\theta$  and  $f_K\theta^c = p_2^*\theta^c$ .*

## 2. INDEPENDENT CLASSES OF $H^*(\mathrm{WU}_q)$

We study the following example, which is a complexification of Baker's example [2, Example 1(a)] (see also [4] for the case where  $n = 3$  and  $k = 2$ ). If  $n = k = 1$ , then it is an example of Roussarie [6].

**Example 2.1.** Let  $G = \mathrm{SL}(k+n; \mathbb{C})$ , where  $n > k > 0$  or  $n = k = 1$ ,  $H = \{(a_{ij}) \in G \mid a_{ij} = 0 \text{ if } i > k \text{ and } j \leq k\}$ ,  $K = \mathrm{S}(\mathrm{U}(k) \times \mathrm{U}(n))$  and  $T = T^{k+n-1}$  the maximal torus in  $G$  realized as diagonal matrices. Then, the left cosets of  $H$  induce transversely holomorphic foliations of complex codimension  $kn$  on  $\Gamma \backslash G/T$  and  $\Gamma \backslash G/K$ , where  $\Gamma$  is a discrete subgroup of  $G$  such that  $\Gamma \backslash G/K$  is a closed manifold.

We have a complexification of [2, Theorem 5.3] as follows. If  $I = \{i_1, \dots, i_l\}$  then we set  $\tilde{u}_I = \tilde{u}_{i_1} \cdots \tilde{u}_{i_l}$ . We denote  $\tilde{u}_I$  also by  $u_{i_1, \dots, i_l}$ . If  $I = \emptyset$  then we set  $\tilde{u}_I = 1$  and regard  $i_1 = +\infty$ . We define  $h_I$  in a similar way.

**Theorem 2.2.** *Let  $I = \{i_1, \dots, i_l\}$  and suppose that  $k < i_1 < \dots < i_l \leq n$ . The classes of the form  $\tilde{u}_{1, \dots, k} \tilde{u}_I v_1^{kn} \bar{v}_1^{kn}$  are non-trivial and linearly independent in  $H^*(\Gamma \backslash \mathrm{SL}(k+n; \mathbb{C})/T)$ . These classes are rigid classes, namely, rigid under deformations of foliations if  $k = 1$  and  $i_1 > 2$ .*

*Proof.* We retain the above notations. We denote by  $E_{ij}$  the element of  $\mathfrak{gl}(k+n; \mathbb{C})$  such that the  $(i, j)$ -entry is 1 and the others are 0. Let  $\{\omega_{ij}\}$  be the dual basis of  $\{E_{ij}\}$  for  $\mathfrak{gl}(k+n; \mathbb{C})$ . We define a splitting  $\sigma: \mathfrak{g}/\mathfrak{h} \rightarrow \mathfrak{g}$  by setting  $\sigma([E_{ij}]) = E_{ij}$ , where  $j \leq k < i$ . If we set  $\omega = {}^t(\omega_{k+1,1}, \dots, \omega_{k+n,1}, \dots, \omega_{k+1,k}, \dots, \omega_{k+n,k})$ ,

then  $\ker \omega = \mathfrak{h}$ . We set  $\theta_0 = \begin{pmatrix} \omega_{k+1,k+1} & \cdots & \omega_{k+1,k+n} \\ \vdots & & \vdots \\ \omega_{k+n,k+1} & \cdots & \omega_{k+n,k+n} \end{pmatrix}$ ,  $I_{ij} = E_{ij} \otimes I_n = \begin{pmatrix} O_n & \cdots & O_n \\ \vdots & I_n & \vdots \\ O_n & \cdots & O_n \end{pmatrix}$  for  $1 \leq i, j \leq k$ , and  $\theta = \overbrace{\theta_0 \oplus \cdots \oplus \theta_0}^{k\text{-times}} - \sum_{1 \leq i, j \leq k} \omega_{ij} I_{ij}$ . Then,

we have  $d\omega = -\theta \wedge \omega$ . As in [1, Example 3.3.6],  $\theta^u = \frac{\theta - t\bar{\theta}}{2}$  induces a unitary connection with respect to the Hermitian metric given by  $([A], [B]) \mapsto \text{tr}({}^t A \bar{B})$ . Hence we can compute complex secondary classes as follows. We denote by  $v_i(\theta)$  the  $i$ -th Chern form calculated by using a connection  $\theta$ . The Chern-Simons form calculated by the  $i$ -th Chern polynomial and connections  $\theta_1, \theta_2$  is denoted by  $\Delta_{c_i}(\theta_1, \theta_2)$ . Let  $G_0 = \text{SU}(k+n)$  and  $f_K$  as above. Then, by Proposition 1.5,  $f_K v_i(\theta) = v_i(p_1^* \theta + p_2^* \theta)$  and  $f_K \bar{v}_i(\theta) = f_K v_i(\bar{\theta}) = v_i(p_2^* \theta^c)$ . Similarly,  $f_K \tilde{u}_i(\theta, \theta^u) = \Delta_{c_i}(p_1^* \theta + p_2^* \theta, p_1^* \theta^u + p_2^* \theta^u) - \Delta_{c_i}(p_2^* \theta^c, p_2^* \theta^{uc})$ . We now set  $X_{ij} = E_{ij} - E_{ji}$ ,  $Y_{ij} = \sqrt{-1}(E_{ij} + E_{ji})$  and  $K_k = \sqrt{-1}(E_{11} - E_{kk})$ . Let  $\beta_{ij}, \gamma_{ij}$  and  $\alpha_k$  be their duals. If  $i \neq j$ , then  $\omega_{ij} = \beta_{ij} + \sqrt{-1}\gamma_{ij}$ . We have  $(p_2^* \theta^c)_{kk} = \sqrt{-1}(-\alpha_1 + \alpha_k)$  and  $(p_2^* \theta^c)_{ij} = p_2^* \beta_{ij} - \sqrt{-1}p_2^* \gamma_{ij}$ , where  $k > 1$  and  $i \neq j$ . On the other hand, we have

$$\begin{aligned} v_1(\theta) &= \frac{\sqrt{-1}}{2\pi} d \left( k \sum_{j=k+1}^{k+n} \omega_{jj} - n \sum_{j=1}^k \omega_{jj} \right) = \frac{-(k+n)\sqrt{-1}}{2\pi} \sum_{j=1}^k d\omega_{jj} \\ &= \frac{(k+n)\sqrt{-1}}{2\pi} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \omega_{i,j+n} \wedge \omega_{j+n,i}. \end{aligned}$$

As  $\beta_{ji} = -\beta_{ij}$  and  $\gamma_{ji} = \gamma_{ij}$ , we have

$$f_K \bar{v}_1(\theta) = \frac{(k+n)}{\pi} \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} \beta_{i,j+n} \wedge \gamma_{i,j+n}.$$

Hence  $f_K \bar{v}_1(\theta)^{kn} = \left( \frac{k+n}{\pi} \right)^{nk} \bigwedge_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} (\beta_{i,j+n} \wedge \gamma_{i,j+n})$ . The right hand side is equal to  $C \text{vol}_N$ , where  $C$  is a non-zero real number and  $\text{vol}_N$  denotes the volume form of  $N = \text{SU}(k+n)/\text{S}(\text{U}(k) \times \text{U}(n))$ . On the other hand,  $p_1^* \theta$  (resp.  $p_1^* \theta^u$ ) is the complexification of the real Bott connection (resp. metric connection with respect to the metric obtained by restriction to the real part) for the foliations modelled on  $\text{SL}(k+n; \mathbb{R})$  with leaves  $\{gH_{\mathbb{R}}\}_{g \in \text{SL}(k+n; \mathbb{R})}$ , where  $H_{\mathbb{R}} = H \cap \text{SL}(k+n; \mathbb{R})$ . Let  $\chi_{K\mathbb{R}}$  be the characteristic mapping from  $W_q^{\mathbb{R}}$  to  $H^*(\mathfrak{sl}(k+n; \mathbb{R}); \mathbb{R})$  obtained in the same way as in Theorem 1.3 (see [2] for details). We regard  $\chi_{K\mathbb{R}}$  as a mapping from  $W_q^{\mathbb{R}} \otimes \mathbb{C}$  to  $H^*(G_0)$  by complexification and denote it again by  $\chi_{K\mathbb{R}}$ . Then  $f_{K^*} \chi_K(\omega) = C \chi_{K\mathbb{R}}(\omega_{\mathbb{R}}) \otimes \text{vol}_N$ ,

where  $\omega = \tilde{u}_{1,\dots,k}\tilde{u}_{i_1,\dots,i_l}v_1^{kn}\bar{v}_1^{kn}$ ,  $\omega_{\mathbb{R}} = h_{1,\dots,k}h_{i_1,\dots,i_l}c_1^{kn} \in W_q^{\mathbb{R}}$  and  $C$  is the constant as above. If  $K$  is replaced by  $T$ , then there is a natural projection  $\pi: \Gamma \backslash G/T \rightarrow \Gamma \backslash G/K$  and  $\mathcal{F}_T = \pi^* \mathcal{F}_K$ . If we denote by  $\pi^*$  the natural mapping from  $H^*(\mathfrak{sl}(k+n; \mathbb{C})_{\mathbb{R}}, \mathfrak{k})$  to  $H^*(\mathfrak{sl}(k+n; \mathbb{C})_{\mathbb{R}}, \mathfrak{t})$  by abuse of notations, then  $\chi_T(\omega) = \pi^*(\chi_K(\omega))$  by the naturality. If we denote by  $\pi'$  the projection from  $G_0/T$  to  $G_0/K$ , then  $f_{T*} \circ \chi_T \circ \pi^* = (\text{id} \otimes \pi'^*) \circ f_{K*} \circ \chi_K$ . It is well-known that  $\pi'^*$  is injective on the subalgebra generated by the Chern classes so that  $\chi_T(\omega)$  is also non-trivial. Therefore, the first part follows from Baker's theorem [2, Theorem 5.3]. Finally, the cochain  $\tilde{u}_{1,\dots,k}\tilde{u}_I(v_1^{kn+1}\bar{v}_1^{kn-1} + v_1^{kn}\bar{v}_1^{kn} + v_1^{kn-1}\bar{v}_1^{kn+1})$  is mapped to the cocycle  $\tilde{u}_{1,\dots,k}\tilde{u}_I v_1^{kn}\bar{v}_1^{kn}$  under  $\rho: \text{WU}_{q+1} \rightarrow \text{WU}_q$ , and is closed if  $k = 1$  and  $i_1 > 2$ . Hence  $\tilde{u}_{1,\dots,k}\tilde{u}_I v_1^{kn}\bar{v}_1^{kn}$  represents a rigid class if  $k = 1$  and  $i_1 > 2$ .  $\square$

We can also show the following by an essentially the same argument as above, using [8, Theorem 5.37] (see also [2, Theorem 5.9]) instead of [2, Theorem 5.3].

**Theorem 2.3.** *Let  $k = 1$  in Example 2.1 and  $I = \{i_1, \dots, i_l\}$ , and let  $q(v_1, \dots, v_n)$  and  $r(\bar{v}_1, \dots, \bar{v}_n)$  be monomials of degree  $2i$  and  $2n$ , respectively. Then, we have the following.*

- 1) *We fix  $r(\bar{v}_1, \dots, \bar{v}_n)$  and denote it by  $\bar{r}$ . Then, the classes of the form  $\tilde{u}_{n-i+1}\tilde{u}_I q(v_1, \dots, v_n)\bar{r}$ ,  $n - i + 1 < i_1 < \dots < i_l \leq n$ , are linearly independent in  $H^*(\Gamma \backslash \text{SL}(n+1; \mathbb{C})/T)$ . These classes are multiplies of the classes  $\tilde{u}_1\tilde{u}_I v_1^n \bar{v}_1^n$  in  $H^*(\Gamma \backslash \text{SL}(n+1; \mathbb{C})/T)$ .*
- 2) *The classes of the form  $\tilde{u}_I q(v_1, \dots, v_n)r(\bar{v}_1, \dots, \bar{v}_n)$ , where  $n - i + 1 < i_1 < \dots < i_l \leq n$ , are trivial in  $H^*(\Gamma \backslash \text{SL}(n+1; \mathbb{C})/T)$ .*

Note that the classes in 2) are rigid ones. Some of the classes in 1) are also rigid. Suppose that  $q(v_1, \dots, v_n)$  is of the form  $v_j q'(v_1, \dots, v_n)$ , where  $n - i + 1 + j < i_1 < \dots < i_l$ . Suppose also that  $\bar{r}$  is of the form  $\bar{v}_k r'(\bar{v}_1, \dots, \bar{v}_n)$ , where  $k + 1 < i_1$ . In the both cases, we regard  $i_1 = +\infty$  and the conditions are satisfied if  $I = \emptyset$ . If we set  $\omega = \tilde{u}_j \tilde{u}_I q'(v_1, \dots, v_n) \bar{v}_{n-i+1} \bar{r} + \tilde{u}_{n-i+1} \tilde{u}_I q(v_1, \dots, v_n) \bar{r} + \tilde{u}_k \tilde{u}_I v_{n-i+1} q(v_1, \dots, v_n) r'(\bar{v}_1, \dots, \bar{v}_n)$ , then  $\omega$  determines an element of  $H^*(\text{WU}_{n+1})$  which is mapped to  $\tilde{u}_{n-i+1} \tilde{u}_I q(v_1, \dots, v_n) \bar{r}$ . The condition seems artificial but classes of the form  $\tilde{u}_1 \tilde{u}_I v_1^n \bar{v}_1^n$  satisfy it if  $i_1 > 2$ . We also remark that  $\sqrt{-1} \tilde{u}_1 v_1^n \bar{v}_1^n$  represents the Godbillon-Vey class.

We continue to study Example 2.1 with  $k = 1$  and  $n = q$ . We assume that  $1 < i_1 < \dots < i_l \leq q$ . If  $l > 1$  then we also assume that  $i_1 + i_2 > q + 1$ . We set

$$\begin{aligned} \xi_q &= \tilde{u}_1(v_1^q + v_1^{q-1}\bar{v}_1 + \dots + \bar{v}_1^q), \\ \eta &= \eta_{i_1, \dots, i_l, q} = \tilde{u}_{i_1, \dots, i_l} v_1^q + \sum_{\substack{1 \leq j \leq q \\ 1 \leq m \leq l}} (-1)^{m-1} \tilde{u}_{1, i_1, \dots, \widehat{i_m}, \dots, i_l} v_1^{q-j} \bar{v}_1^{j-1} \bar{v}_{i_m}. \end{aligned}$$

We have  $\xi_q \eta = \tilde{u}_{1,i_1,\dots,i_l} v_1^q \bar{v}_1^q$ . The class  $\xi_q$  is the imaginary part of the Bott class and is a variable class. On the other hand,  $\eta$  is a rigid class if  $i_1 + i_2 > q + 2$ . We do not know if  $\eta$  is a variable class if  $i_1 + i_2 = q + 2$ . The cocycle  $\eta$  is closed as follows. We have

$$\begin{aligned} d\eta &= \sum_{m=1}^l (-1)^{m-1} \tilde{u}_{i_1,\dots,\widehat{i_m},\dots,i_l} v_1^q (v_{i_m} - \bar{v}_{i_m}) + \sum_{m=1}^l (-1)^{m-1} \tilde{u}_{i_1,\dots,\widehat{i_m},\dots,i_l} (v_1^q - \bar{v}_1^q) \bar{v}_{i_m} \\ &\quad \pm \sum_{\substack{1 \leq j \leq q \\ \widehat{m}, \widehat{n}}} \tilde{u}_{1,i_1,\dots,\widehat{i_n},\dots,\widehat{i_m},\dots,i_l} v_1^{q-j} \bar{v}_1^{j-1} \bar{v}_{i_m} (v_{i_n} - \bar{v}_{i_n}) \\ &= - \sum_{m=1}^l (-1)^{m-1} \tilde{u}_{i_1,\dots,\widehat{i_m},\dots,i_l} v_1^q \bar{v}_{i_m} + \sum_{m=1}^l (-1)^{m-1} \tilde{u}_{i_1,\dots,\widehat{i_m},\dots,i_l} v_1^q \bar{v}_{i_m} \\ &\quad \pm \sum_{\substack{1 \leq j \leq q \\ \widehat{m}, \widehat{n}}} \tilde{u}_{1,i_1,\dots,\widehat{i_n},\dots,\widehat{i_m},\dots,i_l} v_1^{q-j} \bar{v}_1^{j-1} \bar{v}_{i_m} (v_{i_n} - \bar{v}_{i_n}) \end{aligned}$$

in  $WU_{q+1}$  because  $i_1 > 1$ . If  $l = 1$ , then the last term does not appear so that  $d\eta = 0$  in  $WU_{q+1}$ . If  $l > 1$ , then we have  $(q-j) + i_n + (j-1) + i_m \geq i_1 + i_2 + q$ . Hence,  $\eta$  is closed in  $WU_{q+1}$  if  $i_1 + i_2 > q + 2$  and in  $WU_q$  if  $i_1 + i_2 > q + 1$ .

By applying Theorem 2.2 with  $k = 1$  and  $n = q$ , we deduce the following.

**Corollary 2.4.** *The product of  $\xi_q$  with  $\eta_{i_1,\dots,i_l,q}$ , where  $1 < i_1 < \dots < i_l \leq q$  and either  $l = 1$  or  $l > 1$  and  $i_1 + i_2 > q + 1$ , are non-trivial in  $H^*(\Gamma \backslash \mathrm{SL}(q+1; \mathbb{C})/T; \mathbb{C})$ . The family of classes  $\{\xi_q, \eta_{i_1,\dots,i_l,q}\}$  is linearly independent. The class  $\xi_q$  is variable. The classes  $\eta_{i_1,\dots,i_l,q}$  with  $l = 1$  or  $i_1 + i_2 > q + 2$  are rigid.*

*Proof.* By taking the product with  $\xi_q$ , the family  $\{\eta_{i_1,\dots,i_l,q}\}$ , where  $1 < k \leq q$ ,  $1 < i_1 < \dots < i_l \leq q$ ,  $l > 1$  and  $i_1 + i_2 > q + 1$ , is mapped to a family which consists of linearly independent classes by Theorem 2.2. Hence  $\{\xi_q, \eta_{i_1,\dots,i_l,q}\}$  consists of linearly independent classes.  $\square$

*Remark 2.5.* Let  $k = 2$  and  $n = 3$  in Example 2.1, and  $\xi_6$  and  $\eta_{2,6}$  as above. Then  $\xi_6 \eta_{2,6} = \tilde{u}_1 \tilde{u}_2 v_1^6 \bar{v}_1^6$ , which is non-trivial by Theorem 2.2. Therefore  $\xi_6$  and  $\eta_{2,6}$  are non-trivial. On the other hand, if we set  $\mu = \tilde{u}_1 \tilde{u}_2 (v_1^6 \bar{v}_1^5 + v_1^5 \bar{v}_1^6)$ , then  $\mu$  is closed and  $\mu v_1 = \tilde{u}_1 \tilde{u}_2 v_1^6 \bar{v}_1^6$ . Hence  $\mu$  and  $v_1$  are also non-trivial. Actually  $v_1^6$  is a generator of  $H^{12}(\mathrm{SU}(5)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)))$ . On the other hand, by examining the degree of generators of  $H^*(\mathrm{SU}(5)) \otimes H^*(\mathrm{SU}(5)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(3)))$ , we see that the Godbillon-Vey class  $\tilde{u}_1 v_1^6 \bar{v}_1^6$  is trivial. We have  $\tilde{u}_1 v_1^6 \bar{v}_1^6 = \xi_6 v_1^6$ . Hence, unlike the above cases, a product of non-trivial classes yields a trivial class.

Let  $q = 2q' - 2$  with  $q' > 1$ . Let  $\mathcal{R}$  be a subset of  $H^*(W_q) \subset H^*(W_q^{\mathbb{C}})$  defined by  $\mathcal{R} = \{u_{2,i_2,\dots,i_s} v_2^{q'-1} | 2 < i_2 < \dots < i_s \leq q\} \cup \{u_{q',i_2,\dots,i_s} v_{q'} | q' < i_2 < \dots < i_s \leq q\}$ . The family  $\mathcal{R}$  is a set of rigid secondary classes studied by Hurder in

[7]. Indeed,  $\mathcal{R}$  consists of linearly independent secondary classes. The families in Theorems 2.2 and 2.3 differ from  $\mathcal{R}$ . At present we know only the following classes  $\alpha_q$  and  $\beta_q$  as elements of  $\text{WU}_q$  which belongs to the subspace of  $H^*(\text{W}_q^{\mathbb{C}})$  spanned by  $\mathcal{R}$  and  $\overline{\mathcal{R}}$ . Let  $\alpha_{q'} = \tilde{u}_2(v_2^{q'-1} + \cdots + \bar{v}_2^{q'-1})$  and  $\beta_{q'} = \tilde{u}_{q'}(v_{q'} + \bar{v}_{q'})$ . Note that  $\alpha_2 = \beta_2$ . These classes are rigid classes also in  $H^*(\text{WU}_q)$ . We have the following.

**Corollary 2.6.**  $\alpha_2$  is non-trivial in  $H^7(\Gamma \backslash \text{SL}(3; \mathbb{C})/T; \mathbb{C})$ . If  $q' > 2$ , then  $\alpha_{q'}$  and  $\beta_{q'}$  are linearly independent in  $H^{4q'-1}(\Gamma \backslash \text{SL}(q+1; \mathbb{C})/T; \mathbb{C})$ .

*Proof.* Let  $\gamma = \tilde{u}_1 v_2^{q'-1} + \tilde{u}_2(v_2^{q'-2} + \cdots + \bar{v}_2^{q'-2})\bar{v}_1$ . Then,  $\gamma$  is closed and  $\gamma\alpha_{q'} = \tilde{u}_1 \tilde{u}_2 v_2^{q'-1} \bar{v}_2^{q'-1}$ . This class is non-trivial in  $H^{4q'+4}(\Gamma \backslash \text{SL}(q+1; \mathbb{C})/T; \mathbb{C})$  by Theorem 2.3. On the other hand, we have  $\alpha_{q'}\beta_{q'} = \tilde{u}_2 \tilde{u}_{q'}(v_2^{q'-1} \bar{v}_{q'} + v_{q'} \bar{v}_2^{q'-1})$ . Let  $\delta = \tilde{u}_1(v_1^{2q'-2} \bar{v}_1^{q'-2} + \cdots + v_1^{q'-2} \bar{v}_1^{2q'-2})$ . If  $q' > 2$ , then  $\delta\alpha_{q'} = 0$  because  $(q'-2+2q'-2)+2(q'-1) = 5q'-6 > 2q = 4q'-4$ . On the other hand,  $\delta\beta_{q'} = \tilde{u}_1 \tilde{u}_{q'}(v_1^{2q'-2} \bar{v}_1^{q'-2} \bar{v}_{q'} + v_1^{q'-2} v_{q'} \bar{v}_1^{2q'-2})$ . By Theorem 2.3, there is a non-zero real number  $C$  such that  $\tilde{u}_1 \tilde{u}_{q'} v_1^{2q'-2} \bar{v}_1^{q'-2} \bar{v}_{q'} = \tilde{u}_1 \tilde{u}_{q'} v_1^{q'-2} v_{q'} \bar{v}_1^{2q'-2} = Ch_1 h_{q'} c_1^q \otimes c_1^q$ . Hence  $\delta\beta_{q'}$  is non-trivial.  $\square$

*Remark 2.7.* If  $q' > 3$ , then  $\alpha_{q'}\beta_{q'}v_{q'-3} = \tilde{u}_2 \tilde{u}_{q'} v_{q'-3} v_{q'} \bar{v}_2^{q'-1}$ , which is non-trivial in  $H^*(\Gamma \backslash \text{SL}(q+1; \mathbb{C})/T; \mathbb{C})$  by Theorem 2.3.

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, 3–8–1  
KOMABA, MEGURO-KU, TOKYO 153–8914, JAPAN

*E-mail address:* asuke@ms.u-tokyo.ac.jp