Semiclassical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space

Dedicated to Professor Tokuzo Shiga on the occasion of his sixtieth birthday

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Abstract

We study a semiclassical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space. Key results are semiboundedness theorem of the Schrödinger operator, Laplace-type asymptotic formula and IMS localization formula. We also make a remark on semiclassical problem of a Schrödinger operator on a path space over a Riemannian manifold.

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1 Introduction

The Schrödinger operator $H = -\hbar^2\Delta + U$ stands for the Hamiltonian of quantized finitely many particles moving in the potential U in a Euclidean space. The quantum phenomena are reduced to the corresponding classical ones under the semiclassical limit $\hbar \to 0$ and there have been many researches on the analysis. However, to the author's knowledge, there are not so many mathematically rigorous results in semiclassical analysis in models in infinite dimensional spaces as well as quantum field models. Here I note just a few pioneering works, Sjöstrand [30, 31], Arai [6], Dobrokhotov and Kolokoltsov [11]. In this paper, we consider infinite dimensional Schrödinger-type operators and study the asymptotics of the bottom of the spectrum under the semiclassical limit.

First, let us recall finite dimensional results. We consider $H_\lambda = -\Delta + \lambda^2 U$ instead of $-\hbar^2\Delta + U$, where λ is a large parameter. Here we assume that U is a nonnegative smooth function and $\liminf_{|x|\to\infty} U(x) = \infty$ for simplicity. The lowest eigenvalue $E_0(\lambda)$ of H_λ is the ground state energy and it is a basic problem to study the behavior of $E_0(\lambda)$ under the semiclassical limit $\lambda \to \infty$. Intuitively, the ground state localizes near the neighborhood of the minima of U as $\lambda \to \infty$. Then by approximating U by quadratic functions near minima, we obtain a family of quantum Hamiltonians of harmonic oscillators. The divergence order of $E_0(\lambda)$ is determined by these operators. This result is proved rigorously by Combes, Duclos and Seiler [9]

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and Simon [27]. Actually, they determined the divergence order of the nth eigenvalue $E_n(\lambda)$ for all n. Previous to these works, the degeneracy problem of the first and the second eigenvalues which is related to tunneling phenomena were studied by Harrell [14] and Jona-Lasinio et al. [18]. Further studies were made by Simon [28], Helffer and Sjöstrand[17]. On the other hand, Witten [32] considered a supersymmetric Hamiltonian and proved the Morse inequality by using the semiclassical behavior. More precisely, let S be a Morse function on a compact Riemannian manifold X and consider the Witten complex $d_\lambda = e^{-\lambda S/2} d e^{\lambda S/2}$ which is defined on L^2-space of differential forms on X with respect to the Riemannian volume. Let d_λ^* be the adjoint operator and set $\Box_\lambda = d_\lambda d_\lambda^* + d_\lambda^* d_\lambda$ which is called the Witten Laplacian. By a consequence of semiclassical analysis, the number of eigenvalues of $\Box_\lambda$ acting on p-forms which remain finite under $\lambda \to \infty$ is dominated by the number of critical points of S whose indices are p. This implies the Morse inequality. Also in the same paper, Witten proposed studying the corresponding problem in the case of quantum field theory, that is, in infinite dimensional cases. Typical examples of infinite dimensional manifolds are loop spaces, pinned path spaces over a compact Riemannian manifold. There exist natural probability measures, the Brownian bridge measures, on them and many probabilists have been interested in proving a Hodge-Kodaira-type theorem on loop spaces by using the measures. The Brownian bridge measure on the pinned path space with a constraint $\gamma(0) = x, \gamma(1) = y$ is formally written as $d\nu_{x,y,\lambda}(\gamma) = Z_\lambda^{-1} \exp\left(-\lambda S(\gamma)\right) d\gamma$, where $S(\gamma) = \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2 dt$, Z_λ is a normalizing constant and $d\gamma$ denotes a fictitious Riemannian volume on the pinned path space. Note that the energy function $S(\gamma)$ is a Morse function for certain x and y. The exterior differential operator d and the adjoint operator $d^*_{\nu_{x,y,\lambda}}$ and $\Box_{x,y,\lambda} = d d^*_{\nu_{x,y,\lambda}} + d^*_{\nu_{x,y,\lambda}} d$ can be defined as closable operators on L^2 space with respect to $d\nu_{x,y}$ for some cases. Formally, by using the unitary transformation, $\Phi_\lambda : \alpha \to Z_\lambda^{-1/2} \exp\left(-\frac{\lambda}{2} S(\gamma)\right) \cdot \alpha$ between $L^2(d\nu_{x,y})$ and $L^2(d\gamma)$, $\Box_{x,y,\lambda}$ is unitarily equivalent to the formal corresponding Witten Laplacian. Some discussions on this topic can be found in [4]. I think that the study of the semiclassical behavior of $\Box_{x,y,\lambda}$ is interesting from a mathematical point of view although this is not related with physical model directly. This is one motivation to study semiclassical analysis in infinite-dimensional spaces.

The Schrödinger-type operator studied in this paper is a self-adjoint operator, $-L_{\lambda,V}$ on an abstract Wiener space which has a physical meaning. In fact, it is a perturbed Hamiltonian on an abstract Boson Fock space whose one-particle Hamiltonian A is identity. Let (B, H, μ) be an abstract Wiener space and L be the Ornstein-Uhlenbeck operator on $L^2(B, \mu)$. Let V be a Borel measurable function on B and set $V_\lambda(\phi) = \lambda V\left(\lambda^{-1/2}\phi\right)$. $L_{\lambda,V}$ is given by $-L_{\lambda,V} = -L + V_\lambda$ on $L^2(B, \mu)$. The aim of this paper is to determine the divergence order of $E_0(\lambda) = \inf \sigma(-L_{\lambda,V})$ when $\lambda \to \infty$. The semiclassical problem for the operator $-L + \lambda^2 V$ was studied in [3]. We note that Arai [6] studied the semiclassical limit of the partition function of the Hamiltonian in the case where A^{-1} is a compact operator which fits in with $P(\phi)$ type model on a finite volume region. In this paper, we consider the case where $A = I$ only but the study of general cases might be much more interesting.

To explain the meaning of the scaling of V, let us consider the case where H is a d-dimensional Euclidean space. In this case, $d\mu(x) = \varphi^2(x)dx$, where $\varphi(x) = \left(\frac{1}{2\pi}\right)^{d/4} \exp(-|x|^2/4)$. Then $-\lambda L_{\lambda,V}$ is unitarily equivalent to the Schrödinger operator on $L^2(\mathbb{R}^d, dx)$ such that

$$-H_{\lambda,V} = -\Delta + \lambda^2 \left(\frac{|x|^2}{4} + V(x)\right) - \frac{d}{2}\lambda. \tag{1.1}$$

The unitary transformation is given by $-\lambda L_{\lambda,V} = M_{\varphi}^{-1} S_{\lambda}^{-1}(-H_{\lambda,V}) S_{\lambda} M_{\varphi}$. Here, $M_{\varphi} f = \varphi \cdot f$ and $S_{\lambda} f(x) = \lambda^{d/4} f\left(\lambda^{1/2} x\right)$. Therefore, $-\lambda L_{\lambda,V}$ on $L^2(B,\mu)$ is formally unitarily equivalent to the infinite-dimensional Hamiltonian $-\Delta + \lambda^2 \left(\frac{1}{4}\|\phi\|_H^2 + V(\phi)\right) - \frac{\lambda}{2} \dim H$ on $L^2(H, d\phi)$, where $d\phi$ denotes "the Lebesgue measure" on H. Now we explain the semiclassical behavior of the lowest eigenvalue of $H_{\lambda,V}$ in finite dimensions more precisely. Assume that the minima of $U(x) = \frac{|x|^2}{4} + V(x)$ form a finite set $\{h_1, \ldots, h_n\}$ and take the minimum value 0. Also let us denote the Hessian of $\frac{1}{2}V(x)$ at h_j by K_j. Assume that the Hessian of $U(x)$ at h_j, $\frac{1}{4}I + K_j$ are nondegenerate for all $1 \le j \le n$ and $\liminf_{|x|\to\infty} U(x) > 0$.

Then by the previous mentioned works,

$$\lim_{\lambda\to\infty} E_0(\lambda) = \frac{1}{2} \min_{1\le j\le n} \operatorname{tr}\left(\sqrt{I + 4K_j} - I\right). \tag{1.2}$$

We will prove the same asymptotics in infinite-dimensional cases.

The organization of this paper is as follows. In Section 2, we will introduce assumptions on a potential function V and define a Schrödinger operator $-L_{\lambda,V}$. Note that the assumptions are standard ones in finite-dimensional cases which we already mentioned. Also we recall the semiboundedness theorem of the Schrödinger operator which is called the GNS(=Glimm, Nelson and Segal) bound [26]. After that, main theorem (Theorem 2.6) will be stated. In Section 3, we approximate $U(\phi) = \frac{1}{4}\|\phi\|_H^2 + V(\phi)$ by quadratic functions near the minimizers and obtain a family of approximate Schrödinger operators. By using the ground states of them as trial functions, we will prove the upper bound estimate of $E_0(\lambda)$ in Section 4. In Section 5, we will prove a rough lower bound estimate $\liminf_{\lambda\to\infty} E_0(\lambda) > -\infty$. Most of part of our proof of main theorem proceeds in the parallel way as the finite dimensional cases as in [27]. However this rough estimate as well as precise lower bound estimate are nontrivial because our "true" potential function is $U(\phi)$ and a part of it is hidden in the measure μ. Also there is a renormalized part $-\frac{\lambda}{2} \dim H$. These difficulties are overcome by using Schilder's classical Laplace asymptotic formula and the GNS bound. Note that the scaling $\lambda V(\phi/\sqrt{\lambda})$ is meaningless on function spaces over curved space. Therefore, we will explain another unitarily equivalent representation of $-H_{\lambda,V}$ in infinite dimensions. We will consider such kind of operator in Remark 5.3 in Section 5. See also [4]. In Section 6, we will prove the lower bound estimate by using the rough estimate in Section 5 and IMS(=Ismagilov, Morgan, Sigal, Simon) localization formula [27]. In Section 7, we will present examples. We might expect tunneling phenomena in such examples.

In this paper, we consider differentiable potential functions in the sense of Fréchet. Note that in our analysis, we need some continuity property of the potential function. In a recent preprint [5], I make use of Lyons' continuity theorem [23] of solutions of stochastic differential equations in a problem in infinite dimensional spaces which have difficulties coming from the discontinuity of the solutions with respect to usual topologies of Wiener spaces. I think that such kind of regularity properties play important role in the semiclassical analysis on path spaces over Riemannian manifolds.

2 Preliminaries and main result

Let (B, H, μ) be an abstract Wiener space. That is, B is a real separable Banach space and H is a real separable Hilbert space continuously embedded in B. μ is the unique Gaussian measure on B satisfying that for all $h \in B^*$,

$$\int_B \exp\left[\sqrt{-1}_{B^*}(h,\phi)_B\right] d\mu(\phi) = \exp\left(-\frac{1}{2}\|h\|_H^2\right).$$

Here we use the natural embedding and the identification, $H \simeq H^* \supset B^*$. Let us introduce the following assumptions on potential functions on B.

Assumption 2.1 Let V be a continuous function on B and set $V_\lambda(\phi) = \lambda V(\phi/\sqrt{\lambda})$.
(A1) Let $U(\phi) = \frac{1}{4}\|\phi\|_H^2 + V(\phi)$. Then $\min_{\phi\in H} U(\phi) = 0$ and the minima form a finite set $N = \{h_1, \ldots, h_n\}$.
(A2) V is a C^2 function in a neighborhood of N in B. (Then the symmetric operator $K_i = \frac{1}{2}D^2V(h_i)$ is a trace class operator on H for all i. See Theorem 4.6 in page 83 in [19]). Assume that $\frac{1}{4}I_H + K_i$ is a strictly positive self-adjoint operator.

For $\phi \in H$, let

$$R_i(\phi) = U(\phi) - \frac{1}{2}D^2U(h_i)(\phi - h_i, \phi - h_i). \tag{2.1}$$

Then actually R_i is a continuous function on B. See Lemma 2.3 (1). We consider assumptions which depend on positive numbers R and ε.
(A3(ε, R)) There exists a constant $\xi(R)$ such that

$$|R_i(\phi)| \le \xi(R)\|\phi - h_i\|_B^{2+\varepsilon} \qquad \text{for } \|\phi\|_B < R. \tag{2.2}$$

(A3(ε)) (A3(ε, R)) holds for all $R > 0$.
(A4) For any $\varepsilon > 0$ and $R > 0$, we have

$$\inf\{U(\phi) \mid d_B(\phi) \ge \varepsilon, \|\phi\|_B \le R, \phi \in H\} =: \theta(\varepsilon, R) > 0, \tag{2.3}$$

where $d_B(\phi) = \min\{\|\phi - h_i\|_B \mid 1 \le i \le n\}$.
(A5) For any $\lambda > 0$,

$$V_\lambda \in L^1(B, \mu). \tag{2.4}$$

(A6) There exists a positive number $\alpha > 2$ such that

$$\limsup_{\lambda\to\infty} \lambda^{-1} \log E\left[e^{-\alpha V_\lambda(\phi)}\right] < \infty. \tag{2.5}$$

Remark 2.2 (1) Actually (A4) follows from (A1) if we take another Banach space B_0 instead of B. We explain it. There exists a separable Banach space B_0 such that $H \subset B_0 \subset B$ and the inclusion maps are compact. Furthermore $\mu(B_0) = 1$ and (B_0, H, μ) itself is an abstract Wiener space. Define $\theta'(\varepsilon, R)$ by replacing B by B_0 in the definition of $\theta(\varepsilon, R)$. Then we have $\theta'(\varepsilon, R) > 0$ for any $\varepsilon, R > 0$. Assume $\theta'(\varepsilon, R) = 0$. Then there exists $\{\phi_n\}$ such that $\sup_n \|\phi_n\|_{B_0} \le R$, $d_{B_0}(\phi_n) \ge \varepsilon$ and $\lim_{n\to\infty}\left\{\frac{1}{4}\|\phi_n\|_H^2 + V(\phi_n)\right\} = 0$. Then there exists a subsequence $\{\phi_{n(k)}\}$ such that $\lim_{k\to\infty}\phi_{n(k)} = \psi$ in the norm of B. So $\lim_{k\to\infty} V(\phi_{n(k)})$ exists. So $\sup_k \|\phi_{n(k)}\|_H < \infty$. Taking subsequence $\{\phi_{m(k)}\}$ again, we see that $\lim_{k\to\infty}\phi_{m(k)} = \psi$ in B_0, strongly and $\lim_{k\to\infty}\phi_{m(k)} = \psi$ in H, weakly. Since $d_{B_0}(\psi) \ge \varepsilon$ and $U(\psi) = 0$, this contradicts with (A1). For simplicity, we assume (A4).
(2) If V is a C^3-function on B, then by replacing B by B_0 in (1), the assumption (A3(1)) holds.
(3) Number 2 in (A6) has a special meaning. It is the same number 2 which will appear in logarithmic Sobolev inequality (2.9) and GNS bound in Lemma 2.4. We need (A6) for the Large deviation result in the proof of Lemma 5.2.

We will give examples of potential functions in Section 7. As a consequence of the assumptions, we have the following lemma.

Lemma 2.3 (1) *For all* $\phi \in H$,

$$R_i(\phi) = V(\phi) - V(h_i) - DV(h_i)(\phi - h_i) - \frac{1}{2}D^2V(h_i)(\phi - h_i, \phi - h_i). \tag{2.6}$$

In particular, $R_i(\phi)$ *can be extended to a continuous function on* B.
(2) $h_i \in B^*$ $(1 \le i \le n)$.

Proof. Since h_i is a critical point of U, we have for all $\phi \in H$,

$$\frac{1}{2}(\phi, h_i) + DV(h_i)(\phi) = 0. \tag{2.7}$$

This implies $h_i \in B^*$. Since $\frac{1}{4}\|\phi\|_H^2$ is a quadratic function,

$$\begin{aligned} R_i(\phi) &= U(\phi) - U(h_i) - DU(h_i)(\phi - h_i) - \frac{1}{2}DU^2(h_i)(\phi - h_i, \phi - h_i) \\ &= V(\phi) - V(h_i) - DV(h_i)(\phi - h_i) - \frac{1}{2}D^2V(h_i)(\phi - h_i, \phi - h_i). \end{aligned} \tag{2.8}$$

Since the right-hand side is a continuous function on B, (2.6) holds for all $\phi \in B$. ■

Here we recall the definition of the Schrödinger operator $-L_{\lambda,V} = -L + V_\lambda$. Before doing so, we recall semi-boundedness theorem (Lemma 2.4). This estimate is standard in quantum field theory and it is proved by using the hypercontractivity of the Ornstein-Uhlenbeck semigroup or the equivalent logarithmic Sobolev inequality:

$$\int_B u(\phi)^2 \log\left(u(\phi)^2/\|u\|_{L^2(\mu)}^2\right) d\mu(\phi) \le 2\int_B |Du(\phi)|_H^2 d\mu. \tag{2.9}$$

We refer the readers to Theorem 7 in [13] for the proof.

Lemma 2.4 (GNS bound) *Assume that* $E[e^{-2V_\lambda}] < \infty$ *and* $V_\lambda \in L^1(B, \mu)$. *Consider a densely defined symmetric form:*

$$\mathcal{D} := \mathbb{D}_2^1(B) \cap L^\infty(B, \mu), \tag{2.10}$$

$$\mathcal{E}_\lambda(u, v) := \int_B (Du(\phi), Dv(\phi))_H d\mu(\phi) + \int_B V_\lambda(\phi)u(\phi)v(\phi)d\mu(\phi) \quad (u, v \in \mathcal{D}). \tag{2.11}$$

Then $\mathcal{E}_\lambda$ *is a closable form such that, for any* $u \in \mathcal{D}$ *with* $\|u\|_{L^2(\mu)} = 1$,

$$\mathcal{E}_\lambda(u, u) \ge -\frac{1}{2}\log\left(\int_B e^{-2V_\lambda(\phi)}d\mu(\phi)\right). \tag{2.12}$$

From now on, we always assume that V_λ satisfies the assumptions in the lemma above.

Definition 2.5 *Let* $-L_{\lambda,V}$ *be the semi-bounded self-adjoint operator which corresponds to the smallest closed extension of* $\mathcal{E}_\lambda$. *Let* $E_0(\lambda) = \inf \sigma(-L_{\lambda,V})$.

It is easy to see that the domain of $\mathcal{E}_\lambda$ is nothing but $\mathbb{D}_2^1(B)\cap L^2(|V_\lambda|\mu)$. See [2]. Actually it is known that $E_0(\lambda)$ is a simple eigenvalue and the corresponding eigenfunction is called the ground state. However we do not use these general results in this paper. We use the existence of the ground state for the quadratic approximate Schrödinger operator in Section 3. In that case, we have the explicit expression of the ground state. The estimate in Lemma 2.4 and Laplace asymptotic formula implies $\liminf_{\lambda\to\infty} E_0(\lambda) > -\infty$. This rough estimation is a first step to prove the precise asymptotics below which is a main result of this paper.

Theorem 2.6 *Assume* (A1),(A2), (A3(ε)),(A4),(A5),(A6). *Then*

$$\lim_{\lambda\to\infty} E_0(\lambda) = \frac{1}{2}\min_{1\le i\le n} \mathrm{tr}\left(\sqrt{I_H+4K_i}-I_H\right). \tag{2.13}$$

We will use the following Fernique's inequality several times:
There exist positive constants C and C' such that for all $R>0$,

$$\mu\left(\|\phi\|_B \ge R\right) \le Ce^{-C'R^2}. \tag{2.14}$$

3 Approximate Schrödinger operators

Let K be a trace class self-adjoint operator on H and assume that I_H+4K is strictly positive. For $h\in H$, let us consider a Schrödinger operator:

$$-L_{h,K} = -L - \frac{1}{2}(\phi,h) + \frac{1}{4}\|h\|_H^2 + \left(K(\phi-h),(\phi-h)\right)_H. \tag{3.1}$$

Let

$$\begin{aligned}\varphi_{h,K}(\phi) &= \det(I_H+4K)^{1/8}\exp\left\{\left(-\frac{1}{4}\left(\sqrt{I_H+4K}-I_H\right)(\phi-h),(\phi-h)\right)\right\}\\ &\quad\times\exp\left\{\left(\frac{1}{2}(\phi,h)-\frac{1}{4}\|h\|_H^2\right)\right\}\end{aligned} \tag{3.2}$$

and $f_{h,K}(\phi)=\log\varphi_{h,K}(\phi)$.

Lemma 3.1 *Let $\mu_{h,K}$ be the Gaussian measure on B whose covariance operator is $(I_H+4K)^{-1/2}$ and mean is h. Then we have the following.*
(1) *$\mu_{h,K}$ and μ are equivalent to each other and the density is given by $\frac{d\mu_{h,K}}{d\mu}(\phi)=\varphi_{h,K}(\phi)^2$. Moreover, there exists a positive number δ such that $\varphi_{h,K}\in L^{2+\delta}(B,\mu)$.*
(2) *$\varphi_{h,K}$ is in the domain of $-L$ and it holds that*

$$-L_{h,K}\varphi_{h,K} = \frac{1}{2}\mathrm{tr}\left(\sqrt{I_H+4K}-I_H\right)\varphi_{h,K} \tag{3.3}$$

Proof. (1) Let us consider the linear transformation $S\phi=\phi+\left((I_H+4K)^{1/4}-I_H\right)\phi$. Note that S is an invertible operator. Let $\tilde{\mu}_S$ be the image measure of μ by the map $\phi\to S^{-1}\phi$. Then, for any $k\in H$,

$$\int_B (\phi,k)_H^2 d\tilde{\mu}_S = \left(S^{-2}k,k\right)_H = \left((I_H+4K)^{-1/2}k,k\right). \tag{3.4}$$

So $\tilde{\mu}_S = \mu_{0,K}$. Therefore, $\mu_{h,K}$ is the image measure of μ by the map $\phi \to S^{-1}\phi + h$. By the transformation law of the Gaussian measure (see Theorem 5.4 in page 141 in [19]), we obtain the desired results. Now we prove the latter part. Note that

$$-\frac{1}{4}\left(\sqrt{I_H + 4K} - I_H\right) \quad = \quad -K(\sqrt{I_H + 4K} + I_H)^{-1}. \tag{3.5}$$

Since $I_H + 4K$ is strictly positive, $\frac{1+\eta}{2}\left(\sqrt{I_H + 4K} - I_H\right) + \frac{1}{2}I_H$ is also a strictly positive operator for sufficiently small positive η. This implies $\varphi_{h,K} \in L^{2(1+\eta)}(B,\mu)$.

(2) By $\varphi_{h,K} \in L^{2+\delta}(B,\mu)$, it is easy to see that $\varphi_{h,K}$ is in the domain of $-L$ and

$$\begin{aligned} -L\varphi_{h,K} &= \left(-Lf_{h,K} - |Df_{h,K}|^2\right)\varphi_{h,K} \\ &= \left(\frac{1}{2}(\phi,h)_H - \frac{1}{4}\|h\|_H^2 - (K(\phi-h),(\phi-h))_H\right)\varphi_{h,K} \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\sqrt{I_H+4K} - I_H\right)\varphi_{h,K}. \end{aligned} \tag{3.6}$$

■

Since $\varphi_{h,K}(\phi) > 0$ for almost all ϕ, $\varphi_{h,K}$ is the ground state of $-L_{h,K}$. $-L_{h,K}$ is an approximate operator of more general Schrödinger operator. Indeed, $-L_{k_i,K_i}$ is the quadratic approximate Schrödinger operator of $-L_{\lambda,V}$ at $k_i := \sqrt{\lambda}h_i$, where $h_i \in N$ and $K_i = \frac{1}{2}D^2V(h_i)$. To be more precise, we prove the following lemma.

Lemma 3.2 (1) *For any smooth cylindrical function f, it holds that*

$$-L_{\lambda,V}f = -Lf + \left(\frac{1}{4}\|k_i\|_H^2 - \frac{1}{2}(\phi,k_i)_H + (K_i(\phi-k_i),(\phi-k_i))_H\right)f + \lambda R_i\left(\lambda^{-1/2}\phi\right)f. \tag{3.7}$$

(2) *Let $g \in \mathrm{D}(\mathcal{E}_\lambda)$. Moreover, we assume that Dg and g belong to $L^{2+\delta}(B,\mu)$ for some $\delta > 0$. Then we have*

$$\begin{aligned} \mathcal{E}_\lambda(g,g) &= \int_B |Dg(\phi) - g(\phi)Df_{k_i,K_i}(\phi)|_H^2 d\mu(\phi) + \lambda\int_B R_i(\lambda^{-1/2}\phi)g(\phi)^2 d\mu(\phi) \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\sqrt{I_H+4K_i} - I_H\right)\int_B g(\phi)^2 d\mu(\phi). \end{aligned} \tag{3.8}$$

Proof. (1) By (2.6),

$$\begin{aligned} V\left(\frac{\phi}{\sqrt{\lambda}}\right) &= V(h_i) + DV(h_i)(\lambda^{-1/2}\phi - h_i) + \frac{1}{2}D^2V(h_i)\left((\lambda^{-1/2}\phi - h_i), \lambda^{-1/2}\phi - h_i\right) \\ &\quad + R_i(\lambda^{-1/2}\phi) \\ &= V(h_i) + \frac{1}{2}\|h_i\|_H^2 - \frac{1}{2}\left(\frac{\phi}{\sqrt{\lambda}}, h_i\right)_H + \lambda^{-1}(K_i(\phi-k_i),(\phi-k_i))_H \\ &\quad + R_i(\lambda^{-1/2}\phi). \end{aligned} \tag{3.9}$$

In (3.9), we have used that for all $\phi \in B$,

$$-\frac{1}{2}\left(\lambda^{-1/2}\phi, h_i\right)_H = -\frac{1}{2}\|h_i\|_H^2 + DV(h_i)\left(\lambda^{-1/2}\phi - h_i\right)$$

which follows from (2.7). Thus, noting $\frac{1}{4}\|h_i\|_H^2 + V(h_i) = 0$, we get the desired result.

(2) We have

$$\begin{aligned}
&\int_B |Dg(\phi) - g(\phi)Df_{k_i,K_i}(\phi)|_H^2 d\mu(\phi) \\
&\quad = \int_B |Dg(\phi)|_H^2 d\mu(\phi) - \int_B \left(Df_{k_i,K_i}(\phi), D(g(\phi)^2)\right)_H d\mu(\phi) + \int_B |Df_{k_i,K_i}(\phi)|_H^2 g(\phi)^2 d\mu(\phi) \\
&\quad = \int_B |Dg(\phi)|_H^2 d\mu(\phi) + \int_B \left(Lf_{k_i,K_i}(\phi) + |Df_{k_i,K_i}(\phi)|_H^2\right) g(\phi)^2 d\mu(\phi) \\
&\quad = \int_B |Dg(\phi)|_H^2 d\mu(\phi) + \int_B \left(\frac{1}{4}\|k_i\|_H^2 - \frac{1}{2}(\phi, k_i)_H + (K_i(\phi - k_i), (\phi - k_i))_H\right) g(\phi)^2 d\mu(\phi) \\
&\qquad -\frac{1}{2}\mathrm{tr}\left(\sqrt{I_H + 4K_i} - I_H\right)\int_B g(\phi)^2 d\mu(\phi). \qquad (3.10)
\end{aligned}$$

This and (3.7) imply (3.8).

■

4 Upper bound estimate

We prove that

Lemma 4.1 *Assume* (A1), (A2), (A3(ε)), (A5). *Then it holds that*

$$\limsup_{\lambda\to\infty} E_0(\lambda) \le \min_{1\le i\le n} \frac{1}{2}\mathrm{tr}\left(\sqrt{I_H + 4K_i} - I_H\right). \qquad (4.1)$$

Proof. Take a positive number ε' such that $(2+\varepsilon)(1-\varepsilon') > 2$. By Fernique's theorem, there exists $C > 0$ such that

$$\mu_{k_i,K_i}\left(\|\phi - k_i\|_B \ge \lambda^{\frac{\varepsilon'}{2}}\right) = \mu_{0,K_i}\left(\|\phi\|_B \ge \lambda^{\varepsilon'/2}\right) \le e^{-C\lambda^{\varepsilon'}}. \qquad (4.2)$$

Let χ be a C^∞ function such that $\chi(t) = 1$ for $|t| \le 1$ and $\chi(t) = 0$ for $|t| \ge 2$. Let us take a trial function $\Phi_\lambda(\phi) = C_\lambda \varphi_{k_i,K_i}(\phi)\chi(\|\phi - k_i\|_B^2 \lambda^{-\varepsilon'})$. Here C_λ is the normalizing constant such that $\|\Phi_\lambda\|_{L^2(\mu)} = 1$. By (4.2), $\lim_{\lambda\to\infty} C_\lambda = 1$. Note that (3.8) holds by replacing g by Φ_λ. By Fernique's inequality,

$$\begin{aligned}
&\int_B |D\Phi_\lambda(\phi) - \Phi_\lambda(\phi)Df_{k_i,K_i}(\phi)|_H^2 d\mu(\phi) \\
&\quad \le C\int_B \chi'(\|\phi - k_i\|_B^2\lambda^{-\varepsilon'})^2 \lambda^{-2\varepsilon'}\|\phi - k_i\|_B^2 d\mu_{k_i,K_i}(\phi) \le e^{-C'\lambda^{\varepsilon'}}. \qquad (4.3)
\end{aligned}$$

Now we estimate $\lambda\left(R_i(\lambda^{-1/2}\phi)\Phi_\lambda(\phi), \Phi_\lambda(\phi)\right)_{L^2(\mu)}$.

$$\begin{aligned}
&\left|\lambda(R_i(\lambda^{-1/2}\phi)\Phi_\lambda(\phi), \Phi_\lambda(\phi))_{L^2(\mu)}\right| \\
&\quad \le \lambda\int_{\|\lambda^{-1/2}\phi - h_i\|_B \le \sqrt{2}\lambda^{(-1+\varepsilon')/2}} R_i(\lambda^{-1/2}\phi)\Phi_\lambda(\phi)^2 d\mu(\phi) \qquad (4.4) \\
&\quad \le C''\lambda\cdot\lambda^{\frac{(\varepsilon'-1)(2+\varepsilon)}{2}}. \qquad (4.5)
\end{aligned}$$

In (4.4), we have used the support property of χ. In (4.5), we have used (2.2). This completes the proof. ■

5 Rough lower bound estimate

In this section, by combining Lemma 2.4 and the Laplace asymptotic formula, we give a lower bound on $E_0(\lambda)$.

Lemma 5.1 (Rough lower bound) *Suppose* (A1), (A2), (A4), (A5), (A6). *Let* $R > \max_{1\le i\le n}\|h_i\|_B$ *and assume* (A3(ε, R)). *Let* $\rho_{\kappa,B}(\phi) = \kappa\min(d_B(\phi),1)^2$, *where* κ *is a nonnegative number.*
(1) *There exists a positive number* κ *satisfying the following:*

(a) $\min\{U(\phi) - \rho_{\kappa,B}(\phi) \mid \phi \in H\} = 0$,

(b) *The zero point set of* $U - \rho_{\kappa,B}$ *is* $N = \{h_1, \ldots, h_n\}$,

(c) *For any* $\varepsilon > 0$ *and* $R > 0$, $\inf\{U(\phi) - \rho_{\kappa,B}(\phi) \mid d_B(\phi) \ge \varepsilon, \|\phi\|_B \le R, \phi \in H\} > 0$,

(d)

$$\int_B e^{-2(K_i\phi,\phi)+3\kappa\|\phi\|_B^2} d\mu(\phi) < \infty \qquad \textit{for all } 1 \le i \le n. \tag{5.1}$$

(2) *Here we assume that* R *is sufficiently large positive number. Let* κ *be a nonnegative number satisfying the assumptions in* (1). *Let* $E_{0,\kappa}(\lambda)$ *be the lowest eigenvalue of the Schrödinger operator* $-L + V_\lambda(\phi) - \lambda\rho_{\kappa,B}(\lambda^{-1/2}\phi)$. *Then it holds that*

$$\liminf_{\lambda\to\infty} E_{0,\kappa}(\lambda) \ge -\frac{1}{2}\log\left(\sum_{i=1}^n \int_B \exp\Big[-2(K_i\phi,\phi) + 2\kappa\|\phi\|_B^2\Big] d\mu(\phi)\right) > -\infty. \tag{5.2}$$

Proof of Lemma 5.1 (1) Note that $\int_B \exp\Big[-2(K_i\phi,\phi)\Big]d\mu(\phi) = \det(I_H + 4K_i)^{-1/2} < \infty$. So by Fernique's theorem, (5.1) holds for sufficiently small κ. We prove (a), (b) and (c) for sufficiently small κ. Let M_B be a number such that $\|\phi\|_B \le M_B\|\phi\|_H$ for all $\phi \in H$ and ν be the minimum of the bottoms of the spectrum of $\frac{1}{4}I_H + K_i$ for $1 \le i \le n$. Let δ be a positive number such that $\delta < \min\left(\frac{1}{2}\min_{i\ne j}\|h_i - h_j\|_B, 1\right)$. By the definition of R_i, for $\phi \in H$ with $\|\phi - h_i\|_B \le \delta$, it holds that

$$\begin{aligned} U(\phi) - \rho_{\kappa,B}(\phi) &= \left(\left(\frac{1}{4}I_H + K_i\right)(\phi - h_i), (\phi - h_i)\right) + R_i(\phi) - \rho_{\kappa,B}(\phi) \\ &\ge \nu M_B^{-1}\|\phi - h_i\|_B^2 - \xi(\|h_i\|_B + \delta)\|\phi - h_i\|_B^{2+\varepsilon} - \kappa\|\phi - h_i\|_B^2 \\ &\ge \left(\nu M_B^{-1} - \xi(\|h_i\|_B + \delta)\delta^\varepsilon - \kappa\right)\|\phi - h_i\|_B^2. \end{aligned} \tag{5.3}$$

Also for $\phi \in H$ with $d_B(\phi) \ge \delta$ and $\|\phi\|_B \le R$, it holds that $U(\phi) \ge \theta(\delta, R)$. In addition to the assumption above on δ, assume that $\nu M_B^{-1} - \xi(\|h_i\|_B + \delta)\delta^\varepsilon > 0$. Then for κ satisfying $\kappa < \min\left(\nu M_B^{-1} - \xi(\|h_i\|_B + \delta)\delta^\varepsilon, \theta(\delta, R)\right)$, (a), (b) and (c) hold for $U - \rho_{\kappa,B}$.

(2) By Lemma 2.4, it suffices to prove the following Laplace asymptotic formula. ■

Lemma 5.2 *We assume the same assumptions in Lemma* 5.1 *for* V. *Let* κ *be a nonnegative number satisfying the assumptions in* (1) *in the lemma above. Then we have*

$$\lim_{\lambda\to\infty} \int_B e^{-2V_\lambda(\phi)+2\lambda\rho_{\kappa,B}(\lambda^{-1/2}\phi)} d\mu(\phi) = \sum_{i=1}^n \int_B \exp\Big[-2(K_i\phi,\phi) + 2\kappa\|\phi\|_B^2\Big] d\mu(\phi). \tag{5.4}$$

This was proved essentially in Theorem B in Schilder [25]. For more general results, we refer the readers to Ben Arous [7], Kusuoka and Stroock [20], [21]. We omit the proof.

Remark 5.3 First, we give a remark on the scaling of V. Let $\mu_\lambda(\cdot) = \mu(\sqrt{\lambda}\cdot)$. This is the Gaussian measure on B whose covariance operator is I_H/λ. By the unitary transformation $f(\phi) \to f(\sqrt{\lambda}\phi)$ from $L^2(B,\mu)$ onto $L^2(B,\mu_\lambda)$, $-L_{\lambda,V}$ is unitarily equivalent to $\lambda^{-1}(D^*_{\mu_\lambda}D + \lambda^2 V(\phi))$. Here $D^*_{\mu_\lambda}$ denotes the adjoint operator of D with respect to the inner product of $L^2(B, d\mu_\lambda)$. The scaling $\lambda V(\phi/\sqrt{\lambda})$ cannot be defined on path spaces over Riemannian manifolds but the probability measure corresponding to μ_λ exists and we can formulate the semiclassical problems. Now let us consider a rough lower bound estimate on $E_0(\lambda)$ on a path space over a Riemannian manifold. Let (M,g) be a complete Riemannian manifold whose Ricci curvature is bounded. Let μ_λ be the Brownian motion measure on $P_x(M) := C([0,1] \to M \mid \gamma(0) = x)$ such that

$$\begin{aligned}&\mu_\lambda\left(\gamma(t_1) \in dx_1, \ldots, \gamma(t_m) \in dx_m\right)\\ &\quad = \left(\prod_{i=1}^{m} p\left(\lambda^{-1}(t_i - t_{i-1}), x_{i-1}, x_i\right)\right) dx_1 \cdots dx_m,\end{aligned} \tag{5.5}$$

where $t_0 = 0$, $x_0 = x$, $p(t,x,y) = e^{t\Delta/2}(x,y)$ and Δ is the Laplace-Beltrami operator. Let us define the tangent space along γ by the Levi-Civita connection and denote the H-derivative by D. Let V be a continuous function on $P_x(M)$. Let $-L_\lambda := D^*_{\mu_\lambda}D$ and $-L_{\lambda,V} := \lambda^{-1}(-L_\lambda + \lambda^2 V)$. As already noted, this operator is unitarily equivalent to $-L + \lambda V(\lambda^{-1/2}\phi)$ on $L^2(B,\mu)$ when M is a Euclidean space. Let $E_0(\lambda) = \inf \sigma(-L_{\lambda,V})$. We refer the readers to [1] for the notations below.

Let $\mathfrak{F}_t$ be the augmented filtration of $\sigma(\gamma_s \mid 0 \le s \le t)$ and take a smooth function F on $P_x(M)$. Then, the following martingale representation holds.

$$E_{\mu_\lambda}\left[F \mid \mathfrak{F}_t\right] = E_{\mu_\lambda}\left[F\right] + \int_0^t \left(H(\gamma)_s, db(s)\right)_{T_xM}. \tag{5.6}$$

where $b(t) = \int_0^t \tau(\gamma)_s^{-1} \circ d\gamma(s)$ and $\tau(\gamma)_t$ denotes the stochastic parallel translation and

$$H(\gamma)_t := E\left[\dot{h}(\gamma)_t - \frac{1}{2\lambda}\int_t^1 \left(M^\lambda(\gamma)_t^{-1}\right)^* M^\lambda(\gamma)_s^* \overline{\mathrm{Ric}}(\gamma)_s \dot{h}(\gamma)_s ds \,\middle|\, \mathfrak{F}_t\right] \tag{5.7}$$

$$h(\gamma)_t := DF(\gamma)_t \tag{5.8}$$

$M^\lambda(\gamma)_t$ is the operator satisfying

$$\frac{d}{dt}M^\lambda(\gamma)_t = -\frac{1}{2\lambda}\overline{\mathrm{Ric}}(\gamma)_t M^\lambda(\gamma)_t \tag{5.9}$$

$$M(\gamma)_0 = I. \tag{5.10}$$

Note that $b(t)$ is a Brownian motion such that $E_{\mu_\lambda}[(b(t),b(s))_{T_xM}] = \frac{\dim M}{\lambda} \cdot t \wedge s$. (5.6) can be proven by the same method as in [8] by using the integration by parts formula on $(P_x(M), \mu_\lambda)$. Then, by the same proof of logarithmic Sobolev inequality as in [8], we have

$$E_{\mu_\lambda}\left[F^2 \log\left(F^2/\|F\|^2_{L^2(\mu_\lambda)}\right)\right] \le \frac{2}{\lambda}\left(1 + \frac{C_\lambda}{\lambda}\right) E_{\mu_\lambda}\left[|DF|_H^2\right]. \tag{5.11}$$

Here, C_λ is a positive number depending on the norm of Ric such that $\limsup_{\lambda\to\infty} C_\lambda < \infty$. By the estimate in Theorem 7 in [13] again, we have

$$E_0(\lambda) \geq -\frac{\lambda}{2(\lambda + C_\lambda)} \log \left(\int_{P_x(M)} e^{-2\lambda(1+\frac{C_\lambda}{\lambda})V(\gamma)} d\mu_\lambda(\gamma) \right). \tag{5.12}$$

Consequently, under suitable assumptions on V (e.g., $\min\left\{\frac{1}{4}\int_0^1 \|\dot{\gamma}(t)\|^2 dt + V(\gamma)\right\} = 0$, non-degeneracy of the Hessian at minimizers, etc), Laplace type asymptotics formulae [7],[20], [21] imply the boundedness of the right-hand side in (5.12) as $\lambda \to \infty$. However, note that it is not obvious to see that $\limsup_{\lambda\to\infty} E_0(\lambda) < \infty$ under the assumptions on V above which seems to be natural. The detailed study will appear in elsewhere.

6 Proof of the lower bound estimate

Now we prove the lower bound estimate.

Lemma 6.1 *Assume* (A1),(A2),(A3(ε)),(A4),(A5),(A6). *Then,*

$$\liminf_{\lambda\to\infty} E_0(\lambda) \geq \min_{1\leq i\leq n} \frac{1}{2} \mathrm{tr}\left(\sqrt{I_H + 4K_i} - I_H\right). \tag{6.1}$$

This lemma and Lemma 4.1 imply Theorem 2.6.

Proof of Lemma 6.1. Let $\chi(t)$ be a smooth nonnegative function such that $\chi(t) = 1$ for $|t| \leq 2$, $\chi(t) = 1 - \exp\left(-\frac{1}{t^2-4}\right)$ for $2 \leq t \leq 3$ and $\chi(t) = 0$ for $|t| \geq 4$. Also we assume $\chi'(t) \leq 0$ for $t \geq 0$. Take a positive number ε' such that $(2+\varepsilon)(1-\varepsilon') > 2$.

Let $J_i(\phi) = \chi\left(\lambda^{-\varepsilon'}\|\phi - k_i\|_B^2\right)$ and $J_0(\phi) = \left(1 - \sum_{i=1}^n J_i(\phi)^2\right)^{1/2}$ for sufficiently large λ. Here, $k_i = \sqrt{\lambda} h_i$. Take a smooth cylindrical function u with $\|u\|_{L^2(\mu)} = 1$. Then, by an elementary calculation, we have

$$\begin{aligned} \mathcal{E}_\lambda(u,u) &= \sum_{i=0}^n \mathcal{E}_\lambda(J_i u, J_i u) \\ &\quad - \sum_{i=0}^n \int_B |DJ_i(\phi)|_H^2 u(\phi)^2 d\mu(\phi) \end{aligned} \tag{6.2}$$

which is called the IMS localization formula [27]. We give estimates on each term.

$$|DJ_i(\phi)|_H^2 \leq C\chi'\left(\lambda^{-\varepsilon'}\|\phi - k_i\|_B^2\right)^2 \lambda^{-2\varepsilon'}\|\phi - k_i\|_B^2 \leq C\lambda^{-\varepsilon'}. \tag{6.3}$$

As for DJ_0, noting that there exists a positive constant C such that $\chi'(t)^2 \leq C(1 - \chi(t)^2)$ for any $t \in \mathbb{R}$,

$$|DJ_0(\phi)|_H^2 \leq C \cdot \sum_{i=1}^n \frac{J_i(\phi)^2 |DJ_i(\phi)|^2}{1 - \sum_{i=1}^n J_i(\phi)^2} \leq C'\lambda^{-\varepsilon'}. \tag{6.4}$$

Let $1 \leq i \leq n$. We use the notation in Lemma 3.2. By (3.8),

$$\begin{aligned}\mathcal{E}_\lambda(J_i u, J_i u) &\geq \lambda \int_B R_i\left(\lambda^{-1/2}\phi\right) J_i(\phi)^2 u^2(\phi) d\mu(\phi) \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\sqrt{I_H + 4K_i} - I_H\right) \int_B J_i(\phi)^2 u(\phi)^2 d\mu(\phi). \end{aligned} \tag{6.5}$$

If $J_i(\phi) \neq 0$, then by (A3(ε)), $R_i(\lambda^{-1/2}\phi) \leq C'\lambda^{-\frac{1}{2}(2+\varepsilon)(1-\varepsilon')}$. So we get

$$\lambda \left| \int_B R_i\left(\frac{\phi}{\sqrt{\lambda}}\right) J_i(\phi)^2 u^2(\phi) d\mu(\phi) \right| \leq C'\lambda \cdot \lambda^{-\frac{1}{2}(2+\varepsilon)(1-\varepsilon')}. \tag{6.6}$$

Thus, for $1 \leq i \leq n$, we obtain for some $0 < \delta < 1$,

$$\begin{aligned}\mathcal{E}_\lambda(J_i u, J_i u) &\geq \frac{1}{2}\mathrm{tr}\left(\sqrt{I_H + 4K_i} - I_H\right) \int_B J_i(\phi)^2 u(\phi)^2 d\mu(\phi) \\ &\quad - C'' \cdot \lambda^{-\delta}. \end{aligned} \tag{6.7}$$

Now we estimate $\mathcal{E}_\lambda(J_0 u, J_0 u)$. Let $\rho_{\kappa,B}$ be the function in Lemma 5.1 (2). Take a sufficiently large $C > 0$. Then by applying the rough lower bound (5.2),

$$\begin{aligned}\mathcal{E}_\lambda(J_0 u, J_0 u) &= \mathcal{E}(J_0 u, J_0 u) + \int_B \left(V_\lambda(\phi) - \lambda\rho_{\kappa,B}\left(\frac{\phi}{\sqrt{\lambda}}\right)\right) J_0(\phi)^2 u(\phi)^2 d\mu(\phi) \\ &\quad + \lambda \int_B \rho_{\kappa,B}\left(\frac{\phi}{\sqrt{\lambda}}\right) J_0(\phi)^2 u(\phi)^2 d\mu(\phi) \\ &\geq -C\int_B u(\phi)^2 J_0(\phi)^2 d\mu(\phi) + \lambda \int_B \rho_{\kappa,B}\left(\frac{\phi}{\sqrt{\lambda}}\right) J_0(\phi)^2 u(\phi)^2 d\mu(\phi) \\ &:= I_1 + I_2. \end{aligned} \tag{6.8}$$

We estimate the second term. Noting when $J_0 \neq 0$, $\rho_{\kappa,B}(\lambda^{-1/2}\phi) \geq C'\lambda^{-(1-\varepsilon')}$ for sufficiently large λ. Therefore, we have

$$I_2 \geq C'\lambda^{\varepsilon'} \int_B u(\phi)^2 J_0(\phi)^2 d\mu(\phi) \tag{6.9}$$

Hence

$$\mathcal{E}_\lambda(J_0 u, J_0 u) \geq (C'\lambda^{\varepsilon'} - C) \int_B J_0(\phi)^2 u(\phi)^2 d\mu(\phi). \tag{6.10}$$

Consequently, combining all the above and $\sum_{i=0}^n J_i(\phi)^2 = 1$, we obtain

$$\mathcal{E}_\lambda(u, u) \geq \frac{1}{2} \min_{1\leq i\leq n} \mathrm{tr}\left(\sqrt{I_H + 4K_i} - I_H\right) - C\lambda^{-\delta}, \tag{6.11}$$

where $0 < \delta < 1$. This completes the proof. ■

Corollary 6.2 *Assume that V_0 is a C^3 function on B satisfying that there exist $a, b \geq 0$ such that*

$$\sup_{\|\phi\|_B \leq R} \|D^3 V_0(\phi)\|_{L(B\times B\times B)} < \infty \qquad \textit{for all } R > 0, \tag{6.12}$$

$$V_0(\phi) \geq -a\|\phi\|_B - b \qquad \textit{for all } \phi \in B. \tag{6.13}$$

Then we have the following.
(1) *Let* $U_0(\phi) = \frac{1}{4}\|\phi\|_H^2 + V_0(\phi)$. *Then the minimizers of* $U_0(\phi)$ *exist.*
(2) *Let* $V(\phi) = V_0(\phi) - \min U_0(\phi)$. *For this* V, (A6) *holds for all* $\alpha > 0$.
(3) *Further assume* (A1), (A2), (A5) *for* V *in* (2). *Then* (2.13) *holds for* $-L_{\lambda,V}$.

Proof. Let $m := \inf U_0(\phi)$. Take a sequence $\{\phi_n\} \subset H$ such that $\lim_{n\to\infty} U_0(\phi_n) = m$. Since the norm of H is stronger than that of B, $\lim_{\|\phi\|_H\to\infty} U_0(\phi) = \infty$. So we may assume that $\sup_n \|\phi_n\|_H < \infty$. Then there exists a subsequence $\{\phi_{n(k)}\}_{k=1}^\infty$ such that $\phi_{n(k)} \to \phi_\infty \in H$ weakly and strongly in B. Then by the continuity of V, we have $U_0(\phi_\infty) = m$. This implies (1). We prove (2). By [22], for H-Lipschitz continuous function F with $|F(\phi+h) - F(\phi)| \le C\|h\|_H$ for all $\phi \in B, h \in H$ and $E[F] = 0$, we have $\int_B e^{\alpha F} d\mu \le e^{\frac{C^2\alpha^2}{2}}$. Take a positive number M_B such that $\|\phi\|_B \le M_B\|\phi\|_H$ for all $\phi \in H$. Let $\alpha > 0$. By the assumption, $-\alpha V_\lambda(\phi) \le a\alpha\sqrt{\lambda}\|\phi\|_B + (b+m)\alpha\lambda$. Therefore

$$E[e^{-\alpha V_\lambda(\phi)}] \le \exp\left[\left\{\frac{(aM_B\alpha)^2}{2}\lambda + \alpha\sqrt{\lambda}\Big(aE[\|\phi\|_B] + \sqrt{\lambda}(b+m)\Big)\right\}\right] \tag{6.14}$$

which proves (A6) for all α. We prove (3). By the Taylor expansion, we see that (A3(1)) holds. We prove (A4). Set $D_\delta = \{\phi \in H \mid d_B(\phi) \ge \delta\}$. Assume that there exists a sequence $\{\phi_n\}_{n=1}^\infty \subset D_\delta$ such that $\lim_{n\to\infty} U(\phi_n) = 0$. Then $\sup_n \|\phi_n\|_H < \infty$. So by the same argument as in (1), a subsequence $\{\phi_{n(k)}\}_{k=1}^\infty$ converges to $\phi_\infty \in H$ weakly and strongly in B and $U(\phi_\infty) = 0$. But $\phi_\infty \in D_\delta$. This is a contradiction. Consequently, our main theorem implies the conclusion. ■

7 Examples

Let us consider a classical Wiener space. That is, the Cameron-Martin subspace H is the space of H^1-path $\phi = \{\phi(t)\}_{0\le t\le 1}$ with values in $\mathbb{R}$ starting at 0. The norm is given by $\|\phi\|_H^2 = \int_0^1 \dot{\phi}(t)^2 dt$. Let B be the Banach space consisting of continuous functions on $[0,1]$ with $\phi(0) = 0$. The norm is given by $\|\phi\|_B = \sup_{0\le t\le 1} |\phi(t)|$. Now let us introduce a potential function V. Take a C^3-function W on $\mathbb{R}$ satisfying that there exist positive constants C_i such that $-C_1|x| - C_2 \le W(x) \le C_3|x|^{C_4} + C_5$ for all $x \in \mathbb{R}$. Define

$$V_0(\phi) = \int_0^1 W(\phi(t))dt, \tag{7.1}$$

$$U_0(\phi) = \frac{1}{4}\|\phi\|_H^2 + V_0(\phi) \tag{7.2}$$

Note that V_0 is a C^3 function on B.

Lemma 7.1 (1) V_0 *satisfies* (6.12) *and* (6.13).
(2) *Set* $m = \min U_0(\phi)$, $V(\phi) = V_0(\phi) - m$ *and* $U(\phi) = \frac{1}{4}\|\phi\|_H^2 + V(\phi)$. *If* (A1) *and* (A2) *hold for this* V, *then* (2.13) *holds for* $-L_{\lambda,V}$.

Proof. (1) This is obvious.
(2) This is an immediate consequence of Corollary 6.2. ■

It is well-known that (A1) and (A2) holds for U defined by $W(x) = a(x^2-1)^2$ when a is sufficiently large. In this case, the minimizers are two points set $\{\phi_0, -\phi_0\}$ and $\phi_0(t) > 0$

holds for all $t > 0$. We give a self-contained proof of it below for slightly more general potential functions. The decomposition formula (7.11) seems to be new and the fact that the minimizers are two point set naturally follows from the formula. However, of course, the conclusion holds for more general potential functions. See Remark 7.3. In these symmetric cases, it might be an interesting problem to consider the tunneling phenomena.

Theorem 7.2 *Suppose that W satisfies the following.*
(H1) *There exists a nonnegative C^3 function $f(x)$ on $\mathbb{R}$ such that $W(x) = f(x^2)$.*
(H2) *There exists a unique $x_0 > 0$ such that $f(x_0) = 0$.*
(H3) $\inf_x f''(x) > 0$.
(H4) *There exists $C > 0$ and $p \in \mathbb{N}$ such that*

$$|f(x)| \le C(1 + |x|)^p. \tag{7.3}$$

Then, the following hold.

(1) *Let $U^a(\phi) = \frac{1}{4}\|\phi\|_H^2 + aV(\phi)$. Then for sufficiently large a, the minimizers of U^a is a two point set $\{\pm\phi_0\}$.*

(2) *For sufficiently large a,* (A1), (A2) *holds for the potential function $V^a = a \cdot V - \min_\phi U^a(\phi)$.*

(3) *For sufficiently large a,* (2.13) *holds for $-L_{\lambda,V^a}$.*

Remark 7.3 (A1) and (A2) hold for W satisfying that
(C1) $W \in C^2(\mathbb{R})$ and $W(x) = W(-x)$ for $x \in \mathbb{R}$,
(C2) $W(x) > 0$ for $x \neq \pm x_0$ and $W(\pm x_0) = 0$,
(C3) $W''(\pm x_0) > 0$.

The above remark is due to Professor Kazunaga Tanaka. As for potential functions in Theorem 7.2, we can give different proof of Theorem 2.6 without using the rough lower bound estimate in Lemma 5.1 since we have explicit expression (7.11). But the proof cannot work for the potential functions satisfying (C1), (C2) and (C3) in which case the expression like (7.11) probably does not hold.

In the rest of this section, we prove Theorem 7.2.

Lemma 7.4 *Let $U^a(\phi)$ be the function in Theorem* 7.2. *For sufficiently large a, $U^a(\phi)$ has at least two minimizers $\phi_0(\cdot)$ and $-\phi_0(\cdot)$.*

Proof. Let $l(t) = t\sqrt{x_0}$. Then

$$U^a(l) = \frac{1}{4}x_0 + \frac{a}{\sqrt{x_0}}\int_0^{\sqrt{x_0}} W(u)du.$$

On the one hand, $U^a(0) = aW(0)$. By the assumption on f, $W(x)$ $(-\sqrt{x_0} \le x \le \sqrt{x_0})$ has maximum at 0. So for sufficiently large a, $U^a(l) < U^a(0)$. So 0 is not the minimizer. Since $U^a(\phi) = U^a(-\phi)$, there exist at least two minimizers. ■

Lemma 7.5 (1) *For any C^2 functions $\phi, k \in H$,*

$$DU_0(\phi)(k) \;=\; \frac{1}{2}\dot{\phi}(1)k(1) + \int_0^1 \left(-\frac{1}{2}\ddot{\phi}(t) + W'(\phi(t))\right) k(t)dt. \tag{7.4}$$

(2) *For any* $\phi, k_1, k_2 \in H$*,*

$$D^2U_0(\phi)(k_1,k_2) = \frac{1}{2}\int_0^1 \dot{k}_1(t)\dot{k}_2(t)dt + \int_0^1 W''(\phi(t))k_1(t)k_2(t)dt. \tag{7.5}$$

(3) *Each critical point* ϕ *of* U_0 *is a* C^2 *function and satisfies that*

$$-\frac{1}{2}\ddot{\phi}(t) + q(\phi(t))\phi(t) = 0 \tag{7.6}$$

with Dirichlet-Neumann boundary condition $\phi(0) = \dot{\phi}(1) = 0$*. Here* $q(x) = W'(x)/x$*.*
(4) *Assume that nonzero minimizer exists. Then there exists a* C^2 *minimizer such that* $h(t) > 0$ *for all* $t > 0$*. (We call this minimizer a positive minimizer.)*
(5) *Assume the existence of a positive* C^2 *minimizer* h*. Let* $-H_q = -\frac{1}{2}\frac{d^2}{dt^2} + q(h(t))$ *be the Schrödinger operator with Dirichlet-Neumann boundary condition* $\phi(0) = \dot{\phi}(1) = 0$*. Then* $\inf\sigma(-H_q) = 0$ *and* 0 *is the simple eigenvalue and the eigenfunction is* h*.*

Proof. (1) and (2) is proved by the standard calculation.
(3) This follows from (1) and the elliptic regularity of the Laplacian.
(4) Suppose that ϕ is a minimizer. Then $\phi_1(\cdot) = |\phi(\cdot)| \in H$ is also a minimizer. So (7.6) holds in distribution sense. So by the elliptic regularity of the Laplacian, ϕ_1 is a C^2 function and (7.6) holds in classical sense. By the maximum principle, for any $t > 0$, $\phi_1(t) > 0$.
(5) This follows from the fact that the eigenfunction corresponding to the lowest eigenvalue is almost nonnegative or nonpositive.
■

By using Lemma 7.5 (2), we have

Lemma 7.6 *Let* h *be a critical point of* U_0*. Then the Hessian is given by*

$$\frac{1}{2}D^2U_0(h) = \frac{1}{4}I_H + K_h,$$

where K_h *is the trace class operator such that*

$$(K_h\phi)(t) = \frac{1}{2}\int_0^t \left(\int_s^1 W''(h(u))\phi(u)du.\right) ds. \tag{7.7}$$

Remark 7.7 Note that if h is a critical point, then $-h$ is also a critical point and the Hessian at $-h$ is also $\frac{1}{4}I_H + K_h$.

We calculate the error term when $U_0(\phi) - U_0(h)(= U_0(\phi) - U_0(-h))$ is approximated by the quadratic Taylor expansion.

Lemma 7.8 *Let* h *be a critical point of* U_0*.*
(1) *Let* R_h *be the function such that*

$$R_h(\phi) = V_0(\phi) - V_0(h) - DV_0(h)(\phi - h) - \left(\frac{1}{2}D^2V_0(h)(\phi - h), (\phi - h)\right). \tag{7.8}$$

Then we have

$$R_h(\phi) = \int_0^1 \frac{1}{2} f''\left(h(t)^2\right)(\phi(t)-h(t))^3(\phi(t)+3h(t))dt$$
$$+\int_0^1 dt \int_0^1 ds \int_0^s du \int_0^u f'''\left(h(t)^2+\tau(\phi(t)^2-h(t)^2)\right) d\tau \left(\phi(t)^2-h(t)^2\right)^3 . \quad (7.9)$$

(2) *Let*

$$(T_h\phi)(t) = \frac{1}{2}\int_0^t \left(\int_s^1 q(h(u))\phi(u)du\right) ds. \quad (7.10)$$

Then

$$U(\phi) - U(h) = \left(\left(\frac{1}{4}I_H + T_h\right)(\phi-h),(\phi-h)\right)_H + \tilde{R}_h(\phi), \quad (7.11)$$

where

$$\tilde{R}_h(\phi) = \int_0^1 \left(\int_0^1 ds \int_0^s f''\left(h(t)^2+u(\phi(t)^2-h(t)^2)\right) du\right)\left(\phi(t)^2-h(t)^2\right)^2 dt. \quad (7.12)$$

Proof. (1) We have

$$\begin{aligned}
R_h(\phi) &= \int_0^1 \Big\{W(\phi(t)) - W(h(t)) - W'(h(t))(\phi(t)-h(t)) \\
&\quad -\frac{1}{2}W''(h(t))(\phi(t)-h(t))^2\Big\}dt \\
&= \int_0^1 \Big\{f(\phi(t)^2) - f(h(t)^2) - 2h(t)f'(h(t)^2)(\phi(t)-h(t)) \\
&\quad -f'(h(t)^2)(\phi(t)-h(t))^2 \\
&\quad -2h(t)^2 f''(h(t)^2)(\phi(t)-h(t))^2\Big\}dt \\
&= \int_0^1 \left(f(\phi(t)^2) - f(h(t)^2) - f'(h(t)^2)(\phi(t)^2-h(t)^2) - \frac{1}{2}f''(h(t)^2)(\phi(t)^2-h(t)^2)^2\right) dt \\
&\quad + \int_0^1 \frac{f''(h(t)^2)}{2}(\phi(t)-h(t))^3(\phi(t)+3h(t))dt. \qquad (7.13)
\end{aligned}$$

By using the Taylor expansion, we complete the proof.

(2) Using (2.6), we have

$$
\begin{aligned}
&U(\phi)-U(h)\\
&\qquad=\frac{1}{2}D^2U_0(h)\left((\phi-h),(\phi-h)\right)\\
&\qquad+V_0(\phi)-\left\{V_0(h)+DV_0(h)(\phi-h)+\frac{1}{2}D^2V_0(h)\left((\phi-h),(\phi-h)\right)\right\}\\
&=\quad\left(\left(\frac{1}{4}I_H+T_h\right)(\phi-h),(\phi-h)\right)_H\\
&\qquad+\int_0^1\left(W(\phi(t))-W(h(t))-W'(h(t))(\phi(t)-h(t))-\frac{1}{2}q(h(t))(\phi(t)-h(t))^2\right)dt\\
&=\quad\left(\left(\frac{1}{4}I_H+T_h\right)(\phi-h),(\phi-h)\right)_H\\
&\qquad+\int_0^1\left\{f(\phi(t)^2)-f(h(t)^2)-f'(h(t)^2)(\phi(t)^2-h(t)^2)\right\}dt \qquad (7.14)
\end{aligned}
$$

So by the Taylor's theorem, we complete the proof. ■

T_h and K_h have the following properties.

Lemma 7.9 (1) *T_h is a trace class self-adjoint operator.*
(2) *Assume h is a positive minimizer of U_0. Then $-\frac{1}{4}$ is the lowest simple eigenvalue of T_h and h is the corresponding eigenfunction.*
(3) *Assume h is a positive minimizer. Then $\frac{1}{4}I_H+K_h$ is a strictly positive self-adjoint operator.*
(4) *Assume the existence of nonzero minimizer of U_0. Then positive minimizer h is unique and the set of minimizers is $\{h,-h\}$.*

Proof. (1) This is a standard fact.

(2) Suppose $T_h\phi=\xi\phi$. Then ϕ is a C^2 function and satisfies that $\xi\ddot{\phi}(t)+\frac{1}{2}q(h(t))\phi(t)=0$. So

$$
-\xi\|\phi\|_H^2+\frac{1}{2}\int_0^1 q(h(t))\phi(t)^2dt=\frac{1}{2}\left(-H_q\phi,\phi\right)_{L^2}-\left(\frac{1}{4}+\xi\right)\|\phi\|_H^2=0. \qquad (7.15)
$$

Since the Schrödinger operator $-H_q$ is nonnegative operator, $\xi\ge -1/4$ is valid. Also if $\xi=-1/4$, then Lemma 7.5 implies $\phi=c\cdot h$.

(3) Noting $W''(x)-q(x)=4x^2f''(x^2)$, we see that

$$
\frac{1}{4}I_H+K_h=\frac{1}{4}I_H+T_h+T_1, \qquad (7.16)
$$

where $(T_1\phi_1,\phi_2)_H=2\int_0^1 h(t)^2f''(h(t)^2)\phi_1(t)\phi_2(t)dt$. Let $\inf\sigma((\frac{1}{4}I_H+T_h)|_{\{h\}^\perp})=\kappa$. Then by (2), $\kappa>0$. Take ψ such that $(\psi,h)=0$ with $\|\psi\|_H=1$ and set $\phi=xh+y\psi$. By the nonnegativity of T_1, we have

$$
\left(\left(\frac{1}{4}I_H+K_h\right)\phi,\phi\right)_H\ge\max\left\{\kappa y^2,Ax^2-2B|xy|\right\}.
$$

Here $A=(T_1h,h)_H>0$ and $B=|(T_1h,\psi)|$. Since $\max\left\{\kappa y^2,Ax^2-2B|xy|\mid x^2+y^2=1\right\}>0$, the proof is completed.

(4) Now the uniqueness is obvious by the expression (7.11), (7.12), $\inf f''(x) > 0$ and $\frac{1}{4}I_H + T_h \geq 0$. ■

Proof of Theorem 7.2. This follows from the lemma above immediately. ■

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