Functional limit theorem for intermittent interval maps

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Interval maps with indifferent fixed points have been studied as models of intermittent phenomena, such as intermittent transitions to turbulent flow in convective fluid. In this context, the occupations near indifferent fixed points correspond to long regular or *laminar phases*, while the occupations away from them correspond to short irregular or *turbulent* bursts. There have been many studies of scaling limits of the occupations near and away from them, e.g., [1, 7, 8, 10, 4, 6]. In this talk, we present a functional and joint-distributional refinement of them, based on [5]. It is motivated particularly by [2, 3, 9].

We impose the following assumption from now on:

Assumption. An interval map $T : [0, 1] \to [0, 1]$ satisfies the following conditions:

(1) (for simplicity) T is point-symmetric, i.e., $Tx = 1 - T(1 - x)$, $x \in (1/2, 1]$.

(2) the restriction $T|_{[0,1/2]} : [0, 1/2] \to [0, 1]$ is a C^2-bijective map.

(3) $T0 = 0$, $T'0 = 1$ and $T''x > 0$, $x \in (0, 1/2)$.

Note that 0 and 1 are indifferent fixed points of T. We know that T has a unique (up to scalar multiplication) σ-finite invariant measure $\mu(\mathrm{d}x)$ equivalent to the Lebesgue measure $\mathrm{d}x$. From now on, let us fix $\delta \in (0, 1/2)$. Then, it holds that $\mu([0, \delta)) = \mu((1 - \delta, 1]) = \infty$ and $\mu([\delta, 1 - \delta]) < \infty$. Hence Birkhoff's pointwise ergodic theorem implies

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{\{T^k x \in [\delta,1-\delta]\}} \underset{n\to\infty}{\to} 0 \quad \left(\text{equivalently,} \quad \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{\{T^k x \notin [\delta,1-\delta]\}} \underset{n\to\infty}{\to} 1\right), \quad \text{a.e.} x.$$

Roughly speaking, the orbit $(x, Tx, T^2x, \dots)$ of almost every starting point x is concentrated close to 0 and 1. We are interested in non-trivial scaling limits of occupation times for $[0, \delta), [\delta, 1 - \delta]$ or $(1 - \delta, 1]$. Let us denote by $\varphi(N) = \varphi(N, x)$ the Nth hitting time of $(x, Tx, T^2x, \dots)$ for $[\delta, 1 - \delta]$:

$$\varphi(0) = 0 \quad \text{and} \quad \varphi(N + 1) = \min\{k > \varphi(N) : T^k x \in [\delta, 1 - \delta]\}, \quad N \geq 0.$$

We will denote by $\overline{\mu} = \mu([\delta, 1 - \delta] \cap \cdot)/\mu([\delta, 1 - \delta])$ the normalized restriction of μ over $[\delta, 1 - \delta]$. We now present our main result.

Theorem (S. [5])**.** *Let $\alpha \in (0, 1)$, and let ξ be a $[0, 1]$-valued random variable with $\mathbb{P}[\xi \in \mathrm{d}x] \ll \mathrm{d}x$. Then the following conditions are equivalent:*

(i) $Tx - x = (1-x) - T(1-x)$ *is regularly varying of index* $(1+1/\alpha)$ *at* 0.

(ii) *it holds that*

$$\left(\frac{1}{n}\sum_{k=0}^{\varphi([b_n t])} 1\!\!1_{\{T^k\xi<\delta\}},\ \frac{1}{n}\sum_{k=0}^{\varphi([b_n t])} 1\!\!1_{\{T^k\xi>1-\delta\}} : t\geq 0\right) \xrightarrow[n\to\infty]{\mathrm{d}} \left(S_-^{(\alpha)}(t),\ S_+^{(\alpha)}(t) : t\geq 0\right), \quad \text{in } D,$$

where $b_n := 1/\overline{\mu}[\varphi(1)>n]$, *and* $S_-^{(\alpha)}(t)$ *and* $S_+^{(\alpha)}(t)$ *are i.i.d.* α*-stable subordinators with Lévy measure* $\frac{\alpha}{2}r^{-1-\alpha}\mathrm{d}r$, $r>0$.

(iii) *it holds that*

$$\left(\frac{1}{n}\sum_{k=0}^{[nt]} 1\!\!1_{\{T^k\xi<\delta\}},\ \frac{\Gamma(1-\alpha)}{b_n}\sum_{k=0}^{[nt]} 1\!\!1_{\{T^k\xi\in[\delta,1-\delta]\}},\ \frac{1}{n}\sum_{k=0}^{[nt]} 1\!\!1_{\{T^k\xi>1-\delta\}} : t\geq 0\right)$$
$$\xrightarrow[n\to\infty]{\mathrm{d}} \left(\int_0^t 1\!\!1_{\{Z^{(\alpha)}(s)<0\}}\mathrm{d}s,\ L^{(\alpha)}(t),\ \int_0^t 1\!\!1_{\{Z^{(\alpha)}(s)>0\}}\mathrm{d}s : t\geq 0\right), \quad \text{in } D,$$

where $Z^{(\alpha)}(t)$ *denotes a* $(2-2\alpha)$*-dimensional symmetric Bessel diffusion process starting from the origin, and* $L^{(\alpha)}(t)$ *denotes the local time of* $Z^{(\alpha)}(t)$ *at the origin in the Blumenthal–Getoor normalization.*

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