Resolution of sigma-fields for multiparticle finite-state evolution with infinite past

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Let us consider the stochastic recursive equation

$$X_k = N_k X_{k-1} \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z} \tag{1}$$

with the *observation process* $X = \{X_k\}_{k\in\mathbb{Z}}$ taking values in a set V and with the *noise process* $N = \{N_k\}_{k\in\mathbb{Z}}$ doing in a composition semigroup Σ consisting of mappings from V to itself, where we write fv simply for the evaluation $f(v)$. For a given probability law on Σ, we call the pair $\{X, N\}$ a *μ-evolution* if the equation (1) holds and each N_k has law μ and is independent of $\mathcal{F}^{X,N}_{k-1} := \sigma(X_j, N_j : j \le k-1)$. Our problem here is to resolve the observation $\mathcal{F}^X_k = \sigma(X_j : j \le k)$ into three independent components as

$$\mathcal{F}^X_k = \mathcal{F}^Y_k \vee \mathcal{F}^X_{-\infty} \vee \sigma(U_k) \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z}, \tag{2}$$

where, for each k, the first component $\mathcal{F}^Y_k$ is a sub-σ-field of the noise $\mathcal{F}^N_k$, the second $\mathcal{F}^X_{-\infty} := \bigcap_{k\in\mathbb{Z}} \mathcal{F}^X_k$ is the remote past, and the third U_k is a random variable which is independent of $\mathcal{F}^Y_k \vee \mathcal{F}^X_{-\infty}$. For σ-fields $\mathcal{F}_1, \mathcal{F}_2, \dots$ we write $\mathcal{F}_1 \vee \mathcal{F}_2 \vee \cdots$ for $\sigma(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots)$.

If we assume that the product $N_j N_{j+1} \cdots N_k$ converges $\mathbb{P}$-a.s. as $j \to -\infty$ to some random mapping $\widetilde{N}_k$ and that X_j does to some random variable $X_{-\infty}$, then we obtain

$$\mathcal{F}^X_k \subset \mathcal{F}^{\widetilde{N}}_k \vee \mathcal{F}^X_{-\infty} \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z} \tag{3}$$

with $\mathcal{F}^X_{-\infty} = \sigma(X_{-\infty})$, $\mathbb{P}$-a.s. We notice that, in typical cases, these a.s. convergences fail but the resolution (2) holds with the third random variable U_k being uniform in some sense.

Motivated by Tsirelson's example [2] of a stochastic differential equation without strong solutions, Yor [7] studied this problem in the case $V = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the one-dimensional torus, and $\Sigma = \mathbb{T}$ by identifying $z \in \mathbb{T}$ with the multiplication mapping $w \mapsto zw$. By means of the Fourier series and the martingale convergence theorems, he obtained a complete answer to the resolution problem. Akahori–Uenishi–Yano [1] and Hirayama–Yano [3] generalized Yor's results to compact groups; see also Yano–Yor [6] for a survey on this topic.

We now consider the resolution problem when the state space is a finite set $V = \{1, 2, \dots, \#V\}$ and $\Sigma = \mathrm{Map}(V)$ is the finite composition semigroup of all mappings from V to itself. In Yano [5] we gave a partial answer in the sense that the inclusion

$$\mathcal{F}^X_k \subset \mathcal{F}^N_k \vee \mathcal{F}^X_{-\infty} \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z} \tag{4}$$

holds if and only if $\mathrm{Supp}(\mu)$ is *sync*, i.e., the image $g(V)$ is a singleton for some $g \in \langle \mathrm{Supp}(\mu) \rangle$, where $\langle \mathrm{Supp}(\mu) \rangle$ denotes the subsemigroup of Σ consisting of all finite compositions from $\mathrm{Supp}(\mu)$. Unfortunately, we have not so far obtained a general result nor a counterexample for the resolution of the form (2).

We thus focus on the resolution problem for multiparticle evolutions. For a probability law μ and for $m \in \mathbb{N}$, we mean by an *m-particle μ-evolution* the pair $\{\mathbb{X}, N\}$ of a V^m-valued process $\mathbb{X} = \{\mathbb{X}_k\}_{k\in\mathbb{Z}}$ with $\mathbb{X}_k = (X_k^1, \ldots, X_k^m)$ and a Σ-valued process $N = \{N_k\}_{k\in\mathbb{Z}}$ such that the stochastic recursive equation

$$X_k^i = N_k X_{k-1}^i \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z} \text{ and } i = 1, \ldots, m \tag{5}$$

holds and each N_k has law μ and is independent of $\mathcal{F}_{k-1}^{\mathbb{X},N}$. Choosing

$$m = \inf\{\#g(V) : g \in \langle \mathrm{Supp}(\mu) \rangle\}, \tag{6}$$

we shall give a complete answer to the resolution problem of the form

$$\mathcal{F}_k^{\mathbb{X}} = \mathcal{F}_k^Y \vee \mathcal{F}_{-\infty}^{\mathbb{X}} \vee \sigma(U_k) \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z}. \tag{7}$$

For this purpose, we utilize the *Rees decomposition* from the algebraic semigroup theory, which has played a fundamental role in the theory of infinite products of random variables taking values in topological semigroups; see, e.g., [4] for the details.

References

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