ある無限グラフ上の因子に関するリーマン・ロッホの定理について
(On a Riemann-Roch theorem for divisors on an infinite graph)

厚地　淳,　　金子　宏 (A. Atsuji and H. Kaneko)

1. Riemann-Roch theorem on a weighted finite graph

Let $G = (V_G, E_G)$ be a connected graph consisting of finite set V_G of vertices and of finite set E_G of edges. We assume that weight $C_{x,y}$ is given at every edge $\{x, y\} \in E_G$.

For every vertex $x \in V_G$, define $N(x) = \{y \in V_G \mid \{x, y\} \in E_G\}$ and $i(x) = \min\{|\sum_{y\in N(x)} f(y)C_{x,y}| \in (0,\infty) \mid f : V_G \to \mathbb{Z}\}$.

notions	probabilistic materials
weight on edges	conductance $C_{x,y}$ between x and y
weight at vertices	$i(x)$
divisor	$D = \sum_{x\in V_G} \ell(x)i(x)1_{\{x\}}$
degree of divisor	$\deg(D) = \sum_{x\in V_G} \ell(x)i(x)$
canonical divisor	$K_G = \sum_{x\in V_G}\{\sum_{y\in N(x)} C_{x,y} - 2i(x)\}1_{\{x\}}$
Laplacian of f at $x \in V_G$	$\Delta f(x) = \sum_{y\in N(x)} C_{x,y}(f(x) - f(y))$
Euler-like characteristic	$\mathfrak{e}_{(V,C)} = \sum_{x\in V_G} i(x) - \sum_{\{x,y\}\in E_G} C_{x,y}$

A divisor $D = \sum_{x\in V_G} \ell(x)i(x)1_{\{x\}}$ is said to be effective, if $\ell(x) \geq 0$ for all $x \in V_G$. We need also the canonical divisor $K_G = \sum_{x\in V_G}\{\sum_{y\in N(x)} C_{x,y} - 2i(x)\}1_{\{x\}}$ and the family of total orders on V_G denoted by $\mathcal{O}$. For each $O \in \mathcal{O}$, its inverted total order $\overline{O}$ is defined by $x <_{\overline{O}} y$ for any $x, y \in V_G$ satisfying $y <_O x$. We introduce the divisor

$$\nu_O(x) = \sum_{y\in N(x), y<_O x} C_{x,y} - i(x), \quad x \in V_G$$

of degree $-\mathfrak{e}_{(V,C)} = \sum_{\{x,y\}\in E_G} C_{x,y} - \sum_{x\in V_G} i(x)$ admitting only non-effective equivalent divisors.

We introduce an equivalence between divisors D and D' and notation for the equivalence class given by

$$D \sim D' \Leftrightarrow D' = D + \Delta f \text{ for some } \mathbb{Z}\text{-valued function} f,$$
$$|D| = \{D' \mid D' \text{ is effective and equivalent with } D\}.$$

For any divisor D and non-negative integer k, we take

$$E_k(D) = \{ \text{ effective divisors } E \mid \deg\,(E) = i_{(V,C)}k \text{ satisfying } |D - E| \neq \emptyset\}$$

to define the dimension $r(D)$ of the divisor D by

$$r(D) = \begin{cases} -i_{(V,C)}, & \text{if } E_0(D) = \emptyset, \\ \max\{i_{(V,C)}k \mid E_k(D) \text{ consists of all effective divisors of degree } i_{(V,C)}k\}, & \text{otherwise.} \end{cases}$$

Theorem (Riemann-Roch theorem on weighted finite graph). For any divisor D,

$$r(D) - r(K_G - D) = \deg(D) + \mathfrak{e}_{(V,C)}.$$

Similarly to M. Baker and S. Norine's article [1], we can prove this assertion, the corner stones of which are the following assertions:

(RR) For each divisor D, there exists an $O \in \mathcal{O}$ such that either $|D|$ or $|\nu_O - D|$ is empty.

Proposition 1 (RR) implies $r(D) = \left(\min_{D' \sim D, O \in \mathcal{O}} \deg^+(D' - \nu_O)\right) - i_{(G,C)}$ for any divisor D, where $i_{(G,C)} = \min\{|\sum_{x \in V_G} \ell(x)i(x)| \in (0,\infty) \mid \ell : V_G \to \mathbb{Z}\}$ and $\deg^+(D) = \sum_{\ell(x)>0} \ell(x)i(x)$ for divisor $D = \sum_{x \in V_G} \ell(x)i(x)1_{\{x\}}$.

2. Riemann-Roch theorem in an infinite graph

Throughout this section we consider an infinite graph $G = (V_G, E_G)$ with locally finiteness and finite volume, namely, the function $\#N(x)$ given by $N(x) = \{y \mid \{x,y\} \in E_G\}$ is integer valued and the total volume $m(V) = \sum_{x \in V} m(x)$ given by $m(x) = \sum_{y \in N(x)} C_{x,y}$ is finite.

For any pair $\{x,y\}$ of distinct elements in V_G, we define the graph metric $d(x,y)$ between x, y by $d(x,y) = \min\{k \in \mathbb{N} \mid \{z_0, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-1}, z_k\} \in E_G$ for some $z_1, \ldots, z_{k-1} \in V_G$ with $z_0 = x, z_k = y\}$. We fix a reference vertex $v_0 \in V_G$ and take the sphere $S_k = \{y \in V_G \mid d(v_0, y) = k\}$ centered at the reference vertex v_0 with radius $k \in \mathbb{N}$ with respect to the graph metric d.

We consider a divisor $D = \sum_{x \in V_G} \ell(x)i(x)1_{\{x\}}$ on V_G satisfying $\sum_{x \in V_G} |\ell(x)|i(x)1_{\{x\}} < \infty$. We take an exhaustion sequence $G_1 \subset G_2 \subset \cdots$ of subgraphs of $G = (V_G, E_G)$ determined by $V_n = \{a \in V_G \mid d(v_o, a) \leq n\}$, $E_n = \{\{a,b\} \in E_G \mid a, b \in V_n\}$ and $G_n = (V_n, E_n)$ for each $n \in \mathbb{N}$.

We make an attempt to extend the Riemann-Roch theorem on finite graphs to one on an infinite graph by finding such sufficient conditions that sequence $\{r_n(D)\}$ consisting of so-called dimension of D on each G_n converges as n tends to ∞ for any divisor $D = \sum_{x \in V_G} \ell(x)i(x)1_{\{x\}}$ with finiteness of its support $\mathrm{supp}[D] = \{x \in V_G \mid \ell(x) \neq 0\}$. We will propose several conditions on the infinity of G for controlling the dimensions of the divisor by closely looking at the Laplacian naturally associated with $\{C_{x,y}\}$.

As a result, after redefinitions of the dimension $r(D)$, the canonical divisor K_G and Euler-like characteristic $\mathfrak{e}_{(V,C)}$, we can assert the same identity as in Theorem as a Riemann-Roch theorem for divisor $D = \sum_{x \in V_G} \ell(x)i(x)1_{\{x\}}$ on V_G with $\sum_{x \in V_G} |\ell(x)|i(x)1_{\{x\}} < \infty$ on an infinite graph satisfying specific conditions.

References

[1] M. Baker and S. Norine, Riemann-Roch and Abel-Jacobi theory on a finite graph, Advances in Mathematics, Volume 215, Issue 2, Pages 766-788.