Default functions and value distribution of holomorphic maps

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[Plan of my talk]

§1 Default functions: definition and basic properties

§2 Submartingale properties of subharmonic functions : Symmetric diffusion cases

§3 L^1- Liouville properties of subharmonic functions

§4 Liouville theorems for strongly subharmonic functions related to Yamabe equations

§5 Liouville theorems for holomorphic maps

§6 Picard type theorem for meromorphic functions functions

Default is ...

Dictionaries say:

Oxford dictionary;

1. Failure to fulfil an obligation, especially to repay a loan or appear in a law court.

Mirriam-Webster;

3. economics :a failure to pay financial debts,

e.g. ” was in default on her loan”, ”mortgage defaults”

§1 Default functions: definition and basic properties.

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

Let M_t be a continuous local martingale. i.e. $\exists T_n \uparrow \infty$ (stopping time) s.t. $M_t^{T_n} = M_{T_n \wedge t}$ is a true continuous martingale for each n.

M_t is a (true) martingale
$\Leftrightarrow E[M_T] = E[M_0]$ for $\forall T$: bounded stopping time.

Def. If M_t is not a true martingale, we say M_t is a strictly local martingale.

"Local" property of M_t :

$$\gamma_T(M) := E[M_0] - E[M_T].$$

is called a default function (Elworthy- X.M.Li-Yor('99)).

Rem. If M_t is positive, $\gamma_T(M) \geq 0$ ($\because$) M_t is a supermartingale).

Default formula : Assume that $E[|M_T|] < \infty, E[|M_0|] < \infty$ for a stopping time T and $\{M^-_{T\wedge S}; S:$ stopping times $\}$ is uniformly integrable. Set $M^*_t := \sup_{0\le s\le t} M_s$.

$$E[M_T : M^*_T \le \lambda] + \lambda P(M^*_T > \lambda) + E[(M_0 - \lambda)_+] = E[M_0].$$

Letting $\lambda \to \infty$,

$$\gamma_T(M) = \lim_{\lambda\to\infty} \lambda P(\sup_{0\le t\le T} M_t > \lambda).$$

Another quantity: $\sigma_T(M)$

Def.

$$\sigma_T(M) := \lim_{\lambda\to\infty} \lambda P(\langle M\rangle_T^{1/2} > \lambda).$$

Theorem (Elworthy-Li-Yor, Takaoka('99)) Assume that $E[|M_T|] < \infty, E[|M_0|] < \infty$.

$$\exists \gamma_T(M) = \sqrt{\frac{\pi}{2}} \sigma_T(M).$$

Moreover $M_t^T := M_{T \wedge t}$ is a uniformly integrable martingale iff $\gamma_T(M) = \sigma_T(M) = 0$.

See also Azema-Gundy -Yor('80), Galtchouk-Novikov('97).

[Example]

R_t : d-dimensional Bessel process(= the modulus of Brownian motion on R^d; $R_t = |B_t|$).

If $d > 2$, then R_t^{2-d} is a strictly local martingale.

As for default function, if $R_0 = r$,

$$\gamma_t(R^{2-d}) = \frac{1}{2^\nu \Gamma(\nu)} \int_0^t \frac{du}{u^{1+\nu}} \exp(-\frac{r^2}{2u}),$$

where $d = 2(1+\nu)$.

If $d = 2$, $\log R_t$ is a strictly local martingale.

$$\gamma_t(\log R) = \frac{1}{2} \int_0^t \frac{du}{u} \exp(-\frac{r^2}{2u}).$$

[submartingale case]

Let $X_t = X_0 + M_t + A_t$ where M is a local martingale and A is an adapted increasing process.

Lem.(Default function for submartingale)

If X is positive and $E[A_T] < \infty$,

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda P(\sup_{0\le t\le T} X_t > \lambda) &= \lim_{\lambda\to\infty} \lambda P(\sup_{0\le t\le T} M_t > \lambda) \\ &= E[X_0] - E[X_T] + E[A_T].\end{aligned}$$

Example (stochastic Jensen's formula).

Let Z_t : BM$(\mathbf{C})$ with $Z_0 = o$, $\tau_r = \inf\{t > 0 : |Z_t| > r\}$

and f be a non-constant holomorphic function on $\mathbf{C}$.

Set $X_t := \log|f(Z_{\tau_r \wedge t}) - a|^{-2}$: a local martingale bounded below.

$$\lim_{\lambda\to\infty} \lambda P(\sup_{0<t<\tau_r} X_t > \lambda) = \sum_{f(\zeta)=a,\ |\zeta|<r} 2\log\frac{r}{|\zeta|}.$$

From this we can see an essential relationship between Nevanlinna theory and complex Brownian motion (Carne(86), A.(95)).

§2 Submartingale property of subharmonic functions.

[Settings]

Let M be a separable, metrizable, locally compact space and m a Radon measure whose support is M.

(X_t, P_x) be a symmetric diffusion process with generator L defined from the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(m)$.

$\mu_{\langle u \rangle}$ denotes the energy measure of u such that

$$\int_M f(x) d\mu_{\langle u \rangle}(x) = 2\mathcal{E}(uf, u) - \mathcal{E}(u^2, f), \quad f \in \mathcal{F} \cap C_o(M).$$

Define

$$\mu_{\langle u,v \rangle} = \frac{1}{2}(\mu_{\langle u+v \rangle} - \mu_{\langle u \rangle} - \mu_{\langle v \rangle}).$$

Then

$$\mathcal{E}(u, v) = \frac{1}{2}\mu_{\langle u,v \rangle}(M) \quad (u, v \in \mathcal{F}_b).$$

If $\mu_{\langle u,v\rangle}$ is absolutely continuous w.r.t.m,

$$\Gamma(u,v) := \frac{d\mu_{\langle u,v\rangle}}{dm}.$$

Assume that

- **$(\mathcal{E}, \mathcal{F})$ is a strongly local, irreducible regular Dirichlet form.**
- **(AC) the transition probability $p(t,x,dy)$ is absolutely continuous w.r.t. m for $\forall t > 0, \forall x$.**
- **(EXH) there exists a nonnegative exhaustion function $r(x)$ (i.e. $\{r(x) < r\}$: rel.cpt for $\forall r \geq 0$) such that $\Gamma(r,r)$ is bounded.**
- **(CON) (X_t, P_x) is conservative.**

Typical Example : Brownian motion on a complete, connected Riemannian manifold $\mathcal{M}$. $L = \dfrac{1}{2}\Delta, \Gamma(u,u) = \dfrac{1}{2}|\nabla u|^2, r(x) = d(o,x)$, $m =$ Riemannian volume $dv, p(t,x,dy) = p(t,x,y)dv(y)$ where $p(t,x,y)$ is the heat kernel of $\partial/\partial t - \dfrac{1}{2}\Delta$.

$\mathcal{F} = H_0^1(\mathcal{M}) = \overline{C_0^\infty(\mathcal{M})}^{\mathcal{E}_1}$ where $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + ||u||^2_{L^2(m)}$.

Note $\mathcal{C} = C_0^\infty(\mathcal{M})$. This satisfies the assumptions (AC), (EXH). If the Ricci curvature of $\mathcal{M}$ satisfies

$$Ric \geq -Cr(x)^2 - C,$$

for some $C > 0$, (CON) holds.

[subharmonic function]

Def. u is (L-)subharmonic if $u \in \mathcal{F}_{loc} \cap L^{\infty}_{loc}(M)$ and $\mathcal{E}(\phi, u) \leq 0$ for $\forall \phi \geq 0, \phi \in \mathcal{F}$ with compact support.

Let

$$\mathcal{U} := \{u : \text{ a positive } L\text{-subharmonic function } \mid E_x[u(X_t)] < \infty (\forall t > 0) a.e.x\}.$$

It is well-known that $u(X_t)$ is a local submartingale if $u \in \mathcal{U}$:

$$\tilde{u}(X_t) - \tilde{u}(x) = M_t^{[u]} + A_t^{[u]} \qquad P_x\text{-a.s.}$$

Def. Submartingale property of subharmonic functions We say that u has the L-submartingale property if $\tilde{u}(X_t)$ is a continuous submartingale under P_x for a.e. x.

Def. Default function of $u(X_T)$

$$N_x(T,u) = \lim_{\lambda\to\infty} \lambda P_x(\sup_{0\le s\le T} \tilde{u}(X_s) > \lambda).$$

We consider the condition for the default function to be vanishing when $T = t$.

Theorem 0. Let $B(r) := \{r(x) < r\}$.

If $u \in \mathcal{U}$ and

$$\liminf_{r\to\infty} \frac{1}{r^2}(\log \int_{B(r)} u^\alpha dm + \log m(B(r)) < \infty$$

for some $\alpha > 2$, then u has the L-submartingale property. i.e. $\tilde{u}(X_t)$ is a submartingale under P_x for a.e.x.

sketch of proof.

1°. Let $\tau_r = \inf\{t > 0 | X_t \notin B(r)\}$. If

$$\lim_{r\to\infty} E_x[\tilde{u}(X_{\tau_r}) : \tau_r < t] = 0,$$

then $N_x(t,u) = 0$.

2°. Estimate $E_x[\tilde{u}(X_{\tau_r})]$.

Lem. **Let $x_0 \in M, \eta > 0$. If u is a positive L-subharmonic function, there exists a constant $C(x_0)$ such that for r large enough,**

$$E_{x_0}[\tilde{u}(X_{\tau_r})] \le C(x_0)\left(\int_{B(r(\eta+1))} u(x)^2 dm\right)^{1/2} + C(x_0).$$

3°. Estimate $P_x(\tau_r < t)$.

Lem. (Takeda's inequality) Fix $r_0 > 0$. If $r > r_0$, there exists $c > 0$ such that

$$\int_{B(r_0)} P_y(\tau_r < t)dm(y) \le const.\frac{vol(B(r+1))}{r} e^{-\frac{cr^2}{t}},$$

4°. $N_{x_0}(t_0, u) = 0$ for some x_0, t_0 implies $N_x(t, u) = 0$ for $\forall t > 0$ and a.e.x.

[Brownian motion case]

When $\mathcal{M}$ is a complete Riemannian manifold and (X_t, P_x) is Brownian motion on $\mathcal{M}$, the Ricci curvature controls the conditions in the above theorem.

Theorem. If there exists a constant $C > 0$ such that $Ric \geq -Cr(x)^2 - C$ and a positive subharmonic function u satisfies

$$\liminf_{r \to \infty} \frac{1}{r^2} \log \int_{B(r)} u(x) dv(x) < \infty,$$

then u has the Δ-submartingale property.

§3. L^1 Liouville theorem.

[Known results]

1-1. L^p**-Liouville theorem: (Yau '76, P.Li-Schoen '84) If $\mathcal{M}$ is a complete Riemannian manifold and a positive Δ-subharmonic function u is L^p-integrable for $p > 1$, u is constant.**

1-2. Generalization in the context of Dirichlet form (T.Sturm, '94).
Under our setting, if a positive L-subharmonic u satisfies

$$\int^{\infty} \frac{r\,dr}{\int_{B(r)} u^p dm} = \infty$$

for some $p > 1$, then u is constant.

2. L^1-Liouville theorem. Let $\mathcal{M}$ be a complete Riemannian manifold and u a positive Δ-subharmonic function.

Ricci curvature condition (P.Li '84)

If $\mathcal{M}$ is a complete Riemannian manifold satisfying $Ric \geq -Cr(x)^2 - C$ for some $C > 0$ and u is L^1, then u is constant.

3. Weighted L^p-Liouville theorem. (Nadirashvili '85)

If $\displaystyle\int_{\mathcal{M}} \frac{f(u(x))}{r(x)^2+1} dv(x) < \infty$ for a nonnegative function on $[0, \infty)$ satisfying $\displaystyle\int_0^\infty 1/f(t)dt < \infty$, then u is constant

[L^1-Liouville theorem and submartingale property]

We do not assume the conservativeness of (X_t, P_x) in the following proposition.

Prop. **If u is a positive, integrable L-subharmonic function and u has the submartingale property, then u is constant a.e. Namely vanishing of default function of u implies L^1-Liouville theorem.**

[(Counter)Example]

Example 1 (recurrent case).

The following example is originally due to Li-Schoen. We give a little modification. Let $\overline{M}$ be a compact 2-dim Riemannian manifold without boundary, equipped with a metric ds_0^2,
$\Delta_{\overline{M}}$ is the Laplacian defined from ds_0^2 and
$\overline{X}$ Brownian motion on $\overline{M}$ with its generator $\frac{1}{2}\Delta_{\overline{M}}$. Fix $o \in \overline{M}$. Set

$$g(o,x) = 2\pi \int_0^\infty (p(t,o,x) - \frac{1}{vol(\overline{M})})dt + C,$$

where $p(t,x,y)$ is the transition density of $\overline{X}$ and C is a positive constant such that $g(o,x) > 0$ for all $x \in \overline{M} \setminus \{o\}$. Remark that
$g(o,x) \sim \log \frac{1}{d_{\overline{M}}(o,x)^2} \quad (d_{\overline{M}}(o,x) \to 0)$. **Note**

$$\frac{1}{2}\Delta_{\overline{M}} g(o,x) = -2\pi\delta_o(x) + \frac{1}{Vol(\overline{M})}.$$

Let M be $\overline{M} \setminus \{o\}$. Take σ be a smooth function on M s.t.

$$\sigma(x) \sim t^{-1}(\log\frac{1}{t})^{-1}(\log\log\frac{1}{t})^{-\alpha} \text{ with } 1/2 < \alpha < 1$$

when $t = d_{\overline{M}}(o,x) \to 0$.

Define a metric $ds^2 = \sigma^2 ds_0^2$ on M. Note that Laplacian Δ_M defined from ds^2 has a form

$$\Delta_M = \sigma^{-2}\Delta_{\overline{M}},$$

where $\Delta_{\overline{M}}$ is defined from ds_0^2. Let X_t be Brownian motion on M with its generator $\frac{1}{2}\Delta_M$. Then X_t is a time changed process of $\overline{X}_t$ which is recurrent. Hence X_t is recurrent, in particular, conservative.

(M, ds^2) satisfies

- complete and stochastically complete.
- M is of finite volume w.r.t ds^2.
- u is a nonnegative smooth subharmonic function on M and integrable w.r.t. ds^2.
- the curvature $\sim -const.r^{\frac{2\alpha}{1-\alpha}} = -cr^{2+\epsilon}$ as $r \to \infty$ ($\epsilon = (4\alpha - 2)/(1-\alpha) > 0$).

From these facts we see $u(X_t)$ is a strictly local submartingale and L^1-Liouville property of M fails.

Example 2 (transient case).

Let M be a unit disc($\{|z| < 1\}$)$\setminus\{o\}$ in C.

Take a (non-degenerate) conformal metric g:

$$g \sim \begin{cases} ds^2 \text{ around } o \ (ds^2 \text{ as in Example 1}), \\ \text{the Poincaré metric near } |z| = 1. \end{cases}$$

Then g is complete on M and $\log volB(r) = O(r)$. The Brownian motion defined from g is a time-change of a hyperbolic Brownian motion. Set

$u(z) := -\log(2|z| \wedge 1)) = (-\log|z|) \vee \log 2 - \log 2 \geq 0.$

u is a nonnegative integrable subharmonic function w.r.t. the volume defined from g.

[Our results]

Theorem 1. Suppose u is an L-subharmonic function. $u_+ := \max\{u, 0\}$.

i) Assume there exists $\alpha > 2$ and $0 \leq p < 1$ such that

$$\liminf_{r \to \infty} \frac{1}{r^{2(1-p)}} \log\{m(B(r)) \int_{B(r)} u_+(x)^\alpha dm(x)\} < \infty.$$

If

$$\int_{\mathcal{M}} \frac{|u(x)|}{(1 + r(x))^{2p}} dm(x) < \infty,$$

then u is constant a.e.

ii) Assume there exists $\alpha > 2$ such that

$$\liminf_{r\to\infty} \frac{1}{(\log r)^2} \log\{m(B(r)) \int_{B(r)} u_+(x)^\alpha dm(x)\} < \infty.$$

If

$$\int_{\mathcal{M}} \frac{|u(x)|}{1+r(x)^2} dm(x) < \infty,$$

then u is constant a.e.

Rem. If $\mathcal{M}$ is a complete Riemannian manifold, u is a Δ-subharmonic function and $Ric \geq -Cr(x)^2 - C$, then the assumption of Theorem 1 with $p = 0$ is satisfied. It implies P.Li's theorem.

[Brownian motion case]

When $\mathcal{M}$ is a complete Riemannian manifold and u is a Δ-subharmonic function, using Ricci curvature condition enables us to simplify the results as follows.

Theorem 2 (A. 2016, 2017 manuscripta math.). Suppose $Ric \geq -k(r(x))$.

Let u be a smooth subharmonic function on M.

i) Assume that $k(r)$ is non-decreasing and there exists $0 \leq p \leq 1/2$ such that $\liminf_{r\to\infty} \frac{k(r)}{r^{2(1-2p)}} < \infty$. If

$$\int_M \frac{|u(x)|}{(1+r(x))^{2p}} dv(x) < \infty, \quad \text{then } u \text{ is constant.}$$

ii) Assume that $k(r)$ is regularly varying or moderately monotone, and there exists

$0 \leq p < 1$ such that
$\liminf_{r\to\infty} \frac{1}{r^{2(1-p)}}\{k(r)^{1/2} + \int_1^r k(t)^{1/2}dt + \log vol(\{r(x) < r\})\} < \infty$. If

$$\int_M \frac{|u(x)|}{(1+r(x))^{2p}} dv(x) < \infty, \quad \text{then } u \text{ is constant.}$$

iii) Assume that $k(r)$ is regularly varying or moderately monotone,
and $\liminf_{r\to\infty} \frac{1}{(\log r)^2}\{k(r)^{1/2} + \int_1^r k(t)^{1/2}dt + \log vol(\{r(x) < r\})\} < \infty$.

If

$$\int_M \frac{|u(x)|}{1+r(x)^2} dv(x) < \infty, \quad \text{then } u \text{ is constant.}$$

Proof of Theorem 1. As for the case of $p = 0$ directly from the submartingale property for $u(X_t)$. For the other case use time-change argument as follows. Let $\rho(t)$ is a non-increasing, positive function on $(0, \infty)$ such that $\int_0^\infty \rho(t)^{1/2} dt = \infty$. Y_t defined by

$$Y_t = X_{\zeta_t^{-1}} \text{ with } \zeta_t = \int_0^t \rho(r(X_s))ds.$$

Note that Y_t has a generator $\frac{1}{2}\rho(r(x))^{-1}L$ which becomes a self-adjoint operator on $L^2(\rho(r(x))dm)$. Define an exhaustion function $\theta(x)$ on $\mathcal{M}$ by

$$\theta(x) = \int_0^{r(x)} \sqrt{\rho(s)}ds.$$

Then $\Gamma(\theta, \theta)$ is bounded. Thus our argument as before is available. Take $\rho(t) = (1+t)^{-2p}$ with $0 \leq p < 1$ in case of i) and with $p = 1$ in case of ii).

§4. Liouville type theorems for strongly subharmonic functions.

Takegoshi ('06) and Pigola-Rigoli-Setti('03) showed :

Theorem (Takegoshi('06), Pigola-Rigoli-Setti('03) ($b < 2$)) **Let M be a non-compact complete Riemannian manifold and $v(r)$ denote the volume of a geodesic ball of radius $r > 0$ with center x_0. $r(x) := d(x_0, x)$. If there exist $u \in C^2(M), C > 0, a > 0, \delta > 0$ such that $\{u > \delta\} \neq \emptyset$ and**

$$\Delta u(x) \geq \frac{Cu(x)^{a+1}}{(1+r(x))^b} \quad \text{on } \{u > \delta\} \qquad (*)$$

holds for $b \leq 2$, then $\liminf_{r\to\infty} \dfrac{\log v(r)}{r^{2-b}} = \infty \; (b < 2)$,

$$\liminf_{r\to\infty} \frac{\log v(r)}{\log r} = \infty \; (b = 2).$$

Takegoshi called a function satisfying (*) a strongly subharmonic function. This inequality is related to Yamabe type differential inequality :

$$\Delta u(x) + k(x)u(x) \geq l(x)u(x)^{1+a}.$$

cf. Yamabe's equation : Let $f : (M, g) \to (N, h)$ be a conformal immersion such that $f^*h = u^{4/(m-2)}g$ $(m \geq 3)$, $f^*h = ug$ $(m = 2)$. Then u satisfies :

$$c_m \Delta_M - s_g u + K_{f^*h} u^{(m+2)/(m-2)} = 0 \ (m \geq 3),$$
$$\Delta \log u - s_g + K_{f^*h} u = 0 \ (m = 2),$$

where $c_m = 4(m-1)/(m-2)$, s_g and K_{f^*h} are scalar curvatures of g and f^*h, respectively.

We can extend the above result by Takegoshi and Pigola et. al to the case of our symmetric diffusion case.

Theorem 3. Let ρ be a non-increasing, positive continuous function on $\mathbb{R}$ s.t. $\int_0^\infty \sqrt{\rho}(t)dt = \infty$. Set $\Phi(t) := \int_0^t \sqrt{\rho}(s)ds$. If $u \in \mathcal{F}_{loc}$ satisfies that $\{u > \delta\} \neq \emptyset$ for some $\delta > 0$ and

$$Lu(x) \geq \rho(r(x))u^{a+1}(x) \quad \text{on } \{u > \delta\}$$

holds for some $a > 0$, then $\liminf_{r\to\infty} \frac{\log m(\Phi(r(x)) < r)}{\Phi(r)^2} = \infty$.

Cor. Takegoshi's theorem holds replacing the conclusion in the case of $b = 2$ by $\liminf_{r\to\infty} \frac{\log v(r)}{(\log r)^2} = \infty$.

Proof of Theorem 3. By time-change argument it is sufficient to consider the case that $\rho = 1$. The problem can be deduced to consider u satisfying

$$Lu \geq Cu^{a+1} \tag{**}$$

on M.

Lemma. If $u \in \mathcal{F}_{loc}$ satisfies (**), there exists a constant $C_1 > 0$ such that

$$\int_{B(r)} u(x)^{2+a} dm(x) \leq C_1 V(2r),$$

where $V(r) := m(\{x | r(x) \leq r\})$.

Hence by Theorem 0, if u satisfies (**) and

$$\liminf_{r\to\infty} \frac{\log V(r)}{r^2} < \infty,$$

then u has L-submartingale property.

§5. Liouville theorems for holomorphic maps.

Let $\mathcal{M}$ be a complete Kähler manifold, $\mathcal{N}$ a Hermitian manifold, and $f : \mathcal{M} \to \mathcal{N}$ a holomorphic map. $R(x) := \inf_{\xi \in T_x\mathcal{M}, ||\xi||=1} Ric(\xi, \xi)$, $R_-(x) := \max\{0, -R(x)\}, B(r) := \{x \in M | r(x) < r\}$, $K(y)$: holomorphic bisectional curvature of $\mathcal{N}$.

Let ρ be a non-increasing, positive continuous function on $\mathbb{R}$ s.t. $\int_0^\infty \sqrt{\rho}(t)dt = \infty$ as Theorem 3.

Theorem 4. Assume Brownian motion on $\mathcal{M}$ is transient. If $K(f(x)) \leq -\rho(r(x))$, $\int_M R_-(x)dv(x) < \infty$ and

$$\liminf_{r\to\infty} \frac{1}{\Phi(r)^2} \log vol(B(r)) < \infty,$$

then f is constant, where $\Phi(r) = \int_0^r \sqrt{\rho(t)}dt$.

Cor. If $\int_{\mathcal{M}} R_-(x)dv(x) < \infty$ and

$$\liminf_{r\to\infty} \frac{1}{r^2} \log vol(B(r)) < \infty,$$

then every bounded holomorphic function on $\mathcal{M}$ is constant.

Rem.

- This type result is originally due to Li and Yau(1990) where they treated the case that ρ was constant.

- Pigola-Rigoli-Setti('08) showed the above theorem when $\rho(t) = ct^{-b}$ $(b < 2)$.

- In recurrent cases Theorem 4 does not always hold. In particular, it does not hold when $\dim_{\mathbf{C}} M = 1$.

Idea of proof. Let $e(x) := tr_{g_{\mathcal{M}}} f^* g_{\mathcal{N}}$ (energy density of f). Chern-Lu formula implies

$$\frac{1}{2}\Delta \log e(x) \geq -K(f(x))e(x) - R_-(x) \text{ if } e(x) \neq 0.$$

Then the problem can be deduced to

Theorem 5. Suppose that L-diffusion (X_t, P_x) is transient and that

$$\liminf_{r\to\infty} \frac{\log m(B(r))}{r^2} < \infty,$$

where $B(r) = \{x \in M \mid r(x) < r\}$. If $a > 0$ and a nonnegative $u \in \mathcal{F}_{loc} \cap L^{\infty}_{loc}(M)$ satisfies

$$L \log u(x) \geq u(x)^a - g(x), \tag{***}$$

where g is a nonnegative m-integrable function, then $u = 0$.

Transience assumption is effectively used as follows:

Lemma. **If u satisfies (***) and**

$$\log w(x) = \log u(x) - E_x[\int_0^\infty g(X_s)ds],$$

then w satisfies

$$Lw \geq w^{1+a}.$$

Then the problem can be deduced to Theorem 3.

§6. Picard type theorems.

Consider the value distribution of meromorphic functions on negatively curved Kähler manifolds. Let f be a nonconstant meromorphic function on M: Kähler manifold i.e. $f : M \to \mathrm{P}^1(\mathrm{C})$ holomorphic.

$[x, y]$ denotes the chordal distance on $\mathrm{C} \cup \{\infty\} \cong \mathrm{P}^1(\mathrm{C})$ defined by

$$[x, y] = \begin{cases} \dfrac{|x-y|}{\sqrt{|x|^2+1}\sqrt{|y|^2+1}} & (x, y < \infty) \\ \dfrac{1}{\sqrt{|x|^2+1}} & (y = \infty). \end{cases}$$

Let (X_t, P_x) be a Brownian motion defined from the Kähler metric. In this section we consider default function $\log[f(X_t), a]^{-2}$ $(a \in \mathrm{P}^1(\mathrm{C}))$.

Assume (X_t, P_x) is conservative (i.e. (M, g) is stochastically complete).

Define

$$\tilde{m}_x(t,a) = E_x[\log[f(X_t),a]^{-2}],$$

$$\tilde{N}_x(t,a) = \lim_{\lambda\to\infty} \lambda P_x(\sup_{0\le s\le t} \log[f(X_s),a]^{-2} > \lambda),$$

$$\tilde{T}_x(t) = E_x[\int_0^t e(X_s)ds]$$

provided that $f(x) \neq a$. As before by Ito's formula, we have an analogy of the First Main Theorem of Nevanlinna theory:

$$\tilde{m}_x(t,a) - \log[f(x),a]^{-2} + \tilde{N}_x(t,a) = \tilde{T}_x(t)$$

provided that $f(x) \neq a$ and $\tilde{T}_x(t) < \infty$.

Let $r(x)$ be a distance function from a reference point on M and, set $R(x) = \inf_{|\xi|=1,\ \xi\in T_x M} \mathrm{Ric}(\xi,\xi)$ and $B(r) = \{r(x) < r\}$.

Lemma. Assume $R(x) \geq -Cr(x)^2 - C$ for some $C > 0$. If f omits $a \in \mathrm{P}^1(\mathrm{C})$ and

$$\liminf_{r \to \infty} \frac{1}{r^2} \log \int_{B(r)} e(x) dv(x) < \infty,$$

then $\tilde{N}_x(t, a) = 0$ for $\forall t > 0$, a.e. x.

Theorem 5 (A. 2017 Forum Math.) Let M be a complete Kähler manifold whose sectional curvature is non-positive and its Ricci curvature satisfies

$$R(x) \geq -Cr(x)^{\beta} - C \text{ for } \beta < 2.$$

Let f be a nonconstant meromorphic function on M, $a_1, a_2, \ldots, a_q$ distinct points of $\mathrm{P}^1(\mathrm{C})$ and $x \in M$ such that $f(x) \neq a_j$ $(j = 1, \ldots, q)$. Assume that f cannot omit any sets of positive logarithmic capacity. Then

(i) f omits at most two points or

(ii)

$$\sum_{i=1}^{q} \tilde{m}_x(t, a_i) + \tilde{N}_1(t, x) \leq 2\tilde{T}_x(t) + \tilde{N}_x(t, \mathrm{Ric}) + O(\log \tilde{T}_x(t))$$

holds for $t \in (0, \infty)$ except for a set of finite Lebesgue measure, where

$$\tilde{N}_x(t, \mathrm{Ric}) = -E_x[\int_0^t R(X_s)ds].$$

Rem. 1) The assumption that f cannot omit any sets of positive logarithmic capacity implies $\tilde{T}_x(t) \to \infty$ $(t \to \infty)$.

2) The case when $\tilde{T}_x(t) = \infty$ for a finite $t > 0$ is included in the case (i).

3) When $dim_{\mathbf{C}} M = 1$, the negativity assumption of the sectional curvature of M can be removed.

Cor. Let M and f be as above.

If

$$\alpha := \limsup_{t\to\infty} \frac{\tilde{N}_x(t, \mathrm{Ric})}{\tilde{T}_x(t)} < \infty,$$

then f can omit at most $2 + \alpha$ points.

Examples. 1) Let M be a Riemann surface of finite total curvature and $Ric \geq -Cr^{\beta} - C$ for some $\beta < 2$. Then we have

$$\alpha = \lim_{t\to\infty} \frac{\tilde{N}_x(t, \mathrm{Ric})}{\tilde{T}_x(t)} = \frac{K_M}{e(f)},$$

where K_M is the total curvature of M and $e(f) = \int_M e(x)dV(x)$ $(\leq \infty)$.

Hence

$$\#(\mathbf{P}^1(\mathbf{C}) \setminus f(M)) \leq 2 + \frac{K_M}{e(f)}.$$

2) Assume that $dim_{\mathbf{C}} M \geq 2$, $Sect_M \leq 0$, $\int_M R_- dv < \infty$ and $Ric \geq -Cr^{\beta} - C$ for some $\beta < 2$. If X is transient, $\tilde{\mathrm{N}}_x(\infty, \mathrm{Ric}) < \infty$. Hence

$$Cap(\mathrm{P}^1(\mathrm{C}) \setminus f(M)) = 0 \text{ implies } \#(\mathrm{P}^1(\mathrm{C}) \setminus f(M)) \leq 2.$$

If X is recurrent,

$$\#(\mathrm{P}^1(\mathrm{C}) \setminus f(M)) \leq 2 + \frac{\int_M R_- dv}{e(f)}$$

as before.