Witten Laplacian on pinned path group and its expected semiclassical behavior

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1 Introduction

Let (X, g) be a d-dimensional compact Riemannian manifold and let f be a Morse function on X. That is, the set of critical points of f is a finite set $\{c_1, \ldots, c_k\}$ and the Hessian there are nondegenerate. Let $d_\lambda = e^{-\lambda f/2} d e^{\lambda f/2}$. Here d denotes the exterior differential operator on X. Taking an adjoint of d_λ on $L^2(\wedge T^*X, dx)$ (dx is the Riemannian volume), we see $d_\lambda^* = e^{\lambda f/2} d^* e^{-\lambda f/2}$ explicitly. d_λ defines an elliptic complex which is called a Witten complex. Let $\Box_\lambda = d_\lambda^* d_\lambda + d_\lambda d_\lambda^*$. Then we see that

$$\Box_\lambda \alpha = \Box\alpha + \frac{\lambda^2}{4}|\nabla f(x)|^2\alpha + \frac{\lambda}{2}\sum_{i,j=1}^{d}\left[\mathrm{ext}(e_i^*), \mathrm{int}(e_j^*)\right]\nabla_{e_i}\nabla_{e_j} f(x)\alpha, \tag{1.1}$$

where $\Box = dd^* + d^*d$ and $\{e_i\}_{i=1}^n$ denotes an orthonormal system on T_xX. Let $\Box_\lambda|_p$ denote the restriction of $\Box_\lambda$ on p-form. Let $\sigma(\Box_\lambda|_p)$ be the spectral set of $\Box_\lambda|_p$. By using the semiclassical analysis, we see that

$$\lim_{\lambda\to\infty}\frac{\inf\sigma(\Box_\lambda|_p)}{\lambda} = \inf_{j,I}\left(\sum_{i\in I, \xi_i(j)>0}|\xi_i(j)| + \sum_{i\notin I, \xi_i(j)<0}|\xi_i(j)|\right), \tag{1.2}$$

where $\{\xi_i(j)\}_{i=1}^d$ denotes the eigenvalues of $\nabla^2 f$ at the critical point c_j $(1 \le j \le k)$ and I runs all subset of $\{1, \ldots, d\}$ with $|I| = p$. This is not stated explicitly but it is not difficult to check it by the result in [28], [19]. See also [8]. In [8], the Riemannian metric is taken to be flat near critical points, but this is not necessary as noted in [28]. If there are no critical points whose indices are p, then the right-hand side in (1.2) is positive. Therefore p-th Betti number should be 0 by Hodge-Kodaira's theorem. Further consideration implies the Morse inequality. We refer the reader to [8] for the detail. Now let us consider an unitary transformation $\Phi_\lambda : L^2(\wedge T^*X, e^{-\lambda f}dx) \to L^2(\wedge T^*X, dx)$ as $\Phi_\lambda\alpha = e^{-\lambda f/2}\alpha$. We write $d\mu_\lambda = e^{-\lambda f}dx$. Then by using this unitary transformation, $\Box_\lambda$ is unitarily equivalent to the differential operator

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$\Box_{\mu_\lambda} = d^*_{\mu_\lambda} d + dd^*_{\mu_\lambda}$ on $L^2(\wedge T^*X, e^{-\lambda f}dx)$. Here $d^*_{\mu_\lambda}$ denotes the adjoint of d with respect to the inner product of $L^2(\wedge T^*X, d\mu_\lambda)$. Therefore the semiclassical behavior of the spectrum of $\Box_{\mu_\lambda}$ is the same as $\Box_\lambda$. Under some condition on the curvature and the Morse function f, (1.2) still holds for noncompact Riemannian manifolds.

Now let us consider a pinned path space over a compact Riemannian manifold M. That is, let $X = P_{x,y}(M)$ which is a space of continuous path γ from $[0,1]$ to M with $\gamma(0) = x$ and $\gamma(1) = y$. Let $E(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2 dt$ for H^1-path $\gamma \in P_{x,y}(M)$. Then by taking x, y appropriately, E is a Morse function on the subset consisting of H^1-paths and the critical points (=geodesics) are isolated. Let ν_λ be the pinned Brownian motion measure on X such that

$$\begin{aligned}\nu_\lambda\left(\gamma(t_1) \in dx_1, \ldots, \gamma(t_m) \in dx_m\right) \\ = p(\lambda^{-1}, x, y)^{-1}\left(\prod_{i=1}^{m+1} p\left(\lambda^{-1}(t_i - t_{i-1}), x_{i-1}, x_i\right)\right) dx_1 \cdots dx_m.\end{aligned} \tag{1.3}$$

Here $t_0 = 0$, $t_{m+1} = 1$, $x_0 = x$, $x_{m+1} = y$, $p(t,x,y) = e^{t\Delta/2}(x,y)$ and Δ is the Laplace-Beltrami operator. Then formally, it is often written as

$$d\nu_\lambda(\gamma) = C_\lambda e^{-\lambda E(\gamma)} d\gamma_\lambda. \tag{1.4}$$

Here $d\gamma_\lambda$ denotes the fictitious Riemannian volume element which may depend on λ. Pinned measure ν_λ does exist although $d\gamma_\lambda$ does not exist. We have differential calculus based on the pinned measure which is an extension of the Malliavin calculus on the classical Wiener space. Then taking the unitary equivalence between $\Box_\lambda$ and $\Box_{\nu_\lambda}$ into accounts, one may consider $\Box_{\nu_\lambda}$ on $L^2(\wedge T^*X, d\nu_\lambda)$ as the mathematically well-defined Hodge-Kodaira-Witten Laplacian. Motivated by this, the author [2], [3] studied semiclassical behaviors of spectrum of Schrödinger type operators on Wiener spaces. Note that this problem is related with semiclassical problems in Euclidean field theory [7]. We note that this view point was used by Eberle [11] to study the spectral gap problem of the Ornstein-Uhlenbeck operator acting on functions on loop space. Even if in such a heuristic level, we have to think what is the Riemannian metric on X since the Riemannian volume is defined by the Riemannian metric. In probability theory, the tangent space at the path γ is defined by choosing a Riemmanian connection. We refer the reader to [9], [26] for the tangent space and the Riemannian metric. Andersson and Driver [6] takes the infinite dimensional Riemmanian volume into accounts and give a rigorous meaning to the formal expression (1.4) for the path space $P_x(M)$ which consists of continuous paths starting at x. Let us recall their results. The Riemannian metric on $P_x(M)$ which is defined by the Levi-Civita connection induces the Riemmanin metric on a finite dimensional approximate submanifold X_n and it defines the Riemmanian volume dx_n on X_n. Let us consider the restriction of $E(\gamma)$ to submanifold X_n and consider the weighted probability measure $d\mu_n = C_n \cdot e^{-E(x)}dx_n$ on X_n. Then μ_n converges to the Brownian motion measure. Eberle [11] studied the asymptotics of the gap of spectrum of the finite dimension version of the Ornstein-Uhlenbeck operator acting on functions on X_n and consider the limit $n \to \infty$. However, the change of the limit was not studied.

Next serious problem is in the definition of the exterior differential operator d. The definition of d depends on the choice of the tangent space at each γ and on the choice of the Riemannian connection. Different Riemannian connection defines nonequivalent Hilbert space structure at each tangent space. If the Riemannian curvature of the connection is not zero, then the commutator of the vector field $[X,Y]$ is not in the tangent space. Consequently we cannot define exterior differential operator d in this case. Concerning this difficulty, Léandre ([22], [23],[24],[21]) defined a regularized exterior differential operator and a Witten Laplacian and gave many conjectures on them. Also he computed an operator which is obtained by the semiclassical limit of the Witten Laplacian. Also we note another approach of Elworthy and Li [12] for the exterior differential operator on path spaces. Here we consider the case where M is a compact Lie group G with bi-invariant Riemannian metric and consider the right-invariant trivial connection. Then the curvature tensor is 0 and this connection defines a right-invariant Riemannian metric on the pinned path space. This Riemannian metric and its Levi-Civita connection on the space of H^1-loops are studied by Freed [15]. We refer the reader to [25], [5],[16],[17], [29],[13],[1] for analysis based on the connection and the Brownian motion measure on continuous path spaces over G. Since the curvature is 0, for the Hilbert (Riemannian) structure at each tangent space, the Lie bracket of the vector field is well-defined and so is the definition of the exterior differential operator. But the Riemannian metric is different from the metric which is defined by the Levi-Civita connection on G. Thus the result in [6] cannot be applied directly in this case. Andersson and Driver's finite dimensional approximation may converge to the pinned measure but this is not studied yet to the author's knowledge. In this paper, we give a heuristic explanation on the expression (1.4) in Section 2 based on Malliavin calculus and coarea formula [4], [14]. Note that the "Riemannian volume element" $d\gamma_\lambda$ does not depend on λ in the argument.

After introducing the Riemannian metric, we can calculate the eigenvalues of the Hessian of the energy functional $E(\gamma)$ at critical points. Note that the calculation at critical points does not need the connection on the pinned path space differently from that at general points. Hence, we do not discuss about the connection on path spaces. The main aim of this paper is to calculate the eigenvalues and prove the strict positivity of the quantity corresponding to the right-hand side in (1.2) when p is odd number in the case of $G = SU(n)$. By this, one may expect that the bottom of spectrum of $\Box_{\nu_\lambda}$ diverges when $\lambda \to \infty$ and there are no harmonic p-forms for odd p. Bott's theorem [27] tells us that the index of the energy functional is even at each critical points and so $\sum_{i\in I,\xi_i(j)>0} |\xi_i(j)| + \sum_{i\notin I,\xi_i(j)<0} |\xi_i(j)| > 0$ for fixed I and j. Therefore, our task is to prove a uniform lower bound on them.

2 Right invariant Riemannian metric and heuristic explanation of (1.4)

Let G be a compact Lie group and g be a bi-invariant Riemannian metric on G. g is defined by the Ad-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g} = T_eG$. We denote the Lie bracket on $\mathfrak{g}$ by $[\cdot,\cdot]$. Set

$$\begin{aligned} X &= H^1(G) = H^1([0,1] \to G \mid \gamma(0) = e), && (2.1)\\ X_a &= H^1_a(G) = H^1([0,1] \to G \mid \gamma(0) = e, \gamma(1) = a). && (2.2)\end{aligned}$$

Then X and X_a are Hilbert manifolds modeled on the Hilbert space H, H_0 respectively. Here

$$\begin{aligned} H &= H^1(\mathfrak{g}) = H^1([0,1] \to \mathfrak{g} \mid h(0) = 0) && (2.3)\\ H_0 &= H^1_0(\mathfrak{g}) = H^1([0,1] \to \mathfrak{g} \mid h(0) = h(1) = 0). && (2.4)\end{aligned}$$

The norm of $h \in H$ is given by $\|h\|^2 = \int_0^1 |\dot{h}(t)|^2_{\mathfrak{g}} dt$. The tangent vector at γ is a vector field along $\gamma(\cdot)$. The tangent space as the Hilbert manifold at γ is given by

$$T_\gamma X = \left\{ k : [0,1] \to TG \mid k(t) \in T_{\gamma(t)}G \text{ and } (R_{\gamma(\cdot)})_*^{-1} k(\cdot) \in H \right\}, \qquad (2.5)$$

where $R_a b = ba$ for $a, b \in G$. $T_\gamma X_a$ is the subset of $T_\gamma X$ with $k(1) = 0$. Thus, the tangent bundles of X and X_a are trivial and we identify the tangent spaces with H and H_0 respectively. The vector field Z_k corresponding to $k \in H$ (or H_0) acts on the function f by

$$Z_k f(\gamma) = \lim_{\varepsilon \to \infty} \frac{f\left(e^{\varepsilon k(\cdot)}\gamma(\cdot)\right) - f(\gamma)}{\varepsilon}, \tag{2.6}$$

where $e^{\varepsilon k(t)}$ is defined by the exponential map, $\exp : \mathfrak{g} \to G$. Let us define a Riemannian metric on X by

$$(k, k)_{T_\gamma X} = \|(R_{\gamma(\cdot)})_*^{-1} k(\cdot)\|_H^2. \tag{2.7}$$

The Riemannian metric on X_a is defined in the same way. These Riemannian metric are called the right invariant H^1-Riemannian metrics. Left invariant Riemannian metric $\langle \cdot, \cdot \rangle$ is defined by

$$\langle k, k \rangle = \|(L_{\gamma(\cdot)})_*^{-1} k(\cdot)\|_H^2, \tag{2.8}$$

where $L_a b = ab$. These metrics were studied by Freed [15]. Let $h \in H$ and consider the following differential equations on G.

$$\begin{aligned} \dot\gamma(t) &= (R_{\gamma(t)})_* \dot h(t) && (2.9)\\ \dot\gamma(t) &= (L_{\gamma(t)})_* \dot h(t) && (2.10)\\ \gamma(0) &= e, && (2.11) \end{aligned}$$

We denote the solutions to (2.9) and (2.10) by $I_R(h)$ and $I_L(h)$ respectively. Also we denote $h_R = I_R^{-1}(\gamma)$ and $h_L = I_L^{-1}(\gamma)$. The map $h \to I_R(h)$ defines a diffeomorphism between two Hilbert manifolds H and X. Also note that the energy of $I_L(h)$ and $I_R(h)$ are equal to $\|h\|_H^2$. Noting $\frac{d}{dt}\left(\gamma(t)^{-1}\right) = -\gamma(t)^{-1}\dot\gamma(t)\gamma(t)^{-1}$, we have for $h \in H$,

$$\frac{d}{dt}\left(Ad(\gamma(t)^{-1})h(t)\right) = Ad(\gamma(t)^{-1})\dot h(t) - Ad(\gamma(t)^{-1})[\dot h_R(t), h(t)].$$

Since

$$\frac{d}{dt}\left((L_{\gamma(t)})_*^{-1} k(t)\right) = \frac{d}{dt}\left(Ad(\gamma(t)^{-1})((R_{\gamma(t)})_*)^{-1} k(t)\right), \tag{2.12}$$

we have

$$\langle k, k \rangle = \int_0^1 \left| \frac{d}{dt}\left(R_{\gamma(t)}\right)_*^{-1} k(t)) - \left[\dot h_R(t), (R_{\gamma(t)})_*^{-1} k(t)\right] \right|^2 dt. \tag{2.13}$$

The linear operator $K_\gamma : h \to \int_0^\cdot [\dot h_R(t), h(t)]dt$ is a Hilbert-Schmidt operator on H and has no eigenvalues which is easily checked by observing that K_γ is a Volterra type operator. These imply that the right-invariant and left-invariant Riemannian metric is equivalent on X. However differently from finite dimensional cases, we cannot consider “bi-invariant Riemannian metric” since there are no Haar measure on X. By the same definition as in (2.7) and (2.8), the Riemannian metric and H^1-tangent space are defined on continuous pinned path space too. In such cases, the tangent space at “generic paths” are different corresponding to the metric. Below we adopt the right-invariant Riemannian metric. For $h, k \in H$, let $\phi(t, h, k) = I_L(h + k)(t)I_L(h)(t)^{-1}$. Then $\phi(t, h, k)$ satisfies

$$\phi(t, h, k)^{-1}\dot\phi(t, h, k) = Ad(I_L(h)(t))\dot h(t) \qquad (0 \le t \le 1).$$

So we have

$$\left(R_{I_L(h)(t)}\right)_*^{-1} D_k I_L(h)(t) = \int_0^t Ad\left(I_L(h)(t)\right)\dot{k}(s)ds. \tag{2.14}$$

D denotes the H-derivative on H. This implies I_L is a Riemannian isometry mapping from H onto X. Now consider a submanifold with codimension $\dim \mathfrak{g}$ in H:

$$S_a = \{h \in H \mid I_L(h)(1) = a\}. \tag{2.15}$$

There is a Riemannian metric g_a induced by the Riemannian metric $\| \|_H$ on H. Then by (2.14),

Theorem 2.1 *I_L is a Riemannian isometry between (S_a, g_a) and (X_a, g_R).*

Let us consider the stochastic case. Let $P(\mathfrak{g})$ be the space of continuous paths $x(\cdot)$ on $[0,1]$ with values in $\mathfrak{g}$ starting at 0. There exists a probability measure μ_λ on $P(\mathfrak{g})$ such that

$$\int_{P(\mathfrak{g})} e^{\sqrt{-1}(\gamma,h)_H} d\mu_\lambda(\gamma) = \exp\left(-\frac{\|h\|^2}{2\lambda}\right). \tag{2.16}$$

This measure is the Gaussian measure whose covariance operator is $\lambda^{-1}I_H$. Let $P(G)$ be the space of continuous paths γ on G satisfying $\gamma(0) = e$. We denote by $P_a(G)$ the subset of $P(G)$ with $\gamma(1) = a$. The differential equations (2.9) and (2.10) can be interpreted as the Stratonovich differential equation for the Brownian motion path on $\mathfrak{g}$. Then I_R and I_L are Itô's map. We denote $B(\cdot) = I_L^{-1}(\gamma)(\cdot)$ and $b(\cdot) = I_R^{-1}(\gamma)(\cdot)$ which are called the Itô-Cartan development. We refer the precise meaning to [20]. We denote by $p(t,x,y)$ the heat kernel of the heat semigroup of $e^{t\Delta/2}$ where Δ is the Laplace-Beltrami operator. Define a probability measure on $P(\mathfrak{g})$ by $d\mu_{a,\lambda}(B) = c_{\lambda,a}\delta_a(I_R(B)(1))d\mu_\lambda(B)$. Here $c_{\lambda,a} = p\left(\lambda^{-1}, e, a\right)^{-1}$. Then we see that the image measure of $\mu_{a,\lambda}$ by I_R is the pinned Brownian motion measure $\nu_{a,\lambda}$ on $P_a(G)$. See [30]. Formally, $\mu_{a,\lambda}$ is a measure on $S_a \subset H$. Now we recall the coarea formula in finite dimension [14].

Theorem 2.2 *Let F be a smooth mapping from $\mathbb{R}^n$ to a compact Riemannian manifold M. Assume that there exists an open subset $U \subset M$ such that $dF(x) : T_x\mathbb{R}^n \to T_F(x)M$ is surjective for any $x \in F^{-1}(U)$. Let ρ be a continuous function on M with $\operatorname{supp} \rho \subset U$. Then for any $z \in U$, $F^{-1}(z)$ is a smooth submanifold. Let $dv_z(x)$ be the induced volume element on $F^{-1}(z)$. Then for any smooth nonnegative function φ on $\mathbb{R}^n$ it holds that*

$$\int_{\mathbb{R}^n} \rho\left(F(x)\right)\varphi(x)dx = \int_M dz\ \rho(z)\left\{\int_{F^{-1}(z)} \frac{\varphi(x)dv_z(x)}{\sqrt{\det\left(dF(x)dF(x)^*\right)}}\right\}. \tag{2.17}$$

Now, we consider the case where $\mathbb{R}^n = H$ and $M = G$ formally. By the definition of the pull-back of the delta function by the Wiener functional I_L (see also [4]), we have for any smooth function ψ on $P(\mathfrak{g})$ in the sense of Malliavin,

$$\int_{P(\mathfrak{g})} \rho(I_L(B)(1))\psi(B)d\mu_\lambda(B) = \int_G da\ \rho(a)\left\{\int_{P(\mathfrak{g})} \psi(B)\delta_a(I_L(B)(1))d\mu_\lambda(B)\right\}, \tag{2.18}$$

where da denotes the Riemannian volume element on G. Formally it holds that

$$d\mu_\lambda(h) = \left(\frac{\lambda}{2\pi}\right)^{\dim H/2} \exp\left(-\frac{\lambda}{2}\int_0^1 |\dot{h}(t)|^2 dt\right) dh, \tag{2.19}$$

where dh is the "Lebesgue measure" on H. By (2.14), we see

$$DI_L(h)(1)DI_L(h)(1)^* = \mathrm{Id}_{T_{I_L(h)(1)}}. \tag{2.20}$$

Therefore, the coarea formula above implies formally that

$$d\mu_{a,\lambda}(B) = \left(\frac{\lambda}{2\pi}\right)^{\dim H/2} c_{\lambda,a} \exp\left(-\frac{\lambda}{2}\int_0^1 |\dot{h}(t)|^2 dt\right) dv_a(h), \tag{2.21}$$

where $dv_a(h)$ denotes the formal induced Riemannian volume on S_a. Take a function ρ on $P_a(G)$. Then

$$\begin{aligned}
\int_{P_a(G)} \rho(\gamma) d\nu_{a,\lambda}(\gamma) &= \int_{S_a} \rho\left(I_L(B)\right) d\mu_{a,\lambda}(B) \\
&= \int_{S_a} \rho\left(I_L(h)\right) \left(\frac{\lambda}{2\pi}\right)^{\dim H/2} c_{\lambda,a} \exp\left(-\frac{\lambda}{2}\int_0^1 |\dot{h}(t)|^2 dt\right) dv_a(h) \\
&= \int_{S_a} \rho\left(I_L(h)\right) \left(\frac{\lambda}{2\pi}\right)^{\dim H/2} c_{\lambda,a} \exp\left(-\frac{\lambda}{2}\int_0^1 |\dot{I}_L(h)(t)|^2 dt\right) dv_a(h) \\
&= \int_{P_a(G)} \rho\left(\gamma\right) \left(\frac{\lambda}{2\pi}\right)^{\dim H/2} c_{\lambda,a} \exp\left(-\frac{\lambda}{2}\int_0^1 |\dot{\gamma}(t)|^2 dt\right) d\gamma_a. \qquad (2.22)
\end{aligned}$$

Here $d\gamma_a$ denotes the induced Riemannian volume element on X_a. In (2.22), we use the isometry property of I_L, that is $I_L^* d\gamma_a = dv_a$. (2.22) implies that

$$d\nu_{a,\lambda}(\gamma) = \left(\frac{\lambda}{2\pi}\right)^{\dim H/2} c_{\lambda,a} \cdot \exp\left(-\frac{\lambda}{2}\int_0^1 |\dot{\gamma}(t)|^2 dt\right) d\gamma_a \tag{2.23}$$

which we want to show. Of course, the above calculation is formal and the Andersson and Driver's type theorem is desired. Note that this expression is not contradicted with their results

$$d\nu_{a,\lambda}(\gamma) = C'_\lambda \cdot \exp\left(-\frac{\lambda}{2}\int_0^1 |\dot{\gamma}(t)|^2 dt\right) d_{LC}\gamma. \tag{2.24}$$

Here $d_{LC}\gamma$ denotes the Riemannian volume element which is induced by the Riemannian metric g_{LC} which is defined by the Levi-Civita connection on G. (Actually they proved it for Brownian motion measure without conditioning $\gamma(1) = a$.) That is, the right-invariant Riemannain metric and g_{LC} is different but it is possible that the induced volume element on total space coincides and in fact it happens in finite dimensional cases too.

3 Calculation of the Hessian of the energy functional and its spectrum

Let

$$E(\gamma) = \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2_{T_{\gamma(t)}G} dt. \tag{3.1}$$

This is a smooth functional on X_a. $Z_h Z_k E(\gamma)$ is a continuous symmetric form on H_0 if γ is a geodesic, namely, $\gamma(t) = e^{tv}$, where $v \in \mathfrak{g}$. Here, we identify H_0 and $T_\gamma X_a$. So there exists a self-adjoint operator Q_v such that $(Q_v h, k)_{H_0} = Z_h Z_k E(\gamma)$. We calculate Q_v explicitly.

Lemma 3.1 (1)

$$\frac{1}{2}\|h_R\|_H^2 = E(\gamma). \tag{3.2}$$

(2) *Let* $\gamma(t) = e^{tv}$. *Then* $Q_v = I_{H_0} + T_v$ *where*

$$(T_v h)(t) = \int_0^t [h(s), v]ds - t\int_0^1 [h(s), v]ds. \tag{3.3}$$

(3) *Let* $\{\xi_i\}_{i=1}^l$ *be the positive eigenvalues of the symmetric operator* $\mathrm{ad}(v)^2$ *on* $\mathfrak{g}$ *counting the multiplicity, where* $\mathrm{ad}(v)u = [v, u]$ *for* $v, u \in \mathfrak{g}$. *Then nonzero eigenvalues of* T_v *are*

$$\left\{\pm\frac{\sqrt{\xi_i}}{2\pi m},\ \pm\frac{\sqrt{\xi_i}}{2\pi m}\ \Big|\ m \in \mathbb{N}, 1 \le i \le l\right\} \tag{3.4}$$

counting the multiplicity.

Proof. (1) This holds because the Riemmanian metric on G is right invariant.

(2) First, let $\gamma \in X_a$. Suppose h is a C^1-path. Below, we calculate as if G is a matrix group. But this is not a restriction. In this case, the exponential map coincides with the exponential of the matrices. Thus, we have for almost all s,

$$\begin{aligned}
\frac{d}{ds}I_R^{-1}\left(e^{\varepsilon h}\gamma\right)(s) &= \frac{d}{ds}\left(e^{\varepsilon h(s)}\right)e^{-\varepsilon h(s)} + Ad\left(e^{\varepsilon h(s)}\right)\dot{h}_R(s) \\
&= \left[\varepsilon\dot{h}(s) + \sum_{n=2}^{\infty}\frac{\varepsilon^n}{n!}\sum_{l=0}^{n-1}h(s)^l\dot{h}(s)h(s)^{n-l-1}\right]e^{-\varepsilon h(s)} \\
&\quad + Ad\left(e^{\varepsilon h(s)}\right)\dot{h}_R(s).
\end{aligned} \tag{3.5}$$

Integrating both sides with respect to s from 0 to t and taking derivatives with respect to ε, we have

$$Z_h h_R = h + \int_0^{\cdot} [h(s), \dot{h}_R(s)]ds. \tag{3.6}$$

That is, $I_R^{-1}(\gamma)$ is a H-valued smooth function. This formula is found in (4.44) in [16]. Using this,

$$Z_h\int_0^1 |\dot{h}_R(t)|^2 dt = 2\int_0^1\left(\dot{h}(t), \dot{h}_R(t)\right)dt + 2\int_0^1\left([h(s), \dot{h}_R(s)], \dot{h}_R(s)\right)ds \tag{3.7}$$

$$= 2\int_0^1\left(\dot{h}(s), \dot{h}_R(s)\right)ds \tag{3.8}$$

Therefore

$$Z_k Z_h E(\gamma) = \int_0^1\left(\dot{h}(s), \dot{k}(s)\right)ds + \int_0^1\left(\dot{h}(s), [k(s), \dot{h}_R(s)]\right)ds. \tag{3.9}$$

If $\gamma(t) = e^{tv}$, then $\dot{h}_R(t) = v$. So this implies (2).

(3) First note that $T_v = -S \cdot \mathrm{ad}(v)$, where S is an operator on H_0 such that $(Sh)(t) = \int_0^t h(s)ds - t\int_0^1 h(s)ds$. Then we have $Se_k = (2\pi k)^{-1}f_k$ and $Sf_k = -(2\pi k)^{-1}e_k$ where $e_k(t) =$

$\sin(2\pi kt)/(\sqrt{2}\pi k)$, $f_k(t) = \frac{\cos(2\pi kt)-1}{\sqrt{2}\pi k}$ and $\{e_k, f_k\}_{k=1}^{\infty}$ constitutes a complete orthonormal system of $H_0^1([0,1] \to \mathbb{R})$. Since $\mathrm{ad}(v)$ is a skewsymmetric operator on $\mathfrak{g}$, there exists an orthonormal system $\{u_i\}_{i=1}^{l}, \{v_i\}_{i=1}^{l}$ and $\{w_i\}_{i=1}^{d-2l}$ such that $\mathrm{ad}(v)u_i = \sqrt{\xi_i}v_i$, $\mathrm{ad}(v)v_i = -\sqrt{\xi_i}u_i$ and $\mathrm{ad}(v)w_j = 0$. Here $d = \dim \mathfrak{g}$. Thus $\{e_k u_i + f_k v_i, e_k v_i - f_k u_i\}$ are eigenfunctions with the eigenvalue $\sqrt{\xi_i}/(2\pi k)$ and $\{e_k u_i - f_k v_i, e_k v_i + f_k u_i\}$ are eigenfunctions with the eigenvalue $-\sqrt{\xi_i}/(2\pi k)$ and $\{e_k w_i, f_k w_i\}_{1 \le i \le d-2l, k \in \mathbb{N}}$ are eigenfunctions whose eigenvalue is 0. This completes the proof. ■

4 The case of $SU(n)$

In this section, let $G = SU(n)$. Then the Lie algebra is $\mathfrak{su}(n)$. Ad-invariant inner product on $\mathfrak{su}(n)$ is given by $(A, B) = \mathrm{tr}AB^*$. We denote the diagonal matrix whose (i,i) element is η_i by $D[\eta_1, \ldots, \eta_n]$. Also we denote the matrix whose (i,j) element is 1 and other elements are 0 by E_{ij}. We take and fix an element $v = D\left[\sqrt{-1}\lambda_1, \ldots, \sqrt{-1}\lambda_n\right] \in \mathfrak{su}(n)$ such that $\sum_{i=1}^n \lambda_i = 0$ and $\lambda_i - \lambda_j \notin 2\pi\mathbb{Z}$. Let

$$a = e^v = D\left[e^{\sqrt{-1}\lambda_1}, \ldots, e^{\sqrt{-1}\lambda_n}\right]. \tag{4.1}$$

We can identify all geodesics joining e and a under the above assumptions.

Lemma 4.1 *Let $A \in \mathfrak{su}(n)$ and assume $e^A = a$. Then, there exists $k_i \in \mathbb{Z}$ with $\sum_{i=1} k_i = 0$ such that*

$$A = D\left[\sqrt{-1}(\lambda_1 + 2\pi k_1), \ldots, \sqrt{-1}(\lambda_n + 2\pi k_n)\right]. \tag{4.2}$$

Proof. Since $A \in \mathfrak{su}(n)$, there exists $U \in U(n)$ and $\eta_i \in \mathbb{R}$ such that

$$U^* A U = \sqrt{-1} D\left[\eta_1, \ldots, \eta_n\right]. \tag{4.3}$$

Then

$$U^* e^A U = D\left[e^{\sqrt{-1}\eta_1}, \ldots, e^{\sqrt{-1}\eta_n}\right]. \tag{4.4}$$

By taking U appropriately, we see that $e^{\sqrt{-1}\eta_j} = e^{\sqrt{-1}\lambda_j}$. So $\eta_j = \lambda_j + 2\pi k_j$. By (4.4), U and $D\left[e^{\sqrt{-1}\lambda_1}, \ldots, e^{\sqrt{-1}\lambda_n}\right]$ are commutative. Since $\left\{e^{\sqrt{-1}\lambda_i}\right\}$ are distinct, U is also a diagonal matrix. By (4.3), we complete the proof. ■

By this lemma, we see that there is one to one correspondence between $V \subset \mathbb{Z}^n$ which consists of $\{k_i\}_{i=1}^n$ with $\sum_{i=1}^n k_i = 0$ and the space of geodesics joining e and a. Let $P_a(G)^{cr}$ denote the subset of geodesics of $P_a(G)$.

Theorem 4.2 *Let $\{\xi_i(c)\}_{i=1}^{\infty}$ be all eigenvalues of the Hessian of $E(\cdot)$ at the geodesic c which corresponds to $\{k_i\} \in V$ counting the multiplicity. Let S be the set which is obtained by removing all 1 from $\{\xi_i(c)\}$.*
(1) *We have*

$$S = \left\{1 \pm \frac{|\lambda_i - \lambda_j + 2\pi(k_i - k_j)|}{2m\pi}, 1 \pm \frac{|\lambda_i - \lambda_j + 2\pi(k_i - k_j)|}{2m\pi} \;\Big|\; m \in \mathbb{N}, 1 \le i, j \le n\right\}. \tag{4.5}$$

In particular, a is not a conjugate point of e along any geodesics and the multiplicity of all eigenvalues except 1 are even number.

(2) *Let $l(c)$ be the length of c corresponding to $v = \{k_i\}_{i=1}^n \in V$. Then $l(c) = \left\{\sum_{i=1}^n (\lambda_i + 2\pi k_i)^2\right\}^{1/2}$.*
(3) *Let $p \in \mathbb{N}$ and for a geodesic $c \in P_a(G)^{cr}$, set*

$$\theta_p(c) = \inf_{I \subset \mathbb{N}, |I|=p} \sum_{i \in I, \xi_i(c)>0} |\xi_i(c)| + \sum_{i \notin I, \xi_i(c)<0} |\xi_i(c)|. \tag{4.6}$$

If p is odd, then

$$\inf_{c \in P_a(G)^{cr}} \theta_p(c) > 0. \tag{4.7}$$

Proof. (1) In this case, we can see the eigenvalues and eigenvectors explicitly. Let $e_{ij} = \sqrt{2}^{-1}(E_{ij} - E_{ji})$, $f_{ij} = -\sqrt{-2}^{-1}(E_{ij} + E_{ji})$ and $g_k = \sqrt{-2}^{-1}(E_{11} - E_{kk})$ $(2 \le k \le n)$. These constitute a complete orthonormal system. Let $A = D\left[\sqrt{-1}(\lambda_1 + 2\pi k_1), \ldots, \sqrt{-1}(\lambda_n + 2\pi k_n)\right]$. Then $\mathrm{ad}(A)e_{ij} = \{(\lambda_i - \lambda_j) + 2\pi(k_i - k_j)\} f_{ij}$, $\mathrm{ad}(A)f_{ij} = -\{(\lambda_i - \lambda_j) + 2\pi(k_i - k_j)\} e_{ij}$, $\mathrm{ad}(A)g_k = 0$ hold. Hence, (4.5) is a consequence of Lemma 3.1 (3). Since $\lambda_i - \lambda_j \notin 2\pi\mathbb{Z}$, second assertion follows from (4.5). The third assertion also follows immediately from (4.5).

(2) This follows from the definition of the Riemannian metric.

(3) The results in (1) imply that $\theta_p(c) > 0$ for odd p. Thus, it is sufficient to prove that

$$\lim_{l(c) \to \infty} \theta_p(c) = \infty, \tag{4.8}$$

since for any R, the number of geodesics whose length are less than R is finite. Take a positive number R_0 such that $\left\{\sum_{1 \le i \le n} |\lambda_i|^2\right\}^{1/2} \le R_0$. By (2), for any large number R, except finite number of geodesics, $l(c) \ge R$ holds. For these geodesics, it holds that $\left(\sum_{1 \le i \le n} (2\pi k_i)^2\right)^{1/2} \ge R - R_0$. So $|2\pi k_*| := \max_{1 \le i \le n} |2\pi k_i| \ge \frac{R - R_0}{n}$. Since $\sum_{i=1}^n k_i = 0$, there exists at least one k_i whose sign is different from k_*'s. So it holds that $\max_{1 \le i < j \le n} |2\pi(k_i - k_j)| \ge \frac{R-R_0}{n} + 2\pi$. Let us take a positive number K. The above results and (4.5) imply that the number of negative eigenvalues of E at c less than $-K$ diverges uniformly when $l(c) \to \infty$. This proves (4.8) and completes the proof. ■

Remark 4.3 (1) (2) implies that the set of critical points is an isolated set in X_a.
(2) Note that the Ricci curvature of $SU(n)$ is positive. Let $m(c)$ be the index at c, that is the total number of the negative eigenvalues of the Hessian at c. By the Morse theory [27], we see that $\lim_{l(c) \to \infty} m(c) = \infty$ for a compact Riemannian manifold with positive Ricci curvature. Therefore (4.8) might hold in such cases.

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