Minimal surfaces and associated martingales

Robert Neel

Department of Mathematics
Lehigh University

November 23, 2018
Tokyo one-day workshop on stochastic analysis and geometry
The University of Tokyo

Acknowledgments

I would like to thank everyone involved in arranging this workshop, and inviting me to it, especially Professors Kawabi and Aida.

Brownian motion

There are a few ways of introducing Brownian motion on a manifold:

- it's the continuous version of an isotropic random walk on a manifold.
- it's the continuous Markov process with transition density given by the heat kernel.
- it solves the martingale problem for half the Laplacian; that is, if B_t is Brownian motion and f is smooth and compactly supported, then $f(B_t) - f(x_0) - \int_0^t \frac{1}{2}\Delta f(B_s)\, ds$ is a martingale.

It follows from this last point that if h is harmonic then $h(x_0)$ is given by the expectation of $h(B_\sigma)$ where B_t is Brownian motion started at x_0 and σ is a (bounded) stopping time. Two familiar examples are integrating h against the heat kernel and integrating h against harmonic measure.

Rank-n martingales

A rank-n martingale is a (continuous) process X_t on a smooth, m-dimensional manifold M, possibly defined up to some explosion time ζ, which locally (in space and time) satisfies an SDE of the form

$$dX_t = \sum_{i=1}^{n} v_{i,t}\, dW_t^i$$

where $(v_{1,t}, \ldots, v_{n,t})$ is a continuous, adapted n-tuple of orthonormal vectors (in $T_{X_t}M$). (We assume that $1 \leq n < m$.)

Infinitesimally, it's a BM on $\Lambda_t = \operatorname{span}\{v_{1,t}, \ldots, v_{n,t}\}$. We think of such a process as really being determined by Λ_t, but having an associated frame is often convenient for computations.

Rank-n martingales in geometry

This notion provides a unifying theme to a few geometric objects (formally, at least), and allows us to study "coarse" properties of such objects all at one time.

- Brownian motion on a minimal submanifold (meaning the mean curvature vector vanishes) gives rise to a rank-n martingale in the ambient space.
- This generalizes to the natural diffusion along the mean curvature flow, backward in time.
- The diffusion associated to a rank-n sub-Riemannian structure (realized as the restriction of a Riemannian metric) with the property that the sub-Laplacian is a sum-of-squares of coordinate vector fields in normal coordinates also gives a rank-n martingale. (Though this is a bit artificial.)

Martingales of bounded dilation

There is a relationship with martingales of bounded dilation, as studied by Kendall, Darling, Arnaudon, etc. Indeed, any rank-n martingale with $n \geq 2$ is a special case of a martingale of bounded dilation (in fact, of 1-bounded dilation and already essentially on the intrinsic time scale). These arise in studying the harmonic mapping problem, especially to prove generalized Picard little theorems.

However, we will be working in different geometric contexts than that of the harmonic mapping problem. In the following result about transience on a Cartan-Hadamard manifold, our methods are closer to those of March and Hsu for BM.

Course results motivated by minimal submanifolds

Let M be a (smooth, complete) Cartan-Hadamard manifold of dimension m. Choose $n \in \{1, \ldots, m-1\}$.

Markvorsen and Palmer (2003) prove that if N is a complete, n-dimensional, minimally immersed submanifold of M then if either of the following hold

- $n = 2$ and the sectional curvatures of M are bounded above by $-a^2 < 0$, or
- $n \geq 3$,

we have that N is transient. They prove this using nontrivial capacity estimates.

We wish to extend this (in a few directions), and to do so using elementary stochastic methods.

Transience results

Theorem (N. '14)

Let X_t be a rank-n martingale on a Cartan-Hadamard manifold M, as above. Then if either of the following two conditions hold:

1. $n = 2$ *and, in polar coordinates around some point,* M *satisfies (for some* $\varepsilon > 0$ *and* $R > 1$*) the curvature estimate*
$$K(r, \theta, \Sigma) \leq -\frac{1+2\varepsilon}{r^2 \log r} \quad \text{for } r > R \text{ (and for all } \theta \text{ and } \Sigma \ni \partial_r\text{)}$$
2. $n \geq 3$,

we have that X_t is transient.

So we re-prove Markvorsen and Palmers' results when $n \geq 3$, and weaken the curvature condition for $n = 2$. Further, this also applies to BM along backwards mean curvature flow and certain sub-Riemannian structures (although the significance is less clear).

Angular behavior

Theorem (Kendall '90s-ish, applied to rank-n martingales)

If M is a Cartan-Hadamard manifold of dimension $m \geq 3$ and with sectional curvatures pinched between two negative constants, and X_t is a rank-n martingale on M with $2 \leq n < m$, then $\theta_t = \theta(X_t)$ converges, almost surely, as $t \to \zeta$. Further, the distribution of $\theta(\zeta)$ on $\mathbb{S}^{m-1}$ is "genuinely random."

In our context, we made a first effort at relaxing the upper bound:

More angular behavior

Theorem (N. '14)

Suppose that M is Cartan-Hadamard manifold of dimension $m \geq 4$, and that M is radially symmetric around some point p. Let (r, θ) be polar coordinates around p, and let X_t be a rank-n martingale, for $3 \leq n < m$. Further, assume that M satisfies the curvature estimate

$$-a^2 \leq K(r, \theta, \Sigma) \leq -\frac{2+\varepsilon}{r^2} \quad \text{when } r > R \text{, and for all } \theta \text{ and } \Sigma \ni \partial_r,$$

for some $a > 0$, $\varepsilon > 0$, and $R > 1$. Then we have that $\theta_t = \theta(X_t)$ converges, almost surely, as $t \to \zeta$, and this limit θ_ζ is not a point mass. Further, for any $0 < \delta < 1$, there exists ρ (depending only on M and n) such that, if $r_0 > \rho$, then $\theta_\zeta \in B_\delta(\theta_0) \subset \mathbb{S}^{m-1}$ with probability at least $1 - \delta$.

Consequences

Theorem (N. '14)

Let N be an n-dimensional, properly immersed minimal submanifold (in M), corresponding to either of the above situations. Then N admits a non-constant, bounded, harmonic function.

Also, a sub-Riemannian structure of the type we've been discussing, in either situation, admits a non-constant, bounded, harmonic function (relative to the sub-Laplacian). Further, in the second case, we can solve some extrinsic version of the Dirichlet problem at infinity.

Finer results for minimal surfaces

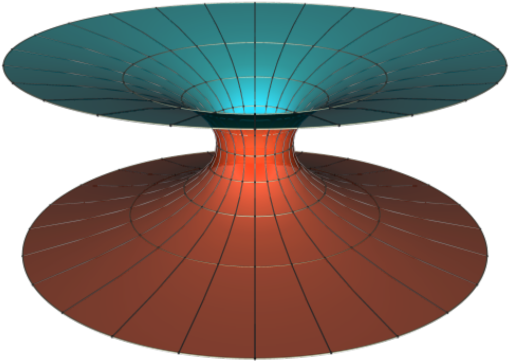
For now, we consider the case when M is a minimal surface in $\mathbb{R}^3$.

The/a plane is the simplest such surface; we'll see more in a moment. We assume M is stochastically complete; this means that Brownian motion never explodes.

Two important families of minimal surfaces:

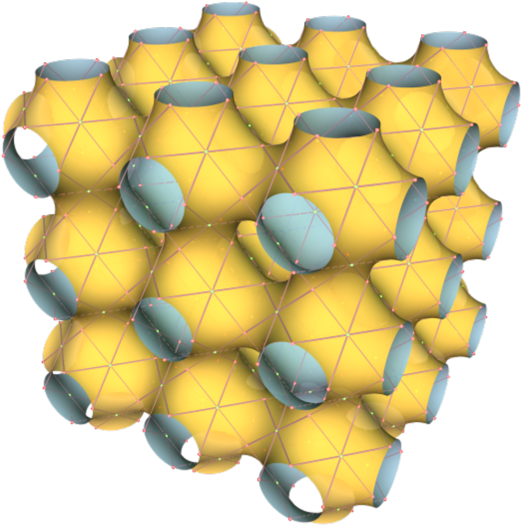
- Complete minimal surfaces of bounded curvature are stochastically complete.
- Properly immersed (complete) minimal surfaces are stochastically complete.

Example: The catenoid



- Properly embedded
- Bounded (Gauss) curvature
- Recurrent for Brownian motion

Example: The Schwarz P surface



- Properly embedded
- Bounded curvature
- Triply periodic (invariant under translations by $\mathbb{Z}^3$)
- Transient for Brownian motion

Less picturesque possibilities

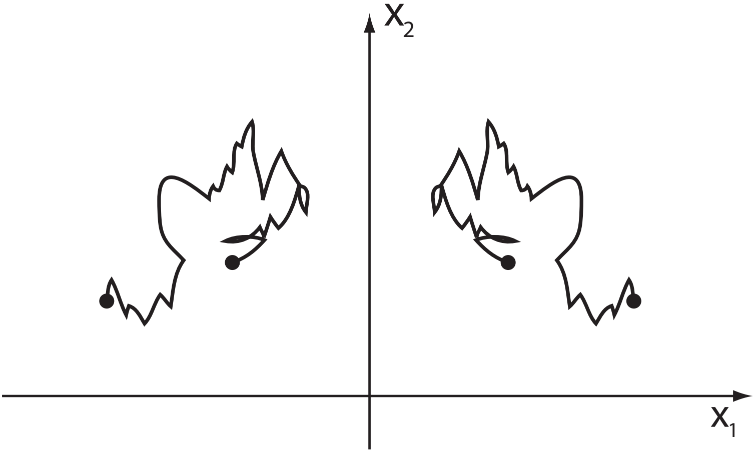
- There exists a proper, conformal, minimal immersion of the open disk into $\mathbb{R}^3$ (Morales, 2003). Obviously, this is transient with lots of non-constant bounded harmonic functions.
- There exists a *non-proper*, conformal, minimal immersion of the disk into a ball (Nadirashvili, 1996). Brownian motion almost surely blows up in finite time.

The Gauss map

- The Gauss map $M \to \mathbb{S}^2$ associates to each point its unit normal.
- It is (anti-) conformal, with the area distortion given by K, the Gauss curvature.
- Thus, the Gauss map composed with Brownian motion is a time-changed spherical Brownian motion, with time change given by the integral of $-K$.

Thus, not only is BM on M a rank-2 martingale in $\mathbb{R}^3$, but we also have control over the evolution of Λ_t. This extra structure beyond what we have for an arbitrary rank-2 martingale allows us to study finer properties of minimal surfaces.

Mirror coupling in $\mathbb{R}^2$



Kendall and Cranston (though see von Renesse and Kuwada) give a version on Riemannian manifolds; it's useful for spectral gap estimates, etc.

Mirror couplings for minimal surfaces

- Let M and N be stochastically complete minimal surfaces; a coupled Brownian motion is a process (x_t, y_t) on $M \times N$ such that x_t and y_t are Brownian motions.
- Our goal is to get x_t and y_t to couple in $\mathbb{R}^3$ in finite time.
- Intuitively, we want to develop an extrinsic analogue of the Kendall-Cranston mirror coupling.

- If M and N are different and x_t and y_t couple with positive probability, then M and N intersect.
- If $M = N$ is embedded and x_t and y_t couple almost surely, then M admits no non-constant bounded harmonic functions.
- Both of the above situations have analogues when M and N have boundary. In fact, the ability to extend results to the case when M and N are allowed to have boundary relatively easily is one of the motivations for using Brownian motion methods.

Intuitive local picture

With a fair amount of technical hassle, one can show that there exists a coupling which is instantaneously "at least barely favorable." While this is always true, some additional global control is needed to get the two particles to actually couple. (Recall that parallel planes exist.)

This situation is somewhat analogous to wanting to show that a function $f : [0, \infty) \to \mathbb{R}$ with $f(0) > 0$ has a zero. Showing $f' \leq 0$ is the borderline infinitesimal condition, but you also need some global condition so that you don't get "stuck" before reaching zero.

Choice of coupling

- Ler $r_t = \mathrm{dist}_{\mathbb{R}^3}(x_t, y_t)$
- Coupling instantaneously determined by an identification of $T_{x_t}M$ and $T_{y_t}N$; these are parametrized by $O(2)$.
- Itô's rule and trigonometry show that we can choose σ pointwise so that r_t is instantaneously modeled on a time-changed Bessel process of dimension two or less.
- Further, the instantaneous time-change is generically positive, and the instantaneous dimension is generically less than 2.

Obstacles to coupling

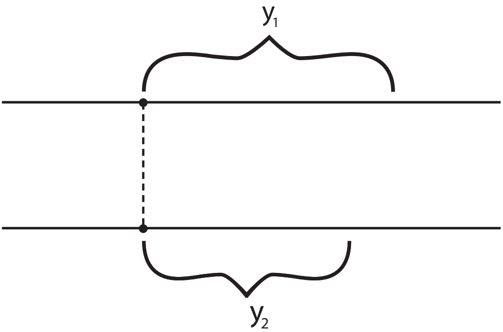
Assuming a coupling corresponding to a choice of σ with these properties exists, the particles will couple barring the following potential problems:

- The time-change might be deficient (the growth of the time-change/quadratic variation is too small), so that r_t converges and never hits zero.
- The domination by a two-dimensional Bessel process isn't strict enough near zero, so that r_t comes arbitrarily close to zero but never hits it.

Thus, the idea is to somehow control the unit normals and $(x_t - y_t)/r_t$ in order to rule out the above problems.

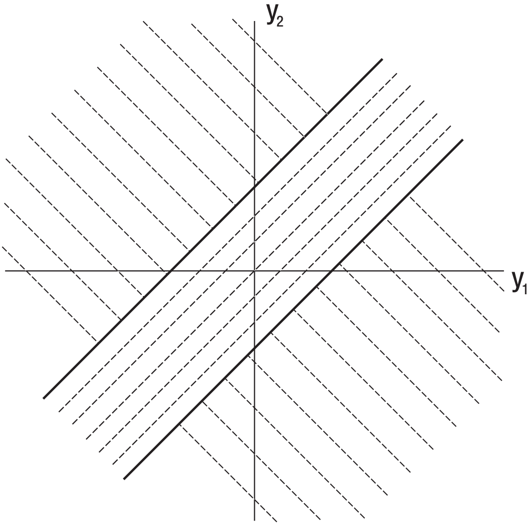
Parallel lines/planes

We illustrate the coupling and the potential problems in a revealing case. Suppose that M and N are parallel planes.



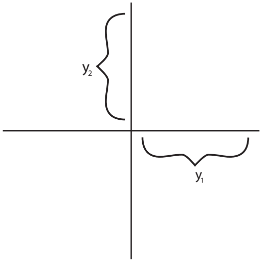
An extrinsic cross-section with coordinates.

A cross-section of $M \times N$

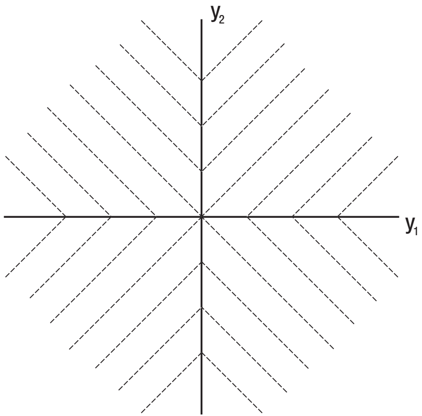


Perpendicular lines/planes

By way of contrast, for perpendicular planes with the particles started in the same cross-section, we see that they couple as follows:



An extrinsic cross-section with coordinates



A cross-section of $M \times N$

A general solution

- The desired process on $M \times N$ can be viewed as the solution to a martingale problem corresponding to a second order operator on $M \times N$.
- As the above suggests, the operator for our coupling is degenerate (rank 2 out of 4) everywhere and discontinuous on some subset.
- Thus, standard existence results do not apply.

The best we can do (which happens to be good enough) is

- Replace the optimal operator with a "good enough" choice with the same qualitative features; this makes the set of discontinuities smaller.
- Use an approximation argument to show existence, though not uniqueness.
- Solution need not be Markov at the discontinuities.

Partial global control

Theorem (Pathwise weak halfspace theorem)

Let M be a stochastically complete minimal surface of bounded curvature. Then if M is not flat, the Gauss sphere process (the composition of the Gauss map with Brownian motion) almost surely accumulates infinite occupation time in every open set of $\mathbb{S}^2$.

This is enough to show that the particles get arbitrarily close, giving …

Strong halfspace theorem for bounded curvature

Theorem (N. '09)

Let M be a stochastically complete, non-flat minimal surface with bounded curvature, and let N be a stochastically complete minimal surface. Then the distance between M and N is zero.

Compare this with

Theorem (Rosenberg '01)

Let M and N be complete minimal surfaces of bounded curvature. Then either M and N intersect, or they are parallel planes.

Maximum principle at infinity

Theorem (Meeks-Rosenberg '08)

Let M and N be disjoint, complete, properly immersed minimal surfaces-with-boundary, at least one of which has non-empty boundary. Then the distance between them satisfies

$$\mathrm{dist}(M,N) = \min\{\mathrm{dist}(M,\partial N), \mathrm{dist}(\partial M, N)\}.$$

Theorem (N. '09)

Let M and N be stochastically complete minimal surfaces-with-boundary, at least one of which has non-empty boundary, such that $\mathrm{dist}(M,N) > 0$. *If M has bounded curvature and is not a plane, then*

$$\mathrm{dist}(M,N) = \min\{\mathrm{dist}(M,\partial N), \mathrm{dist}(\partial M, N)\}.$$

A conjecture

Conjecture (Sullivan)

A complete, properly embedded minimal surface admits no non-constant, positive harmonic functions.

Previously known cases:

- The theorem obviously holds if M is recurrent.
- Meeks, Pérez, and Ros (2006) have proved this under additional symmetry assumptions (double or triple periodicity, various conditions about the quotient by the isometry group having finite topology).

A partial result

Theorem (N. '09)

Let M be a complete, properly embedded minimal surface of bounded curvature. Then M has no non-constant bounded harmonic functions.

- Again, bounded curvature means the particles get arbitrarily close, either infinitely often or until they couple.
- Meeks and Rosenberg's tubular neighborhood theorem implies that when the particles are close, the situation is uniformly close to the mirror coupling on $\mathbb{R}^2$.
- Then eventually the particles couple, almost surely.

Settling a conjecture of Meeks

Theorem (N. '08 (only arXiv) & '12, Impera-Pigola-Setti '13)

Any minimal graph (that is, a complete minimal surface-with-boundary, the interior of which is a graph over some planar region), other than a plane, is parabolic in the sense that any bounded harmonic function is determined by its boundary values.

- Here the "global control" used to make the coupling work is that the Gauss sphere process converges on a graph.
- Parabolicity is equivalent to Brownian motion almost surely hitting the boundary in finite time.
- In particular, such surfaces have non-empty boundary.
- Weitsman (2008) proved this assuming the domain was finitely connected. Other partial/related results were obtained by López and Pérez, and Meeks and Pérez.
- Impera, Pigola, and Setti give an analytic proof and some related results.

Example: The Sherk surface